Document Version

Accepted versión

Citation for published version :

Rodríguez Abella, Á. (2023). Interconnection of Lagrange–Dirac systems through nonstandard interaction structures. International Journal of Geometric Methods in Modern Physics, 20(07), 2350124. https://doi.org/10.1142/S0219887823501244

General rights

This manuscript version is made available under the CC-BY-NC-ND 4.0 licence (https://web.upcomillas.es/webcorporativo/RegulacionRepositorioInstitucionalComillas.pdf).

Take down policy

If you believe that this document breaches copyright please contact Universidad Pontificia Comillas providing details, and we will remove access to the work immediately and investigate your claim

INTERCONNECTED LAGRANGE–DIRAC SYSTEMS WITH NONSTANDARD INTERACTION STRUCTURE

ÁLVARO RODRÍGUEZ ABELLA

Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), Calle Nicolás Cabrera, 13–15, Madrid, 28049, Madrid, Spain.

Abstract. The Lagrange–Dirac interconnection theory has been developed for primitive subsystems coupled by a standard interaction Dirac structure, i.e., a structure of the form $D_{int} = \Sigma_{int} \oplus \Sigma_{int}^{\circ}$, where $\Sigma_{int} \subset T(T^*Q)$ is a regular distribution, $\Sigma_{int}^{\circ} \subset T^*(T^*Q)$ is its annihilator and Q is the configuration manifold of the theory. In this work, we extend this theory to allow for parameter-dependent subsystems coupled by nonstandard interaction Dirac structures. This is done, first, by using the Dirac tensor product and, then, by using interaction forces. Both approaches are shown to be equivalent, and also equivalent to a variational principle. After that, we demonstrate the relevance of this generalization by investigating three applications. Firstly, an electromechanical system is modelled; namely, a piston driven by an ideal DC motor through a scotch-yoke mechanism. Secondly, we relate the interconnection theory to the Euler–Poincaré–Suslov reduction. More specifically, we show that the reduced system may be regarded as an interconnected Lagrange–Dirac system with parameters. The nonholonomic Euler top is presented as a particular instance of this situation. Lastly, control interconnected systems are defined and a control for a planar rigid body with wheels is designed.

control theory; Dirac structure; implicit Lagrangian system; interconnection; nonholonomic mechanics; reduction by symmetries.

1. Introduction

The use of Dirac structures [1] in geometric mechanics provides a unifying framework for treating degenerate systems with nonholonomic constraints both from the Lagrangian and the Hamiltonian points of view [2; 3]. In the same way, Dirac mechanics allows for interconnecting such systems through ports [4–6]. This is essential to model complex multi-physical systems, such as electric circuits or chemical reactions. More recently, the thermodynamics of non-simple systems [7] and nonequilibrium thermodynamics [8] has also been studied by means of interconnection of Dirac systems. On the other hand, the well-known theory of reduction by symmetries in Lagrangian [9] and Hamiltonian mechanics [10] has been extended to Dirac mechanics [11; 12]. Nevertheless, the interplay between interconnection and reduction has not been explored yet (see section 5).

The aim of this work is to extend the interconnection theory for Lagrange–Dirac systems developed in [6] to systems with parameters and external forces. Furthermore, we consider nonstandard Dirac structures to couple the primitive systems. Any Dirac structure, D_M, on a smooth manifold, M, is characterized by a distribution, $\Delta_M \subset TM$, and a skew-symmetric bilinear form, Ω_{Δ_M}, on Δ_M (see Proposition 2.1 below). We say that D_M is *standard* if $\Omega_{\Delta_M} = 0$, in which case $D_M = \Delta_M \oplus \Delta_M^{\circ}$, where $\Delta_M^{\circ} \subset T^*M$ is the annihilator of Δ_M. Otherwise, D_M is said to be *nonstandard*. The use of nonstandard interaction structures has not been developed in the previous literature on interconnection of Lagrange–Dirac systems. This generalization is significant, since there are physical systems that are coupled through this type of interaction structures, such as electromechanical systems. We

E-mail address: `alvrod06@ucm.es (ORCID ID: 0000-0003-0889-0450)`.

illustrate this by modelling a piston driven by an ideal DC motor through a scotch-yoke mechanism. On the other hand, in the context of reduction of (possibly nonholonomic) Lagrange–Dirac systems on Lie groups [11], we show that the reduced equations may be regarded as the equations of an interconnected system, where the (nonstandard) interaction structure is defined by the the adjoint representation of the corresponding Lie algebra. To conclude, control interconnected systems are defined by means of a control map that modifies the dynamics of the original system, and a control for a planar rigid body with wheels is designed.

The paper is structured as follows. In section 2 we briefly recall the geometric tools needed in the forthcoming development, namely: the notion of Dirac structure on a smooth manifold, the tensor product of Dirac structures and Lagrange–Dirac mechanics (including the parameter-dependent case). Section 3 presents the main results of the work. More specifically, we couple a family of Lagrange–Dirac systems with parameters using a nonstandard interaction Dirac structure that may be parameter-dependent too. The interconnection may be regarded from two viewpoints: by means of the Dirac tensor product and by means of the interaction forces that each system exerts on the others. After presenting both approaches, we show that they are equivalent, and also equivalent to the Lagrange–D'Alembert–Pontryagin variational principle. Next, we present three applications of this theory in section 4. Firstly, we study a piston driven by an ideal DC motor through a scotch-yoke mechanism. This provides an example of several primitive subsystems with parameters coupled through a nonstandard interconnection structure. Secondly, we recall the Euler–Poincaré–Suslov reduction theory and we reinterpret the reduced system as an interconnected Lagrange–Dirac system, where the interaction Dirac structure is defined by the adjoint representation of the Lie algebra. This situation is illustrated with the Euler top subject to a nonholonomic constraint. Lastly, we define control interconnected Lagrange–Dirac systems and we design a control for a planar rigid body with wheels. To conclude, the main contributions of the paper are summarized in section 5, and some future research directions are proposed.

Throughout the work, the space of parameters, S, includes every possible value for the constants of the system. Since the parameters are fixed before computing the dynamical equations, S may be a general set, although in some cases it may have additional structure. For instance, in the context of reduction with advected parameters [13; 14] (see also section 5), S is required to be a vector space and the parameters become dynamical variables after reduction.

Henceforth, we work with finite dimensional smooth manifolds, and every map is assumed to be smooth unless otherwise stated. Since S is not a smooth manifold in general, any map depending on the parameters is assumed to be smooth *after* fixing the parameter. Moreover, the tangent and cotangent bundles of a smooth manifold, M, are denoted by $\pi_{TM} : TM \to M$ and $\pi_{T^*M} : T^*M \to M$, respectively. Given a vector space V, we denote by V^* its dual space and by $\langle \cdot, \cdot \rangle$ the corresponding dual pairing. At last, the family of sections of a vector bundle, $\pi_U : U \to M$, is denoted by $\Gamma(\pi_U)$.

2. Preliminaries

Let us briefly recall some fundamental notions about Dirac structures on manifolds and their use to describe implicit Lagrangian systems. See [2; 6] for an extended development of these concepts. Here and throughout the work, the constraint distributions considered do not need to be integrable. In other words, Dirac mechanics is valid for both holonomic and nonholonomic systems. In addition, every Lagrangian is possibly degenerate although not explicitly stated.

2.1. Dirac structures on manifolds.

Definition 2.1. *A* Dirac structure *on a smooth manifold M is a vector subbundle $D_M \subset TM \oplus T^*M$ such that*

$$\langle \alpha_1, v_2 \rangle + \langle \alpha_2, v_1 \rangle = 0, \qquad (v_1, \alpha_1), (v_2, \alpha_2) \in D_M(x),\ x \in M.$$

Furthermore, D_M is said to be integrable *if*

$$\langle \pounds_{v_1}\alpha_2, v_3\rangle + \langle \pounds_{v_2}\alpha_3, v_1\rangle + \langle \pounds_{v_3}\alpha_1, v_2\rangle = 0$$

for each $(v_i,\alpha_i)\in\Gamma(\pi_{D_M})$, $1\le i\le 3$, *where* $\pounds$ *denotes the Lie derivative.*

The family of all Dirac structures on M is denoted by $\mathrm{Dir}\,(M)$. The integrability condition is the Dirac analogue of the closure condition for presymplectic structures or the Jacobi identity for Poisson brackets. In the following, the Dirac structures utilized are not assumed to be integrable, since this condition is too restrictive (cf. [6; 15]).

Given another smooth manifold N, a Dirac structure $D_N\in\mathrm{Dir}\,(N)$ and a smooth map $f: M\to N$, the *pull-back of D_N by f* is defined as

$$f^*D_N = \Big\{(v_x,\alpha_x)\in TM\oplus T^*M \mid \exists\beta_{f(x)}\in T^*_{f(x)}N \text{ such that } \alpha_x = (df)^*_x(\beta_{f(x)}),\ ((df)_x(v_x),\beta_{f(x)})\in D_N\Big\}\in\mathrm{Dir}\,(M).$$

Let $\Delta_M\subset TM$ be a regular distribution on M and $\Omega_M\in\Omega^2(M)$ be a 2-form on M. Then the Dirac structure on M defined by Δ_M and Ω_M is the following Dirac structure,

$$D_M = \left\{(v_x,\alpha_x)\in TM\oplus T^*M \mid v_x\in\Delta_M,\ \alpha_x - \Omega_M^\flat(v_x)\in\Delta_M^\circ\right\},$$

where $\Delta_M^\circ\subset T^*M$ denotes the annihilator of Δ_M and $\Omega_M^\flat: TM\to T^*M$ is given by $\left\langle\Omega_M^\flat(v_x),w_x\right\rangle = \Omega_M(v_x,w_x)$ for each $v_x,w_x\in T_xM$, $x\in M$. Reciprocally, any Dirac structure is defined by some distribution $\Delta_M\subset TM$ and some skew-symmetric bilinear form $\Omega_{\Delta_M}:\Delta_M\times_M\Delta_M\to\mathbb{R}$, where $\times_M$ denotes the fibered product, as stated in the following result, which can be found in [1, §2.2] (see also [6, §4.8]).

Proposition 2.1. *Let $D_M\in\mathrm{Dir}\,(M)$. Then D_M is the Dirac structure on M defined by the distribution $\Delta_M=\pi_1(D_M)$, where $\pi_1: TM\oplus T^*M\to TM$ is the projection onto the first component, and the skew-symmetric bilinear form Ω_{Δ_M} on Δ_M given by $\Omega_{\Delta_M}(x)(v_x,w_x)=\langle\alpha_x,w_x\rangle$ for each $v_x,w_x\in\Delta_M(x)$ and $x\in M$, being $\alpha_x\in T_x^*M$ any element such that $(v_x,\alpha_x)\in D_M(x)$.*

If $\Omega_{\Delta_M}=0$, then $D_M=\Delta_M\oplus\Delta_M^\circ$ and it is called a *standard* Dirac structure. Otherwise, D_M is said to be *nonstandard*.

2.1.1. *Direct sum and Dirac tensor product of Dirac structures.* Let us define the direct sum and the Dirac tensor product of Dirac structures, as presented in [6]. The direct sum of the Dirac structures $D_i\in\mathrm{Dir}\,(M_i)$, $1\le i\le n$, is the usual Direct sum as vector bundles, which is denoted by $D_1\oplus\cdots\oplus D_n$. Observe that it is a vector bundle over $M=M_1\times\cdots\times M_n$. As a matter of fact, $D_1\oplus\cdots\oplus D_n\in\mathrm{Dir}\,(M)$. Moreover, it is integrable if so are D_i, $1\le i\le n$. Although the Direct sum is appropriate to add separate physical subsystems, it does not model the interaction between them. To that end, we need to define the Dirac tensor product.

Definition 2.2. *Let $D_a,D_b\in\mathrm{Dir}\,(M)$. The* Dirac tensor product *of D_a and D_b is defined as*

$$D_a\boxtimes D_b = \big\{(v_x,\alpha_x)\in TM\oplus T^*M \mid \exists\beta_x\in T_x^*M \text{ such that } (v_x,\alpha_x+\beta_x)\in D_a,\ (v_x,-\beta_x)\in D_b\big\}.$$

This is how Dirac tensor product is introduced in [5; 6; 16], but there are equivalent definitions (cf. [17]).

Theorem 2.1. *Let $D_a,D_b\in\mathrm{Dir}\,(M)$ and $K=\{(v_x,v_x)\in T(M\times M)\mid x\in M\}\oplus T^*(M\times M)$. If $(D_a\oplus D_b)\cap K$ has locally constant rank, then $D_a\boxtimes D_b\in\mathrm{Dir}\,(M)$. Furthermore, if D_a and D_b are integrable, then so is $D_a\boxtimes D_b$.*

Let $\Delta_a=\pi_1(D_a)$ and $\Delta_b=\pi_1(D_b)$. The previous Theorem ensures that $D_a\boxtimes D_b$ is a Dirac structure provided $\Delta_a\cap\Delta_b$ has locally constant rank, i.e., it is a regular distribution.

2.2. **Lagrange–Dirac mechanics.** Let Q be a smooth manifold, $\Delta_Q \subset TQ$ be a regular distribution (which is known as the *constraint distribution*) and $\Omega \in \Omega^2(T^*Q)$ be the canonical symplectic form, and consider the lifted distribution, $\Delta_{T^*Q} = (d\pi_{T^*Q})^{-1}(\Delta_Q) \subset T(T^*Q)$. The Dirac structure induced by Δ_Q is the Dirac structure on T^*Q defined by Δ_{T^*Q} and Ω, i.e.,

$$D_{\Delta_Q} = \left\{(v_{p_q}, \alpha_{p_q}) \in T(T^*Q) \oplus T^*(T^*Q) \mid v_{p_q} \in \Delta_{T^*Q},\ \alpha_{p_q} - \Omega^\flat(v_{p_q}) \in \Delta_{T^*Q}^\circ\right\}.$$

If $\Delta_Q = TQ$, it is known as the *canonical Dirac structure on* T^*Q.

Let $L : TQ \to \mathbb{R}$ be a (possibly degenerate) Lagrangian and $\mathbb{F}L : TQ \to T^*Q$ be the corresponding Legendre transform, we denote by

$$\mathfrak{D}L = \gamma_Q \circ dL : TQ \to T^*(T^*Q)$$

the Dirac differential of L, where $\gamma_Q = \Omega^\flat \circ (\kappa_Q)^{-1} : T^*(TQ) \to T^*(T^*Q)$ and $\kappa_Q : T(T^*Q) \to T^*(TQ)$ is the natural diffeomorphism as defined in, for example, [2, §4].

Definition 2.3. *A* Lagrange–Dirac system *on Q, also known as* implicit Lagrangian system*, is a triple $\left(L, D_{\Delta_Q}, X\right)$, where $L : TQ \to \mathbb{R}$ is a Lagrangian, D_{Δ_Q} is the Dirac structure on T^*Q induced by the constraint distribution $\Delta_Q \subset TQ$, and $X : T^*Q \to T(T^*Q)$ is a vector field on T^*Q, satisfying*

$$(X(p_q), \mathfrak{D}L(v_q)) \in D_{\Delta_Q}(p_q), \qquad v_q \in TQ,\ p_q = \mathbb{F}L(v_q) \in T^*Q.$$

Observe that when Δ_Q is integrable, we have a *holonomic* system. Otherwise, the system is *nonholonomic*. A *solution curve* for an implicit Lagrangian system $\left(L, D_{\Delta_Q}, X\right)$ is a curve $c : [t_0, t_1] \to TQ$ such that $\mathbb{F}L \circ c : [t_0, t_1] \to T^*Q$ is an integral curve of X.

In order to write the local expression of the equations, let $U \subset Q$ be an open trivializing set for TQ and T^*Q, and let V be the typical fiber of TQ. For the sake of simplicity, we write $U = Q$. This way, locally we have $TQ = Q \times V$ and $T^*Q = Q \times V^*$. We denote $(q, v) \in TQ$ and $(q, p) \in T^*Q$. The Lagrange–Dirac equations locally read

$$p = \frac{\partial L}{\partial v}(q, v), \qquad \dot{q} = v \in \Delta_Q(q), \qquad -\frac{\partial L}{\partial q}(q, v) + \dot{p} \in \Delta_Q^\circ(q).$$

Remark 2.1. *For each $(q, p) \in T^*Q$, we write $X(q, p) = (q, p, \dot{q}, \dot{p})$ for certain $\dot{q} \in V$ and $\dot{p} \in V^*$. In fact, they are maps $\dot{q} : T^*Q \to V$ and $\dot{p} : T^*Q \to V^*$, but we do not explicit the arguments in order to maintain the notation simple. We will use this notation without further mention when working with the local expression of vector fields.*

Recall that the Lagrange–Dirac equations may also be obtained variationally (see [3]).

Definition 2.4. *The* Lagrange–D'Alembert–Pontryagin principle *for a Lagrangian $L : TQ \to \mathbb{R}$ is defined as*

$$\delta \int_{t_0}^{t_1} \left(L(v_q) + \langle p_q, \dot{q} - v_q \rangle\right) dt = 0$$

for curves $(v_q, p_q) : [t_0, t_1] \to TQ \oplus T^*Q$ projecting onto $q = \pi_{TQ} \circ v_q : [t_0, t_1] \to Q$ such that $\dot{q} \in \Delta_Q$, variations $\delta q \in \Delta_Q$ with fixed endpoints, i.e., $\delta q(t_0) = \delta q(t_1) = 0$, and arbitrary variations $(\delta v_q, \delta p_q)$.

Proposition 2.2. *The variational equations given by the Lagrange–D'Alembert–Pontryagin principle are exactly the Lagrange–Dirac equations.*

2.3. **Lagrange–Dirac mechanics with parameters.** To conclude this introductory section, we extend the Lagrange–Dirac mechanics introduced above for systems with parameters.

Definition 2.5. *A* Dirac structure on a smooth manifold M with parameters in S*, also known as* parameter-dependent Dirac structure on M*, is a map $D_M : S \to \mathrm{Dir}\,(M)$.*

From Proposition 2.1, a parameter-dependent Dirac structure, $D_M : S \to \mathrm{Dir}\,(M)$, is defined by

(i) A parameter-dependent regular distribution, $\Delta_M : S \to \operatorname{Sub}(TM)$, where $\operatorname{Sub}(TM)$ denotes the family of vector subbundles of $\pi_{TM} : TM \to M$, i.e., the family of regular distributions on M.
(ii) A parameter-dependent skew-symmetric bilinear form, $\Omega_{\Delta_M}(s) : \Delta_M(s) \times_M \Delta_M(s) \to \mathbb{R}$, $s \in S$.

The pull-back, direct sum and tensor product of Dirac structures are straightforwardly extended to Dirac structures with parameters.

Let Q be a smooth manifolds. A parameter-dependent constraint distribution, $\Delta_Q : S \to \operatorname{Sub}(TQ)$, induces a parameter-dependent Dirac structure on T^*Q,

$$D_{\Delta_Q(s)} = \big\{(v_{p_q}, \alpha_{p_q}) \in T(T^*Q) \oplus T^*(T^*Q) \mid \\ v_{p_q} \in \Delta_{T^*Q}(s),\ \alpha_{p_q} - \Omega^\flat(v_{p_q}) \in \Delta^\circ_{T^*Q}(s)\big\},$$

for each $s \in S$, where $\Delta_{T^*Q}(s) = (d\pi_{T^*Q})^{-1}(\Delta_Q(s))$.

In the same vein, a parameter-dependent Lagrangian is a map $L : TQ \times S \to \mathbb{R}$. Observe that, for each $s \in S$, the restriction $L_s = L|_{TQ\times\{s\}} : TQ \to \mathbb{R}$ is a (standard) Lagrangian. In particular, we may consider its Dirac differential, $\mathfrak{D}L_s = \gamma_Q \circ dL_s : TQ \to T^*(T^*Q)$, as well as its Legendre transform $\mathbb{F}L_s : TQ \to T^*Q$. Similarly, a parameter-dependent vector field is a map $X : T^*Q \times S \to T(T^*Q)$. We denote $X_s = X|_{T^*Q\times\{s\}}$ for each $s \in S$.

Definition 2.6. *Let $L : TQ\times S \to \mathbb{R}$ be a parameter-dependent Lagrangian, $\Delta_Q : S \to \operatorname{Sub}(TQ)$ be a parameter-dependent constraint distribution, and $X : T^*Q\times S \to T(T^*Q)$ be a parameter-dependent vector field. The triple $\big(L, D_{\Delta_Q}, X\big)$ is said to be a* Lagrange–Dirac system on Q with parameters in S, *also known as* parameter-dependent Lagrange–Dirac system on Q *or* parameter-dependent implicit Lagrangian system on Q, *if for each $s \in S$ it satisfies*

$$(X_s(p_q), \mathfrak{D}L_s(v_q)) \in D_{\Delta_Q(s)}(p_q), \qquad v_q \in TQ,\ p_q = \mathbb{F}L_s(v_q) \in T^*Q.$$

A solution curve for a Lagrange–Dirac system with parameters $\big(L, D_{\Delta_Q}, X\big)$ is a map $c : [t_0, t_1] \times S \to TQ$ such that $\mathbb{F}L_s \circ c_s : [t_0, t_1] \to T^*Q$ is an integral curve of X_s for each $s \in S$, where $c_s = c|_{[t_0,t_1]\times\{s\}}$.

In a trivialization, $TQ = Q\times V$ and $T^*Q = Q \times V^*$, the Lagrange–Dirac equations with parameters read

$$p = \frac{\partial L_s}{\partial v}(q, v), \qquad \dot q = v \in \Delta_Q(s)(q), \qquad -\frac{\partial L_s}{\partial q}(q, v) + \dot p \in \Delta^\circ_Q(s)(q).$$

Observe that the condition for the velocities to belong to the constraint distribution, $v_q \in \Delta_Q(s)(q)$, is already included in the equations.

Remark 2.2 (External forces)**.** *The previous setting may be extended to non-conservative systems. Namely, a* parameter-dependent external force *is a bundle morphism, $F^{ext} : TQ \times S \to T^*Q$, covering the identity,* id_Q. *As usual, we denote $F_s = F|_{TQ\times\{s\}} : TQ \to T^*Q$ for each $s \in S$. The corresponding* parameter-dependent Lagrange–Dirac equations with external forces *are given by*

$$\big(X_s(p_q), \mathfrak{D}L_s(v_q) - (d\pi_{T^*Q})^*\left(F^{ext}_s(v_q)\right)\big) \in D_{\Delta_Q(s)}(p_q),$$

for each $v_q \in TQ$, $p_q = \mathbb{F}L_s(v_q) \in T^*Q$ and $s \in S$. Locally, if we write $F^{ext}_s(q, v) = (q, f^{ext}_s(q, v))$, $(q, v) \in TQ$, $s \in S$, then the equations read

$$p = \frac{\partial L_s}{\partial v}(q, v), \qquad \dot q = v \in \Delta_Q(s)(q), \qquad -\frac{\partial L_s}{\partial q}(q, v) + \dot p - f^{ext}_s(q, v) \in \Delta^\circ_Q(s)(q).$$

3. Interconnected Lagrange–Dirac systems with parameters

Here we present the main theoretical results of the paper. Namely, we interconnect several Lagrange–Dirac systems with parameters through a nonstandard interaction Dirac structure that may be parameter-dependent too. Firstly, the interconnection is performed by using the Dirac tensor product and, then, by using interaction forces. Next, we show that both approaches are equivalent when the forces are appropriately chosen. After that, we present the variational point of view, which leads to the same dynamical equations for the interconnected system. Lastly, we relate the interconnected Lagrange–Dirac systems introduced here with the backward input–output systems defined in [15].

3.1. **Interaction structure and interconnected structure.** Let Q be a smooth manifold. Given a parameter-dependent regular distribution, $\Sigma_Q : S \to \mathrm{Sub}\,(TQ)$, and a parameter-dependent 2-form, $\Omega_Q : S \to \Omega^2(Q)$, their pull-backs to T^*Q are considered,

$$\Sigma_{int} = (d\pi_{T^*Q})^{-1}(\Sigma_Q) : S \to \mathrm{Sub}\,(T(T^*Q)), \quad \Omega_{int} = \pi_{T^*Q}^*(\Omega_Q) : S \to \Omega^2(T^*Q).$$

The *parameter-dependent interaction Dirac structure* is the Dirac structure on T^*Q with parameters in S defined by Σ_{int} and Ω_{int}, i.e., for each $s \in S$ and $p_q \in T^*Q$ we have

$$\begin{aligned} D_{int}(s)(p_q) = \big\{ \left(v_{p_q}, \alpha_{p_q}\right) \in T_{p_q}(T^*Q) \oplus T_{p_q}^*(T^*Q) \mid \\ v_{p_q} \in \Sigma_{int}(s)(p_q),\ \alpha_{p_q} - \Omega_{int}^\flat(s)(v_{p_q}) \in \Sigma_{int}^\circ(s)(p_q)\big\}. \end{aligned} \quad (1)$$

Observe that $D_{int}(s) = \pi_{T^*Q}^*(D_Q(s))$, where $D_Q : S \to \mathrm{Dir}\,(Q)$ is the Dirac structure with parameters defined by Σ_Q and Ω_Q.

Definition 3.1. *Let $\Delta_Q, \Sigma_Q : S \to \mathrm{Sub}\,(TQ)$ and $\Omega_Q : S \to \Omega^2(Q)$, and let $\Sigma_{int} = (d\pi_{T^*Q})^{-1}(\Sigma_Q)$ and $\Omega_{int} = \pi_{T^*Q}^*(\Omega_Q)$. The* parameter-dependent interconnected Dirac structure *on T^*Q is defined as*

$$D = D_{\Delta_Q} \boxtimes D_{int} : S \to \mathrm{Dir}\,(T^*Q).$$

Recall that nor Σ_Q neither Δ_Q are supposed to be integrable. Therefore, the corresponding systems may be either holonomic or nonholonomic.

Proposition 3.1. *If $\Delta_Q \cap \Sigma_Q : S \to \mathrm{Sub}\,(TQ)$, then $D = D_{\Delta_Q} \boxtimes D_{int} : S \to \mathrm{Dir}\,(T^*Q)$. Moreover, it is defined by $\Delta_D = \Delta_{T^*Q} \cap \Sigma_{int}$ and $\Omega_D = (\Omega + \Omega_{int})|_{\Delta_{T^*Q} \cap \Sigma_{int}}$, where $\Omega : S \to \Omega^2(T^*Q)$ is the constant map that assigns the canonical symplectic form $\Omega \in \Omega^2(T^*Q)$ to each $s \in S$.*

Proof. Let $s \in S$. The first part is straightforward, since the condition $\Delta_Q \cap \Sigma_Q : S \to \mathrm{Sub}\,(TQ)$ ensures that $\Delta_Q(s) \cap \Sigma_Q(s) \subset TQ$ is a regular distribution. For the second part, it is immediate that $\Delta_D(s) = \Delta_{T^*Q}(s) \cap \Sigma_{int}(s) \subset T(T^*Q)$. On the other hand, the 2-form $\Omega_D(s) : \Delta_D(s) \times \Delta_D(s) \to \mathbb{R}$ is given by $(v_{p_q}, w_{p_q}) \mapsto \langle \alpha_{p_q}, w_{p_q}\rangle$, where $(v_{p_q}, \alpha_{p_q}) \in D(s)$. Again, by definition of Dirac tensor product there exists $\beta_{p_q} \in T_{p_q}^*(T^*Q)$ such that $(v_{p_q}, \alpha_{p_q} + \beta_{p_q}) \in D_{\Delta_Q}(s)$ and $(v_{p_q}, -\beta_{p_q}) \in D_{int}(s)$. Since $D_{\Delta_Q}(s)$ is defined by $\Delta_{T^*Q}(s)$ and Ω, the first condition reads $\alpha_{p_q} + \beta_{p_q} - \Omega^\flat(v_{p_q}) \in \Delta_{T^*Q}^\circ(s)$. Likewise, since $D_{int}(s)$ is defined by $\Sigma_{int}(s)$ and $\Omega_{int}(s)$, the second condition says that $-\beta_{p_q} - \Omega_{int}^\flat(s)(v_{p_q}) \in \Sigma_{int}^\circ(s)$. By gathering both expressions, we obtain $\alpha_{p_q} - \Omega^\flat(v_{p_q}) - \Omega_{int}^\flat(s)(v_{p_q}) \in \Delta_{T^*Q}^\circ(s) + \Sigma_{int}^\circ(s) = (\Delta_{T^*Q}(s) \cap \Sigma_{int}(s))^\circ$. Hence, we conclude

$$\Omega_D(s)(v_{p_q}, w_{p_q}) = \Omega(v_{p_q}, w_{p_q}) + \Omega_{int}(s)(v_{p_q}, w_{p_q}), \qquad v_{p_q}, w_{p_q} \in \Delta_D(s).$$

□

3.1.1. *Local expression.* Let us find the local expression of the interconnected Dirac structure with parameters. Let $TQ = Q\times V$ be a trivialization of the tangent bundle, as in the last part of section 2.2. This leads to $T(T^*Q) = (Q\times V^*)\times(V\times V^*)$ and $T^*(T^*Q) = (Q\times V^*)\times(V^*\times V)$. In addition, for each $s\in S$, $\Delta_{T^*Q}(s)$ and $\Sigma_{int}(s)$ are the pull-backs of distributions on Q, which enable us to write

$$\Delta_D(s) = \{(q,p,\delta q,\delta p)\in T(T^*Q) \mid (q,\delta q)\in\Delta_Q(s)\cap\Sigma_Q(s)\}$$

and

$$\Delta_D^\circ(s) = \{(q,p,\alpha,w) \mid (q,\alpha)\in(\Delta_Q(s)\cap\Sigma_Q(s))^\circ,\ w=0\}\,.$$

Recall that $(\Delta_Q(s)\cap\Sigma_Q(s))^\circ = \Delta_Q^\circ(s)+\Sigma_Q^\circ(s)$, since we are dealing with finite dimensional manifolds. On the other hand, for each $(q,p,\delta q,\delta p)\in T(T^*Q)$ the canonical symplectic form is locally given by $\Omega^\flat(q,p,\delta q,\delta p) = (q,p,-\delta p,\delta q)$. In the same fashion, if we write

$$\Omega_Q^\flat(s)(q,\delta q) = (q,\varpi_s(q,\delta q)), \qquad (q,\delta q)\in TQ, \tag{2}$$

for some (local) function $\varpi_s : TQ\to V^*$, then its pull-back is given by $\Omega_{int}^\flat(s)(q,p,\delta q,\delta p) = (q,p,\varpi_s(q,\delta q),0)$. As a result,

$$\Omega_D^\flat(s)(q,p,\delta q,\delta p) = (q,p,-\delta p+\varpi_s(q,\delta q),\delta q), \qquad (q,p,\delta q,\delta p)\in\Delta_{T^*Q}(s)\cap\Sigma_{int}(s).$$

In short, for each $(q,p)\in T^*Q$ and $s\in S$, the interconnected Dirac structure with parameters locally read

$$\begin{aligned} D(s)(q,p) = &\\ \Big\{((\delta q,\delta p),(\alpha,w))\in T_{(q,p)}(T^*Q)\oplus T^*_{(q,p)}(T^*Q) \mid \delta q\in\Delta_Q(s)(q)\cap\Sigma_Q(s)(q),&\\ \alpha+\delta p-\varpi_s(q,\delta q)\in\Delta_Q^\circ(s)(q)+\Sigma_Q^\circ(s)(q),\ w=\delta q\Big\}.& \end{aligned} \tag{3}$$

3.2. **Interconnection via Dirac tensor product.** For $1\le i\le n$, let $\left(L_i, D_{\Delta_{Q_i}}, X_i\right)$ be a Lagrange–Dirac system with parameters in S_i (possibly degenerate, possibly nonholonomic), which we call *parameter-dependent primitive subsystem.* The configuration manifold and the space of parameters of the interconnected system are given by

$$Q = Q_1\times\cdots\times Q_n, \qquad S = S_1\times\cdots\times S_n. \tag{4}$$

Definition 3.2. *Let $D_{int} = \pi_{T^*Q}^*(D_Q) : S\to \mathrm{Dir}\,(T^*Q)$ be an interaction Dirac structure with parameters as in* (1)*. The* interconnected Lagrange–Dirac system on Q *is defined to be* (L,D,X)*, where $L = L_1+\cdots+L_n : TQ\times S\to\mathbb{R}$, $D = \left(D_{\Delta_{Q_1}}\oplus\cdots\oplus D_{\Delta_{Q_n}}\right)\boxtimes D_{int}$ and $X : T^*Q\times S\to T(T^*Q)$ satisfy the* interconnected Lagrange–Dirac equations with parameters*,*

$$(X_s(p_q),\mathfrak{D}L_s(v_q))\in D(s)(p_q), \qquad v_q\in TQ,\ p_q = \mathbb{F}L_s(v_q)\in T^*Q,\ s\in S.$$

Note that $D_{\Delta_{Q_1}}\oplus\cdots\oplus D_{\Delta_{Q_n}} = D_{\Delta_Q}$, where $\Delta_Q = \Delta_{Q_1}\oplus\cdots\oplus\Delta_{Q_n}$. In the following, we will assume that $\Delta_Q\cap\Sigma_Q : S\to\mathrm{Sub}\,(TQ)$, which ensures that D is a Dirac structure with parameters.

A solution curve for the interconnected Lagrange–Dirac system (L,D,X) is a map $c:[t_0,t_1]\times S\to TQ$ such that $\mathbb{F}L_s\circ c_s : [t_0,t_1]\to T^*Q$ is an integral curve of X_s for each $s\in S$, where $c_s = c|_{[t_0,t_1]\times\{s\}}$.

Proposition 3.2. *Let $TQ = Q\times V$ be a trivialization of the tangent bundle. Then the interconnected Lagrange–Dirac equations with parameters locally read*

$$p = \frac{\partial L_s}{\partial v}(q,v), \quad \dot q = v\in\Delta_Q(s)(q)\cap\Sigma_Q(s)(q),$$
$$-\frac{\partial L_s}{\partial q}(q,v)+\dot p-\varpi_s\,(q,\dot q)\in\Delta_Q^\circ(s)(q)+\Sigma_Q^\circ(s)(q).$$

Proof. The result can be easily checked from (3) by writing $X_s(q,p) = (q,p,\dot q,\dot p)$ and by the recalling that the Dirac differential is locally given by

$$\mathfrak{D}L_s(q,v) = \left(q, \frac{\partial L_s}{\partial v}(q,v), -\frac{\partial L_s}{\partial q}(q,v), v\right), \qquad (q,v) \in TQ,\ s \in S.$$

□

Note that, given $s \in S$, the interaction Dirac structure is locally described by the map $\varpi_s : Q \times V \to V^*$, as well as the family of vector spaces, $\{\Sigma_Q(s)(q) \subset V \mid q \in Q\}$. Therefore, there are $2\dim Q$ decoupled equations,

$$p = \frac{\partial L_s}{\partial v}(q,v), \quad \dot q = v,$$

together with $2\dim Q$ coupled equations,

$$\begin{aligned} v &\in \Delta_Q(s)(q) \cap \Sigma_Q(s)(q), \\ -\frac{\partial L_s}{\partial q}(q,v) + \dot p - \varpi_s(q,\dot q) &\in \Delta_Q^\circ(s)(q) + \Sigma_Q^\circ(s)(q). \end{aligned}$$

Remark 3.1 (External forces)**.** *As in Remark 2.2, we may consider non-conservative primitive subsystems. Let $F_i^{ext} : TQ_i \times S_i \to T^*Q_i$ be a parameter-dependent external force, $1 \le i \le n$. The* total parameter-dependent external force *is the map $F^{ext} : TQ \times S \to T^*Q$ defined as*

$$F_s^{ext}(v_q) = \left((F_1^{ext})_{s_1}(v_{q_1}), \dots, (F_n^{ext})_{s_n}(v_{q_n})\right),$$

for each $v_q = (v_{q_1}, \dots, v_{q_n}) \in TQ$ and $s = (s_1, \dots, s_n) \in S$. This way, the interconnected, parameter-dependent Lagrange–Dirac equations with external forces *are given by*

$$\left(X_s(p_q), \mathfrak{D}L_s(v_q) - (d\pi_{T^*Q})^*\left(F_s^{ext}(v_q)\right)\right) \in D(s)(p_q),$$

for each $v_q \in TQ$, $p_q = \mathbb{F}L_s(v_q) \in T^*Q$ and $s \in S$. Locally, if we write $F_s^{ext}(q,v) = (q, f_s^{ext}(q,v))$, $(q,v) \in TQ$, $s \in S$, then the equations read

$$\begin{aligned} p = \frac{\partial L_s}{\partial v}(q,v), &\qquad \dot q = v \in \Delta_Q(s)(q) \cap \Sigma_Q(s)(q), \\ -\frac{\partial L_s}{\partial q}(q,v) + \dot p - \varpi_s(q,\dot q) - f_s^{ext}(q,v) &\in \Delta_Q^\circ(s)(q) + \Sigma_Q^\circ(s)(q). \end{aligned}$$

Remark 3.2 (Relation with backward input–output systems)**.** *Alternate approaches of Dirac systems can be found in the literature, as in [15], where a new geometric framework to deal with systems that may gain or lose energy through ports was described. This approach requires more general definitions of Dirac structures, such as Dirac structures on vector bundles and coisotropic structures. It can be checked that the* backward input–output systems *(BIO-systems) introduced there to describe systems without energy exchange (closed systems) are the Hamiltonian counterpart of the interconnected Lagrange–Dirac systems presented in the present work provided the Lagrangian is* hyperregular*, i.e., its Legendre transform is an isomorphism. In other words, every interconnected Lagrange–Dirac system (without parameters) has an equivalent BIO-system associated.*

3.3. **Interconnection via interaction forces.** Now we model the interaction by means of the interaction forces that the systems exert on each other. As above, for $1 \le i \le n$, let $\left(L_i, D_{\Delta_{Q_i}}, X_i\right)$ be a primitive subsystem with parameters in S_i. Elements of $Q = Q_1 \times \dots \times Q_n$ and $S = S_1 \times \dots \times S_n$ are denoted by $q = (q_1, \dots, q_n)$ and $s = (s_1, \dots, s_n)$, respectively. Likewise, we denote $v_q = (v_{q_1}, \dots, v_{q_n}) \in TQ$ and $p_q = (p_{q_1}, \dots, p_{q_n}) \in T^*Q$.

Definition 3.3. *Let $D_{int} = \pi_{T^*Q}^*(D_Q) : S \to \mathrm{Dir}\,(T^*Q)$ be a parameter-dependent interaction Dirac structure as in (1). The* interconnected Lagrange–Dirac system on Q *is defined as the union of the following forced Lagrange–Dirac systems,*

$$(L_i, D_{\Delta_{Q_i}}, X_i, F_i), \qquad 1 \le i \le n,$$

where the bundle morphisms $F_i : TQ \times S \to T^*Q_i$ are known as the parameter-dependent interaction forces *and they satisfy*

$$\left((d\pi_{T^*Q})_{p_q}(X_s(p_q)), F_s(v_q)\right) \in D_Q(s)(q), \qquad (v_q, p_q) \in TQ \oplus T^*Q,\ s \in S, \tag{5}$$

where $X : T^*Q \to T(T^*Q)$ is given by $p_q = (p_{q_1}, \ldots, p_{q_n}) \mapsto (X_1(p_{q_1}), \ldots, X_n(p_{q_n}))$, and $F = (F_1, \ldots, F_n) : TQ \times S \to T^*Q$ is known as the parameter-dependent total interaction force.

Proposition 3.3. *Let $TQ = Q \times V$ be a trivialization of the tangent bundle. For each $(q, v, p) \in TQ \oplus T^*Q$ and $s \in S$, we write $X_s(q,p) = (q, p, \dot{q}, \dot{p})$ and $F_s(q,v) = (q, f_s(q,v))$ for some (local) function $f : TQ \times S \to V^*$. Then (5) locally reads*

$$\dot{q} \in \Sigma_Q(s)(q), \quad f_s(q,v) - \varpi_s(q,v) \in \Sigma_Q^\circ(s)(q).$$

Proof. The projection is locally given by $\pi_{T^*Q}(q,p) = q$, whence $(d\pi_{T^*Q})_{(q,p)}(X_s(q,p)) = (q, \dot{q})$. Besides, from the local expression (2) it is straightforward that

$$D_Q(s)(q) = \left\{(v,p) \in T_qQ \oplus T_q^*Q \mid v \in \Sigma_Q(s)(q),\ p - \varpi_s(q,v) \in \Sigma_Q^\circ(s)(q)\right\}$$

and we conclude. □

The following result ensures that Definitions 3.2 and 3.3 are equivalent.

Theorem 3.1. *For each $1 \le i \le n$, let $\left(L_i, D_{\Delta_{Q_i}}, X_i\right)$ be a primitive subsystem with parameters in S_i, and $D_{int} = \pi_{T^*Q}^*(D_Q)$ be an interaction Dirac structure on T^*Q with parameters in $S = S_1 \times \cdots \times S_n$. Let $L = L_1 + \cdots + L_n$ and $D = D_{\Delta_Q} \boxtimes D_{int}$, where $\Delta_Q = \Delta_{Q_1} \oplus \cdots \oplus \Delta_{Q_n}$. Then the equation*

$$(X_s(p_q), \mathfrak{D}L_s(v_q)) \in D(s)(p_q) \tag{6}$$

holds if and only if there exists a bundle morphism $F = (F_1, \ldots, F_n) : TQ \times S \to T^*Q$ such that the system of equations

$$\begin{cases} \left((X_i)_s(p_{q_i}), \mathfrak{D}(L_i)_s(v_{q_i}) - (d\pi_{T^*Q_i})^*\left((F_i)_s(v_q)\right)\right) \in D_{\Delta_{Q_i}}(s)(p_{q_i}), & 1 \le i \le n \\ \left((d\pi_{T^*Q})_{p_q}(X_s(p_q)), F_s(v_q)\right) \in D_Q(s)(q) \end{cases} \tag{7}$$

holds, for each $(v_q, p_q) = (v_{q_1}, \ldots, v_{q_n}, p_{q_1}, \ldots, p_{q_n}) \in TQ \oplus T^*Q$ such that $p_q = \mathbb{F}L(v_q)$ and each $s \in S$.

Proof. By definition of Dirac tensor product, Eq. (6) implies that there exists $\phi_s \in T_{p_q}^*(T^*Q)$ such that $(X_s(p_q), \phi_s) \in D_{int}(s)(p_q)$ and $(X_s(p_q), \mathfrak{D}L_s(v_q) - \phi_s) \in D_{\Delta_Q}(s)(p_q)$. By recalling that $D_{int} = \pi_{T^*Q}^*(D_Q)$, from the first condition we know that there exists $\psi_s \in T_q^*Q$ such that $\phi_s = (d\pi_{T^*Q})^*(\psi_s)$ and $\left((d\pi_{T^*Q})_{p_q}(X_s(p_q)), \psi_s\right) \in D_Q(s)(q)$. Subsequently, if we define $F_s(v_q) = \psi_s$, it satisfies the second equation of (7). Moreover, the second condition now reads

$$(X_s(p_q), \mathfrak{D}L_s(v_q) - (d\pi_{T^*Q})^*(F_s(v_q))) \in D_{\Delta_Q}(s)(p_q). \tag{8}$$

Now we use that $Q = Q_1 \times \cdots \times Q_n$ and $L = L_1 + \cdots + L_n$, which enables us to write $\mathfrak{D}L_s(v_q) = (\mathfrak{D}(L_1)_s(v_{q_1}), \ldots, \mathfrak{D}(L_n)_s(v_{q_n}))$. In the same way, observe that $\pi_{T^*Q} = (\pi_{T^*Q_1}, \ldots, \pi_{T^*Q_n})$ and $D_{\Delta_Q}(s)(p_q) = D_{\Delta_{Q_1}}(s)(p_{q_1}) \oplus \cdots \oplus D_{\Delta_{Q_n}}(s)(p_{q_n})$. Therefore, (8) may be restated as

$$\left((X_i)_s(p_{q_i}), \mathfrak{D}(L_i)_s(v_{q_i}) - (d\pi_{T^*Q_i})^*((F_i)_s(v_q))\right) \in D_{\Delta_{Q_i}}(s)(p_{q_i}), \qquad 1 \le i \le n,$$

which is, precisely, the first equation of (7). Lastly, observe that

$$\mathbb{F}L_s(v_q) = (\mathbb{F}(L_1)_s(v_{q_1}), \ldots, \mathbb{F}(L_n)_s(v_{q_n}))\,.$$

□

Remark 3.3. *The interconnection of Lagrange–Dirac systems developed in [6] is a particular case of the theory presented above when $\Omega_Q = 0$ and $S = \varnothing$, that is, when $D_{int} = \pi^*_{T^*Q}(\Sigma_Q \oplus \Sigma_Q^\circ)$ is a standard interaction structure and there are no parameters. In this situation, the local equations given in Proposition 3.2 read*

$$p = \frac{\partial L}{\partial v}(q,v), \quad \dot{q} = v \in \Delta_Q(q) \cap \Sigma_Q(q), \quad -\frac{\partial L}{\partial q}(q,v) + \dot{p} \in \Delta_Q^\circ(q) + \Sigma_Q^\circ(q).$$

The last equation implies that there exists a (local) function $F : TQ \to \Sigma_Q^\circ \subset T^*Q$ such that

$$-\frac{\partial L}{\partial q}(q,v) + \dot{p} - F(q,v) \in \Delta_Q^\circ(q),$$

which is, exactly, the interaction force.

3.4. **Variational viewpoint.** The dynamical equations for an interconnected system can also be obtained variationally. Namely, we employ the Lagrange–D'Alembert–Pontryagin principle with interaction forces. As above, let $\left(L_i, D_{\Delta_{Q_i}}, X_i\right)$ be a primitive subsystem with parameters in S_i, $1 \le i \le n$, and denote $Q = Q_1 \times \cdots \times Q_n$ and $S = S_1 \times \cdots \times S_n$. Let $D_{int} = \pi^*_{T^*Q}(D_Q) : S \to \mathrm{Dir}\,(T^*Q)$ be a parameter dependent interaction Dirac structure as given in (1), and $F = (F_1, \ldots, F_n) : TQ \times S \to T^*Q$ be the total interaction force defined in (5).

Definition 3.4. *In the previous conditions, let $\alpha = (v_q, p_q) : [t_0, t_1] \times S \to TQ \oplus T^*Q$ be a parameter-dependent curve on the Pontryagin bundle of Q projecting onto $q = \pi_{TQ} \circ v_q : [t_0, t_1] \times S \to Q$. For each $s \in S$, the* Lagrange–D'Alembert–Pontryagin principle *for the interconnected system is given by*

$$\delta \int_{t_0}^{t_1} \Big(L_s\big(v_q(t)\big) + \big\langle p_q(t), \dot{q}(t) - v_q(t)\big\rangle\Big)dt + \int_{t_0}^{t_1} \Big\langle F_s\big(v_q(t)\big), \delta q(t)\Big\rangle dt = 0 \tag{9}$$

for arbitrary variations $\delta\alpha = (\delta v_q, \delta p_q)$ such that $\dot{q} \in \Delta_Q \cap \Sigma_Q$ and $\delta q \in \Delta_Q \cap \Sigma_Q$ with fixed endpoints, i.e., $\delta q(t_0) = \delta q(t_1) = 0$.

As stated before, this variational principle leads to the same equations for the interconnected system.

Proposition 3.4. *The Lagrange–D'Alembert–Pontryagin equations obtained from the variational principle (9) with the interaction forces (5) are equivalent to the equations of the interconnected Lagrange–Dirac system (6).*

Proof. Let $s \in S$. We work with the local expressions, which enable us to write $\alpha = (q, v, p) : [t_0, t_1] \times S \to Q \times V \times V^*$. Furthermore, we write $F_s(q,v) = (q, f_s(q,v))$ and $\Omega_Q^\flat(s)(q, \delta q) = (q, \varpi_s(q, \delta q))$ for each $(q,v), (q, \delta q) \in TQ$, as above. By taking variations $\delta q \in \Delta_Q(s) \cap \Sigma_Q(s)$ in (9), we get

$$\frac{\partial L_s}{\partial q}(q,v) - \dot{p} + f_s(q,v) \in \Delta_Q^\circ(s)(q) + \Sigma_Q^\circ(s)(q),$$

where we have used integration by parts and the boundary conditions, $\delta q(t_0) = \delta q(t_1) = 0$. Making use of Proposition 3.3, this equation can be equivalently written as

$$\frac{\partial L_s}{\partial q}(q,v) - \dot{p} + \varpi_s(q,v) \in \Delta_Q^\circ(s)(q) + \Sigma_Q^\circ(s)(q).$$

Analogously, we take free variations δv, yielding

$$\frac{\partial L_s}{\partial v}(q,v) - p = 0.$$

At last, free variations δp lead to

$$\dot{q} = v \in \Delta_Q(s)(q) \cap \Sigma_Q(s)(q),$$

where we have used that $\dot{q} \in \Delta_Q \cap \Sigma_Q$. In short, we have obtained the local expression of the equations for the interconnected Lagrange–Dirac system given in Proposition 3.2. □

Observe that the Lagrange–D'Alembert–Pontryagin principle (9) for the interconnected system is the sum of the Lagrange–D'Alembert–Pontryagin principle (recall Definition 2.4) for each primitive system together with the interaction forces, which model the coupling between the primitive systems, as well as the restriction of the variations and velocities to the constraint distribution.

4. Applications

Let us consider three applications of the previous theory with the purpose of bringing its significance to light. To begin with, an electromechanical system is analyzed. After that, the relation between interconnection and Euler–Poincaré–Suslov–Dirac reduction is explored. Lastly, control interconnected systems are introduced and an example is presented.

4.1. **Piston driven by an ideal DC motor through a scotch-yoke mechanism.** An ideal DC motor consists of an armature coil immersed in a constant magnetic field. When an electric current flows through the coil, it experiences a Lorentz force, causing a torque. This yields an electromechanical coupling that is not of the standard form (cf. [6]). On the other hand, a scotch-yoke mechanism is a device that converts rotational movement into linear movement, or vice versa. The longitude of the rod gives a relation between the linear and the angular displacements that can be regarded as a mechanical coupling. In this example we consider an ideal DC motor driving a piston through a scotch-yoke mechanism, as depicted in Figure 1. The primitive subsystems are presented in Table 1.

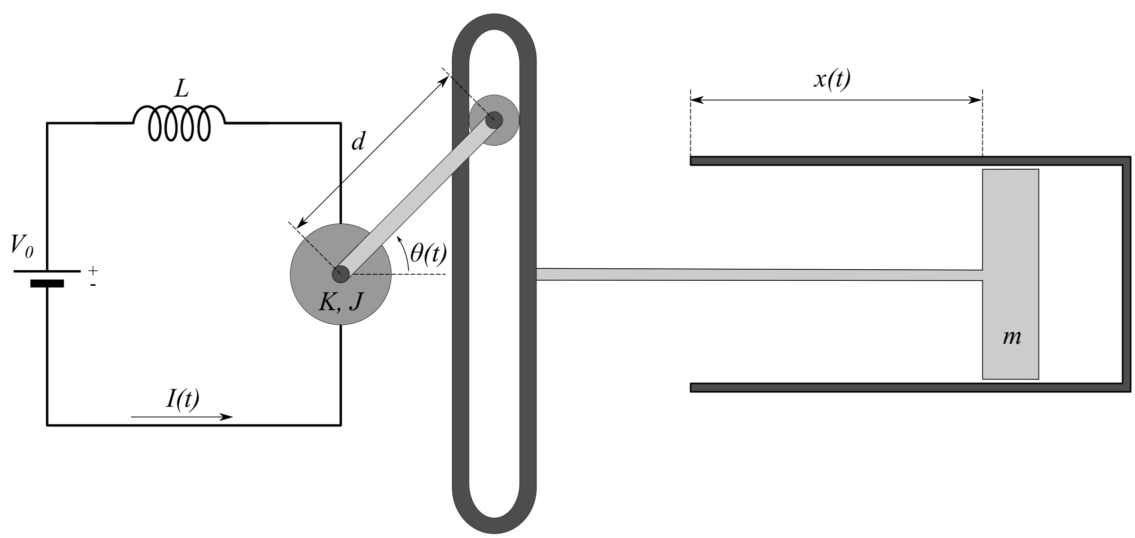

FIGURE 1. Piston driven by an ideal DC motor through a scotch-yoke mechanism. The coil of inductance L represents the internal coil of the motor.

The coupling of the primitive systems is given by the following physical conditions:

(i) The torque produced on the coil due to the magnetic field is related to the electric current through a parameter, K, known as the motor torque constant (cf. [18, Chapter 3]): $\tau = KI$.

Primitive system	Electric circuit	Rotating armature	Piston
Conf. manifold	$Q_1 = \mathbb{R}$	$Q_2 = \mathbb{S}^1$	$Q_3 = \mathbb{R}$
Coordinate	Charge, q	Angular displacement, θ	Linear displacement, x
Velocity	Electric current, I	Angular velocity, ω	Linear velocity, v
Momentum	Voltage, V	Torque, τ	Linear momentum, p
Parameter	Inductance, L	Moment of inertia, J	Mass, m
Lagrangian	$L_1 = LI^2/2$	$L_2 = J\omega^2/2$	$L_3 = mv^2/2$

Table 1. Primitive subsystems of a piston driven by an ideal DC motor through a scoth-yoke mechanism. The configuration space of the interconnected system is $Q = \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}$ with coordinates (q, θ, x).

(ii) The linear displacement of the piston is determined by the angular displacement of the motor through the longitude of the rod, d. Hence, the linear velocity is determined by the angular velocity: $v = -\omega d \sin\theta$.

In addition, we assume that the motor is powered by a battery that provides a constant voltage to the electric circuit, V_0, which we regard as a parameter. For the sake of brevity, we denote the parameters of the system by

$$s = (V_0, L, J, m, K, d) \in S = (\mathbb{R}^+)^6,$$

and we identify $T_q Q_1 = T_q\mathbb{R} = \mathbb{R}$ and $T_q^* Q_1 = T_q^*\mathbb{R} = \mathbb{R}$, and analogous for $Q_2 = \mathbb{S}^1$ (in this case, this identification only holds locally) and $Q_3 = \mathbb{R}$. The coupling conditions may be encoded in the parameter-dependent Dirac structure defined by

$$\Sigma_Q(s)(q,\theta,x) = \left\{(I,\omega,v) \in T_{(q,\theta,x)}Q \mid v = -\omega d \sin\theta\right\}$$

and

$$\Omega_Q(s)(q,\theta,x) = K dq \wedge d\theta$$

for each $s = (V_0, L, J, m, K, d) \in S$ and $(q,\theta,x) \in Q$. On the other hand, the primitive subsystems have no constraints, i.e., $\Delta_Q = TQ$. In addition, for each $(q,\theta,x) \in Q$, $(I,\omega,v) \in T_{(q,\theta,x)}Q$ and $s = (L, J, m, K, d) \in S$, the total Lagrangian with parameters is given by

$$L_s((q,\theta,x),(I,\omega,v)) = \frac{1}{2}\left(LI^2 + J\omega^2 + mv^2\right).$$

Lastly, the battery can be regarded as a constant external force,

$$F_s^{ext}((q,\theta,x),(I,\omega,v)) = (V_0, 0, 0).$$

Proposition 4.1. *The interconnected Lagrange–Dirac equations with parameters for the piston driven by an ideal DC motor through a scotch-yoke mechanism read as*

$$\begin{cases} V = LI, & \dot{q} = I, & v + \omega d \sin\theta = 0, \\ \tau = J\omega, & \dot{\theta} = \omega, & \dot{V} + K\dot{\theta} = V_0, \\ p = mv, & \dot{x} = v, & \dot{\tau} - K\dot{q} = \dot{p} d \sin\theta. \end{cases}$$

Proof. It is easy to check that

$$\Sigma_Q^\circ(s)(q,\theta,x) = \left\{(V,\tau,p) \in T^*_{(q,\theta,x)}Q \mid V = 0,\ \tau = pd\sin\theta\right\}$$

and

$$\Omega_Q^\flat(s)(q,\theta,x)(I,\omega,v) = (-K\omega, KI, 0).$$

The vector field is denoted by

$$X_s((q,\theta,x),(V,\tau,p)) = \big((q,\theta,x),(V,\tau,p),(\dot q,\dot\theta,\dot x),(\dot V,\dot\tau,\dot p)\big)$$

for each $(V,\tau,p) \in T^*_{(q,\theta,x)}Q$. At last, observe that the partial derivatives of the Lagrangian are given by

$$\frac{\partial L_s}{\partial(q,\theta,x)} = (0,0,0)\,, \qquad \frac{\partial L_s}{\partial(I,\omega,v)} = (LI, J\omega, mv)\,.$$

The result is now a straightforward computation from Proposition 3.2 by taking into account the external force $F_s^{ext}((q,\theta,x),(I,\omega,v)) = V_0$ (recall Remark 3.1). ☐

This example illustrates how the interconnection theory for Lagrange–Dirac systems with parameters and nonstandard interaction structure developed above can be used to model systems with electromechanical couplings. Furthermore, it could be extended to more complicated systems by considering more elements. For instance, in [19, §4], the authors added damping coefficients for both the electric part (a resistor) and the mechanical part (a spring).

4.2. **Euler–Poincaré–Suslov–Dirac systems as interconnected systems.** The Euler–Poincaré–Suslov–Dirac reduction, also known as implicit Euler–Poincaré–Suslov reduction, deals with implicit Lagrangian systems with nonholonomic constraints defined on Lie groups. Here we will show that the reduced system may be regarded as an interconnected system with parameters.

4.2.1. *Euler–Poincaré–Suslov–Dirac reduction.* Let us briefly recall the reduction procedure for this kind of systems, as presented in [11, §7]. Let G be a Lie group and $\Delta_G \subset TG$ be a left invariant constraint distribution, i.e., $(dL_h)_g(\Delta_G(g)) = \Delta_G(hg)$ for each $g \in G$, where $L_h : G \to G$ denotes the left multiplication by $h \in G$. We may transfer Δ_G to the trivialized space $G \times \mathfrak{g}^*$ as follows,

$$\Delta_{G\times\mathfrak{g}^*} = (d\pi_1)^{-1}(\Delta_G) \subset T(G\times\mathfrak{g}^*),$$

where $\pi_1 : G\times\mathfrak{g}^* \to G$ is the projection onto the first component. The invariance of Δ_G leads to the invariance of $\Delta_{G\times\mathfrak{g}^*}$, where the left action is given by $h\cdot(g,\mu) = (hg,\mu)$ for each $h,g \in G$ and $\mu \in \mathfrak{g}^*$. Moreover, $\Delta_{G\times\mathfrak{g}^*}$ induces a Dirac structure on $G\times\mathfrak{g}^*$, which we denote by $\tilde D_{\Delta_G} \in \mathrm{Dir}\,(G\times\mathfrak{g}^*)$. It is also left invariant and, thus, it yields a Dirac structure on the corresponding quotient.

Theorem 4.1 ([11, Theorem 7.2])**.** *For a fixed $\mu \in \mathfrak{g}^*$, the Dirac structure $\tilde D_{\Delta_G}$ induces a Dirac structure $[\tilde D_{\Delta_G}](\mu) \in \mathrm{Dir}\,(V)$, where $V = \mathfrak{g}\oplus\mathfrak{g}^*$, known as the* reduced Dirac structure*. Furthermore, it is given*

$$[\tilde D_{\Delta_G}](\mu) = \Big\{((\xi,\rho),(\nu,\eta)) \in V\oplus V^* \mid \xi \in \mathfrak{g}^{\Delta_G},\ \nu+\rho-ad^*_\xi(\mu) \in \big(\mathfrak{g}^{\Delta_G}\big)^\circ,\ \xi=\eta\Big\},$$

where $\mathfrak{g}^{\Delta_G} = \Delta_G(e) \subset T_eG = \mathfrak{g}$, being $e \in G$ the identity element, and $ad^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ is the coadjoint representation of $\mathfrak{g}$.

Let $L : TG \to \mathbb{R}$ be a left invariant Lagrangian density. By means of the left multiplication, we may identify $TG \simeq G\times\mathfrak{g}$, yielding a map $\tilde L : G\times\mathfrak{g} \to \mathbb{R}$ that is also left invariant. Likewise, the map $\gamma_G : T^*(TG) \to T^*(T^*G)$ may also be transferred to the reduced spaces, $\tilde\gamma_G : (G\times\mathfrak{g})\times(\mathfrak{g}\oplus\mathfrak{g}) \to (G\times\mathfrak{g}^*)\times V^*$. This way, the Dirac differential of $\tilde L$ is given by $\mathfrak{D}\tilde L = \tilde\gamma_G \circ d\tilde L : G\times\mathfrak{g} \to (G\times\mathfrak{g}^*)\times V^*$. The invariance of the previous maps leads to the *reduced Dirac differential*,

$$[\mathfrak{D}\tilde L] : \mathfrak{g} \to \mathfrak{g}^*\times V^*.$$

On the other hand, thanks to the invariance of $\tilde L$, it is well-defined the *reduced Lagrangian*, $l : \mathfrak{g} \to \mathbb{R}$, as $l(\xi) = \tilde L(g,\xi)$ for each $\xi \in \mathfrak{g}$ and $g \in G$. Observe that its Legendre transform is a map, $\mathbb{F}l : \mathfrak{g} \to \mathfrak{g}^*$.

Lastly, let $X : T^*G \to T(T^*G)$ be a left invariant vector field. After transferring it to the trivialized spaces, $\tilde X : G\times\mathfrak{g}^* \to (G\times\mathfrak{g}^*)\times V$, it descends to the quotient thanks to the invariance,

$$[\tilde X] : \mathfrak{g}^* \to \mathfrak{g}^*\times V.$$

Theorem 4.2 ([11, Proposition 8.3]). *Let (L, D_{Δ_G}, X) be an implicit Lagrangian system on a Lie group G, and assume that $L : TG \to \mathbb{R}$ and $\Delta_G \subset TG$ are left invariant by the action of G. Then $X : T^*G \to T(T^*G)$ is also left invariant. Therefore, the reduced system is given by $\left(l, [\tilde{D}_{\Delta_G}], [\tilde{X}]\right)$ and it satisfies the* implicit Euler–Poincaré–Suslov equations*, also called* Euler–Poincaré–Suslov–Dirac equations*,*

$$\left([\tilde{X}](\mu), [\mathfrak{D}\tilde{L}](\xi)\right) \in [\tilde{D}_{\Delta_G}](\mu), \qquad \xi \in \mathfrak{g}, \ \mu = \mathbb{F}l(\xi) \in \mathfrak{g}^*.$$

Furthermore, if we denote $[\tilde{X}](\mu) = (\mu, \eta, \dot{\mu})$, then the reduced equations are given by

$$\mu = \frac{\partial l}{\partial \xi}, \quad \eta = \xi \in \mathfrak{g}^{\Delta_G}, \qquad \dot{\mu} - ad_\eta^*(\mu) \in \left(\mathfrak{g}^{\Delta_G}\right)^\circ. \tag{10}$$

4.2.2. *Relation to interconnected systems.* We will show that the reduced system may be regarded as a (degenerate) interconnected system on G with parameters in $\mathfrak{g}^*$. In order to utilize the theory developed above, we make use the (global) left trivializations $TG \simeq G \times \mathfrak{g}$ and $T^*G \simeq G \times \mathfrak{g}^*$. In addition, the family of Dirac structures given in Theorem 4.1 can be regarded as a Dirac structure on $G \times \mathfrak{g}^*$ with parameters in $\mathfrak{g}^*$. More specifically, we define

$$\hat{D}(\mu)(g, \dot{\mu}) = [\tilde{D}_{\Delta_G}](\mu), \qquad \mu \in \mathfrak{g}^*, \ (g, \dot{\mu}) \in G \times \mathfrak{g}^*. \tag{11}$$

For each $\mu \in \mathfrak{g}^*$, let $\Sigma_G(\mu) = G \times \mathfrak{g}^{\Delta_G}$ and

$$\Omega_G(\mu) : G \times \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \qquad (g, \xi, \eta) \mapsto -\langle \mu, [\xi, \eta] \rangle,$$

where $[\cdot, \cdot]$ is the Lie bracket of $\mathfrak{g}$. Let D_G be the Dirac structure on G with parameters in $\mathfrak{g}^*$ defined by Σ_G and Ω_G, i.e.,

$$D_G(\mu)(g) = \left\{(\xi, \rho) \in V \mid \xi \in \mathfrak{g}^{\Delta_G}, \ \rho - \Omega_G^\flat(\mu) \in \left(\mathfrak{g}^{\Delta_G}\right)^\circ\right\}, \qquad \mu \in \mathfrak{g}^*, \ g \in G. \tag{12}$$

On the other hand, we denote by $\Omega_{G\times\mathfrak{g}^*} \in \Omega^2(G \times \mathfrak{g}^*)$ the pull-back of the canonical symplectic form $\Omega \in \Omega^2(T^*G)$ by the left trivialization. Let $D_{G\times\mathfrak{g}^*} \in \mathrm{Dir}\,(G \times \mathfrak{g}^*)$ be the Dirac structure induced by $\Omega_{G\times\mathfrak{g}^*}$ (the constraint distribution being the whole $T(G \times \mathfrak{g}^*) \simeq (G \times \mathfrak{g}^*) \times V$). We may regard $D_{G\times\mathfrak{g}^*}$ as a (constant) Dirac structure with parameters in $\mathfrak{g}^*$, i.e.,

$$D_{G\times\mathfrak{g}^*}(\mu)(g, \dot{\mu}) = \left\{((\xi, \rho), (\nu, \eta)) \in V \oplus V^* \mid (\nu, \eta) = \Omega_{G\times\mathfrak{g}^*}^\flat(\xi, \rho)\right\}, \tag{13}$$

for each $\mu \in \mathfrak{g}^*$ and $(g, \dot{\mu}) \in G \times \mathfrak{g}^*$.

Proposition 4.2. *The reduced Dirac structure (11) is a parameter-dependent interconnected Dirac structure. Namely,*

$$\hat{D} = D_{G\times\mathfrak{g}^*} \boxtimes D_{int} : \mathfrak{g}^* \to \mathrm{Dir}\,(G \times \mathfrak{g}^*),$$

where $D_{G\times\mathfrak{g}^*}$ is given by (13) and $D_{int} = \pi_1^*(D_G) : \mathfrak{g}^* \to \mathrm{Dir}\,(G \times \mathfrak{g}^*)$, with D_G given by (12).

Proof. Let $\mu \in \mathfrak{g}^*$. By definition of the coadjoint representation, we have $\Omega_G(\mu)^\flat(g, \xi) = \left(g, ad_\xi^*(\mu)\right)$. Indeed,

$$\left\langle \Omega_G^\flat(\mu)(g, \xi), (g, \eta) \right\rangle = -\langle \mu, [\xi, \eta] \rangle = \left\langle ad_\xi^*(\mu), \eta \right\rangle, \qquad (g, \xi), (g, \eta) \in G \times \mathfrak{g}.$$

Likewise, the canonical symplectic form is given by $\Omega_{G\times\mathfrak{g}^*}^\flat(\mu)(g, \dot{\mu}, \xi, \rho) = (g, \dot{\mu}, -\rho, \xi)$ for each $(g, \dot{\mu}, \xi, \rho) \in T(G \times \mathfrak{g}^*)$. By using (3), for each $(g, \dot{\mu}) \in G \times \mathfrak{g}^*$

$$\begin{aligned}\left(D_{G\times\mathfrak{g}^*} \boxtimes D_{int}\right)(\mu)(g, \dot{\mu}) = \Big\{((\xi, \rho), (\nu, \eta)) \in V \oplus V^* \mid \\ \xi = \eta \in \mathfrak{g}^{\Delta_G}, \ \nu + \rho - ad_\xi^*(\mu) \in \left(\mathfrak{g}^{\Delta_G}\right)^\circ\Big\},\end{aligned}$$

which agrees with the structure given in Theorem 4.1, and we conclude. ∎

Next, we regard the reduced Lagrangian as a parameter-dependent Lagrangian on $G\times\mathfrak{g}$. Namely, we define

$$\hat{l}:(G\times\mathfrak{g})\times\mathfrak{g}^*\to\mathbb{R},\qquad (g,\xi,\mu)\mapsto l(\xi), \tag{14}$$

and analogously for the reduced vector field,

$$\hat{X}:(G\times\mathfrak{g}^*)\times\mathfrak{g}^*\to(G\times\mathfrak{g}^*)\times V,\qquad (g,\dot{\mu},\mu)\mapsto\big(g,[\tilde{X}](\mu)\big)=(g,\mu,\eta,\dot{\mu}).$$

Theorem 4.3. *Let (L,D_{Δ_G},X) be a left invariant Lagrange–Dirac system on a Lie group G, and consider*

(i) the corresponding reduced system, $\big(l,[\tilde{D}_{\Delta_G}],[\tilde{X}]\big)$, and

(ii) the parameter-dependent interconnected Lagrange–Dirac system defined above, $\big(\hat{l},\hat{D},\hat{X}\big)$.

Then the Euler–Poincaré–Suslov–Dirac equations for $\big(l,[\tilde{D}_{\Delta_G}],[\tilde{X}]\big)$ are equivalent to the interconnected Lagrange–Dirac equations with parameters for $\big(\hat{l},\hat{D},\hat{X}\big)$.

Proof. Let $\mu\in\mathfrak{g}^*$. Observe that

$$\frac{\partial\hat{l}_\mu}{\partial g}=0,\qquad \frac{\partial\hat{l}_\mu}{\partial\xi}=\frac{\partial l}{\partial\xi}.$$

Hence, the parameter-dependent interconnected Lagrange–Dirac equations given in Proposition 3.2 read

$$\mu=\frac{\partial l}{\partial\xi}(g,\xi),\quad \eta=\xi\in\mathfrak{g}^{\Delta_G},\quad \dot{\mu}-ad^*_\xi(\mu)\in\big(\mathfrak{g}^{\Delta_G}\big),$$

which are exactly the Euler–Poincare–Suslov–Dirac equations (10), and we conclude. ☐

Note that we have shown that the reduced Dirac structure is a parameter-dependent interconnected Dirac structure on $G\times\mathfrak{g}^*$ in the sense of Definition 3.1. Nevertheless, $\mathfrak{g}^*$ is not supposed to be a product as in (4). In order to regard the reduced system as the interconnection of certain primitive subsystems, we need to assume the following hypothesis:

(i) There is a decomposition $\mathfrak{g}=V_1\times\cdots\times V_n$ for some vector spaces $V_1,\dots,V_n$.

(ii) The reduced Lagrangian can be written as $l=l_1+\cdots+l_n:\mathfrak{g}\to\mathbb{R}$ for certain $l_i:V_i\to\mathbb{R}$, $1\le i\le n$.

Observe that the first condition leads to a decomposition $\mathfrak{g}^*=V_1^*\times\cdots\times V_n^*$. We illustrate this situation in the following example.

4.2.3. *Example: nonholonomic Euler top.* Let us recall the reduction procedure for the Euler top subject to a nonholonomic constraint, as presented in [11; 20], and relate it to the interconnection theory. The configuration space of the system is $G=SO(3)$, the Lagrangian is given by the kinetic energy, and we assume that the system is subject to a constraint distribution $\Delta_{SO(3)}\subset TSO(3)$. The Lagrangian being $SO(3)$-invariant allows for defining the reduced Lagrangian on the corresponding Lie algebra,

$$l:\mathfrak{so}(3)\to\mathbb{R},\qquad \Sigma\mapsto\frac{1}{2}\langle\mathbb{I}\Sigma,\Sigma\rangle,$$

where $\mathbb{I}:\mathfrak{so}(3)\to\mathfrak{so}(3)^*$ is the inertia tensor of the rigid body, which may be regarded as a parameter. Likewise, the reduced constraint distribution is defined by some fixed $\Sigma^\circ\in\mathfrak{so}(3)^*$, as follows

$$\mathfrak{so}(3)^{\Delta_{SO(3)}}=\Delta_{SO(3)}(e)=\{\Sigma\in\mathfrak{so}(3)\mid\langle\Sigma^\circ,\Sigma\rangle=0\}.$$

Recall that we have the identification

$$\hat{\ }:\mathfrak{so}(3)\to\mathbb{R}^3,\qquad \Sigma=\begin{pmatrix}0&-\Sigma_3&\Sigma_2\\ \Sigma_3&0&-\Sigma_1\\ -\Sigma_2&\Sigma_1&0\end{pmatrix}\mapsto\hat{\Sigma}=\big(\Sigma_1,\Sigma_2,\Sigma_3\big).$$

Here and henceforth, we use the standard basis of $\mathbb{R}^3$. This way, the previous map is a Lie algebra isomorphism with the Lie bracket on $\mathbb{R}^3$ given by the cross product, i.e.,

$$\left[\hat{\Sigma}^1, \hat{\Sigma}^2\right] = \hat{\Sigma}^1 \times \hat{\Sigma}^2 = \left(\sum_{i,j=1}^{3} \epsilon_{ijk} \Sigma_i^1 \Sigma_j^2\right)_{1\leq k\leq 3}, \qquad \hat{\Sigma}^l = \left(\Sigma_1^l, \Sigma_2^l, \Sigma_3^l\right) \in \mathbb{R}^3,\ l = 1, 2,$$

where ϵ_{ijk}, $1 \leq i, j, k \leq 3$, are the components of the Levi-Civita tensor, i.e., the structure constants of the Lie algebra $(\mathbb{R}^3, \times)$ in the standard basis. Under this identification, the reduced Lagrangian is given by

$$\hat{l} : \mathbb{R}^3 \to \mathbb{R}, \qquad \left(\Sigma_1, \Sigma_2, \Sigma_3\right) \mapsto \frac{1}{2} \sum_{i,j=1}^{3} \Sigma_i I_{ij} \Sigma_j,$$

where $\hat{\mathbb{I}} = (I_{ij})_{1\leq i,j\leq 3}$ is the inertia tensor. In the same vein, the constraint distribution reads

$$\mathbb{R}^{\Delta_{SO(3)}} = \left\{\hat{\Sigma} = \left(\Sigma_1, \Sigma_2, \Sigma_3\right) \in \mathbb{R}^3 \mid \Sigma_1 \Sigma_1^\circ + \Sigma_2 \Sigma_2^\circ + \Sigma_3 \Sigma_3^\circ = 0\right\}.$$

By rotating the standard basis if necessary, we can assume that $\Sigma_1^\circ = \Sigma_2^\circ = 0$ and $\Sigma_3^\circ \neq 0$. Hence,

$$\mathbb{R}^{\Delta_{SO(3)}} = \left\{\hat{\Sigma} = \left(\Sigma_1, \Sigma_2, \Sigma_3\right) \in \mathbb{R}^3 \mid \Sigma_3 = 0\right\},$$
$$\left(\mathbb{R}^{\Delta_{SO(3)}}\right)^\circ = \left\{\hat{\sigma} = \left(\sigma_1, \sigma_2, \sigma_3\right) \in \mathbb{R}^3 \mid \sigma_1 = \sigma_2 = 0\right\}.$$

The reduced system may be interpreted as a interconnected system whenever the inertia tensor is diagonal, i.e., $I_{ij} = 0$ for $i \neq j$. In such case, we denote $I_i = I_{ii}$ and the reduced Lagrangian is decomposed as $\hat{l} = \hat{l}_1 + \hat{l}_2 + \hat{l}_3$, where $\hat{l}_i : \mathbb{R} \to \mathbb{R}$ is given by $\hat{l}_i(\Sigma) = I_i \Sigma^2/2$, $1 \leq i \leq 3$. By gathering all and by denoting $\hat{M} = (M_1, M_2, M_3)$, the interconnected Lagrange–Dirac equations read

$$M_k = I_k \Sigma_k, \quad \dot{M}_k - \epsilon_{ijk} \Sigma_i M_j = 0, \qquad k = 1, 2,$$
$$M_3 = \Sigma_3 = 0.$$

Observe that if we choose a non-orthonormal basis of $\mathbb{R}^3$, then the corresponding equations are obtained by substituting the Levi-Civita tensor, ϵ_{ijk}, by the corresponding structure constants, c_{ijk}, $1 \leq i, j, k \leq 3$, in such basis.

4.3. **Control of interconnected systems.** Given a mechanical system, control theory [21; 22] deals with the design of different types of inputs, the so-called *controls*, capable of forcing the system to reach a desired dynamics. Here we follow the approach introduced in [23], where the dynamics of the *plant*, i.e., the physical system to be controlled, is described by a Lagrange–Dirac system (an interconnected system in our case), and the control is described by an external force acting on it. After presenting the general notions, we examine the controlled motion of a planar rigid body with wheels.

Definition 4.1. *Let Q be a smooth manifold, and (L, D, X) be an interconnected Lagrange–Dirac system on Q with parameters in S, as in Definition 3.2. Let $\mathcal{W} \subset T^*Q$ be a vector subbundle and $\mathbf{u} : TQ \times S \to \mathcal{W}$ be a bundle morphism covering the identity,* id_Q*. The* control interconnected Lagrange–Dirac system *is the tuple $(L, D, \mathcal{W}, \mathbf{u}, X^c)$, where $X^c : T^*Q \times S \to T(T^*Q)$ satisfies the* controlled interconnected Lagrange–Dirac equations*,*

$$(X_s^c(p_q), \mathfrak{D}L_s(v_q) - (d\pi_{T^*Q})^*(\mathbf{u}_s(v_q))) \in D(s)(p_q),$$

for each $v_q \in TQ$, $p_q = \mathbb{F}L_s(v_q) \in T^*Q$ and $s \in S$, where we denote $\mathbf{u}_s = \mathbf{u}|_{TQ\times\{s\}} : TQ \to \mathcal{W}$, as usual.

In the previous definition, $\mathcal{W} \subset T^*Q$ is known as the *control bundle* and $\mathbf{u} : TQ \times S \to \mathcal{W}$ is known as the *parameter-dependent control map*. The idea of the control is to modify the dynamics of the plant (encoded by X) to get the desired dynamics (encoded by X_c). Observe that the input, $\mathbf{u}$, is a function of the current state of the system, v_q. For this reason, it is known as a *closed-loop* or *feedback control*. An alternative is to choose the control to be a function, $\mathbf{u} : [t_0, t_1] \to \mathcal{W}$. This situation, where the input of the system does not depend on the configuration of the system, is known as an *open loop control*. In any case, the control system is said to be *underactuated* when $\dim \mathcal{W} < \dim T^*Q$, i.e., there are less controls than degrees of freedom (observe that this condition is equivalent to $\dim \mathcal{W}_q < \dim T_q^*Q$ for each $q \in Q$), and it is said to be *fully actuated* when $\dim \mathcal{W} = \dim T^*Q$.

In a trivialization, $TQ = Q \times V$, the control bundle may be written as $\mathcal{W} = Q \times W$ for some vector subspace $W \subset V^*$, and the control map reads $\mathbf{u}_s(q,v) = (q, \mathrm{u}_s(q,v))$ for each $(q,v) \in TQ$ and $s \in S$, for some $\mathrm{u} : Q \times V \times S \to W$. Hence, the controlled equations are a particular instance of Remark 3.1 when the external force is given by the control map,

$$\begin{aligned} p = \frac{\partial L_s}{\partial v}(q,v), \qquad \dot{q} = v \in \Delta_Q(s)(q) \cap \Sigma_Q(s)(q), \\ -\frac{\partial L_s}{\partial q}(q,v) + \dot{p} - \varpi_s(q,\dot{q}) - \mathrm{u}_s(q,v) \in \Delta_Q^\circ(s)(q) + \Sigma_Q^\circ(s)(q). \end{aligned} \tag{15}$$

4.3.1. *Example: control for a planar rigid body with wheels.* Let us design a control for a planar rigid body with wheels, as depicted in Figure 2. The position of the center of mass and the angular orientation may be regarded as different subsystems, as presented in Table 2.

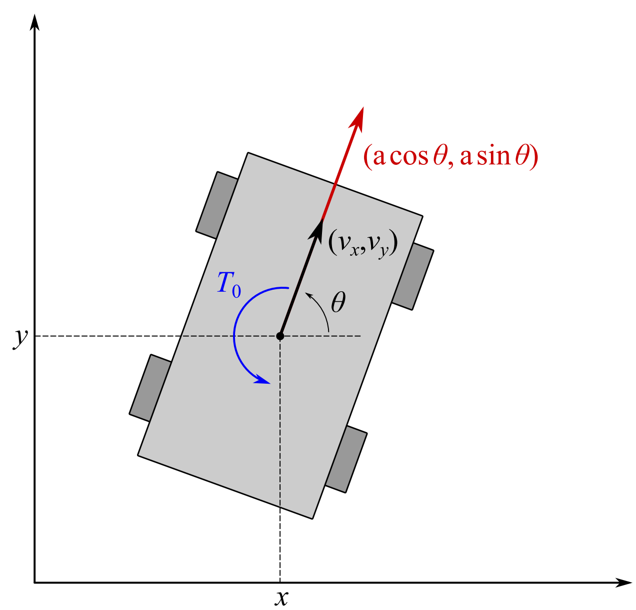

FIGURE 2. Control for a planar rigid body with wheels.

For brevity, the parameters are denoted by $s = (J, m) \in S = (\mathbb{R}^+)^2$, and we identify $T_{(x,y)}Q_1 = T_{(x,y)}\mathbb{R}^2 = \mathbb{R}^2$ and $T^*_{(x,y)}Q_1 = T^*_{(x,y)}\mathbb{R}^2 = \mathbb{R}^2$, and analogous for $Q_2 = \mathbb{S}$ (in this case, the identification is local).

The primitive subsystems are coupled due to the *non-slipping* assumption. This compels the linear velocity to belong to the direction determined by the angular orientation, i.e., the vector (v_x, v_y) must be proportional to $(\cos\theta, \sin\theta)$. As a result, the interaction Dirac structure is defined by

$$\Sigma_Q(s)(x,y,\theta) = \{(v_x, v_y, \omega) \in T_{(x,y,\theta)}Q \mid v_x \sin\theta - v_y \cos\theta = 0\}$$

and $\Omega_Q(s)(x,y,\theta) = 0$ for each $s = (J,m) \in S$ and $(x,y,\theta) \in Q$. Since the primitive subsystems have no constraints, we have $\Delta_Q = TQ$. Moreover, for each $(x,y,\theta) \in Q$, $(v_x, v_y, \omega) \in T_{(x,y,\theta)}Q$ and

Primitive system	Center of mass	Angular orientation
Configuration manifold	$Q_1 = \mathbb{R}^2$	$Q_2 = \mathbb{S}^1$
Coordinate	(x, y)	θ
Velocity	(v_x, v_y)	ω
Momentum	(p_x, p_y)	τ
Parameter	Mass, m	Moment of inertia, J
Lagrangian	$L_1 = m(v_x^2 + v_y^2)/2$	$L_2 = J\omega^2/2$

TABLE 2. Primitive subsystems of a planar rigid body with wheels. The configuration space of the interconnected system is $Q = \mathbb{R}^2 \times \mathbb{S}^1 \simeq SE(2)$, the special Euclidean group in dimension 2, with coordinates (x, y, θ).

$s = (J, m) \in S$, the total Lagrangian is given by

$$L_s((x, y, \theta), (v_x, v_y, \omega)) = \frac{1}{2}(mv_x^2 + mv_y^2 + J\omega^2).$$

Proposition 4.3 (Dynamics without control)**.** *The interconnected Lagrange–Dirac equations with parameters for the planar rigid body with wheels are given by*

$$\begin{cases} p_x = mv_x, & \dot{x} = v_x, & v_x \sin\theta - v_y \cos\theta = 0, \\ p_y = mv_y, & \dot{y} = v_y, & \dot{p}_x \cos\theta + \dot{p}_y \sin\theta = 0, \\ \tau = J\omega, & \dot{\theta} = \omega, & \dot{\tau} = 0. \end{cases}$$

Proof. An easy computation shows that

$$\Sigma_Q^\circ(s)(x, y, \theta) = \{(p_x, p_y, \tau) \in T^*_{(x,y,\theta)}Q \mid p_x \cos\theta + p_y \sin\theta = 0,\ \tau = 0\}.$$

By denoting the vector field as

$$X_s((x, y, \theta), (p_x, p_y, \tau)) = \big((x, y, \theta), (p_x, p_y, \tau), (\dot{x}, \dot{y}, \dot{\theta}), (\dot{p}_x, \dot{p}_y, \dot{\tau})\big)$$

for each $(p_x, p_y, \tau) \in T^*_{(x,y,\theta)}Q$, and by noting that the partial derivatives are given by

$$\frac{\partial L_s}{\partial(x, y, \theta)} = (0, 0, 0), \qquad \frac{\partial L_s}{\partial(v_x, v_y, \omega)} = (mv_x, mv_y, J\omega),$$

the result is straightforward from Proposition 3.2. □

Now we introduce the controls, which consists of an accelerator that produces a force in the direction of θ (in red in Figure 2), and a torque applied at the center of mass (in blue in Figure 2). This way, the open loop control is defined as

$$\mathbf{u} = (\underbrace{a\cos\theta, a\sin\theta}_{\text{Accelerator}}, \underbrace{T_0}_{\text{Torque}}) : [t_0, t_1] \to T^*_{(x,y,\theta)}Q,$$

for some given functions $a, T_0 : [t_0, t_1] \to \mathbb{R}$. In this case,

$$W_{(x,y,\theta)} = \{(p_x, p_y, \tau) \in T^*_{(x,y,\theta)}Q \mid p_x \sin\theta = p_y \cos\theta\} \subset T^*_{(x,y,\theta)}Q$$

is a 2-dimensional subspace for each $(x, y, \theta) \in Q$. Hence, the system is underactuated. At last, the controlled dynamics is obtained from (15).

Proposition 4.4 (Controlled dynamics)**.** *The interconnected Lagrange–Dirac equations with parameters for the controlled planar rigid body are given by*

$$\begin{cases} p_x = mv_x, & \dot{x} = v_x, & v_x \sin\theta - v_y \cos\theta = 0, \\ p_y = mv_y, & \dot{y} = v_y, & \dot{p}_x \cos\theta + \dot{p}_y \sin\theta = a, \\ \tau = J\omega, & \dot{\theta} = \omega, & \dot{\tau} = T_0. \end{cases}$$

5. Conclusions and future work

In the present work, the interconnection of Lagrange–Dirac systems developed in [6] has been extended to include nonstandard interaction structures, external forces and parameters. This has been done from three viewpoints: by means of the Dirac tensor product, by means of interaction forces and from a variational principle. The three approaches have been proven to be equivalent. Moreover, it has been checked that an interconnected Lagrange–Dirac system is the Lagrangian counterpart of a backward input–output system. Furthermore, several applications have been presented to illustrate the importance of this generalization. Firstly, a piston driven by an ideal DC motor through a scotch-yoke mechanism has been analyzed, showing that electromechanical systems require nonstandard interaction structures to be modelled. Next, Euler–Poincaré-Suslov–Dirac systems have been reinterpreted as interconnected systems, where the nonstandard interaction Dirac structure is given by the adjoint representation of the Lie algebra. At last, control interconnected systems have been defined and illustrated by the analysis of a planar rigid body with wheels.

Some future research directions include the following:

(1) If a parameter-dependent primitive subsystem whose configuration manifold is a Lie group, $Q = G$, is G-invariant for each fixed value of the parameter, $s_0 \in S$, then reduction by symmetries may be performed. In such case, S is required to be a vector space. The corresponding reduced system is known as *Euler–Poincaré–Dirac system with advected quantities* (see [13] for the Lagrangian formulation and [14] for the Dirac formulation). In such case, the space of parameters is chosen to be a vector space and the parameters yield dynamic variables of the reduced problem, the so-called advected quantities. It would be interesting to explore the interconnection of invariant subsystems through an invariant interaction structure, thus yielding an invariant interconnected Lagrange–Dirac system, and the corresponding reduction with advected quantities. Moreover, an interconnection theory for Euler–Poincaré–Dirac systems could be formulated in order to have a commutative diagram as follows,

$$\begin{tikzcd} \boxed{\begin{array}{c}\text{Primitive}\\ \text{Lagrange–Dirac}\end{array}} \arrow[r, dashed, "\text{Invariant interaction}", "\text{Dirac structure}"'] \arrow[d, "/G"] & \boxed{\begin{array}{c}\text{Interconnected}\\ \text{Lagrange–Dirac}\end{array}} \arrow[d, "/G"] \\ \boxed{\begin{array}{c}\text{Primitive}\\ \text{Euler–Poincaré–Dirac}\end{array}} \arrow[r, dashed, "\text{Reduced interaction}", "\text{Dirac structure}"'] & \boxed{\begin{array}{c}\text{Interconnected}\\ \text{Euler–Poincaré–Dirac}\end{array}} \end{tikzcd}$$

where the dashed lines mean *interconnection* and the solid lines mean *reduction*.

(2) In continuum mechanics, which include a wide variety of physical systems such as elasticity or fluid dynamics [13; 24–26], the configuration manifold is given by the family of smooth embeddings from a compact, Riemannian manifold, (M, g), with smooth boundary, ∂M, to a boundaryless manifold, B, that is, $Q = \mathrm{Emb}(M, B)$. In addition, the Lagrangian is

defined through a Lagrangian density, i.e.,

$$L : T\,\mathrm{Emb}(M,B) \to \mathbb{R}, \qquad v_\varphi \mapsto L(v_\varphi) = \int_M \mathfrak{L}(v_\varphi, d\varphi)\,\mu_g,$$

where μ_g is the Riemannian volume form on M, and $d\varphi : TM \to TB$ is the tangent map of $\varphi \in \mathrm{Emb}(M,B)$. A generalization of the theory presented here to continuum mechanics would be highly desirable. Some steps in this direction have been made in [27] from the Hamiltonian viewpoint. Furthermore, the relation between interconnection and reduction mentioned above would play a fundamental role, since the Lagrangian of a number of continuous media is invariant either by the (right) action of $\mathrm{Diff}(M)$, which leads to the *spatial representation*, or the (left) action of $\mathrm{Diff}(B)$, which leads to the *convective representation*.

(3) Discrete Dirac structures have been introduced by defining a discrete analog of the Tulczyjew's triple [28; 29], yielding geometric numerical integrators for Lagrange–Dirac systems. In addition, the interconnection of such systems through standard interaction structures have been addressed in [30], where the discrete Dirac tensor product has been introduced. Nevertheless, a general discrete theory of interconnection including the cases developed in this work (nonstandard interaction structures, systems with parameters and external forces, and control theory) is still lacking. On the other hand, the Lagrange–Poincaré–Dirac procedure has been mimicked in the discrete setting [31]. As explained above, the relation between reduction by symmetries and interconnection is significant, so it would be interesting to elaborate a discrete Euler–Poincaré–Dirac reduction theory that includes advected parameters.

Acknowledgments

The author has been supported by a FPU grant from the Spanish Ministry of Science, Innovation and Universities (MICIU).

References

[1] T. Courant. Dirac manifolds. *Trans. Am. Math. Soc.*, 319, 1990.

[2] H. Yoshimura and J. E. Marsden. Dirac structures in Lagrangian mechanics Part I: Implicit Lagrangian systems. *J. Geom. Phys.*, 57(1):133–156, 2006.

[3] H. Yoshimura and J. E. Marsden. Dirac structures in Lagrangian mechanics Part II: Variational structures. *J. Geom. Phys.*, 57(1):209–250, 2006.

[4] J. Cervera, A. J. van der Schaft, and A. Baños. On composition of Dirac structures and its implications for control by interconnection. In Alan Zinober and David Owens, editors, *Nonlinear and Adaptive Control*, pages 55–63, Berlin, Heidelberg, 2003. Springer Berlin Heidelberg.

[5] H. O. Jacobs and H. Yoshimura. Interconnection and composition of Dirac structures for Lagrange-Dirac systems. In *Proc. IEEE Conf. Decis. Control*, pages 928–933, 2011.

[6] H. O. Jacobs and H. Yoshimura. Tensor products of Dirac structures and interconnection in Lagrangian mechanics. *J. Geom. Mech.*, 6(1):67–98, 2014.

[7] H. Yoshimura and F. Gay-Balmaz. A port-Dirac formulation for thermodynamics of non-simple systems. *IFAC-PapersOnLine*, 54:32–37, 2021.

[8] F. Gay-Balmaz and H. Yoshimura. Dirac structures and variational formulation of port-Dirac systems in nonequilibrium thermodynamics. *IMA J. Math. Control. Inf.*, 37, 2020.

[9] H. Cendra, J. E. Marsden, and T. S. Ratiu. Lagrangian reduction by stages. *Mem. Amer. Math. Soc. V*, 152(722), 2001.

[10] J. E. Marsden, G. Misiolek, J. P. Ortega, M. Perlmutter, and T. S. Ratiu. *Hamiltonian Reduction by Stages*, volume 1913 of *Lecture Notes in Mathematics*. Springer, Berlin, Heidelberg, 2007.

[11] H. Yoshimura and J. E. Marsden. Reduction of Dirac structures and the Hamilton-Pontryagin principle. *Reports on Mathematical Physics*, 60(3):381–426, 2007.

[12] H. Yoshimura and J. E. Marsden. Dirac cotangent bundle reduction. *J. Geom. Mech.*, 1, 2009.

[13] D. D. Holm, J. E. Marsden, and T. S. Ratiu. The Euler–Poincaré equations and semidirect products with applications to continuum theories. *Advances in Mathematics*, 137(1):1–81, 1998.

[14] F. Gay-Balmaz and H. Yoshimura. Dirac reduction for nonholonomic mechanical systems and semidirect products. *Advances in Applied Mathematics*, 63:131–213, 2015.

[15] M. Barbero-Liñán, H. Cendra, E. García-Toraño Andrés, and D. Martín de Diego. New insights in the geometry and interconnection of port-Hamiltonian systems. *Journal of Physics A: Mathematical and Theoretical*, 51(37):375201, 2018.

[16] H. O. Jacobs, H. Yoshimura, and J. E. Marsden. Interconnection of Lagrange-Dirac dynamical systems for electric circuits. *AIP Conf. Proc.*, 1281(1):566–569, 2010.

[17] M. Gualtieri. Generalized complex geometry. *Ann. Math.*, 174:75–123, 2010.

[18] A. Hughes and B. Drury. *Electric motors and drives: fundamentals, types and applications.* Newnes, fifth edition edition, 2019.

[19] J. G. Etse, B. M. Luccioni, M. Pucheta, and M. A. Storti. Lagrangian for electromechanical systems. In *Mec. Comput.*, volume XXXVI, pages 1911–1934. Asociación Argentina de Mecánica Computacional, 2018.

[20] D. V. Zenkov and A. M. Bloch. Dynamics of the n-dimensional Suslov problem. *Journal of Geometry and Physics*, 34(2):121–136, 2000.

[21] J. Cortés Monforte. *Geometric, Control and Numerical Aspects of Nonholonomic Systems.* Lecture Notes in Mathematics. Springer Berlin, Heidelberg, first edition edition, 2002.

[22] A. M. Bloch. *Nonholonomic Mechanics and Control.* Interdisciplinary Applied Mathematics. Springer New York, NY, first edition edition, 2003.

[23] H. Yoshimura and J. E. Marsden. Representations of Dirac structures and implicit port-controlled Lagrangian systems. *The Proceedings of the Dynamics & Design Conference*, 2008, 04 2012.

[24] D. Lewis, J. Marsden, R. Montgomery, and T. Ratiu. The Hamiltonian structure for dynamic free boundary problems. *Physica D: Nonlinear Phenomena*, 18(1):391–404, 1986.

[25] Jerrold E. Marsden, Thomas J. R. Hughes, and Donald E. Carlson. *Mathematical foundations of elasticity.* Dover Publications, Inc., 1994.

[26] F. Gay-Balmaz, J. E. Marsden, and T. Ratiu. Reduced variational formulations in free boundary continuum mechanics. *Journal of Nonlinear Science*, 22, 08 2012.

[27] R.P. Ramkrishna Pasumarthy and A. van der Schaft. On interconnections of infinite-dimensional port-Hamiltonian systems. In *Proceedings of the 16th International Symposium on Mathematical Theory of Networks and Systems.* Katholieke Universiteit Leuven, 2004.

[28] M. Leok and T. Ohsawa. Discrete Dirac structures and implicit discrete Lagrangian and Hamiltonian systems. *AIP Conf. Proc.*, 1260, 2010.

[29] M. Leok and T. Ohsawa. Variational and geometric structures of discrete Dirac mechanics. *Found. Comput. Math.*, 11:529–562, 2011.

[30] H. Parks and M. Leok. Variational integrators for interconnected Lagrange–Dirac systems. *J. Nonlinear Sci.*, 27(5):1399–1434, 2017.

[31] A. Rodríguez Abella and M. Leok. Discrete Dirac reduction of implicit Lagrangian systems with abelian symmetry groups. *J. Geom. Mech.*, 15(1):319–356, 2023.