Journal of Computational Dynamics
Vol. 16, 2026, pp. 66-84 (August 4, 2026)
doi:10.3934/jcd.2026015

VARIATIONAL COLLISION INTEGRATORS FOR NONHOLONOMIC LAGRANGIAN SYSTEMS

Álvaro Rodríguez Abella*[1] and Leonardo Colombo[2]

[1]Department of Applied Mathematics, Comillas Pontifical University, Madrid - 28015 Madrid, Spain

[2]Centre for Automation and Robotics (CSIC-UPM), Ctra. M300 Campo Real, Km 0,200, Arganda del Rey - 28500 Madrid, Spain

(Communicated by Sina Ober-Blöbaum)

Abstract. A discrete theory for implicit nonholonomic Lagrangian systems undergoing elastic collisions is developed. It is based on the discrete Lagrange–d'Alembert–Pontryagin variational principle and the dynamical equations thus obtained are the discrete nonholonomic implicit Euler–Lagrange equations together with the discrete conditions for the elastic impact. To illustrate the theory, variational integrators with collisions are built for several examples, including a bouncing ellipse and a nonholonomic spherical pendulum evolving inside a cylinder.

1. **Introduction.** The Lagrange–Dirac formulation of mechanics introduced in [29, 30] constitutes a unifying framework for implicit systems with nonholonomic constraints. The variational approach is based on the Lagrange–d'Alembert–Pontryagin principle, an extension of the Hamilton principle to account for degenerate (implicit) systems and nonholonomic constrains. The versatility of this framework has made it suitable to perform reduction by symmetries [11, 32] and interconnection [13, 23], as well as to extend it to the infinite-dimensional setting [10, 26]. In addition, in [24, 25] it was extended to systems with elastic collisions by following the approach in [9] for unconstrained systems. By regarding external forces as control inputs, implicit port-controlled Lagrangian systems were introduced in [31]. From that perspective, the space of inputs is thus a subbundle of the cotangent bundle of the configuration space of the system.

On the other hand, the literature of numerical schemes for systems with impacts distinguishes two strategies to handle the discontinuities:

1. *Event-driven methods*, which detect the exact moment of impacts. The continuous evolution is simulated by standard ODE solvers. When a collision is detected, an impact law is applied to instantly update the velocities and

2020 *Mathematics Subject Classification.* Primary: 37M15, 70H45; Secondary: 70F25.

Key words and phrases. Lagrangian mechanics, collisions, nonholonomic, Lagrange–d'Alembert–Pontryagin variational principle, discrete dynamics.

The first author was partially supported by grant PID2024-156578NB-I00 funded by MICIU/AEI/10.13039/501100011033/FEDER, EU. The second author acknowledges financial support from Grant PID2022-137909NB-C21 funded by MCIN/AEI/ 10.13039/501100011033.

*Corresponding author: Álvaro Rodríguez Abella.

resume the continuous simulation. This approach is suitable when the time between impacts is long enough, e.g., when Zeno behavior does not appear.

2. *Time-stepping methods*, also known as *event-capturing methods*, which use a fixed time-step to run the simulation and capture the effect of the impacts across the entire step instead of finding the exact impact times. These schemes can handle frequent impacts and grazing contacts.

Our method is based on the approach by Fetecau *et al.* [9] for unconstrained systems, which provides an event-driven method, and it is suitable for (possibly degenerate) mechanical systems with nonholonomic constraints. As it arises from discretizing a continuous theory, the corresponding integrators respect the physical properties of the mechanical systems, such as symplecticity and conservation of momenta, both in the continuous evolution and during impacts. On the other hand, the Moreau–Jean [14, 20] and the nonsmooth RATTLE [3] schemes are event-capturing methods. The Moreau–Jean scheme was originally developed in the framework of measure differential inclusions (MDIs) and it is suitable for elastic and inelastic collisions, friction and constraints. More recently, it has been shown that it comes from a discrete variational principle [4, 5] known as the principle of virtual action and, thus, it inherits the benefits of variational methods. Lastly, the nonsmooth RATTLE is an extension of the classical RATTLE algorithm [1], which is symplectic and suitable for systems with bilateral constraints, to account for systems with impacts.

The discretization of continuous theories is a ubiquitous tool in geometric mechanics to build variational integrators [19]. The discretization of the Lagrange–Dirac mechanics was carried out in [15, 16] (see also [6, 22] for alternative approaches), leading to discrete interconnection [21] and reduction [27].

The aim of this work is to build a discrete analog of the Lagrangian theory for implicit nonholonomic systems with elastic collisions introduced in [24, 25], thus obtaining variational integrators for such systems. In order to solve the collision, the impact time is computed as $\tilde{t} = t_i + \alpha h$, where h is the time-step, t_i is the previous discrete time and $\alpha \in (0, 1)$ is a variational variable of the discrete problem that has to be determined. Once the discrete Hamilton–d'Alembert–Pontryagin variational principle is established, the discrete equations follow from the discrete Lagrange–d'Alembert–Pontryagin principle, and are illustrated by simulating a bouncing ellipse and a nonholonomic spherical pendulum evolving inside a cylinder. Although it is not presented here, we aim to extend our work to include control inputs following the approach in [31]. Namely, external forces are regarded as inputs and added to the system. In addition, the controller may be computed by minimizing some cost functional, yielding optimal control equations [7].

2. **Preliminaries.**

Nonholonomic systems are mechanical systems subject to constraints on velocities that are not derivable from position constraints. These systems typically arise in applications involving rolling contact or sliding. While traditional Lagrangian mechanics is often formulated on the tangent bundle TQ, nonholonomic implicit Lagrangian systems, also known as Lagrange–Dirac systems, utilize the Pontryagin bundle $TQ \oplus T^*Q$.

In this section, we briefly recall the extension of the continuous Lagrange–Dirac theory to a nonsmooth context given in [24, 25]. This generalization accounts for elastic collisions when the configuration manifold has boundary.

Furthermore, we recall the approach of [15, 16] for discretizing implicit nonholonomic Lagrangian systems (without collisions). This scheme is based on the discrete Lagrange–d'Alembert–Pontryagin variational principle and utilizes retractions to discretize the Lagrangian as well as the constrain distribution.

2.1. **Implicit nonholonomic Lagrangian systems with collisions.** Let Q be a smooth manifold with boundary ∂Q. Linear nonholonomic constraints are defined by a regular, nonintegrable distribution $\Delta_Q \subset TQ$, whose annihilator is denoted by $\Delta_Q^\circ \subset T^*Q$.

Given an interval $[\tau_0, \tau_1] \subset$ and $\tilde{\tau} \in (\tau_0, \tau_1)$, the *path space with a unique collision at* $\tilde{\tau}$ is defined as $\Omega(Q, \tilde{\tau}) = \mathcal{T} \times \mathcal{Q}(\tilde{\tau})$. Here, $\mathcal{T}$ consists of smooth reparameterizations $\alpha_T \in C^\infty([\tau_0, \tau_1])$ with $\alpha_T' > 0$, and $\mathcal{Q}(\tilde{\tau})$ consists of piecewise C^2 paths $\alpha_Q \in C^0([\tau_0, \tau_1], Q)$ with only one singularity at $\tilde{\tau}$ such that $\alpha_Q(\tilde{\tau}) \in \partial Q$.

The tangent bundle of $\mathcal{Q}(\tilde{\tau})$ is denoted by $T\mathcal{Q}(\tilde{\tau})$ and, for each $\alpha_Q \in \mathcal{Q}(\tilde{\tau})$ consists of piecewise C^2 paths $\nu_{\alpha_Q} \in C^0([\tau_0, \tau_1], TQ)$ that project to α_Q with only one singularity at $\tilde{\tau}$ such that $\nu_{\alpha_Q}(\tilde{\tau}) \in T_{\alpha_Q(\tilde{\tau})}\partial Q$. Furthermore, $\Delta_{\mathcal{Q}(\tilde{\tau})} \subset T\mathcal{Q}(\tilde{\tau})$ consists of tangent paths that take values on the constraint distribution Δ_Q.

The topological dual bundle of $T\mathcal{Q}(\tilde{\tau})$ is to large to formulate mechanics as $\mathcal{Q}(\tilde{\tau})$ is an infinite-dimensional manifold. Hence, we restrict ourselves to the vector bundle $T^\star\mathcal{Q}(\tilde{\tau})$ whose elements are $\pi_{\alpha_Q} = \mathbb{F}L \circ \nu_{\alpha_Q} \in C^0([\tau_0, \tau_1], T^*Q)$, where $\mathbb{F}L : TQ \to T^*Q$ is the Legendre transform of L (which does not need to be an isomorphism as L may be degenerate).

The dynamics are derived using the *Hamilton–d'Alembert–Pontryagin principle* on the Pontryagin bundle. The corresponding action functional,

$$\mathcal{S} : \Omega(Q, \tilde{\tau}) \times_{\mathcal{Q}(\tilde{\tau})} (T\mathcal{Q}(\tilde{\tau}) \oplus T^\star\mathcal{Q}(\tilde{\tau})) \to \mathbb{R},$$

is defined as

$$\mathcal{S}(\alpha_T, \alpha_Q, \nu_{\alpha_Q}, \pi_{\alpha_Q}) = \int_{t_0}^{t_1} [L(v_\alpha(t)) + p_\alpha(t) \cdot (\dot{q}_\alpha(t) - v_\alpha(t))] \, dt,$$

where we denoted $t_i = \alpha_T(\tau_i)$ for $i = 0, 1$, $q_\alpha = \alpha_Q \circ \alpha_T^{-1}$, $v_\alpha = \nu_{\alpha_Q} \circ \alpha_T^{-1}$ and $p_\alpha = \pi_{\alpha_Q} \circ \alpha_T^{-1}$. In a local trivialization $TQ|_U \simeq U \times V$, we write $(v_\alpha, p_\alpha) \simeq (q, v, p) \in C^0([t_0, t_1], U \times (V \oplus V^*))$.

The *Hamilton–d'Alembert–Pontryagin* variational principle, which consists of free variations with values on the constraint distribution that vanish at the endpoints, gives the *Lagrange–d'Alembert–Pontryagin equations*:

$$\dot{p} - \frac{\partial L}{\partial q}(q, v) \in \Delta_Q^\circ(q), \quad p = \frac{\partial L}{\partial v}(q, v), \quad v = \dot{q} \in \Delta_Q(q),$$

on $\left[t_0, \tilde{t}\right) \cup \left(\tilde{t}, t_1\right]$, where $\tilde{t} = \alpha_T(\tilde{\tau})$, together with the conditions for the *elastic impact*:

$$\begin{aligned}
&p(t^+) - p(t^-) \in \left(T_{q(\tilde{t})}\partial Q\right)^\circ + \Delta_Q^\circ\left(q(\tilde{t})\right), \\
&E\left(q(t^-), v(t^-), p(t^-)\right) = E\left(q(t^+), v(t^+), p(t^+)\right), \\
&v(t^+) = \dot{q}(t^+) \in \Delta_Q\left(q(\tilde{t})\right),
\end{aligned}$$

where t^+ and t^- denote the states immediately after and before the collision, respectively. In the previous expression, the energy of the system $E : TQ \oplus T^*Q \to R$ is given by $E(q, v, p) = p \cdot v - L(q, v)$. On the continuous regime, it follows from the

Lagrange–d'Alembert–Pontryagin equations that it is preserved, i.e., $\dot{E}(q,v,p)=0$. The previous equations are written in a local trivialization of the tangent bundle. See [25, §3.2] for a global version of the results.

2.2. **Retractions and discretization maps.** A *retraction* (cf. [16, Definition 9.1]) on a smooth manifold Q is a map $\mathcal{R}:TQ\to Q$ such that

1. $\mathcal{R}_q(0_q)=q$ for each $q\in Q$, where $\mathcal{R}_q=\mathcal{R}|_{T_qQ}:T_qQ\to Q$ and 0_q is the origin of T_qQ, and
2. $T_{0_q}\mathcal{R}_q=\mathrm{id}_{T_qQ}$ via the identification $T_{0_q}(T_qQ)\simeq T_qQ$.

Given a retraction $\mathcal{R}:TQ\to Q$, we define the following map:

$$\tilde{\mathcal{R}}:TQ\to Q\times Q,\quad v_q\mapsto\tilde{\mathcal{R}}(v_q)=(q,\mathcal{R}_q(v_q)).$$

Note that $\tilde{\mathcal{R}}(0_Q)=\delta_{Q\times Q}$, where we denote $0_Q=\{0_q\in T_qQ\mid q\in Q\}$ and $\delta_{Q\times Q}=\{(q,q)\in Q\times Q\mid q\in Q\}$. The fact that $T_{0_q}\mathcal{R}_q=\mathrm{id}_{T_qQ}$ ensures that $\mathcal{R}_q:T_qQ\to Q$ is invertible around $0_q\in T_qQ$ for each $q\in Q$. Consequently, $\tilde{\mathcal{R}}$ is also invertible around 0_Q, i.e., there exist $\mathcal{U}_{0_Q}\subset TQ$ neighborhood of 0_Q and $\mathcal{U}_{\delta_{Q\times Q}}\subset Q\times Q$ neighborhood of the diagonal $\delta_{Q\times Q}$, such that $\tilde{\mathcal{R}}|_{\mathcal{U}_{0_Q}}:\mathcal{U}_{0_Q}\to\mathcal{U}_{\delta_{Q\times Q}}$ is invertible. The corresponding inverse defines a *discretization map* (cf., for example, [2])

$$\Psi_d=\tilde{\mathcal{R}}|_{\mathcal{U}_{0_Q}}^{-1}:\mathcal{U}_{\delta_{Q\times Q}}\to\mathcal{U}_{0_Q}.$$

Typically, the discretization map also depends on a small parameter $h\in\mathbb{R}^+$ known as the *time-step*. This parameter models the size of the discrete intervals (cf. §3.1 below) In the following, we will make this dependence explicit, thus writing

$$\Psi_d:Q\times Q\times\mathbb{R}^+\to TQ.$$

Note that, for brevity, we write $\mathcal{U}_{\delta_{Q\times Q}}=Q\times Q$, understanding that Ψ_d is only locally defined.

Given a (possibly degenerate) continuous Lagrangian $L:TQ\to\mathbb{R}$, the corresponding discrete Lagrangian is defined as

$$L_d=L\circ\Psi_d:Q\times Q\times\mathbb{R}^+\to\mathbb{R}.$$

2.3. **Discrete nonholonomic constraints.** Let $\Delta_Q\subset TQ$ be a (possibly nonholonomic) constraint distribution on a smooth manifold Q, and $\Delta_Q^\circ\subset T^*Q$ be its annihilator. By denoting $m=\dim Q-\dim\Delta_Q$, there exist $\omega^\mu\in\Omega^1(Q)$, $1\le\mu\le m$, such that

$$\Delta_Q^\circ(q)=\mathrm{span}\{\omega_q^\mu\in T_q^*Q\mid 1\le\mu\le m\},\qquad q\in Q.$$

In order to discretize the constraint distribution, we utilize a retraction $\mathcal{R}:TQ\to Q$ as in §2.2 with the corresponding discretization map $\Psi_d:Q\times Q\times\mathbb{R}^+\to TQ$ (for simplicity, we assume that $\mathcal{U}_{\delta_{Q\times Q}}=Q\times Q$ and $\mathcal{U}_{0_Q}=TQ$). Specifically, for each $1\le\mu\le m$, we introduce the discrete maps (cf. [8, §3])

$$\begin{aligned}\omega_{d+}^\mu&:Q\times Q\times\mathbb{R}^+\to\mathbb{R},\quad(q,q^+,h)\mapsto\omega_{d+}^\mu(q,q^+,h)=\omega_q^\mu(\Psi_d(q,q^+,h)),\\ \omega_{d-}^\mu&:Q\times Q\times\mathbb{R}^+\to\mathbb{R},\quad(q^-,q,h)\mapsto\omega_{d-}^\mu(q^-,q,h)=-\omega_q^\mu(\Psi_d(q,q^-,h)).\end{aligned}$$

Note that, the first argument of the forward-in-time map ω_{d+}^μ is regarded as the base point, whereas the base point is given by the second argument for the backwards-in-time map ω_{d-}^μ. Therefore, for each $h\in\mathbb{R}^+$, the *discrete constraint distributions* are defined as

$$\Delta_Q^{d+}(h)=\{(q,q^+)\in Q\times Q\mid\omega_{d+}^\mu(q,q^+,h)=0,\ 1\le\mu\le m\},$$

$$\Delta_Q^{d-}(h) = \{(q^-, q) \in Q \times Q \mid \omega_{d-}^{\mu}(q^-, q, h) = 0,\ 1 \leq \mu \leq m\}.$$

2.4. **Variational formulation of discrete Dirac mechanics.** Let Q be a vector space, Q^* be its dual space, $L_d : Q \times Q \times \mathbb{R}^+ \to \mathbb{R}$ be a discrete Lagrangian and $\Delta_Q \subset TQ$ be a (possibly nonholonomic) constraint distribution. Let $N \in \mathbb{Z}^+$ be the number of steps and $h \in \mathbb{R}^+$ be the time-step. As usual, there are two possible choices for the discrete action functional arising from the two different types of generating function that can be used to discretize (cf. [16, Definition 8.2]:

(i) A sequence $\{(q_k, q_k^+, p_k) \in Q \times Q \times Q^* \mid 0 \leq k \leq N\}$ is *stationary* or *critical* for the (+)-*discrete Lagrange–d'Alembert–Pontryagin action* if

$$\delta \sum_{k=0}^{N-1} [L_d(q_k, q_k^+, h) + p_{k+1} \cdot (q_{k+1} - q_k^+)] = 0,$$

together with the constraint $(q_k, q_{k+1}) \in \Delta_Q^{d+}(h)$, $0 \leq k \leq N-1$, for free variations such that $\delta q_0 = \delta q_N = 0$ and $\delta q_k \in \Delta_Q(q_k)$, $1 \leq k \leq N-1$.

(ii) A sequence $\{(q_k^-, q_k, p_k) \in Q \times Q \times Q^* \mid 0 \leq k \leq N\}$ is *stationary* or *critical* for the (−)-*discrete Lagrange–d'Alembert–Pontryagin action principle* if

$$\delta \sum_{k=0}^{N-1} [L_d(q_{k+1}^-, q_{k+1}, h) - p_k \cdot (q_k - q_{k+1}^-)] = 0,$$

together with the constraint $(q_{k+1}, q_k) \in \Delta_Q^{d-}(h)$, $0 \leq k \leq N-1$, for free variations such that $\delta q_0 = \delta q_N = 0$ and $\delta q_k \in \Delta_Q(q_k)$, $1 \leq k \leq N-1$.

Remark 2.1 (Intrinsic expressions)**.** The discrete actions introduced above can be made sense intrinsically. Namely, for a general smooth manifold Q, let $\mathcal{U} \subset Q$ be a local coordinate chart, i.e., $\mathcal{U} \simeq \mathbb{R}^n$. Locally, a cotangent vector $p_q \in T^*Q|_{\mathcal{U}}$ reads $p_q \simeq (q, p) \in \mathcal{U} \times (\mathbb{R}^n)^*$. Hence, given a discretization map $\Psi_d : Q \times Q \times \mathbb{R}^+ \to TQ$, we may write

$$\delta \sum_{k=0}^{N-1} \left[L_d(q_k, q_k^+, h) + \left\langle p_{q_k^+}, \Psi_d(q_k^+, q_{k+1}, h) \right\rangle \right] = 0,$$

for each sequence $\{(q_k, p_{q_k^+}) \in Q \times T^*Q \mid 0 \leq k \leq N\}$ and $h \in \mathbb{R}^+$. Note that it is well defined as $\Psi_d(q_k^+, q_{k+1}, h) \in T_{q_k^+}Q$ by construction.

These identifications are valid globally when Q is a vector space. The (−)-case is analogous by considering $(p_{q_k^-}, q_k) \in T^*Q \times Q$.

In order to write the discrete dynamical equations, $D_i L_d : Q \times Q \times \mathbb{R}^+ \to Q^*$ denotes the i-th partial derivative of the discrete Lagrangian for $i = 1, 2$.

Theorem 2.2. *1. A sequence $\{(q_k, q_k^+, p_k) \in Q \times Q \times Q^* \mid 0 \leq k \leq N\}$ is stationary for the* (+)*-discrete Lagrange–d'Alembert–Pontryagin action if it satisfies the* (+)-discrete nonholonomic implicit Euler–Lagrange equations *(also called* (+)-discrete Lagrange–Dirac equations*):*

$$\begin{cases} p_{k+1} = D_2 L_d(q_k, q_k^+, h), & \\ q_{k+1} = q_k^+, & \\ p_k + D_1 L_d(q_k, q_k^+, h) \in \Delta_Q^\circ(q_k), & k \neq 0, \\ (q_k, q_k^+) \in \Delta_Q^{d+}(h), & k \neq 0, \end{cases} \qquad 0 \leq k \leq N-1. \tag{1}$$

2. *A sequence* $\{(q_k^-, q_k, p_k) \in Q \times Q \times Q^* \mid 0 \leq k \leq N\}$ *is stationary for the* $(-)$*-discrete Lagrange–d'Alembert–Pontryagin action if it satisfies the* $(-)$-discrete nonholonomic implicit Euler–Lagrange equations *(also called* $(-)$-discrete Lagrange–d'Alembert–Pontryagin equations*):*

$$\begin{cases} p_k = -D_1 L_d(q_{k+1}^-, q_{k+1}, h), & \\ q_k = q_{k+1}^-, & \\ p_{k+1} - D_2 L_d(q_{k+1}^-, q_{k+1}, h) \in \Delta_Q^\circ(q_{k+1}), & k \neq 0, \\ (q_{k+1}^-, q_{k+1}) \in \Delta_Q^{d-}(h), & k \neq 0, \end{cases} \qquad 0 \leq k \leq N-1. \tag{2}$$

3. **Discrete nonholonomic implicit Lagrangian systems with collisions.** Let Q be a smooth manifold. The *admissible set* is some submanifold $S \subset Q$ with boundary ∂S such that $\dim Q = \dim S$. In the following, Q will be assumed to be a vector space. Therefore, the tangent and cotangent bundles are given by $TQ = Q \times Q$ and $T^*Q = Q \times Q^*$, respectively. For each point on the boundary of the admissible set, $q \in \partial S$, there exists a vector subspace $V_q \subset Q$ such that $T_q \partial S = \{q\} \times V_q$ and $T_q^* \partial S = \{q\} \times V_q^*$. Hence, the tangent and cotangent bundles of ∂S are given by $T\partial S = \bigsqcup_{q \in \partial S} V_q$ and $T^*\partial S = \bigsqcup_{q \in \partial S} V_q^*$. The natural inclusion $\imath_S : \partial S \to S$ induces canonical morphisms

$$T\imath_S : T\partial S \to TQ|_{\partial S} = \partial S \times Q, \qquad \imath_S^* : T^*Q|_{\partial S} = \partial S \times Q^* \to T^*\partial S.$$

3.1. **Discrete configuration and phase spaces.** Let $[t_{\text{init}}, t_{\text{fin}}] \subset \mathbb{R}$ be an interval and $N \in \mathbb{Z}^+$ be the number of steps. The discrete interval is

$$\{t_k = t_{\text{init}} + kh \mid 0 \leq k \leq N\},$$

where $h = (t_{\text{fin}} - t_{\text{init}})/N$ is the time-step. Note that $t_0 = t_{\text{init}}$ and $t_N = t_{\text{fin}}$. We only consider one collision at a fixed step $i \in \{0, \ldots, N-1\}$ (which is assumed to be known) occurring at a fixed time $\tilde{t} = t_i + \alpha h$ for some $\alpha \in (0,1)$ (which is assumed to be unknown). In other words, it is known that the collision takes place within the interval (t_i, t_{i+1}), but the exact impact time $\tilde{t} \in (t_i, t_{i+1})$ is unknown and has to be computed from the variational equations. In practice, the fact that the collision takes place at the i-th interval follows from checking the impact condition at each step (cf. Algorithm 1 below).

Remark 3.1 (Multiple collisions)**.** Although the theoretical results are presented for a single impact for simplicity, the proposed numerical scheme works when the system undergoes several collisions (see Algorithm 1). Specifically, the integrator simulates the continuous system and, at each time step, checks if the impact condition is met, i.e., if the current state is outside the admissible set. In that case, the current state is deleted and the impact time is computed according to Theorem 3.4. After that, the continuous simulation continues until a new impact is detected or the final time is reached.

On the other hand, multiple impacts may occur at a single time-step. This can be handled by decreasing the size of the time-step. If the solution does not undergo Zeno behavior (cf. [25, Remark 3.1]), by compactness it is always possible to pick $h > 0$ small enough such that only one collision occurs at each time-step. This method has the inconvenience that, in general, such $h > 0$ is not known a priori.

Alternatively, our method can be modified to directly address multiple collisions by dividing the impact step (cf. [9, §4.1]). Specifically, let us denote $\tilde{t}_1 = t_i + \alpha_1 h$ the first impact time, which would be obtained as of now, i.e., solving (7) and (8) in Algorithm 1. If there is another impact in $(\tilde{t}_1, t_{i+1})$, then (9) and (10) would provide a non-admissible solution. Let us denote the second impact time by $\tilde{t}_2 = \tilde{t}_1 + \alpha_2 h$ for some $\alpha_2 \in (0, 1-\alpha_1)$. The conservation of the discrete energy and momentum will then give a set of equations analogous to (7) and (8) at $\tilde{t}_2$, thus providing $(\alpha_2, \tilde{q}_2 = q_d(\tilde{t}_2), \tilde{q}_1^+) \in (0, 1-\alpha_1) \times \partial S \times Q$ and $(\tilde{q}_2^+, \tilde{p}_2 = p_d(\tilde{t}_2)) \in S \times Q^*$, respectively. Iteratively, (9) and (10) would be solved. If they provide an admissible solution, then the algorithm would continue with the continuous equations (6). Otherwise, the next impact would take place at time $\tilde{t}_3 = \tilde{t}_2 + \alpha_3 h$ for some $\alpha_3 \in (0, 1-\alpha_2)$, and it would be solved analogously.

Given $\tilde{\alpha} \in (0,1)$, the *discrete interval with a unique collision* is defined as

$$I(i, \tilde{\alpha}) = \{t_0, \dots, t_i, t_i + \tilde{\alpha} h, t_{i+1}, \dots, t_N\}.$$

In the same vein, the *discrete path space* is

$$\Omega_d(S, i, \tilde{\alpha}) = (0,1) \times \mathcal{Q}_d(S, i, \tilde{\alpha}),$$

where

$$\mathcal{Q}_d(S, i, \tilde{\alpha}) = \left\{ q_d : I(i, \tilde{\alpha}) \to S \mid q_d(t_k) \in \operatorname{int} S,\ 0 \le k \le N,\ q_d(t_i + \tilde{\alpha} h) \in \partial S \right\}, \tag{3}$$

with $\operatorname{int} S = S - \partial S$ denoting the interior of S. Note that $\Omega_d(S, i, \tilde{\alpha}) \simeq (0,1) \times (\operatorname{int} S)^{N+1} \times \partial S$. Given $q_d \in \mathcal{Q}_d(S, i, \tilde{\alpha})$, for brevity we denote $q_k = q_d(t_k)$ for each $0 \le k \le N$ and $\tilde{q} = q_d(t_i + \tilde{\alpha} h)$. Analogously, let us introduce the discrete path spaces where the admissible set is disregarded,

$$\mathcal{Q}_d(i, \tilde{\alpha}) = \{ q_d : I(i, \tilde{\alpha}) \to Q \}, \qquad \mathcal{Q}_d^\star(i, \tilde{\alpha}) = \{ p_d : I(i, \tilde{\alpha}) \to Q^* \}.$$

Again, given $p_d \in \mathcal{Q}_d^\star(i, \tilde{\alpha})$, for brevity we denote $p_k = p_d(t_k)$, $0 \le k \le N$, and $\tilde{p} = p_d(t_i + \tilde{\alpha} h)$.

Given a (possibly nonholonomic) constrain distribution $\Delta_Q \subset TQ$, the constrained path spaces are defined as (recall §2.3)

$$\Delta_S^{d+}(i, \tilde{\alpha}, h) = \{ (q_d, q_d^+) \in \mathcal{Q}_d(S, i, \tilde{\alpha}) \times \mathcal{Q}_d(i, \tilde{\alpha}) \mid (q_k, q_k^+), (\tilde{q}, \tilde{q}^+) \in \Delta_Q^{d+}(h),\ 0 \le k \le N \},$$

$$\Delta_S^{d-}(i, \tilde{\alpha}, h) = \{ (q_d^-, q_d) \in \mathcal{Q}_d(i, \tilde{\alpha}) \times \mathcal{Q}_d(S, i, \tilde{\alpha}) \mid (q_k^-, q_k), (\tilde{q}^-, \tilde{q}) \in \Delta_Q^{d-}(h),\ 0 \le k \le N \}.$$

In order to work with variations of a discrete path and address the discrete constraints, we need to consider the corresponding tangent spaces.

$(+)$-case: For each $(q_d, q_d^+) \in \mathcal{Q}_d(S, i, \tilde{\alpha}) \times \mathcal{Q}_d(i, \tilde{\alpha})$, we define

$$T_{(q_d, q_d^+)}(\mathcal{Q}_d(S, i, \tilde{\alpha}) \times \mathcal{Q}_d(i, \tilde{\alpha})) = \left\{ (\delta q_d, \delta q_d^+) : I(i, \tilde{\alpha}) \to Q \times Q \mid \delta \tilde{q} \in V_{\tilde{q}} \right\}.$$

$(-)$-case: For each $(q_d^-, q_d) \in \mathcal{Q}_d(i, \tilde{\alpha}) \times \mathcal{Q}_d(S, i, \tilde{\alpha})$, we define

$$T_{(q_d^-, q_d)}(\mathcal{Q}_d(i, \tilde{\alpha}) \times \mathcal{Q}_d(S, i, \tilde{\alpha})) = \left\{ (\delta q_d^-, \delta q_d) : I(i, \tilde{\alpha}) \to Q \times Q \mid \delta \tilde{q} \in V_{\tilde{q}} \right\}.$$

Lastly, we define the path spaces of constrained variations as subspaces of the previous spaces:

$(+)$-case: For each $(q_d, q_d^+) \in \Delta_S^{d+}$, we define

$$\begin{aligned}\Delta_{TS}^{d+}(i, \tilde{\alpha})(q_d, q_d^+) = &\{ (\delta q_d, \delta q_d^+) \in T_{(q_d, q_d^+)}(\mathcal{Q}_d(S, i, \tilde{\alpha}) \times \mathcal{Q}_d(i, \tilde{\alpha})) \mid \\ &\qquad \delta q_k \in \Delta_Q(q_k),\ 0 \le k \le N,\ \delta \tilde{q} \in \Delta_Q(\tilde{q}) \}.\end{aligned}$$

$(-)$-case: For each $(q_d^-, q_d) \in \Delta_S^{d-}$, we define

$$\begin{aligned}\Delta_{TS}^{d-}(i, \tilde{\alpha})(q_d^-, q_d) = &\{ (\delta q_d^-, \delta q_d) \in T_{(q_d^-, q_d)}(\mathcal{Q}_d(i, \tilde{\alpha}) \times \mathcal{Q}_d(S, i, \tilde{\alpha})) \mid \\ &\qquad \delta q_k \in \Delta_Q(q_k),\ 0 \le k \le N,\ \delta \tilde{q} \in \Delta_Q(\tilde{q}) \},\end{aligned}$$

3.2. **Discrete nonholonomic implicit Euler–Lagrange equations with collisions.** Let $L_d : Q \times Q \times \mathbb{R}^+ \to \mathbb{R}$ be a discrete (possibly degenerate) Lagrangian, where Q is a vector space. Note that the following expressions can be made sense in a general manifold as explained in Remark 2.1.

The two possible choices for the discrete action functional are given by:

(i) The $(+)$-*discrete Lagrange–Pontryagin action*,

$$\mathbb{S}_{d+} : (0,1) \times \mathcal{Q}_d(S, i, \tilde{\alpha}) \times \mathcal{Q}_d(i, \tilde{\alpha})) \times \mathcal{Q}_d^\star(i, \tilde{\alpha}) \to \mathbb{R},$$

is defined as

$$\mathbb{S}_{d+}(\alpha, q_d, q_d^+, p_d) = \sum_{\substack{k=0\\k\neq i}}^{N-1} \left[L_d(q_k, q_k^+, h) + p_{k+1} \cdot (q_{k+1} - q_k^+)\right] + L_d(q_i, q_i^+, \alpha h) \quad (4)$$
$$+ \tilde{p} \cdot (\tilde{q} - q_i^+) + L_d(\tilde{q}, \tilde{q}^+, (1-\alpha)h) + p_{i+1} \cdot (q_{i+1} - \tilde{q}^+).$$

(ii) The $(-)$-*discrete Lagrange–Pontryagin action*,

$$\mathbb{S}_{d-} : (0,1) \times \mathcal{Q}_d(i, \tilde{\alpha}) \times \mathcal{Q}_d(S, i, \tilde{\alpha}) \times \mathcal{Q}_d^\star(i, \tilde{\alpha}) \to \mathbb{R},$$

is defined as

$$\mathbb{S}_{d-}(\alpha, q_d^-, q_d, p_d) = \sum_{\substack{k=0\\k\neq i}}^{N-1} \left[L_d(q_{k+1}^-, q_{k+1}, h) - p_k \cdot (q_k - q_{k+1}^-)\right] + L_d(\tilde{q}^-, \tilde{q}, \alpha h) \quad (5)$$
$$- p_i \cdot (q_i - \tilde{q}^-) + L_d(q_{i+1}^-, q_{i+1}, (1-\alpha)h) - \tilde{p} \cdot (\tilde{q} - q_{i+1}^-).$$

Remark 3.2. Here $\tilde{\alpha}$ is only an auxiliary parameter used to define the path spaces, whereas α is the actual collision time variable determined by the variational equations, i.e., the impact takes place at $\tilde{t} = t_i + \alpha h$.

The following definition introduces the *discrete Hamilton–d'Alembert–Pontryagin principle.*

Definition 3.3. (i) A discrete path

$$\mathsf{c}_{d+} = (\alpha, q_d, q_d^+, p_d) \in (0,1) \times \Delta_S^{d+}(i, \tilde{\alpha}, h) \times \mathcal{Q}_d^\star(i, \tilde{\alpha})$$

is *stationary* (or *critical*) for the $(+)$-discrete Lagrange–d'Alembert–Pontryagin action if it satisfies

$$\mathrm{d}\,\mathbb{S}_{d+}(\mathsf{c}_{d+})(\delta\mathsf{c}_{d+}) = 0,$$

for each $\delta\mathsf{c}_{d+} = (\delta\alpha, \delta q_d, \delta q_d^+, \delta p_d) \in T_\alpha(0,1) \times \Delta_{TS}^{d+}(i, \tilde{\alpha})(q_d, q_d^+) \times \mathcal{Q}_d^\star(i, \tilde{\alpha})$ vanishing at the endpoints $\delta q_0 = \delta q_N = 0$.

(ii) A discrete path

$$\mathsf{c}_{d-} = (\alpha, v_d, q_d, p_d) \in (0,1) \times \Delta_S^{d-}(i, \tilde{\alpha}, h) \times \mathcal{Q}_d^\star(i, \tilde{\alpha})$$

is *stationary* or *critical* for the $(-)$-discrete Lagrange–d'Alembert–Pontryagin action if it satisfies

$$\mathrm{d}\,\mathbb{S}_{d-}(\mathsf{c}_{d-})(\delta\mathsf{c}_{d-}) = 0,$$

for each $\delta\mathsf{c}_{d-} = (\delta\alpha, \delta q_d^-, \delta q_d, \delta p_d) \in T_\alpha(0,1) \times \Delta_{TS}^{d-}(i, \tilde{\alpha})(v_d, q_d) \times \mathcal{Q}_d^\star(i, \tilde{\alpha})$ vanishing at the endpoints $\delta q_0 = \delta q_N = 0$.

The condition $(\delta q_d, \delta q_d^+) \in \Delta_{TS}^{d+}(i, \tilde{\alpha})(q_d, q_d^+)$ implies that $\delta q_k \in \Delta_Q(q_k)$, $0 \leq k \leq N$, and $\delta \tilde{q} \in \Delta_Q(\tilde{q})$, and analogously for $(\delta q_d^-, \delta q_d) \in \Delta_{TS}^{d-}(i, \tilde{\alpha})(q_d^-, q_d)$. These conditions are imposed *after* taking variations inside the summation. We are ready to compute the discrete nonholonomic implicit Euler–Lagrange equations with elastic collisions. Recall that *elastic* means that the energy and momenta of the system are preserved.

Theorem 3.4. *Consider the following discrete paths:*

$$\begin{aligned} \mathsf{c}_{d+} &= (\alpha, q_d, q_d^+, p_d) \in (0,1) \times \Delta_S^{d+}(i, \tilde{\alpha}, h) \times \mathcal{Q}_d^\star(i, \tilde{\alpha}), \\ \mathsf{c}_{d-} &= (\alpha, q_d^-, q_d, p_d) \in (0,1) \times \Delta_S^{d-}(i, \tilde{\alpha}, h) \times \mathcal{Q}_d^\star(i, \tilde{\alpha}). \end{aligned}$$

Then, the following statements hold:

(i) c_{d+} *is stationary for* $\mathbb{S}_{d+}$ *if and only if it satisfies the* $(+)$*-discrete nonholonomic implicit Euler–Lagrange equations* (1) *for* $k \neq i$*, together with the* $(+)$*-discrete conditions for the* elastic impact*:*

$$\begin{cases} D_1 L_d(q_i, q_i^+, \alpha h) + p_i \in \Delta_Q^\circ(q_i), \\ (q_i, q_i^+) \in \Delta_Q^{d+}(h), \\ \tilde{q} = q_i^+, \\ \tilde{q} \in \partial S, \\ D_3 L_d(q_i, q_i^+, \alpha h) = D_3 L_d(\tilde{q}, \tilde{q}^+, (1-\alpha)h), \\ (\imath_S)_{\tilde{q}}^*(D_1 L_d(\tilde{q}, \tilde{q}^+, (1-\alpha)h) + \tilde{p}) \in (\imath_S)_{\tilde{q}}^* \left(\Delta_Q^\circ(\tilde{q})\right), \\ (\tilde{q}, \tilde{q}^+) \in \Delta_Q^{d+}(h), \\ \tilde{p} = D_2 L_d(q_i, q_i^+, \alpha h), \\ p_{i+1} = D_2 L_d(\tilde{q}, \tilde{q}^+, (1-\alpha)h), \\ q_{i+1} = \tilde{q}^+, \end{cases}$$

where $D_3 L_d : Q \times Q \times \mathbb{R}^+ \to \mathbb{R}$ *denotes the partial derivative of the discrete Lagrangian with respect to the third argument.*

(ii) c_{d-} *is stationary for* $\mathbb{S}_{d-}$ *if and only if it satisfies the* $(-)$*-discrete nonholonomic implicit Euler–Lagrange equations* (2) *for* $k \neq i$*, together with the* $(-)$*-discrete conditions for the* elastic impact*:*

$$\begin{cases} (\imath_S)_{\tilde{q}}^*(-D_2 L_d(\tilde{q}^-, \tilde{q}, \alpha h) + \tilde{p}) \in (\imath_S)_{\tilde{q}}^* \left(\Delta_Q^\circ(\tilde{q})\right), \\ (\tilde{q}^-, \tilde{q}) \in \Delta_Q^{d-}(h), \\ q_i = \tilde{q}^-, \\ \tilde{q} \in \partial S, \\ D_3 L_d(\tilde{q}^-, \tilde{q}, \alpha h) = D_3 L_d(q_{i+1}^-, q_{i+1}, (1-\alpha)h), \\ -D_2 L_d(q_{i+1}^-, q_{i+1}, (1-\alpha)h) + p_{i+1} \in \Delta_Q^\circ(q_{i+1}), \\ (q_{i+1}^-, q_{i+1}) \in \Delta_Q^{d-}(h), \\ p_i = -D_1 L_d(\tilde{q}^-, \tilde{q}, \alpha h), \\ \tilde{p} = -D_1 L_d(q_{i+1}^-, q_{i+1}, (1-\alpha)h), \\ \tilde{q} = q_{i+1}^-. \end{cases}$$

Proof. By computing the variation of the (+)-discrete action functional and rearranging terms, we obtain:

$$
\begin{aligned}
&\delta\mathbb{S}_{d+}(\mathsf{c}_{d+})(\delta\mathsf{c}_{d+})\\
&= \sum_{\substack{k=0\\k\neq i-1}}^{N-2} \left(D_1L_d(q_{k+1}, q_{k+1}^+, h) + p_{k+1}\right)\cdot\delta q_{k+1}\\
&\quad + \sum_{\substack{k=0\\k\neq i}}^{N-1} \left(D_2L_d(q_k, q_k^+, h) - p_{k+1}\right)\cdot\delta q_k^+\\
&\quad + \left(D_1L_d(q_i, q_i^+, \alpha h) + p_i\right)\cdot\delta q_i + \left(D_2L_d(q_i, q_i^+, \alpha h) - \tilde{p}\right)\cdot\delta q_i^+\\
&\quad + (\imath_S)^*_{\tilde{q}}\left(D_1L_d(\tilde{q}, \tilde{q}^+, (1-\alpha)h) + \tilde{p}\right)\cdot\delta\tilde{q} + \left(D_2L_d(\tilde{q}, \tilde{q}^+, (1-\alpha)h) - p_{i+1}\right)\cdot\delta\tilde{q}^+\\
&\quad + \sum_{\substack{k=0\\k\neq i}}^{N-1} \delta p_{k+1}\cdot(q_{k+1} - q_k^+) + \delta\tilde{p}\cdot(\tilde{q} - q_i^+) + \delta p_{i+1}\cdot(q_{i+1} - \tilde{q}^+)\\
&\quad + h\left(D_3L_d(q_i, q_i^+, \alpha h) - D_3L_d(\tilde{q}, \tilde{q}^+, (1-\alpha)h)\right)\delta\alpha.
\end{aligned}
$$

The equations follow from $\delta\alpha \in T_\alpha(0,1) \simeq \mathbb{R}$, $\delta q_k \in \Delta_Q(q_k)$, $1 \leq k \leq N-1$, δq_k^+, $\delta\tilde{q}^+ \in Q$, δp_k, $\delta\tilde{p} \in Q^*$, $0 \leq k \leq N$, and $\delta\tilde{q} \in \Delta_Q(\tilde{q}) \cap V_{\tilde{q}}$. The conditions $\tilde{q} \in \partial S$, $(q_k, q_k^+) \in \Delta_Q^{d+}(h)$ and $(\tilde{q}, \tilde{q}^+) \in \Delta_Q^{d+}(h)$ are by construction. Lastly, note that $V_{\tilde{q}}^\circ \cap V_{\tilde{q}}^* = \{0\}$, whence

$$
(\Delta_Q(\tilde{q}) \cap V_{\tilde{q}})^\circ \cap V_{\tilde{q}}^* = \left(\Delta_Q^\circ(\tilde{q}) + V_{\tilde{q}}^\circ\right) \cap V_{\tilde{q}}^* = \Delta_Q^\circ(\tilde{q}) \cap V_{\tilde{q}}^* = (\imath_S)^*_{\tilde{q}}(\Delta_Q^\circ(\tilde{q})).
$$

The $(-)$-discrete case follows from an analog computation. $\square$

As usual, the (+)-discrete equations are utilized to perform forward in time simulations, whereas the $(-)$-discrete equations are appropriate to simulate backwards in time. By utilizing Lagrange multipliers and the definition of Δ_Q^+, this theorem may be implemented as described in Algorithm 1. The backwards in time implementation is analogous. The algorithm checks at each iteration the admissibility of the solution. Whenever the new state is not admissible, it is removed and the conditions for the elastic impact are solved.

The map $\imath_S^*$, which is the pullback of the inclusion $\imath_S : \partial S \to S$, may be regarded as the projection of the restriction of the cotangent bundle of Q to the boundary to the cotangent bundle of the boundary ∂S. In the second equation of (8), it forces the left-hand side to belong to the normal subspace, i.e., the annihilator of $T\partial S$. Alternatively, one may introduce an extra Lagrange multiplier which physically corresponds to the impulse generated by the contact reaction forces. Such multiplier would be subject to a non-negativity constraint to ensure that the reaction force points inwards. This additional unknown together with the sign constraint, would make the algorithm less efficient when solving the impact equations.

4. **Examples.** In this section, we illustrate the theoretical results presented above by describing and simulating several examples. The pseudo-code in Algorithm 1 is implemented in Python and the algebraic equations are solved using the function `fsolve` from the library `scipy.optimize`.

Algorithm 1 Forward implementation of Theorem 3.4.

- **input**: $(q_0, q_0^+) \in S \times Q$, $p_0 = D_2 L_d(q_0, q_0^+, h)$.
- **while** $t_k < t_N$:
  - solve:

$$\begin{cases} p_{k+1} = D_2 L_d(q_k, q_k^+, h), \\ q_{k+1} = q_k^+, \\ D_1 L_d(q_{k+1}, q_{k+1}^+, h) + p_{k+1} = \lambda_{k+1,\mu}\, \omega^\mu_{q_{k+1}}, \\ \omega^\mu_{d+}(q_{k+1}, q_{k+1}^+, h) = 0, \ 1 \leq \mu \leq m, \end{cases} \tag{6}$$

  for $(q_{k+1}, q_{k+1}^+, p_{k+1}) \in Q \times Q \times Q^*$ and $\{\lambda_{k+1,\mu} \in \mathbb{R} \mid 1 \leq \mu \leq m\}$.
  - **if** $q_{k+1} \notin S$:
    * delete: q_k^+, $(q_{k+1}, q_{k+1}^+, p_{k+1})$.
    * solve:

$$\begin{cases} D_1 L_d(q_k, q_k^+, \alpha h) + p_k = \lambda_{k,\mu}\, \omega^\mu_{q_k}, \\ \omega^\mu_{d+}(q_k, q_k^+, h) = 0, \ 1 \leq \mu \leq m, \\ \tilde{q} = q_k^+, \\ \tilde{q} \in \partial S, \end{cases} \tag{7}$$

    for $(\alpha, \tilde{q}, q_k^+) \in (0,1) \times \partial S \times Q$ and $\{\lambda_{k,\mu} \in \mathbb{R} \mid 1 \leq \mu \leq m\}$.
    * solve:

$$\begin{cases} D_3 L_d(q_k, q_k^+, \alpha h) = D_3 L_d(\tilde{q}, \tilde{q}^+, (1-\alpha)h), \\ (\imath_S)^*_{\tilde{q}} \left(D_1 L_d(\tilde{q}, \tilde{q}^+, (1-\alpha)h) + \tilde{p} - \tilde{\lambda}_\mu\, \omega^\mu_{\tilde{q}} \right) = 0, \\ \omega^\mu_{d+}(\tilde{q}, \tilde{q}^+, h) = 0, \ 1 \leq \mu \leq m, \\ \tilde{p} = D_2 L_d(q_k, q_k^+, \alpha h), \end{cases} \tag{8}$$

    for $(\tilde{q}^+, \tilde{p}) \in S \times Q^*$ and $\left\{\tilde{\lambda}_\mu \in \mathbb{R} \mid 1 \leq \mu \leq m\right\}$.
    * solve:

$$\begin{cases} p_{k+1} = D_2 L_d(\tilde{q}, \tilde{q}^+, (1-\alpha)h), \\ q_{k+1} = \tilde{q}^+. \end{cases} \tag{9}$$

    for $(q_{k+1}, p_{k+1}) \in S \times Q^*$.
    * solve:

$$\begin{cases} D_1 L_d(q_{k+1}, q_{k+1}^+, h) + p_{k+1} = \lambda_{k+1,\mu}\, \omega^\mu_{q_{k+1}}, \\ \omega^\mu_{d+}(q_{k+1}, q_{k+1}^+, h) = 0, \ 1 \leq \mu \leq m. \end{cases} \tag{10}$$

    for $q_{k+1}^+ \in Q$ and $\{\lambda_{k+1,\mu} \in \mathbb{R} \mid 1 \leq \mu \leq m\}$.

4.1. **Bouncing particle.** Consider a particle in two dimensions subject to the gravitational force that collides with the floor. The configuration space and the admissible set are $Q = \mathbb{R}^2$ and $S = \{q = (x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, respectively, whose boundary is given by $\partial S = \{\tilde{q} = (x, y) \in \mathbb{R}^2 \mid y = 0\}$. Henceforth, the vectors and covectors are expressed in the following bases: $TQ = \text{span}\{\partial_x, \ \partial_y\}$,

$T^*Q = \operatorname{span}\{dx,\ dy\}$, $T\partial S = \operatorname{span}\{\partial_x\}$ and $T^*\partial S = \operatorname{span}\{dx\}$. The system is unconstrained, i.e., $\Delta_Q = TQ$ and, thus, $\Delta_Q^{d+}(h) = Q \times Q$. By denoting $\tilde{q} = (x, 0) \in \partial S$, we have the following maps:

$$T_{\tilde{q}}\imath_S : T\partial S \to T\mathbb{R}^2|_{\partial S}, \quad \tilde{v} = (\tilde{v}_x) \mapsto T_{\tilde{q}}\imath_S(\tilde{v}) = (\tilde{v}_x, 0),$$
$$(\imath_S^*)_{\tilde{q}} : T^*\mathbb{R}^2|_{\partial S} \to T^*\partial S, \quad p = (p_x, p_y) \mapsto (\imath_S^*)_{\tilde{q}}(p) = (p_x).$$

The Lagrangian of the system $L : T\mathbb{R}^2 \to \mathbb{R}$ consists of the kinetic energy together with the gravitational potential, i.e.,

$$L(q, v) = \frac{1}{2}m(v_x^2 + v_y^2) - mgy,$$

for each $(q, v) = ((x, y), (v_x, v_y)) \in T\mathbb{R}^2$, where $m > 0$ and $g > 0$ are the mass and the gravitational acceleration, respectively. Lastly, we consider the discrete Lagrangian $L_d : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}$ induced by the following discretization map:

$$\Psi_d : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+ \to T\mathbb{R}^2, \quad (q, q^+, h) \mapsto \left(\frac{q + q^+}{2}, \frac{q^+ - q}{h}\right).$$

Namely, it is given by

$$\begin{aligned} L_d(q, q^+, h) &= h\, L\left(\Psi_d(q, q^+, h)\right) \\ &= \frac{1}{2h}m\left((x^+ - x)^2 + (y^+ - y)^2\right) - \frac{1}{2}hmg(y + y^+), \end{aligned}$$

for each $(q, q^+, h) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+$. Therefore, the relation between the continuous and the discrete initial conditions is given by

$$q_0 = q(0) - \frac{h}{2}v(0), \qquad q_0^+ = q(0) + \frac{h}{2}v(0).$$

4.2. **Bouncing 2-dimensional rigid body.** Let us extend the previous example to a 2-dimensional rigid body subject to the gravitational force that collides with the floor and rotates about a perpendicular axis. The configuration space is the Euclidean group in two dimensions, i.e., $Q = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$, where we pick coordinates $q = (\theta, x, y) \in Q$ with θ and (x, y) denoting the angle of rotation and the position of the axis of rotation (which we assume to be inside the body). The system is unconstrained, i.e., $\Delta_Q = TQ$ and, thus, $\Delta_Q^{d+}(h) = Q \times Q$. The distance between the axis and the edge of the body is given by a piecewise smooth function $\phi : [0, 2\pi] \to \mathbb{R}^+$ such that $\phi(0) = \phi(2\pi)$. For instance,

1. $\phi(\theta) = l(|\sin\theta| + |\cos\theta|)$ corresponds to a four-point star-shaped rigid body of length $2l > 0$ rotating about its center [9, Eq. (81)], and
2. $\phi(\theta) = \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}$ corresponds to an ellipse with principal semi-axes of lengths $a, b > 0$ rotating about its center [28, Proposition 2.2].

The admissible set is given by

$$S = \{q = (\theta, x, y) \in \mathrm{SE}(2) \mid y \geq \phi(\theta)\}.$$

It can be checked that the following are bases for the tangent and cotangent bundles of the admissible set, respectively:

$$T\partial S = \operatorname{span}\{e_1 = \partial_x,\ e_2 = \partial_\theta + \dot{\phi}(\theta)\, \partial_y\},$$
$$T^*\partial S = \operatorname{span}\{e^1 = dx,\ e^2 = \psi(\theta)\, (d\theta + \dot{\phi}(\theta)\, dy)\},$$

where $\psi(\theta) = 1/(1+\dot{\phi}(\theta)^2)$. It is readily seen that

$$T\imath_S : T\partial S \to T\mathrm{SE}(2)|_{\partial S}, \quad (q,v) \mapsto T\imath_S(q,v) = v_2\,\partial_\theta + v_1\,\partial_x + \dot{\phi}(\theta)\,v_2\,\partial_y,$$
$$\imath_S^* : T^*\mathrm{SE}(2)|_{\partial S} \to T^*\partial S, \quad (q,p) \mapsto \imath_S^*(q,p) = p_x\,e^1 + (p_\theta + \dot{\phi}(\theta)\,p_y)\,e^2,$$

for each $v = v_1\,e_1 + v_2\,e_2 \in T_q\partial S$ and $p = p_\theta\,d\theta + p_x\,dx + p_y\,dy \in T_q^*\mathrm{SE}(2)$.

The Lagrangian of the system, $L : T\mathrm{SE}(2) \to \mathbb{R}$, consists of the kinetic energy (translational and rotational) together with the gravitational potential, i.e.,

$$L(q,v) = \frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}I_\phi\,v_\theta^2 - mgy, \qquad (q,v) \in T\mathrm{SE}(2),$$

where $m > 0$ and $I_\phi > 0$ are the mass and the moment of inertia with respect to the z axis about the center of the body, respectively, of the object, and $g > 0$ is the gravitational acceleration. The discrete Lagrangian $L_d : \mathrm{SE}(2) \times \mathrm{SE}(2) \times \mathbb{R}^+ \to \mathbb{R}$ is chosen as in the previous example, yielding

$$L_d(q,q^+,h) = \frac{m}{2h}\left((x^+ - x)^2 + (y^+ - y)^2\right) + \frac{I_\phi}{2h}(\theta^+ - \theta)^2 - hmg\frac{y + y^+}{2}.$$

Numerical experiments. For simulations, we considered an ellipse with the following parameters: $m = 1$, $g = 9.8$, $a = 1$ and $b = 0.5$. Recall that the moment of inertia of an ellipse is $I_\phi = m(a^2 + b^2)/4$.

For Figures 1 and 2, we picked $h = 10^{-2}$ as time-step, as well as $q(0) = (\theta(0), x(0), y(0)) = (\pi/2, 0, 3.5)$ and $v_q(0) = (v_\theta(0), v_x(0), v_y(0)) = (-6.5, 2, 0)$ as initial conditions. In Figure 1, the trajectory of the ellipse is simulated for 1.8 seconds, including one impact with the floor. The dashed black line indicates the trajectory of the center of mass, and the dashed red lines indicate the principal axes of the ellipse. In Figure 2, we plot the evolution of the total energy, as well as the rotational and the translational (including potential) energies. As expected, impacts with the floor imprint torques on the ellipsoid that redistribute the energy between the rotational and the translational modes. Consequently, the angular velocity and the maximum height reached by the center of mass are modified after the collision. An animated simulation of the system can be found https://youtu.be/lPvavnEK1Tw.

To analyze the long time behavior of the energy, we chose $h = 10^{-3}$ as time-step, as well as $q(0) = (\pi/2, 0, 5)$ and $v_q(0) = (-10, 2, 0)$ as initial conditions. As can be seen in Figure 3, there is a slight energy fluctuation on the long time, but the total energy remains nearly constant.

4.3. **Nonholonomic spherical pendulum hitting a cylindrical surface.** Let us consider a spherical pendulum (cf. [12, §5.1]) hitting a cylindrical surface. The configuration space of the system is $Q = \mathbb{S}^2$, where we pick spherical coordinates $q = (\theta, \varphi) \in (-\pi/2, \pi/2) \times (-\pi, \pi)$. The Lagrangian, $L : T\mathbb{S}^2 \to \mathbb{R}$ consists of the kinetic energy, together with the gravitational potential,

$$L(\theta,\varphi; v_\theta, v_\varphi) = \frac{1}{2}m\ell^2(v_\theta^2 + v_\varphi^2\sin^2\theta) - mg\ell\cos\theta, \tag{11}$$

where $m, \ell, g > 0$ are the mass and the length of the pendulum, and the gravitational acceleration, respectively, and we denote $v = v_\theta\,\partial_\theta + v_\varphi\,\partial_\varphi \in T_q\mathbb{S}^2$. Suppose that the polar and azimuthal velocities are proportionally related by a function depending only on the polar angle, i.e., $v_\varphi = f(\theta)v_\theta$ for some $f : \mathbb{R} \to \mathbb{R}$ such that $f(\theta + \pi) = f(\theta)$ for each $\theta \in \mathbb{R}$. This assumption provides a nonholonomic constraint. If removed, we obtain the standard spherical pendulum, which may

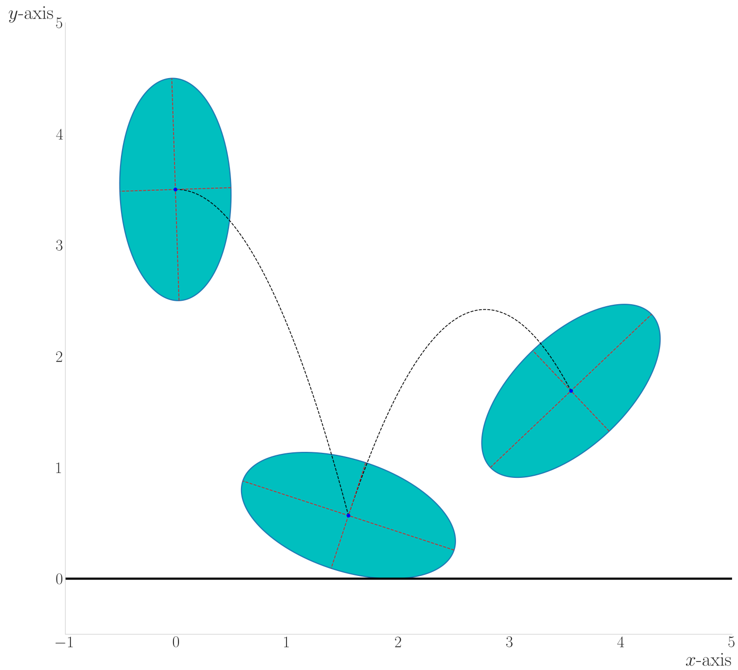

FIGURE 1. Trajectory of an ellipse rotating and colliding with the floor. The initial and final times are $t_0 = 0$ and $t_N = 1.8\,s$.

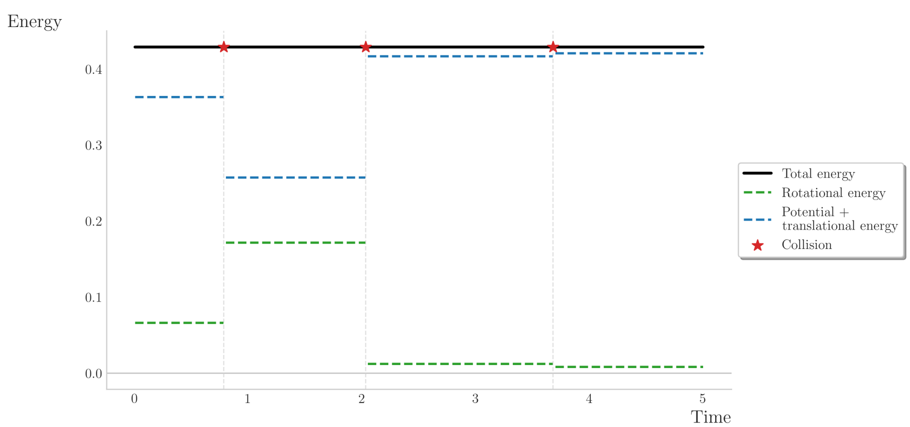

FIGURE 2. Evolution of total, rotational and translational + potential energies of an ellipse rotating and colliding with the floor. The initial and final times are $t_0 = 0\,s$ and $t_N = 5\,s$.

also be simulated using our method. Similarly, for $f \equiv 0$, we recover a standard

1-dimensional pendulum. This results in the following non-holonomic constraint:

$$\Delta_Q = \operatorname{span}\{\partial_\theta + f(\theta)\partial_\varphi\},$$
$$\Delta_Q^\circ = \operatorname{span}\{\omega^1 = f(\theta)d\theta - d\varphi\}.$$

Due to the pendulum moving inside a cylinder, the admissible set reads

$$S = \{(\theta,\varphi) \in \mathbb{S}^2 \mid \ell|\sin\theta| \leq R\},$$

where $R > 0$ is the radius of the cylinder. The tangent and cotangent bundles of ∂S are readily seen to be $T\partial S = \operatorname{span}\{\partial_\varphi\}$ and $T^*\partial S = \operatorname{span}\{d\varphi\}$, respectively, and the annihilator of the tangent bundle is given by $(T\partial S)^\circ = \operatorname{span}\{d\theta\}$. Similarly, the tangent map of the boundary inclusion and its adjoint are given by

$$T\imath_S : T\partial S \to T\mathbb{S}^2|_{\partial S}, \quad (q,v) \mapsto T_q\imath_S(q,v) = v_\varphi\,\partial_\varphi$$
$$\imath_S^* : T^*\mathbb{S}^2|_{\partial S} \to T^*\partial S, \quad (q,p) \mapsto \imath_S^*(q,p) = p_\varphi\,d\varphi,$$

for each $v = v_\varphi\,\partial_\varphi \in T_q\partial S$ and $p = p_\theta\,d\theta + p_\varphi\,d\varphi \in T_q^*\mathbb{S}^2$.

Given a time-step $h \in \mathbb{R}^+$, the retraction may be chosen as

$$\mathcal{R} : T\mathbb{S}^2 \to \mathbb{S}^2, \quad (q,v) \mapsto \mathcal{R}_q(v) = (\theta + h\,v_\theta, \varphi + h\,v_\varphi).$$

For fixed $q = (\theta,\varphi) \in \mathbb{S}^2$, the inverse of $\mathcal{R}_q : T_q\mathbb{S}^2 \to \mathbb{S}^2$ is given by

$$\mathcal{R}_q^{-1}(q^+) = \left(q,\ \frac{\theta^+ - \theta}{h}\,\partial_\theta + \frac{\varphi^+ - \varphi}{h}\,\partial_\varphi\right), \qquad q^+ \in \mathbb{S}^2.$$

As a result, the discrete Lagrangian $L_d : \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R}^+ \to \mathbb{R}$ reads

$$\begin{aligned} L_d(q,q^+,h) &= h\,L\left(q, \mathcal{R}_q^{-1}(q^+)\right) \\ &= \frac{m\ell^2}{2h}\left((\theta^+ - \theta)^2 + (\varphi^+ - \varphi)^2\sin^2\theta\right) - hmg\ell\cos\theta, \end{aligned}$$

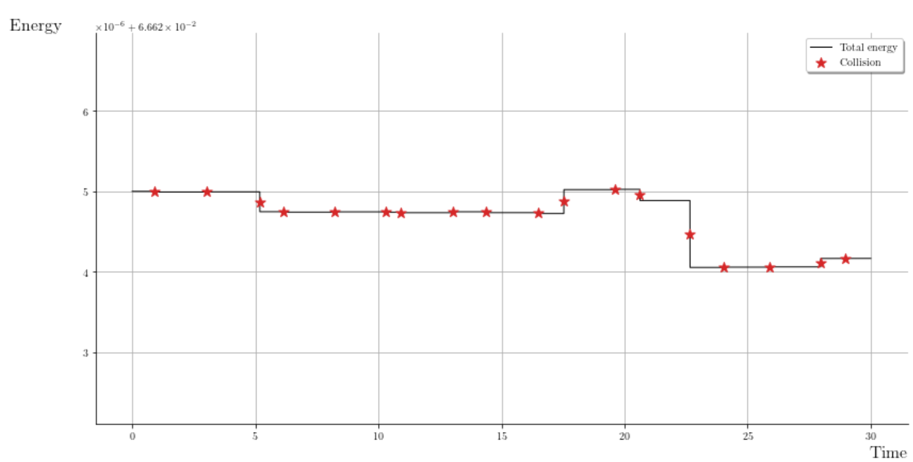

FIGURE 3. Long-time evolution of the total energy of an ellipse rotating and colliding with the floor. The initial and final times are $t_0 = 0\,s$ and $t_N = 30\,s$. The system underwent 18 impacts during that period of time.

for each $(q, q^+, h) \in \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R}^+$. In the same vein, the discrete constraint map $\omega^1_{d+} : \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R}^+ \to \mathbb{R}$ reads

$$\omega^1_{d+}(q, q^+, h) = \omega^1_q(\mathcal{R}_q^{-1}(q^+)) = \frac{1}{h}\left(f(\theta)(\theta^+ - \theta) + \varphi - \varphi^+\right).$$

Numerical experiments. For simulation, we chose the following parameters: $m = 1$, $g = 9.8$, $\ell = 10$, $R = 1$ and $f(\theta) = a \sin\theta$ for some $a > 0$. Similarly, the time-step and initial conditions are chosen as $h = 10^{-3}$, $q(0) = (\theta(0), \varphi(0)) = (0,\ 0)$ and $v(0) = (v_\theta(0), v_\varphi(0)) = (0.25,\ 0)$, respectively. In Figure 4, we depict the evolution of the angles during 5 seconds for $a = 50$. The system underwent three collisions during that period. The motion of the pendulum during that period can be found https://youtube.com/shorts/oBfowpcoUSA?feature=share. Note that the 'wavy' trajectory is due to the nonholonomic constraint.

Lastly, we compared our variational method with standard algorithms; namely:

1. Event detection method with elastic impact, where the continuous dynamics is integrated using an adaptive Runge–Kutta scheme while keeping track of the constraint function $g(\theta) = R - \ell\,|\sin\theta|$. When a zero crossing with negative slope is detected, the exact impact time is located by root-finding and the conditions for an elastic impact are applied.
2. Moreau time-stepping method [14], a first-order, fixed-step scheme that, at each iteration, the free (unconstrained, no impacts) motion is computed and, next, a linear system is solved for impulses that enforce both the nonholonomic constraint and, if a collision is detected, the elastic impact law.

The simulations were run for 10 seconds with $a = 1$, yielding the results shown in Figure 5. The collision times are nearly the same for all methods, up to numerical errors, as can be seen in Table 6. Regarding the energy behavior, the event-driven method provides the most accurate results, as expected from its use of a high-order (5th-order) adaptive Runge–Kutta integrator for the smooth dynamics between impacts. The Moreau time-stepping scheme and our variational integrator both exhibit slight energy fluctuations, but their mean energy remains nearly constant. Despite being only first-order and using a fixed time step, the good behavior of our method is due to symplecticity, which guarantees approximate preservation of the underlying geometric structure and prevents artificial energy drift. In contrast, the comparable energy behavior of the Moreau scheme is a consequence of the implicit velocity update and the perfectly elastic impact law. For more complex applications, we expect our method to be more robust due to its geometric nature.

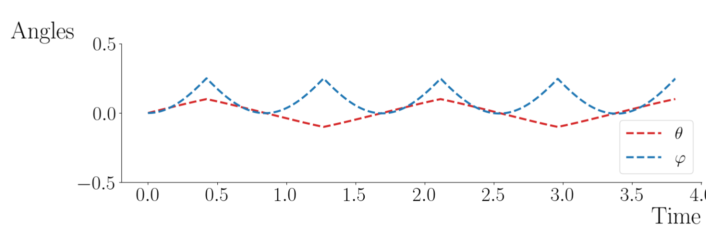

FIGURE 4. Evolution of the angles of a nonholonomic spherical pendulum colliding with a cylindrical surface. The initial and final times are $t_0 = 0\,s$ and $t_N = 5\,s$, and the time-step is $h = 10^{-3}$.

5. **Conclusions and future work.** In this paper, a discrete analog to the nonsmooth variational approach introduced in [24, 25] to treat implicit nonholonomic Lagrangian systems with collisions has been developed, thus extending the discrete theory for unconstrained systems presented in [9]. After giving the discrete Lagrange–d'Alembert–Pontryagin variational principle, the corresponding equations are computed. They consist of the discrete nonholonomic implicit Euler–Lagrange equations obtained in [15, 16], together with the discrete conditions for the elastic impact. The forward in time implementation of these equations is summarized in Algorithm 1. Lastly, some examples illustrating the theory are presented, including a bouncing ellipse and a nonholonomic spherical pendulum, and several simulations are performed to test the validity of the variational integrators.

For future work, we would like to address the following areas:

- **Discrete reduction theory.** By mimicking [25], reduced variational integrators for nonholonomic systems with collisions may be obtained. The removal of superfluous degrees of freedom will lead to significantly faster simulations.
- **Interconnection.** We would like to investigate how the impacts on a subsystem transfer to the rest of them due to the coupling. Furthermore, the interconnection approach will allow for considering systems with open ports that can be utilized to control the system [31]. In turn, the controller of the system may be computed from the optimal control perspective [7], i.e., by

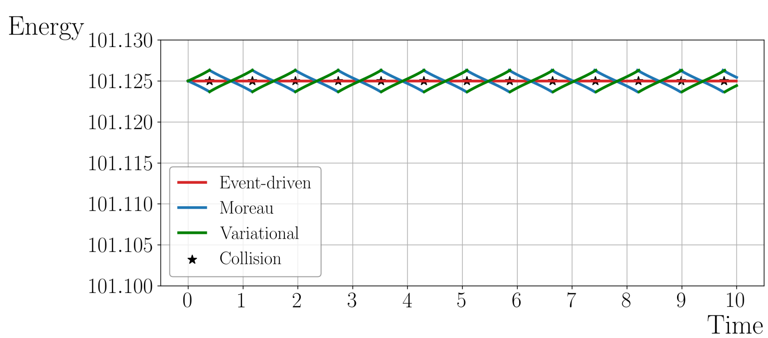

FIGURE 5. Energy comparison for the three methods. The initial and final times are $t_0 = 0\,s$ and $t_N = 10\,s$, and the time-step is $h = 10^{-3}$.

Collision	Event-driven	Moreau	Variational
1	0.39	0.39	0.39
2	1.17	1.17	1.17
3	1.95	1.95	1.96
...	...	...	...
8	8.21	8.19	8.22
9	8.99	8.97	9.00
10	9.77	9.75	9.78

FIGURE 6. Some of the collision times (in seconds) obtained with the three different methods. The initial and final times are $t_0 = 0\,s$ and $t_N = 5\,s$, and the time-step is $h = 10^{-3}$.

minimizing some cost functional. In any case, a critical step will be to ensure that the input preserves the constraints on the state, as well as on its velocity and momentum, at the impact time.

- **Comparison** of the approach followed here to carry out discretization with the alternative proposals presented in [6, 22], testing the performance of the integrators obtained from the different perspectives.
- **Energy dissipation.** The Lagrange–Pontryagin principle principle has to be substituted by the Herglotz principle as in [17, 18], which is appropriate to deal with contact Lagrangians that describe systems with energy dissipation.

REFERENCES

[1] H. Andersen, Rattle: A "velocity" version of the shake algorithm for molecular dynamics calculations, *Journal of Computational Physics*, **52** (1983), 24-34.

[2] M. Barbero-Liñán and D. de Diego, Retraction maps: A seed of geometric integrators, *Found. Comput. Math.*, **23** (2022), 1335–1380.

[3] J. Breuling, G. Capobianco, S. R. Eugster and R. I. Leine, A nonsmooth RATTLE algorithm for mechanical systems with frictional unilateral constraints, *Nonlinear Analysis: Hybrid Systems*, **52** (2024), 101469, 16 pp.

[4] G. Capobianco and S. Eugster, Time finite element based moreau-type integrators, *International Journal for Numerical Methods in Engineering*, **114** (2018), 215-231.

[5] G. Capobianco and S. R. Eugster, A Moreau-type variational integrator, *PAMM*, **16** (2016), 941-944.

[6] M. Caruso, J. FernÁndez and M. Zuccalli, Discrete mechanical systems in a Dirac setting: A proposal, *Actas del XVI Congreso Antonio Monteiro*, (2021), 119-128.

[7] L. Colombo, F. Jiménez and D. M. de Diego, Variational integrators for mechanical control systems with symmetries, *Journal of Computational Dynamics*, **2** (2015), 193-225.

[8] J. Cortés and S. Martínez, Non-holonomic integrators, *Nonlinearity*, **14** (2001), 1365-1392.

[9] R. C. Fetecau, J. E. Marsden, M. Ortiz and M. West, Nonsmooth Lagrangian mechanics and variational collision integrators, *SIAM Journal on Applied Dynamical Systems*, **2** (2003), 381-416.

[10] F. Gay-Balmaz, Á. Rodríguez Abella and H. Yoshimura, Infinite-dimensional Lagrange-Dirac systems with boundary energy flow I: Foundations, 2025.

[11] F. Gay-Balmaz and H. Yoshimura, Dirac reduction for nonholonomic mechanical systems and semidirect products, *Adv. Appl. Math.*, **63** (2015), 131-213.

[12] M. E. E. Irazú, L. Colombo and A. Bloch, Reduction by symmetries of simple hybrid mechanical systems, *IFAC-PapersOnLine*, **54** (2021), 94-99.

[13] H. O. Jacobs and H. Yoshimura, Tensor products of Dirac structures and interconnection in Lagrangian mechanics, *J. Geom. Mech.*, **6** (2014), 67-98.

[14] M. Jean and J. J. Moreau, Dynamics in the presence of unilateral contacts and dry friction: A numerical approach, in *Unilateral Problems in Structural Analysis III. Proceedings of the Third Meeting on Unilateral Problems in Structural Analysis*, Prescudin, Italy, on June 17 to 20, 1985, Wien etc.: Springer-Verlag, (1987), 151-196.

[15] M. Leok and T. Ohsawa, Discrete Dirac structures and implicit discrete Lagrangian and Hamiltonian systems, *AIP Conf. Proc.*, **1260** (2010), 91-102.

[16] M. Leok and T. Ohsawa, Variational and geometric structures of discrete Dirac mechanics, *Found. Comput. Math.*, **11** (2011), 529-562.

[17] A. López-Gordón, L. Colombo and M. de León, Nonsmooth Herglotz variational principle, in *2023 American Control Conference (ACC)*, (2023), 3376-3381.

[18] E. Maciel, I. Ortiz and C. E. Schaerer, A herglotz-based integrator for nonholonomic mechanical systems, *Journal of Geometric Mechanics*, **15** (2023), 287-318.

[19] J. E. Marsden and M. West, Discrete mechanics and variational integrators, *Acta Numerica*, **10** (2001), 357–514.

[20] J. Moreau, Unilateral contact and dry friction in finite freedom dynamics, in *Nonsmooth Mechanics and Applications* (eds. M. J.J. and P. P.D.), International Centre for Mechanical Sciences (Courses and Lectures), Springer, **302** (1988), 1-82, `https://hal.science/hal-01713847`.

[21] H. Parks and M. Leok, Variational integrators for interconnected Lagrange-Dirac systems, *J. Nonlinear Sci.*, **27** (2017), 1399-1434.

[22] L. Peng and H. Yoshimura, Discrete Dirac structures and discrete Lagrange-Dirac dynamical systems in mechanics, *J. Math. Phys.*, **66** (2025), 42 pp.

[23] A. Rodríguez Abella, Interconnection of Lagrange-Dirac systems through nonstandard interaction structures, *Int. J. Geom. Methods Mod. Phys.*, **20** (2023), 2350124, 28 pp.

[24] A. Rodríguez Abella and L. Colombo, Implicit nonholonomic mechanics with collisions, *IFAC-PapersOnLine*, **58** (2024), 71-76.

[25] A. Rodríguez Abella and L. Colombo, Nonsmooth nonholonomic reduction for mechanical systems with collisions, *SIAM J. Appl. Algebra. Geom.*, **9** (2025), 577-602.

[26] A. Rodríguez Abella, F. Gay-Balmaz and H. Yoshimura, Infinite dimensional Lagrange-Dirac mechanics with boundary conditions, in *Geometric Science of Information*, **14072** (2023), 202-211.

[27] A. Rodríguez Abella and M. Leok, Discrete Dirac reduction of implicit Lagrangian systems with abelian symmetry groups, *J. Geom. Mech.*, **15** (2023), 319-356.

[28] K. Tran and M. Leok, Lie group variational collision integrators for a class of hybrid systems, *SIAM J. Appl. Dyn. Syst.*, **24** (2025), 1622-1667, `https://arxiv.org/abs/2310.15356`.

[29] H. Yoshimura and J. Marsden, Dirac structures in Lagrangian mechanics Part I: Implicit Lagrangian systems, *J. Geom. Phys.*, **57** (2006), 133-156.

[30] H. Yoshimura and J. Marsden, Dirac structures in Lagrangian mechanics Part II: Variational structures, *J. Geom. Phys.*, **57** (2006), 209-250.

[31] H. Yoshimura and J. E. Marsden, Representations of Dirac structures and implicit port-controlled Lagrangian systems, *The Proceedings of the Dynamics & Design Conference*, **2008**.

[32] H. Yoshimura and J. Marsden, Reduction of Dirac structures and the Hamilton-Pontryagin principle, *Rep. Math. Phys.*, **60** (2007), 381-426.

Received August 4, 2025; revised April 20, 2026.