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RESEARCH ARTICLE

Dealing with Non-Reciprocal Matrices in the Additive and Fuzzy Preference Relations Theoretical Frameworks

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ABSTRACT
Many multiple-criteria decision aiding methods apply the so-called multiplicative pairwise comparisons, where the comparisons have the form of a ratio expressing how many times one entity is more important (or preferred) than another. Besides the multiplicative system, additive and fuzzy preference relations systems have been proposed for pairwise comparisons in recent decades. These systems are appealing for their intuitive use and natural properties, but they are not as intensively studied as their multiplicative counterpart. Namely, studies on inconsistency and non-reciprocity in particular, in both theoretical frameworks, are rather scarce and fragmented. Therefore, our study focuses on the problem of non-reciprocity in both frameworks and fills the current gaps in its understanding and evaluation. We introduce measures of non-reciprocity in the additive and fuzzy preference relations frameworks compatible with a previously published measure of non-reciprocity in the multiplicative framework, and we show that all measures are specific representations of a general measure of non-reciprocity based on an Alo-group approach. Further on, we show that new measures are endowed with a set of desirable properties. Furthermore, we perform Monte Carlo simulations on randomly generated non-reciprocal matrices both in additive and fuzzy systems and provide percentile tables allowing decision makers to decide whether a level of non-reciprocity of a given PC matrix is acceptable or not.

KEYWORDS
Additive pairwise comparisons; consistency; fuzzy pairwise comparisons; fuzzy preference relations; multiple-criteria decision making; pairwise comparisons; reciprocity.

1. Introduction

Many multiple-criteria decision-making methods, such as the AHP/ANP, BWM, ELECTRE, MACBETH, PAPRIKA, PROMETHE, etc., include as one of their major features pairwise comparisons (PC), that is, comparisons of only two objects at the same time, e.g. Bana et al. (2005); Brans & Vincke (1985); Govindan & Jepsen (2015); Rezaei (2015); Roy (1968); Saaty (1980, 2008); Vaidya & Kumar (2006). These methods have been applied to most areas of human activity, and are subjects of intense research in terms of their properties, including the aggregation of preferences or their consistency.

Since pairwise comparisons provided by humans are often imperfect, numerous studies focused on the problem of their inconsistency, see e.g. Brunelli (2017), Brunelli (2018), Brunelli & Fedrizzi (2015), Gonzalez-Pachon & Romero (2004), Karapetrovic & Rosenbloom (1999), Koczkodaj & Szwarc (2014), Koczkodaj & Urban (2018), Lin et al. (2014), Mazurek (2023), Mazurek & Linares (2023), Vargas (2008), or Wang & Deng (2023).

However, one of the features of pairwise comparison, typically neglected, is the potential non-reciprocity of the judgements expressed by the decision makers. For example, if object A is two times more important than object B, then the reciprocal judgement would be that B is only half as important as A. This reciprocity is typically taken for granted in many of the methods mentioned above. This is understandable, given that it seems rational, natural and intuitive (Harker & Vargas (1987)). However, empirical studies have shown (see the next section) that human experts and decision-makers are often not consistent in their judgments. The explanation of this phenomenon may be related to external circumstances affecting the assessment process, such as changes in the context (e.g. Tversky & Simonson (1993)), unobserved variables, asymmetric transaction costs (e.g. non-reciprocal currency exchange rates), or hysteresis. But without a doubt, the main source of non-reciprocal judgments is the human mind, which is known to be imperfect and susceptible to many cognitive biases.

Reciprocity is an important feature of pairwise comparison matrices, in that, on the one hand, it reflects the rationality of the decision maker, and on the other, if absent, introduces ambiguity in the elicitation of preferences from these matrices. Understanding correctly how the lack of reciprocity behaves and determines preferences is essential for good decision-making. The question is how to deal with this lack of reciprocity in pairwise comparisons: should it be considered a source of important information, or a flaw of the cognitive process, and hence should it be accepted or eliminated? In case it is accepted, should there be acceptable levels?

Motivated by an earlier study of Mazurek & Linares (2023), where these questions were addressed in the multiplicative pairwise comparisons (MPC) framework, we extend the analysis of the previous questions to two alternative pairwise comparisons theoretical frameworks: *additive* and *fuzzy*. Although less used than the MPC framework, additive and fuzzy frameworks are also used for decison making and have their own virtues, hence they deserve a proper analysis.

Our study has several interconnected objectives.

Firstly, we introduce new measures of non-reciprocity both in the *additive pairwise comparisons* (APC) and *fuzzy preference relations* (FPR) frameworks compatible with a previously published measure, and show these three measure are in fact three particular representations of a general measure of non-reciprocity with respect to an Alo-group approach. Secondly, we perform extensive Monte Carlo simulations to provide percentile tables allowing a decision maker to reject or tolerate a given non-reciprocal APC or FPR matrix with respect to the size of the matrix and applied p-metrics.

The previous paragraph also summarizes the novelty of our work: from a theoretical viewpoint, it is the analysis of non-reciprocity in the additive and fuzzy system, and its synthesis into a unified Alo-group approach. The advantage of the Alo-group approach rests, for example, in the fact that when a theorem is proved (or a property defined), then this theorem holds (or this property emerges) in all three main PC theoretical frameworks as well.

From a practical perspective, we provide a decision maker with a simple tool (in the form of percentile tables) to decide whether a given PC matrix is tolerably non-reciprocal or not. This is a clear analogy to the assessment of inconsistency (for example by Saaty's consistency index CI and consistency ratio CR) in the multiplicative pairwise comparisons framework. More specifically, to evaluate the extent of non-reciprocity, we borrow the 10% threshold on inconsistency from Saaty's AHP (recall that non-reciprocity is a special case of inconsistency, hence they should be treated identically). When the non-reciprocity exceeds the 10% threshold, the judgments are considered too irrational (unreliable, imprecise, etc.) and should be revised by an expert, or discarded.

The paper is organized as follows: Section 2 reviews the existing literature; Section 3 discusses algebraic structure of pairwise comparisons in the case of non-reciprocity, Sections 4 and 5 are devoted to additive and fuzzy preference relations frameworks respectively, section 6 provides a unified approach, and finally, numerical Sections 7 provides percentile tables for non-reciprocity measures in both frameworks. Conclusions close the article.

2. Literature review

In the realm of decision-making, as highlighted in the preceding section, the challenge of non-reciprocal matrices often arises. Addressing this challenge is crucial for ensuring accurate and consistent decision outcomes. Essentially, the literature presents four strategies or approaches to manage non-reciprocal matrices:

(1) Discarding information in order to consider only reciprocal matrices.
(2) Transformations to reciprocal matrices. Some methods transform non-reciprocal matrices into reciprocal ones that are subsequently managed and analyzed by standard optimization methods.
(3) Approximations to priority vectors. This approach aims at finding a reciprocal (or more generally, a fully consistent) approximation of a given non-reciprocal matrix.
(4) Developments of a new theoretical framework. This category comprises papers that merge or expand the standard methodology by shifting from the reciprocity paradigm to a more realistic theoretical framework without including non-reciprocity.

An example of the first approach is the study by Saaty (1994), where one of the pairwise comparisons from a non-reciprocal pair are discarded (randomly, as there is no available information to decide which of the two comparisons is the most accurate). As pointed out by Linares et al. (2016), some experts believe that the extra information is unnecessary, because only $(n-1)$ pairwise comparisons are needed (assuming that the decision-maker is perfectly rational and the corresponding directed graph is connected). Indeed, this reciprocity condition is foundational to the AHP, as corroborated by sources like (Harker & Vargas, 1987) and (Saaty, 1986). However, decision-makers might not always behave rationally, so before deciding to discard any information, it is essential to measure how inconsistent the provided preferences are. Thurstone acknowledged in his 1927 Law of Comparative Judgment (Thurstone, 1927) that decision-makers could offer differing comparative judgments on the same stimulus pair over different instances.

Grzybowski (2012) showed by numerical simulations that forcing reciprocity into an MPC matrix may result in poorer estimations of a priority vector. Diaz-Balteiro et al. (2009) discovered inherent non-reciprocity when soliciting identical comparisons at different times, especially in forest management contexts. In their study, participants first completed the upper triangle of the MPC matrix and, after a month, the lower triangle. Notably, none of the acquired MPC matrices were reciprocal. Fülöp et al.

(2012) added to the list of possible causes for non-reciprocity the fact that several teams might be working in the assessments independently, or the fuzziness in the underlying preference relations. They also provided several real-world examples where the reciprocity condition is violated, including double-blind wine tasting.

From an economic perspective, Hovanov et al. (2008) mentioned the case of transaction costs in exchange rates as a reason for non-reciprocity. In different areas, such as the choice to vote for a political party, it was believed traditionally that preferences stayed constant over time. However, de Andres et al. (2020) demonstrated, with the introduction of local and global decision stability measures, that political preferences might shift with greater frequency than once believed. Additionally, Starczewski (2017) established that while the AHP assumes decision-makers may err in comparing pairs of alternatives, the adopted scale can exacerbate these discrepancies.

Mazurek & Linares (2023) have reviewed the most relevant literature on dealing with this inconsistency in MPC matrices. This literature is large, which is reasonable given its popularity. However, other theoretical frameworks for pairwise comparisons exist, which have not received the same level of attention in terms of their reciprocity properties, namely additive pairwise comparisons (APC) and fuzzy preference relations (FPRs), which constitute the focal point of this paper.

Additive pairwise comparisons were introduced by Barzilai & Golany (1990). The additive formulation of pairwise comparisons allows to use of the mathematical apparatus of linear algebra, see e.g. (Fedrizzi et al., 2020). Unfortunately, we have not found any discussions in the literature about the reciprocity of APC matrices.

Fuzzy preference relations were introduced by Orlowsky (1978) and later elaborated in Tanino (1984), Fodor & Roubens (1994) or Herrera et al. (2004). FPRs enable natural pairwise comparisons by dividing a line segment of a unit length into two segments corresponding to two compared entities. Also, FPRs enable the modelling of linguistic preferences, see e.g. Marimin et al. (1998). Traditional methods using fuzzy preference relations (FPRs) are based on the idea that preferences are mutually consistent or reciprocal. However, this assumption does not always align with the intricate and unpredictable situations that experts might face with different opinions in real-world scenarios, including business strategy meetings, urban planning, healthcare, public policy, and so on. In this context, (Liu et al., 2021a,b,c) introduced the concept of additively reciprocal property breaking (ARPB), and in (Liu et al., 2022a,b,c; Luo et al., 2023) the concept of non-reciprocal fuzzy preference relations (NrFPRs) was proposed to capture situations where the traditional reciprocal assumptions of FPRs are not met and also constructing optimization models that can elicit priorities from non-reciprocal matrices. Finally, in (Jiang et al., 2021) a non-reciprocal fuzzy preference relation (NrFPR) and a probabilistic linguistic preference relation (PLPR) were combined to obtain a model capable of expressing partial relations of alternatives (indifference, preference, and incomparability relations) providing a practical application in selecting social donation channels during the COVID-19 outbreaks. It should be noted that non-reciprocal pairwise comparison matrices were also studied in the context of incomplete information, see e.g. Khalid & Beg (2018), Huang et al. (2020) and Zhang (2022). Measures of non-reciprocity in the context of (additive) fuzzy preference relations were recently discussed in (Liu et al., 2022c), (Wang & Deng, 2022), and (Yang, 2022).

As mentioned earlier, the goal of this paper is to fill the research gap in characterizing reciprocity in APC and FPR matrices in a homogeneous manner, which is also compatible with previous work on MPC (Mazurek & Linares, 2023). In the next sections, we proceed to the evaluation of non-reciprocity in APC and FPR systems.

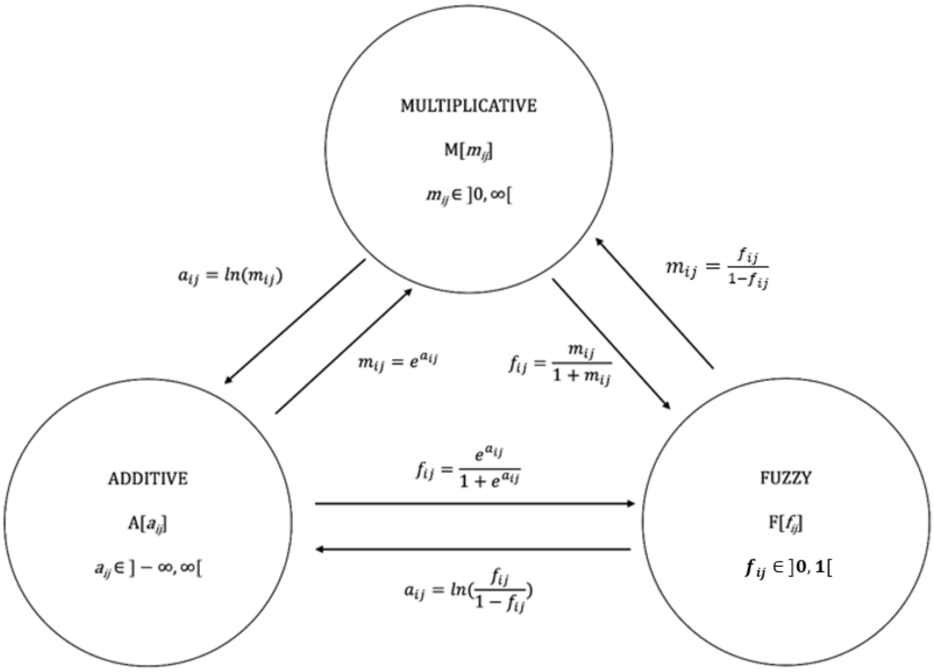

Figure 1. Transformations among the three pairwise comparisons frameworks. All logarithms and exponential functions are taken with respect to the natural base e.

3. Algebraic structure of multiplicative, additive and fuzzy preference relations systems in the case of non-reciprocity

It is well known that multiplicative pairwise comparisons, additive pairwise comparisons and fuzzy preference relations share an identical algebraic structure and form three isomorphic representations of the same *Abelian linearly ordered group (Alo group)*, see e.g. Cavallo & D'Apuzzo (2009), Cavallo & Brunelli (2018), or Kulakowski et al. (2019).

Thereinafter, we show that this is true only when pairwise comparisons are reciprocal.

Definition 3.1. Let $\mathbb{G}$ be a non-empty set equiped with a binary operation $\odot : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$. The set $\mathbb{G}$ is a *group* if the following axioms 1)-3) are satisfied:

(1) Associativity: for all $a, b, c \in \mathbb{G}$ holds $(a \odot b) \odot c = a \odot (b \odot c)$.
(2) Existence of a unique neutral element e: for all $a \in \mathbb{G}$ holds $a \odot e = e \odot a = a$.
(3) Existence of a unique inverse element: for all $a \in \mathbb{G}$, there exists a^{-1} satisfying $a \odot a^{-1} = a^{-1} \odot a = e$, where e is the neutral element.
(4) Commutativity: $a \odot b = b \odot a$ for all $a, b \in \mathbb{G}$.

If the additional axiom (4) is satisfied, the group is called Abelian or commutative.

Group operations are standard multiplication "$\cdot$" in the multiplicative system (group), standard addition "+" in the additive system and the operation "$\otimes$", see Chapter 5, for the fuzzy system. Transformations (isomorphisms) among multiplicative, additive, and fuzzy systems are shown in Figure 1. All systems are endowed with additional properties such as linear ordering or distance, for details see e.g. Cavallo & et al. (2023).

In the following sections, we introduce measures of non-reciprocity in the additive and fuzzy preference relations systems, which correspond to the measure of non-reciprocity already introduced in Mazurek & Linares (2023) for the multiplicative system. Then, we demonstrate that these measures can be unified into a single general measure of non-reciprocity based on the Alo-group approach.

4. Non-reciprocity in the additive framework

In the additive pairwise comparisons system, a preference of i-th object over a j-th object is expressed by a value $a_{ij} \in \mathbb{R}$, where $a_{ij} > 0$ means that the i-th object is preferred over a j-th object, $a_{ij} = 0$ means indifference between both objects and $a_{ij} < 0$ means that the j-th object is preferred over the i-th object. It is common to understand the value of a_{ij} in the sense of "by how much an object i is better (more preferred) than an object j", see also Figure 2.

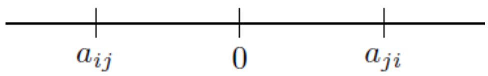

Figure 2. Reciprocal additive pairwise comparisons are antisymmetrical.

Let $A_{n\times n} = [a_{ij}]$ be an additive pairwise comparisons (APC) matrix. The *reciprocity condition* is given as follows:

$$a_{ij} + a_{ji} = 0, \ \ \forall i, j \in \{1, \ldots, n\} \tag{1}$$

The more general *consistency condition* is defined as follows:

$$a_{ij} + a_{jk} = a_{ik}, \ \ \forall i, j, k \in \{1, \ldots, n\} \tag{2}$$

Remark 4.1. Let $A_{n\times n} = [a_{ij}]$, with $a_{ij} \in \mathbb{R}$, be an additive pairwise comparison (APC) matrix of order n, satisfying the consistency condition 2. Then the following properties hold:

(1) Diagonal neutrality: $a_{ii} = 0$ for all $i \in \{1, \ldots, n\}$.
(2) Reciprocity: $a_{ij} + a_{ji} = 0$ for all $i, j \in \{1, \ldots, n\}$.
(3) Zero total sum: If A is reciprocal, then the sum of all its entries satisfies

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} = 0.$$

Conversely, if this total sum differs from zero, then A cannot be reciprocal.

However, the fact that the sum of all matrix elements is equal to 0 does not imply that an APC matrix is reciprocal, as shown in the following counterexample:

$$A = \begin{bmatrix} 0 & 8 & -4 \\ -5 & 0 & 5 \\ -8 & 4 & 0 \end{bmatrix}.$$

The proposed measure of non-reciprocity based on the relation (1) and the definition of a p-norm is introduced as follows.

Definition 4.2. Let $A_{n\times n} = [a_{ij}], a_{ij} \in \mathbb{R}$, be an APC matrix. Let $p \geq 1$. Then the non-reciprocity measure Ξ (for non-diagonal elements) is given as follows:

$$\Xi_p(A) = (\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |a_{ij} + a_{ji}|^p)^{\frac{1}{p}} \ . \tag{3}$$

Further on, the measure (3) can be normalized by the factor (the number of reciprocal pairs) $\frac{n(n-1)}{2}$, where n is the matrix order:

$$\Xi_{p,n}(A) = \frac{2}{n(n-1)}(\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |a_{ij} + a_{ji}|^p)^{\frac{1}{p}} \ . \tag{4}$$

Example 4.3. Consider the following additive PC matrix $A = [a_{ij}]$ and evaluate its non-reciprocity (for $p = 1$) via the measure (3)

$$A = \begin{bmatrix} 0 & 10 & -5 & 15 \\ -8 & 0 & 8 & 22 \\ 4 & -8 & 0 & 13 \\ -15 & -20 & -13 & 0 \end{bmatrix}.$$

From (3), we have: $\Xi_1(A) = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |a_{ij} + a_{ji}| = |10 + (-8)| + |(-5) + 4| + |15 + (-15)| + |8 + (-8)| + |22 + (-20)| + |13 + (-13)| = 5$.

Proposition 4.4. *The measure of non-reciprocity in the additive system, Ξ_p, given by relation (3), is identical to the measure of non-reciprocity $\Theta^{(p)}(M) = \left(\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |\ln(m_{ij} \cdot m_{ji})|^p\right)^{1/p}$ in the multiplicative system, where $M = [m_{ij}]$ denotes the multiplicative pairwise comparison matrix.*

Proof. The transformation from the multiplicative to the additive system (see Figure 1) is given by $m_{ij} = e^{a_{ij}}$. Substituting this into $\Theta^{(p)}(M)$, we get:

$$\Theta^{(p)}(M) = \left(\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |\ln{(e^{a_{ij}} \cdot e^{a_{ji}})}|^p\right)^{1/p} = \left(\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |\ln{(e^{a_{ij}})} + \ln{(e^{a_{ji}})}|^p\right)^{1/p}.$$

Assuming the natural logarithm, we obtain:

$$\Theta^{(p)}(M) = \left(\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} |a_{ij} + a_{ji}|^p \right)^{1/p} = \Xi_p(A),$$

as required. □

It is shown in Mazurek & Linares (2023) that the measure of non-reciprocity in the multiplicative system $\Theta^{(p)}$ satisfies several desirable properties, namely:

- Property 1 (Existence of a unique element representing reciprocity).
- Property 2 (Invariance under permutation of alternatives).
- Property 3 (Monotonicity under reciprocity-preserving mapping).
- Property 4 (Monotonicity on single comparisons).
- Property 5 (Continuity).
- Property 6 (Invariance under inversion of preferences).

Due to simple (*log* and *exp*) transformation between multiplicative and additive systems, also the newly introduced measure $\Xi_{(p)}$ also satisfies all aforementioned properties.

5. Non-reciprocity in the fuzzy preference relations framework

In the fuzzy preference relations (FPR) system, the preference of object i over object j is represented by a value $f_{ij} \in]0,1[$. When $f_{ij} > 0.5$, it indicates that object i is preferred over object j. If $f_{ij} = 0.5$, there is no preference between the two objects, indicating indifference. Conversely, if $f_{ij} < 0.5$, it implies that object j is preferred over object i. For a visual representation, see also Figure 3.

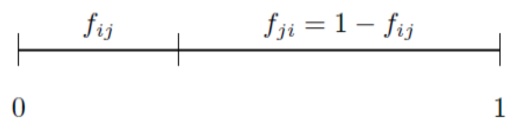

Figure 3. Reciprocal fuzzy preference relations.

Let $F_{n \times n} = [f_{ij}]$ be a fuzzy preference relation (FPR) matrix. The reciprocity condition is defined by

$$f_{ij} + f_{ji} = 1, \;\; \forall i, j \in \{1, \ldots, n\}. \tag{5}$$

The group operation "$\odot$ " is defined as follows: $f_1 \odot f_2 = \frac{f_1 \cdot f_2}{f_1 \cdot f_2 + (1-f_1)\cdot(1-f_2)}$, where $f_1, f_2 \in FPR$, and "$\cdot$" on the right hand side is a standard multiplication.

The generalized consistency condition for the fuzzy preference relation (FPR) is defined as follows (Tanino, 1984):

$$f_{ij} - f_{ik} - f_{kj} + 0.5 = 0, \quad \forall i, j, k \in \{1, \ldots, n\}. \tag{6}$$

Proposition 5.1. *Let $F_{n\times n} = [f_{ij}]$, be a FPR matrix, where $f_{ij} \in]0,1[$, satisfying the generalized consistency condition* (6). *Then:*

(a) $f_{ii} = 0.5$, *for all* $i \in \{1, 2, \ldots, n\}$.
(b) The reciprocity condition (5) *is satisfied.*

Proof. To prove part (a), let $i = j = k$. Then, substituting in (6), we obtain

$$f_{ii} - f_{ii} - f_{ii} + 0.5 = 0,$$

which simplifies to $f_{ii} = 0.5$ for all i.

For part (b), let $i = j \neq k$. Substituting into the consistency condition yields:

$$f_{ii} - f_{ik} - f_{ki} + 0.5 = 0,$$

which simplifies to $f_{ik} + f_{ki} = 1$, for all i, j, k, as required. □

The next proposition is a straightforward consequence of the reciprocity condition.

Proposition 5.2. *Let $F_{n\times n} = [f_{ij}]$ be a reciprocal fuzzy preference relation (FPR) matrix of orden n, where $f_{ij} \in]0,1[$ and $f_{ij} + f_{ji} = 1$, for all $i \neq j$, and $f_{ii} = 0.5$, for all i. Then, the sum of all elements in the matrix is equal to $\frac{n^2}{2}$.*

Proof. The total sum of the elements of $F_{n\times n}$ can be decomposed as

$$\sum_{i,j} f_{ij} \;=\; \sum_{i=1}^{n} f_{ii} \;+\; \sum_{i<j} \big(f_{ij} + f_{ji}\big) \;=\; \tfrac{n}{2} \;+\; \tfrac{n(n-1)}{2} \;=\; \tfrac{n^2}{2}.$$

□

On the contrary, when the sum of all matrix elements differs from $\frac{n^2}{2}$, then an FPR matrix is necessarily non-reciprocal:

Proposition 5.3. *Let $F_{n\times n} = [f_{ij}], f_{ij} \in]0,1[$ be a FPR matrix and $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \neq \frac{n^2}{2}$. Then the matrix F is non-reciprocal.*

However, the fact that the sum of all matrix elements is equal to $\frac{n^2}{2}$ does not imply that a matrix is reciprocal, as shown in the following counterexample:

$$\begin{bmatrix} 0.5 & 0.8 & 0.4 \\ 0.4 & 0.5 & 0.7 \\ 0.4 & 0.3 & 0.5 \end{bmatrix}.$$

To quantify the extent of non-reciprocity, a suitable function is necessary. The only attempts in this direction are the so-called additive reciprocity property-breaking functions proposed in Liu et al. (2021c) and revised in Wang & Deng (2022). However, these functions are defined only for the case $0 \leq f_{ij} + f_{ji} \leq 1; \forall i, j$, thus neglecting the possibility of $f_{ij} + f_{ji} > 1$ for some pair (i, j). Also, the authors do not attempt to investigate the measures' properties or consider transformations (isomorphisms) among multiplicative, additive and fuzzy theoretical frameworks.

Therefore, we propose a new measure of non-reciprocity of FPR matrices without the shortcomings above, based on the definition of a $p-$norm as follows.

Definition 5.4. Let $F_{n\times n} = [f_{ij}]$, with $f_{ij} \in]0,1[$, be a FPR matrix of order n. Let $p \geq 1$. Then the non-reciprocity measure Υ (for non-diagonal elements) is given as follows:

$$\Upsilon_p(F) = (\sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} |log(\frac{f_{ij} \cdot f_{ji}}{(1-f_{ij})(1-f_{ji})})|^p)^{\frac{1}{p}} \ , \tag{7}$$

where the operation "$\cdot$" in (7) is the usual multiplication. Further on, the measure (7) can be normalized by the factor (the number of reciprocal pairs) $\frac{n(n-1)}{2}$, where n is the matrix order:

$$\Upsilon_{p,n}(F) = \frac{2}{n(n-1)} (\sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} |log(\frac{f_{ij} \cdot f_{ji}}{(1-f_{ij})(1-f_{ji})})|^p)^{\frac{1}{p}} \tag{8}$$

Example 5.5. Consider the following fuzzy preference relations matrix $F = [f_{ij}]$ and evaluate its non-reciprocity (for $p = 1$) via the measure (7).

$$F = \begin{bmatrix} 0.5 & 0.4 & 0.8 & 0.7 \\ 0.6 & 0.5 & 0.6 & 0.3 \\ 0.1 & 0.3 & 0.5 & 0.6 \\ 0.4 & 0.5 & 0.4 & 0.5 \end{bmatrix}.$$

From (7) we have: $\Upsilon_1(F) = (|log(\frac{0.4 \cdot 0.6}{(1-0.4)(1-0.6)})| + |log(\frac{0.8 \cdot 0.1}{(1-0.8)(1-0.1)})| + ... = 0 + 0.811 + 0.560 + 0.442 + 0.847 + 0 = 2.660$

Due to the transformations among MPC, APC and FPRs systems shown in Figure 1, the measure $\Upsilon_p(F)$ satisfies desirable properties listed in the previous section.

6. A general measure of non-reciprocity

As mentioned above, multipliciative, additive and fuzzy systems share the same (group) structure, therefore it is natural to ask whether a more general measure of non-reciprocity including Ξ and Υ measures as its particular cases exists.

This measure exists indeed, and is provided in the following definition.

Definition 6.1. Let $\mathbb{G}$ be a group endowed with an operation $\odot$, and let $G = [g_{ij}]$ be a pairwise comparison matrix of the order n. Then the measure of non-reciprocity is given as follows:

$$\Phi_p(G) = (\sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} |\varphi(g_{ij} \odot g_{ji})|^p)^{1/p} \ . \tag{9}$$

The function φ in relation (9) depends on the particular pairwise comparisons system, see Table 1. Note that φ corresponds to transformation relations among PC systems in Figure 1.

Table 1. The general measure of non-reciprocity and measures of non-reciprocity in the additive, multiplicative, and fuzzy systems.

System	φ	Measure
M	log	$\Theta^{(p)}(M) = \left(\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \lvert \log(m_{ij} \cdot m_{ji}) \rvert^p \right)^{1/p}$
A	1	$\Xi_p(A) = \left(\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \lvert a_{ij} + a_{ji} \rvert^p \right)^{1/p}$
F	$\log \frac{g}{1-g}$	$\Upsilon_p(F) = \left(\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \left\lvert \log \left(\frac{f_{ij} \otimes f_{ji}}{1 - f_{ij} \otimes f_{ji}} \right) \right\rvert^p \right)^{1/p}$ $= \left(\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \left\lvert \log \left(\frac{f_{ij} \cdot f_{ji}}{(1-f_{ij})(1-f_{ji})} \right) \right\rvert^p \right)^{1/p}$

The following example illustrates that non-reciprocity evaluation in all three systems leads to identical results.

Example 6.2. Consider the following pairwise comparison matrices M, A and F corresponding to multiplicative, additive and fuzzy representations of identical preferences (each matrix can be converted into another one by formulas in Figure 1). We will evaluate their non-reciprocity for $p = 1$ (and natural logarithm).

$$M = \begin{bmatrix} 1 & 4 & 5 \\ 0.3 & 1 & 0.5 \\ 0.2 & 2 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & log(4) & log(5) \\ log(0.3) & 0 & log(0.5) \\ log(0.2) & log(2) & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 0.5 & 0.8 & 0.833 \\ 0.2308 & 0.5 & 0.333 \\ 0.167 & 0.667 & 0.5 \end{bmatrix}.$$

As can be seen, only one pair with indices $(1, 2)$ is non-reciprocal.
For the multiplicative matrix M we get:
$\Theta^{(1)}(M) = \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \lvert \log(m_{ij} \cdot m_{ji}) \rvert = \lvert log(4 \cdot 0.3) \rvert + 0 + 0 = log(1.2) = 0.182.$
For the additive matrix A we obtain:
$\Xi_1(A) = \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \lvert a_{ij} + a_{ji} \rvert = (\lvert log(4) + log(0.3) \rvert + 0 + 0 = log(1.2) = 0.182.$
For the fuzzy preference relations matrix F we get:
$\Upsilon_1(F) = (\lvert log \frac{0.8 \cdot 0.2308}{(1-0.8)(1-0.2308)} \rvert + 0 + 0 = log(1.2) = 0.182.$
The results, the extent of non-reciprocity, are identical in all systems as expected.

In Mazurek & Linares (2023), percentile tables with respect to non-reciprocity for random multiplicative PC matrices were provided. The applied scale was Saaty's scale from 1/9 to 9, which is the most common among practitioners.

In the next two sections, we provide percentile tables for randomly generated APC and FPRs matrices so that a decision maker can assess non-reciprocity of a given matrix and decide whether the extent of non-reciprocity is acceptable, or not.

For APC matrices, we use the scale from -100 to 100, which is also preferred along with ± 10 scale, see e.g. Guh & al. (2009), in practical applications. Since this scale does not correspond to Saaty's scale, percentile tables for multiplicative and additive case naturally differ. The same applies to percentile tables for FPRs matrices, where matrix elements from the interval $]0, 1[$ were considered.

7. Numerical simulations

In order to provide a decision maker with percentile tables of non-reciprocity, we carried out numerical (Monte Carlo) simulations both for additive and fuzzy preference relation matrices.

We provide our code and data in the following GitHub repository: `https://github.com/LuisAngelCalvoPascual/non_reciprocal`.

7.1. The simulation setting

The simulations, available in `code.ipynb` from the previously mentioned GitHub repository, generated two types of random square matrices of sizes 2x2, 3x3, up to 8x8:

- APC matrices with non-diagonal entries independently sampled from a uniform distribution $U(-100, 100)$.
- FPR matrices, where the diagonal entries were fixed at 0.5, while the off-diagonal elements were independently sampled from a continuous uniform distribution $U(0.0001, 0.9999)$.

The choice of the uniform distribution is in accord with many previous studies on multiplicative PC matrices, see e.g. Alonso & Lamata (2006) or Golden & Wang (1989). Notably, all numerical studies mentioned in Alonso & Lamata (2006) apply a uniform distribution as well.

For each matrix size $n \in \{2, 3, 4, 5, 6, 7, 8\}$, 100,000 matrices were randomly generated.

Next, we calculated Ξ for APC matrices and Υ for FPR matrices, and we applied three different types of norms: L_1 (Manhattan norm), L_2 (Euclidean norm) and L_∞ (Maximum norm). For each simulation setup (matrix type, size, and norm), the following percentiles of the non-reciprocity measures were computed: 0.1%, 1%, 2%, 3%, 4%, 5%, 10%, 20%, 25%, and 50% (median). The percentiles are first computed independently in each of the 10 runs, based on 10,000 random matrices per run. After obtaining the percentile values for each run, they are averaged across the 10 runs to achieve robust and stable estimates of the percentiles. The simulation outputs are saved as separate Excel ('.xlsx') files for each combination of matrix type (FPR or APC), matrix size, and norm. The results are visualised using a separate plotting notebook `plot_pmetrics_percentiles.ipynb`, which reads the generated Excel files, extracts the specified percentile values (1st, 5th, 10th, and 50th percentiles), and plots them as functions of matrix size. The visualization includes three plots for APC matrices and three plots for FPR matrices, in both cases, for the three norms. Each curve demonstrates how the percentile values of the non-reciprocity measures change as the matrix size increases from 2x2 to 8x8. Additionally, to facilitate the analysis, the notebook 'tables.ipynb' processes the individual Excel outputs and generates six consolidated tables: three tables for FPR matrices (one for each norm), and three tables for APC matrices (one for each norm).

7.2. Assessing the reciprocity of random APC matrices

For each matrix order n, a total of 100,000 random matrices were generated. Each matrix was evaluated for non-reciprocity via relation (3) for $p = 1$, $p = 2$ and $p = \infty$ (the logarithm applied was the decadic one).

Results, presented in the form of percentile tables, are summarized in Tables 2-4 and Figure 4. As can be seen, percentile values are expectedly to grow monotonically with increasing n: the larger the order of the matrix, the more significant the value of non-reciprocity for a corresponding percentile.

Now, let us consider an APC matrix of the order $n = 6$ with $\Xi_1 = 564$. Is this matrix tolerably non-reciprocal? From Table 2, it follows that such a matrix is less non-reciprocal than 99% of random APC matrices of the same order; hence, by applying the 10% rule similarly to the AHP, the matrix can be considered acceptably non-reciprocal.

Table 2. Percentile tables of Ξ defined by relation (3) for random APC matrices, $p = 1; 2 \leq n \leq 8$.

percentile	0.1	1	2	3	4	5	10	20	25	50 (median)
$n = 2$	0.11	1.01	1.99	3.03	4.07	5.10	10.31	21.05	26.93	58.66
$n = 3$	18.83	40.90	51.96	60.43	68.32	73.41	96.81	127.05	140.09	195.50
$n = 4$	105.80	155.88	179.89	195.80	208.19	217.84	253.34	301.07	318.09	395.91
$n = 5$	259.57	343.79	376.60	398.80	415.05	428.99	478.02	537.86	562.24	661.69
$n = 6$	484.88	599.19	640.72	670.23	688.82	705.57	767.10	844.26	873.39	996.60
$n = 7$	781.91	918.82	975.30	1005.37	1029.84	1051.59	1126.04	1216.39	1253.53	1396.47
$n = 8$	1148.49	1308.89	1371.39	1409.07	1438.80	1465.61	1549.82	1656.88	1695.67	1860.89

Table 3. Percentile tables of Ξ defined by relation (3) for random APC matrices, $p = 2; 2 \leq n \leq 8$.

percentile	0.1	1	2	3	4	5	10	20	25	50 (median)
$n = 2$	0.10	1.04	2.04	2.98	4.12	5.04	10.32	21.10	26.94	58.64
$n = 3$	13.34	27.58	36.46	42.23	46.96	50.71	66.43	87.13	95.57	132.50
$n = 4$	53.00	80.57	91.87	99.24	105.79	110.73	128.69	150.99	159.85	194.79
$n = 5$	103.51	138.09	151.45	159.10	166.86	171.22	189.55	211.75	220.01	254.34
$n = 6$	163.27	196.44	211.03	219.84	225.39	231.05	249.31	271.05	279.34	312.75
$n = 7$	222.04	256.81	269.32	277.56	285.01	289.68	308.00	329.87	338.59	371.63
$n = 8$	281.42	314.62	328.65	336.85	343.57	348.86	367.06	388.33	396.89	429.47

Table 4. Percentile tables of Ξ defined by relation (3) for random APC matrices, $p = \infty; 2 \leq n \leq 8$.

percentile	0.1	1	2	3	4	5	10	20	25	50 (median)
$n = 2$	0.11	1.00	2.02	2.99	4.08	5.13	10.44	21.32	26.85	58.39
$n = 3$	10.71	22.46	29.34	33.87	37.82	41.29	53.36	70.92	78.34	109.02
$n = 4$	34.06	53.18	61.46	66.90	71.20	74.79	87.34	103.23	109.07	134.04
$n = 5$	58.17	78.59	86.18	91.16	95.11	98.23	109.29	122.94	128.09	148.14
$n = 6$	77.75	97.26	104.16	108.72	112.08	114.82	124.50	136.00	140.80	157.41
$n = 7$	95.37	111.89	117.21	121.75	124.31	127.19	135.42	145.66	149.49	164.07
$n = 8$	106.55	122.14	127.91	131.35	133.94	136.37	143.73	152.71	156.28	168.86

7.3. Assessing the reciprocity of random FPR matrices

For each matrix of the order n, a total of 100,000 random matrices were generated. Each matrix was evaluated for non-reciprocity via relation (7) for $p = 1$, $p = 2$ and $p = \infty$ (the logarithm applied was the decadic one).

Results, presented in the form of percentile tables, are summarized in Tables 5-7 and Figure 4. In general, percentile values are expectedly to grow monotonically with increasing n: the larger the order of the matrix, the larger is the value of non-reciprocity for a corresponding percentile.

Let's consider a FPR matrix of the order $n = 6$ with $\Upsilon_2 = 8.13$, and $p = 2$. Then this matrix would be considered intolerably non-reciprocal, since the value of non-reciprocity equal to 8.13 is higher than the threshold for the 20^{th} percentile (8.02), see Table 6.

Table 5. Percentile tables of Υ defined by relation (7) for random FPR matrices, $p = 1; 2 \leq n \leq 8$

percentile	0.1	1	2	3	4	5	10	20	25	50 (median)
$n = 2$	0.00	0.03	0.06	0.09	0.12	0.15	0.30	0.61	0.75	1.64
$n = 3$	0.59	1.18	1.51	1.76	1.94	2.11	2.72	3.58	3.95	5.65
$n = 4$	2.97	4.43	5.10	5.52	5.90	6.18	7.21	8.60	9.17	11.63
$n = 5$	7.33	9.87	10.71	11.34	11.91	12.29	13.72	15.59	16.40	19.65
$n = 6$	13.99	17.12	18.33	19.21	19.83	20.38	22.21	24.60	25.61	29.66
$n = 7$	22.30	26.48	28.09	29.02	29.83	30.41	32.78	35.63	36.84	41.60
$n = 8$	33.20	37.65	39.64	40.90	41.82	42.66	45.31	48.66	50.01	55.57

Table 6. Percentile tables of Υ defined by relation (7) for random FPR matrices, $p = 2; 2 \leq n \leq 8$

percentile	0.1	1	2	3	4	5	10	20	25	50 (median)
$n = 2$	0.00	0.03	0.06	0.10	0.12	0.15	0.30	0.61	0.76	1.64
$n = 3$	0.41	0.81	1.02	1.19	1.32	1.44	1.85	2.44	2.68	3.80
$n = 4$	1.52	2.27	2.55	2.79	2.97	3.10	3.62	4.31	4.58	5.79
$n = 5$	2.97	3.91	4.25	4.50	4.69	4.84	5.43	6.17	6.44	7.71
$n = 6$	4.57	5.60	6.00	6.26	6.47	6.64	7.26	8.02	8.32	9.59
$n = 7$	6.30	7.33	7.75	8.07	8.27	8.42	9.07	9.84	10.14	11.45
$n = 8$	7.96	9.09	9.52	9.83	10.07	10.25	10.87	11.68	11.98	13.30

Table 7. Percentile tables of Υ defined by relation (7) for random FPR matrices, $p = \infty; 2 \leq n \leq 8$

percentile	0.1	1	2	3	4	5	10	20	25	50 (median)
$n = 2$	0.00	0.03	0.06	0.09	0.12	0.15	0.30	0.60	0.77	1.62
$n = 3$	0.29	0.66	0.82	0.96	1.06	1.15	1.50	1.98	2.19	3.14
$n = 4$	1.01	1.50	1.72	1.86	1.98	2.08	2.44	2.93	3.14	4.06
$n = 5$	1.62	2.18	2.42	2.56	2.68	2.79	3.13	3.63	3.82	4.73
$n = 6$	2.19	2.76	2.98	3.14	3.23	3.34	3.69	4.17	4.36	5.24
$n = 7$	2.66	3.20	3.43	3.58	3.69	3.79	4.13	4.60	4.79	5.65
$n = 8$	3.10	3.59	3.82	3.97	4.07	4.17	4.51	4.97	5.15	6.01

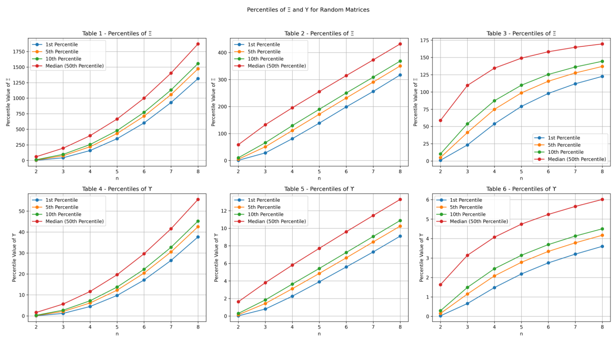

Figure 4. Percentile values for different $p-$metrics. Source: authors.

8. Conclusions

The paradigm of pairwise comparison methods assumes pairwise comparisons are reciprocal (Harker & Vargas, 1987; Saaty, 1977, 1980, 1986). This assumption is based on the notion of rationality and is considered a necessary requirement for consistent judgments. However, recent empirical studies have shown that non-reciprocal matrices appear naturally in many situations and that their root is both subjective (caused by human cognitive bias, lack of knowledge, time pressure, etc.) and objective (Diaz-Balteiro et al., 2009; Fülöp et al., 2012; Hovanov et al., 2008; Linares et al., 2016).

The problem of non-reciprocity in the multiplicative pairwise comparisons framework has been already addressed in the recent study by Mazurek & Linares (2023). The aim of this paper was to extend the study of non-reciprocity in pairwise comparisons to additive and fuzzy preference relations frameworks as well. In both frameworks, new measures of non-reciprocity, Ξ and Υ, satisfying a set of desirable properties, were introduced. These two measures correspond to the previously introduced measure of non-reciprocity in the multiplicative framework. Further on, we derive a unified formula of non-reciprocity based on an Alo-group approach.

In addition, Monte Carlo simulations of randomly-generated non-reciprocal APC and FPR matrices allowed us to provide a decision maker with percentile tables depending on the selected p-metric, the order of the matrix and a given tolerance value, so a decision maker can decide whether a given APC or FPR matrix is acceptably non-reciprocal, or not.

Further research may focus on the analysis of the implications of different p-metrics as well as connections between reciprocity and consistency measured by inconsistency indices proposed for reciprocal PC matrices. Also, generalizations towards pairwise comparisons with uncertainty (e.g. in the form of interval numbers or fuzzy sets), or incomplete pairwise comparisons can be considered.

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Declaration of interest

We have no conflict of interest to declare.

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