Conditions for Equivalence between the Compositional Rule of Inference and the Compatibility Modification Inference

José Villar
Instituto de Investigación Tecnológica,
Universidad Pontificia Comillas,
Alberto Aguilera 23 28015 Madrid Spain
jose@iit.upco.es

Abstract

This paper shows that the compositional rule of inference reduces to the compatibility modification inference, when the antecedent of a fuzzy rule and its input fulfil some properties. Two main types of implications are investigated, those generalising the classical material conditional (residuated and strong implications), called in this paper m-implications, and those generalising the classical Cartesian product (t-norms and pseudo-conjunctions), called t-implications. For m-implications, when their maximum modus ponens generating function is used, the compatibility measure depends obviously on the relationship between input and antecedent, but also on the t-norm the implication comes from. Similarly, for t-implications, also used as their own modus ponens generating function, the compatibility measure depends again on the input and antecedent, but also on the t-norm they come from. As it could be expected, in the first case the compatibility measure is a degree of the inclusion of the input into the antecedent, while in the second one it is a degree of the intersection of both.

1. Introduction

Let P and Q be two fuzzy sets, corresponding to the possibility distributions of two variables u and v, defined over the universes U and V respectively. In this paper we will use $P(u)$, μ_P or P to denote the membership function of a fuzzy set P. Given a rule $P \rightarrow Q$, and given an input (or observation) P', several ways have been proposed to calculate the consequence Q' of the rule. Some of the most important are the well known compositional rule of inference (CRI for short) introduced in [1], and the compatibility modification inference (CMI for short).

CMI, [2] and [3], called approximate analogical reasoning in [4] and plausible reasoning in others works, is an inference mechanism based on the calculation of a measure of compatibility or similarity $S(P',P)$ between the input P' and the antecedent P of the rule, and the use of an implication function I to modify the consequent Q, according to the value of $S(P',P)$. The consequence is then calculated as:

$$Q'(v)=I(S(P'(u),P(u)),Q(v))$$

In [5] CMI using Lukasiewicz implication with a necessity measure is analysed and justified from an axiomatic point of view.

On the other hand, under the CRI the consequence is calculated by means of an implication function I and a modus ponens generating function (*MPGF* for short) M (see [6] and [7]), and the consequent Q' is given by:

$$Q'(v)=\sup_u M(\, P'(u),\, I(P(u),Q(v))\,)$$

Two main types of implications can be distinguished. First of all are those generalising the classical material conditional, that is strong and residuated implications (see [6] or [7]), which are usually well accepted from a strict logical point of view. They will be called m-implications, m standing for material. However, from a computational point of view, other implications have been extensively used in numerical applications (for example in the control field). These implications share the property of generalising the classical Cartesian product, and the main families are the t-norms and the pseudo-conjunctions ([8]). Both of them are the maximum *MPGF* of residuated and strong implications respectively, and will be called t-implications in this paper, t standing for t-norm, since these ones are the

0-7803-3645-3/96 $5.00©1996 IEEE

most commonly used. We will restrict ourselves to study these four families of implications.

2. Fuzzy implications and the compositional rule of inference

We define a fuzzy conditional (conditional for short) as fuzzy relation $R(x,y):X\times Y\to[0,1]$ verifying the following properties:

$$\begin{cases} R: X\times Y\to[0,1] \\ R(1,1)=1 \\ R(1,0)=0 \\ R(x,y) \text{ non decreasing with y} \end{cases}$$

Fuzzy conditionals are generated from fuzzy implications as $R(u,v)=I(P(u),Q(v))$. When the compositional rule of inference is used, a modus ponens generating function M need to be defined, to combine the certainties of the implication and the observation, and benefit from the following properties:

$$M(0,1)=M(1,0)=0$$
$$M(1,1)=1$$
$M(a,b)$ non decreasing with respect to a and b.

If in addition it is:

$$M(x,R(x,y))\le y \text{ (modus ponens inequality)}$$
$$M(1,y)=y \quad\text{and}\quad R(1,y)=y.$$

then the CRI verify the generalised modus ponens (that is if $P'=P$ then $Q'=Q$).

The implications considered in this paper are:

-residuated: $I(x,y)=\sup\{c\in[0,1]/\,T(x,c)\le y\}$
-strong: $I(x,y)=S(1-x,y)=1-T(x,1-y)$
-t-norm: $I(x,y)=T(x,y)$
-pseudo-conjunction: $I(x,y)=\inf\{c\in[0,1]\,/S(1-x,c)\ge y\}$
$=\inf\{c\in[0,1]\,/\,1-T(x,1-c)\ge y\}$

As it can be seen, all of them can be generated from a t-norm. In the sequel we will restrict ourselves to the set of continuous t-norms.

It has been proved in [9] that the CRI can be formulated in an equivalent way by means of fuzzy truth values, as follows:

$$Q'(v)=\tau_{QQ'}(Q(v))=\sup_x\{M(\tau_{PP'}(x),I(x,Q(v)))\}$$

where:

$$\tau_{pp'}(x)=\begin{cases}\sup\{P'(u)\,/\,u\in P^{-1}(\{x\})\} \text{ if } P^{-1}(\{x\})\neq 0\\ 0 \text{ otherwise}\end{cases}$$
$$=\begin{cases}\sup_{u/P(u)=x} P'(u)\\ 0 \text{ if } P^{-1}(x)=0\end{cases}=P(P')$$

which is also called fuzzy compatibility of P with P'.

Figure 1 shows two possible examples of fuzzy compatibilities between an input P' and an antecedent P:

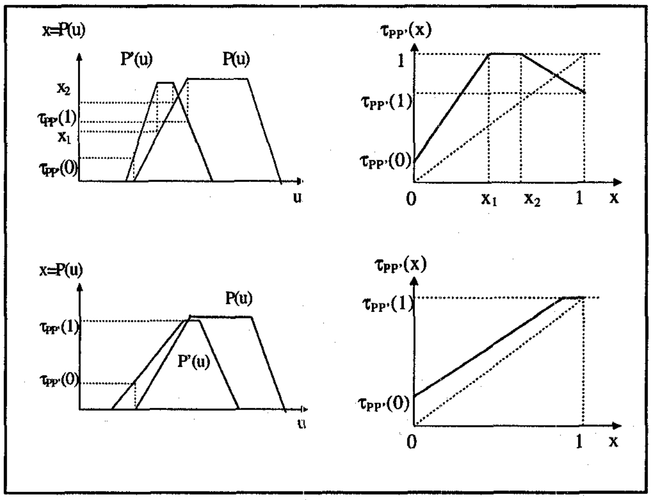

Figure 1. Fuzzy compatibilities

It is interesting to note:

1. if the support of P' is not included in the support of P then $\tau_{PP'}(0)>0$.
2. if P' is normalised then so is $\tau_{PP'}(x)$
3. there is a clear relationship between the slopes of P' and P and the slope of $\tau_{PP'}(x)$

In this paper we will use the fuzzy truth values formulation to present and prove the main results. We will also suppose that all the possibility distributions are normalised.

3. CRI using fuzzy truth values and CMI

When the input is the singleton $P'=\{u_0\}$, the compositional rule of inference reduces to the following inference pattern:

$$Q'(v)=I(\alpha,Q(v))$$

or in terms of fuzzy truth values:

$$\tau_{QQ'}(y)=I(\alpha,y)$$

where $\alpha=P(u_0)$. In this case it is clear that the CRI reduces to the CMI with compatibility measure α.

3.1 m-implications

3.1.1 Residuated implications:

Let I be a residuated implication from a T t-norm, used as *MPGF*. Since I is T-transitive ([10]), then:

$$sup_aT(I(a,b),I(b,c)) \leq I(a,b)$$
$$\Rightarrow sup_xT(I(\alpha,x),I(x,y)) \leq I(\alpha,y)$$

Let $\tau(x)$ be a fuzzy truth value such that $\tau(x)\leq I(\alpha,x)$, and such that $\tau(\alpha)=1$, that is $\tau(\alpha)\geq \mu_{\{\alpha\}}$. Then:

$$sup_xT(\tau(x),I(x,y)) \leq I(\alpha,y)$$

and for $x=\alpha$ the supreme is reached, that is:

$$\tau_{QQ'}(y)=I(\alpha,y)$$

Thus if the input is a truth value verifying $\mu_{\{\alpha\}} \leq \tau(x) \leq I(\alpha,x)$ then the consequence of the rule is given by:

$$\tau_{QQ'}(y)=I(\alpha,y)$$

which reduces to the CMI for a wide range of inputs. Since $\tau(x)$ can be interpreted as the fuzzy compatibility of P given P', it easy to see that α is given by:

$$\alpha = \inf_{x/P'(x)=1} P(x)$$

which can be seen as a pessimistic membership degree of the core of P' in P.

3.1.2 Strong implications:

Let I be a strong implication generated from a t-conorm S (with dual t-norm T) and M its maximum *MPGF* (which is in general a pseudo-conjunction, [6] or [7]). It is well known that if T is an Archimedean t-norm, the following equalities hold:

$$S(x,y)=g^{[-1]}(g(x)+g(y))$$
$$M(x,y)=g^{[-1]}(g(y)-g(1-x))$$
$$I(x,y)=S(1-x,y)=g^{[-1]}(g(1-x)-g(y))$$

where $g(x)$ is an increasing function defined from $[0,+\infty[$ to $[0,1]$ with $g(0)=1$. Using these expressions and given an input $\tau(x)$, the modus ponens inequality, for a conclusion given by $\tau_{QQ'}(z)=I(\alpha,\tau(z))$, becomes:

$$g^{[-1]}(\ g(1-x)+g(y)-g(1-\tau(x))\ ) \leq g^{[-1]}(g(1-\alpha)+g(y))$$

and solving for $\tau(x)$ it is:

$$\tau(x) \leq 1-g^{[-1]}(g(1-x)-g(1-\alpha))=1-M(\alpha,1-x)=I_R^T(\alpha,x)$$

where I_R^T is the r-implication residuated from the t-norm T, dual of S. If in addition $\tau(\alpha)=1$ the supreme is reached. Thus if the input is a truth value verifying $\mu_{\{\alpha\}} \leq \tau(x) \leq I_R^T(\alpha,x)$ then the consequence of the rule is given by:

$$\tau_{QQ'}(y)=I(\alpha,y)$$

which reduces again to the CMI for a wide range of inputs, and again the compatibility measure is given by:

$$\alpha = \inf_{x/P'(x)=1} P(x)$$

When T is the *min* operator, the MP inequality is given by:

$$min(\tau(x),max(1-x,y)) \leq max(1-\alpha,y)$$

and it can be checked that this inequality holds for $\tau(x) \leq I_R^{min}(\alpha,x)$, where:

$$I_R^{min}(\alpha,x) = \begin{cases} 1 & \alpha \leq x \\ & x \end{cases}$$

and if $\tau(\alpha)=1$ the supreme is reached, that is for $\mu_{\{\alpha\}} \leq \tau(x) \leq I_R^{min}(\alpha,x)$ the conclusion is $\tau_{QQ'}(y)=I(\alpha,\tau(y))$ and the CRI reduces again to the CMI.

3.1.3 Remarks:

The compatibility measure obtained for this kind of implications is a pessimistic degree of inclusion of the core of P' into P. Since $\tau(x)$ can be understood as given by:

$$\tau_{PP'}(x) = \begin{cases} \sup_{x/P(u)=x} P'(u) \\ 0 \text{ if } P^{-1}(x)=0 \end{cases} = P(P')$$

the conditions reached for $\tau(x)$ give the relationship P' and P must fulfil for the CRI to behave as the CMI. It is interesting to note that for both strong and residuated implications, the conditions that $\tau(x)$ must verify are the same, that is $\mu_{\{\alpha\}} \leq \tau(x) \leq I_R^T(\alpha,x)$, although the conclusions obtained are different. The following figures represent $I_R^T(\alpha,x)$ for some of the most common m-conditionals:

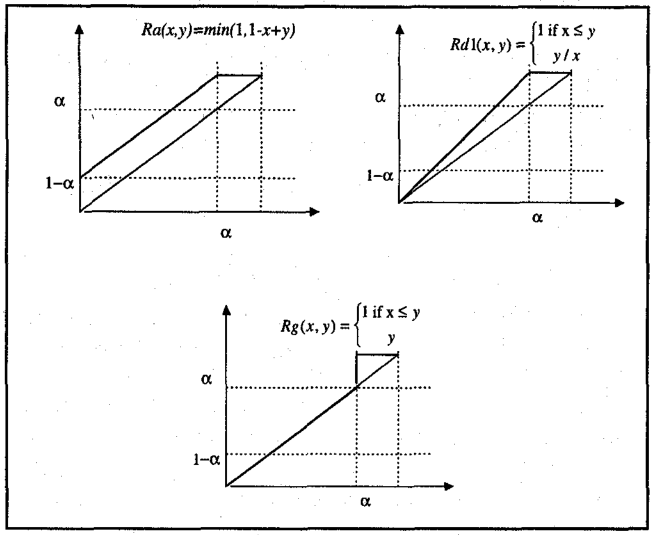

Figure 2. $I_R^T(\alpha,x)$ for the most common m-conditionals

It should also be noted that if the absolute values of the slopes of *P'* are greater or equal than that of *P*, it is always possible to find a truth value generated from the Lukasiewicz implication such that the above conditions are verified, with $\alpha \neq 0$ and $\tau(x)$ not identically equal to 1. In these cases the CRI with Lukasiewicz implication reduces to the CMI and a level of indetermination in the input *P'*, less or equal than $1\text{-}\alpha$, does not modify the conclusion. If *P'* is a more or less accurate measurement, the slopes of *P'* should have in general a big absolute value. However nothing can be said in general about *P*, whose slopes depend on the vagueness of the concept being modelled.

On the other side, for the rest of residuated and strong implications generated from positive t-norms, since $I_R^T(\alpha,0)=0$, the CRI can only reduce to the CMI with $\alpha \neq 0$ when the support of *P'* is included in the support of *P*, which not only depends on the shapes of the membership functions of *P* and *P'*. This means that the equivalence holds only for particular cases. It could be very interesting to investigate if similar even if not equivalent results can be reached for these implications, under both inference mechanisms, when the compatibility measure of the CMI is $\alpha = \inf_{x/P'(x)=1} P(x)$. In fact with residuated implications generated from positive t-norms, when the support of *P'* is not included in the support of *P*, an indetermination level appears in the conclusion when the CRI is used. However the conclusion obtained from the CMI with the same implications does not present any indetermination level. When using strong implications under the same hypothesis it can be shown that the conclusion with the CRI is unknown (that is $Q'(v)=V$) while using the CMI produces known conclusions with an indetermination level.

3.2 t-implications:

3.2.1 t-norms:

Let *T* be a t-norm implication and *M=T* its *MPGF*. For an input $\tau(x)$ the output is given by:

$$sup_x M(\tau(x),T(x,y))$$

The MP inequation, being *T* a continuous t-norm with additive generator *f*, is given by:

$$f^{[-1]}(\,f(\tau(x))+f(f^{[-1]}(f(x)+f(y)))\,) \le f^{[-1]}(f(\alpha)+f(y))$$

and solving for $\tau(x)$ it is:

$$\tau(x) \le f^{[-1]}(f(\alpha)\text{-}f(x))=I(x,\alpha)$$

If $\tau(\alpha)=1$ then the supreme is reached and thus if $\mu_{[\alpha]} \le \tau(x) \le I(x,\alpha)$ the CRI reduces to the CMI with compatibility measure given by:

$$\alpha=sup_x T(\tau(x),x)$$

since $\alpha=sup_x T(\mu_{[\alpha]},x) \le sup_x T(\tau(x),x) \le sup_x T(I(x,\alpha),x) \le \alpha$. This measure can also be calculated as $\alpha=sup\{x\,/\,\tau(x)=1\}$ due to the fact that all the sets are normalised.

A more general result can be reached, taking into account the associativity of t-norms:

$$\tau_{QQ'}(y)=T(\,sup_x(T(\tau(x),x)\,,\,y\,)$$

which holds for any input $\tau(x)$. Notice that it is always possible to find α such that $\tau(x) \le I(x,\alpha)$ (this inequality must hold if the MP inequality holds) but not such that $\mu_{[\alpha]} \le \tau(x) \le I(x,\alpha)$.

3.2.2 Pseudo-conjunctions:

When the pseudo-conjunction comes from a continuous t-norm it is:

$$M(x,y)=g^{[-1]}(g(y)\text{-}g(1\text{-}x))$$

where the function *g* has already been defined. Using *M* as *MPGF*, which is in general a non associative operator, for an input $\tau(x)$ the output is given by:

$$sup_x M(\tau(x),M(x,y))=sup_x\, g^{[-1]}(\, g(M(x,y))-g(1-\tau(x))\,)$$

Solving the MP inequality:

$$g^{[-1]}(\, g(g^{[-1]}(g(y)-g(1-x)))-g(1-\tau(x))\,) \leq g^{[-1]}(g(y)-g(1-\alpha))$$

we obtain:

$$\tau(x) \leq 1-g^{[-1]}(g(1-\alpha)-g(1-x))=1-M(x,1-\alpha)=I_R^T(x,\alpha)$$

If $\tau(\alpha)=1$ then the supreme is reached for $x=\alpha$, $M(\tau(\alpha),M(\alpha,y))=M(1,M(\alpha,y))=M(\alpha,y)$. And then if $\mu_{[\alpha]} \leq \tau(x) \leq I(x,\alpha)$ the CRI reduces to the CMI with compatibility measure given by:

$$\alpha=sup_x T(\tau(x),x)=sup\{\, x \,/\, \tau(x)=1\,\}$$

3.2.3 Remarks:

For this type of implications the compatibility measure obtained is a degree of intersection between P' and the core of P. On the other hand if $\tau(x)$ verify $\mu_{[\alpha]} \leq \tau(x) \leq I_R^T(x,\alpha)$ then CRI reduces to MCI, but the lower bound of $\tau(x)$ is only a sufficient condition. For t-norms it can easily be proved that it is not a necessary condition, and it would be interesting to analyse with more detail the case of pseudo-conjunctions.

The following figures represent $I_R^T(x,\alpha)$ for the most common t-conditionals:

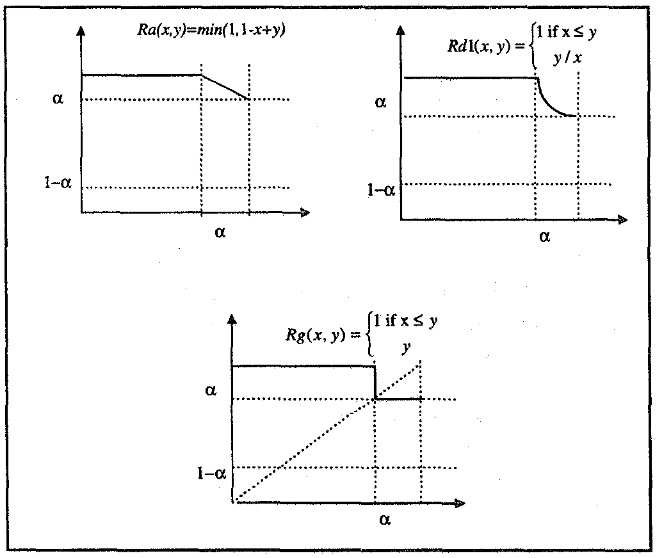

Figure 3: $I_R^T(x,\alpha)$ for the most common t-conditionals

It should be noted that it is always possible to find a truth value $I_R^T(x,\alpha)$ such that $\tau(x) \leq I_R^T(x,\alpha)$ with $\alpha \neq 0$, since $I_R^T(x,1)=1$, but this is not true with $\mu_{[\alpha]} \leq \tau(x) \leq I_R^T(x,\alpha)$. However it is possible to find membership functions for P' and P, such that the required conditions hold, since it is again related with the relatives slopes of both membership functions.

4. Conclusions

When the selected implication is a t-norm or pseudo-conjunction, it has been proved that the CRI reduces to the CMI for any combination of input and antecedent when their shapes verify a certain relationship. However when the implication is a strong or residuated implication, the same result holds only for some particular combinations of input and antecedent, unless Lukasiewicz implication is used. It would be very interesting to carefully analyse, for these situations, how far are the conclusions obtained from both inference mechanisms although some differences have already been pointed out.

Another interesting point is the the flexibility of the compatibility modification since the compatibility measure can be chosen independently of the implication function, and thus special purpose measures can be chosen depending on the specific application. Compatibility modification seems to be easier to understand and calculate, but it could lead to meaningless conclusions, and some care must be taken in the election of the involved operators. On the other hand the compositional rule of inference seems to benefit from a more solid theoretical background, and is usually the most accepted fuzzy inference mechanism. Thus any theoretical link between both could be a step towards a unified inference theory, in the setting of fuzzy logic.

References

[1] A. Zadeh, "Fuzzy Sets", "Fuzzy Sets", Information and Control, vol8, pp338-353, 1965

[2] Valerie Cross, Thomas Sudkamp, "Fuzzy implication and compatibility modification", 0-7803-0614-7/93$03.00, IEEE 1993.

[3] Valerie Cross, Thomas Sudkamp, "Patterns of fuzzy rule-based inference", International Journal of Approximate Reasoning 11, 235-255, 1994

[4] I. B. Turksen, Z. Zhong, "An approximate reasoning schema based on similarity measures and interval valued fuzzy sets", Fuzzy Set and Systems, 34(3), 323-345, 1990.

[5] P. Magrez, P. Smets, "Fuzzy modus ponens: a new model suitable for applications in knowledge-based systems", International Journal of Approximate Reasoning, 4, 181-200, 1989.

[6] Didier Dubois, Henri Prade, "Fuzzy logic and the generalized modus ponens revisited", Cybernetics and Systems, An International Journal, 15, 293-331, 1984.
[7] E. Trillas y L. Valverde, "On modus ponens in fuzzy logic", Proceedings of the fifteenth international symposium of multiple valued logics, 0195-623X/85/0000/0294$01.00 IEEE, 1985.
[8] Didier Dubois, Henri Prade, "Fuzzy sets in approximate reasoning, Part 1: inference with possibility distributions", Fuzzy Sets and Systems, 40, 143-202, 1991.
[9] R. M. Tong, J. Efstathiou, "A critical assesment of thruth functional modification and its use in approximate reasoning", Fuzzy Sets and Systems 7, 103-108, 1982.
[10]E. Trillas y L. Valverde, "On implication and indistinguishability in the setting of fuzzy logic", Management Decision Support Systems Using Fuzzy Sets and Possibilty theory, edited by J. Kacprzyk and R.R. Yager, Verlag TÜV Rheinland, Köln, 198-212, 1985.