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Nonsmooth Nonholonomic Reduction for Mechanical Systems with Collisions*

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Abstract. This paper studies nonsmooth variational problems on principal bundles for nonholonomic systems with collisions taking place in the boundary of the configuration space of the nonholonomic system. In particular, we first extend to a nonsmooth context appropriate for collisions the variational principle for nonholonomic implicit Lagrangian systems to obtain implicit Lagrange–d’Alembert–Pontryagin equations for nonholonomic systems with collisions. After introducing the notion of connection on a principal bundle, we consider Lagrange–Poincaré–Pointryagin reduction by symmetries for systems with collisions.

Key words. mechanics with collisions, nonsmooth implicit Lagrangian systems, Lagrange–Poincaré reduction, nonholonomic systems, symmetries

MSC codes. 49J52, 49S05, 53B05, 53C05

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1. Introduction. Reduction theory is one of the fundamental tools in the study of mechanical systems with symmetries, and it essentially concerns the removal of superfluous degrees of freedom by using the symmetries of the system and relating the resulting equations to the conservation laws given by the Noether theorem. Such symmetries arise when the Lagrangian of the system $L : TQ \rightarrow \mathbb{R}$ is invariant under (the tangent lift of) the action of a Lie group G on the configuration manifold Q . There are two possibilities for the configuration manifold and, thus, for the corresponding reduction theory:

1. *Euler–Poincaré reduction*, i.e., when the configuration manifold is the group itself, $Q = G$, and the action is given by left translations. Hence, the original Lagrangian may be dropped to the quotient space $\ell : (TG)/G \simeq \mathfrak{g} \rightarrow \mathbb{R}$, where $\mathfrak{g}$ denotes the Lie algebra of G . The reduced dynamics is governed by a system of first order ODEs known as the *Euler–Poincaré equations* (cf. [5, 26]):

$$\frac{d}{dt} \left(\frac{\delta \ell}{\delta \xi} \right) = \text{ad}_\xi^* \left(\frac{\delta \ell}{\delta \xi} \right).$$

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This system, together with the reconstruction equation $\xi(t) = g^{-1}(t)\dot{g}(t)$, is equivalent to the Euler–Lagrange equations on G .

2. *Lagrange–Poincaré reduction*, i.e., when Q is an arbitrary manifold and G acts freely and properly on it. In that case, the Hamilton variational principle on Q can be dropped to $\Sigma = Q/G$ with the aid of a principal connection ω on $\pi_{Q,\Sigma} : Q \rightarrow \Sigma$. Such a connection defines a bundle isomorphism:

$$(1.1) \quad (TQ)/G \rightarrow T\Sigma \oplus \tilde{\mathfrak{g}}, \quad [v_q]_G \mapsto (T_q\pi_{Q,\Sigma}(v_q), [q, \omega_q(v_q)]_G),$$

where $\oplus$ denotes the fibered direct sum over Σ , $\tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G$ is the adjoint bundle of $\pi_{Q,\Sigma}$, and $[\cdot]_G$ denotes the equivalence class. Therefore, the reduced Lagrangian reads $\ell : T\Sigma \oplus \tilde{\mathfrak{g}} \rightarrow \mathbb{R}$. Similarly, a curve $q : [t_0, t_1] \rightarrow Q$ can be decomposed into its horizontal part, $\sigma = \pi_{Q,\Sigma} \circ q : [t_0, t_1] \rightarrow \Sigma$, and its vertical part, $\bar{\zeta} = [q, \omega(\dot{q})]_G : [t_0, t_1] \rightarrow \tilde{\mathfrak{g}}$. The reduced dynamics is governed by the so-called Lagrange–Poincaré equations (cf. [10]):

$$\begin{cases} \frac{\nabla^{\omega*}}{dt} \frac{\delta \ell}{\delta \bar{\zeta}} - \text{ad}^*_{\bar{\zeta}} \frac{\delta \ell}{\delta \bar{\zeta}} = 0, \\ \frac{\delta \ell}{\delta \sigma} - \frac{\nabla^{\Sigma*}}{dt} \frac{\delta \ell}{\delta \dot{\sigma}} = \frac{\delta \ell}{\delta \bar{\zeta}} \cdot (i_{\dot{\sigma}} \tilde{F}^{\omega}), \end{cases}$$

where $\tilde{F}^{\omega}$ is the reduced curvature of ω , and $\nabla^{\omega*}/dt$ and $\nabla^{\Sigma*}/dt$ denote the covariant derivatives on the coadjoint and the cotangent bundles, respectively.

Nonholonomic systems. Some mechanical systems, known as constrained systems, have a restriction on the configurations or velocities that they may assume. In particular, nonholonomic systems [4, 27] are, roughly speaking, mechanical systems with constraints on their velocities that are not derivable from position constraints. They arise, for instance, in systems that have rolling contact (e.g., the rolling of wheels without slipping) or certain kinds of sliding contact (such as the sliding of skates). There are some differences between nonholonomic systems and standard Hamiltonian or Lagrangian systems. Among them, we have the following: nonholonomic systems arise from the Lagrange–d’Alembert principle and not from the Hamilton principle; they may preserve the energy of the system as Hamiltonian or Lagrangian systems, but they are not, in general, time-reversible; they do not preserve, in general, the momentum for systems with symmetries (i.e., Noether’s theorem does not apply, in general). A reduction theory for nonholonomic systems was first introduced in [25] for both cases: $Q = G$ and Q is general smooth manifold; and the corresponding Lagrangian version was developed in [7, 8, 9].

Reduction of implicit systems with nonholonomic constraints. Implicit systems are those for which the dynamical equations governing their behavior are implicit differential equations. In the context of geometric mechanics, they appear when the Lagrangian or Hamiltonian of the system is degenerate, as well as in the theory of interconnection, which provides models for multiphysical systems by aggregating simpler subsystems. A unifying framework for the formulation and analysis of implicit Lagrangian systems with nonholonomic constraints was introduced in [33] making use of Dirac structures. The equivalent variational formulation was presented in [34]. In this context, the analogue of the Euler–Poincaré reduction for implicit systems was developed in [35] for both unconstrained and nonholonomic systems, yielding the

Table 1

Types of Lagrangian reduction. In this work, we develop the bottom right cell, i.e., implicit nonholonomic Lagrange–Poincaré reduction. In addition, we utilize the nonsmooth variational approach to address elastic impacts.

Reduction	Lie group	Lie group with advection	Principal bundle
Original Lagrangian	$L : TG \rightarrow \mathbb{R}$	$L : TG \times V^* \rightarrow \mathbb{R}$	$L : TQ \rightarrow \mathbb{R}$
Reduced Lagrangian	$\ell : \mathfrak{g} \rightarrow \mathbb{R}$	$\ell : \mathfrak{g} \times V^* \rightarrow \mathbb{R}$	$\ell : T\Sigma \oplus \tilde{\mathfrak{g}} \rightarrow \mathbb{R}$
Unconstrained	Euler–Poincaré [26]	Euler–Poincaré with advected param. [22]	Lagrange–Poincaré [10]
Unconstrained (implicit)	Implicit Euler–Poincaré [35]	Implicit Euler–Poincaré with advected param. [19]	Implicit Lagrange–Poincaré [36]
Implicit and nonholonomic	Implicit Euler–Poincaré–Suslov [35]	Implicit Euler–Poincaré–Suslov with advected param. [19]	Implicit nonholonomic Lagrange–Poincaré

so-called implicit Euler–Poincaré and implicit Euler–Poincaré–Suslov equations, respectively. The Lagrange–Poincaré reduction, i.e., the principal bundle reduction, for implicit systems was given in [36] only for the unconstrained case. A further generalization was presented [19] to account for nonholonomic systems on Lie groups with broken symmetries, yielding the implicit Euler–Poincaré–Suslov equations with advected parameters. Nevertheless, a general principal bundle reduction theory for implicit Lagrangian systems with nonholonomic constraints is still missing. The different Lagrangian reduction schemes found in the literature are summarized in Table 1.

Mechanical systems with collisions. Systems undergoing collisions constitute a type of hybrid system where the dynamics is confined within a region with boundary. Elastic collisions with that boundary activate constraints on the momentum and energy before and after the impact. Systems with collisions have been extensively treated in the literature since the early days of mechanics (see [6] and references therein for a comprehensive review). More recently, much work has been done on the rigorous mathematical foundation of impact problems [21, 12]. Some applications include bipedal locomotion [32] and convex rigid bodies bouncing on the horizontal plane due to gravity [31]. Nonholonomic systems subject to impacts or impulse effects have been previously studied in [11, 13]. In this paper, we follow the approach of [17], where a nonsmooth variational formulation of Lagrangian systems is provided.

Reduction of systems with collisions. Several types of reduction have been investigated for hybrid systems and, in particular, for systems with impacts. For instance, a Routh reduction scheme for Lagrangian systems was introduced in [2] and further developed in [1] to gain a better understanding of bipedal walking models. This approach was extended to account for cyclic variables in [15] and for time-dependent systems in [14]. Similarly, a symplectic reduction for Hamiltonian and Lagrangian systems was investigated in [16]. Nevertheless, Euler–Poincaré and Lagrange–Poincaré reduction schemes in the context of systems with collisions cannot be found in the previous literature.

1.1. Contributions and plan of the paper. The purpose of the present paper is twofold. First, we provide a general reduction scheme for implicit Lagrangian systems with nonholonomic constraints. The nonholonomic dynamics obtained through the reduction of the Hamilton–Pontryagin principle gives rise to a set of second-order implicit ordinary differential equations. Second, the continuous setting for implicit Lagrangian systems with nonholonomic constraints introduced in [34] is extended to the nonsmooth setting following the approach in [17] to account for elastic impacts. By gathering both goals, we obtain a principal bundle reduction theory for nonholonomic implicit Lagrangian system with collisions from a nonsmooth variational mechanics viewpoint. Therefore, we close the gap of Lagrange–Poincaré reduction theories (i.e., principal bundle reduction theories) for nonholonomic implicit systems with elastic collisions.

This problem has several motivations. Implicit mechanical systems naturally arise when dealing with degenerate Lagrangians or performing interconnections. Furthermore, nonholonomic constraints are ubiquitous in such systems, and collisions frequently occur in physical-world applications, as the system’s allowed states are restricted to specific regions of the configuration space. Robotics applications, such as bipedal walking [1], integrate all these features. In that context, we provide the appropriate framework to model such systems and perform reduction by symmetries, thus simplifying the equations and obtaining conserved quantities.

The remainder of the paper is structured as follows. In section 2 we recall the main facts about implicit nonholonomic Lagrangian systems. Next, in section 3 we present a nonsmooth variational framework to formulate implicit Lagrangian systems with collisions. Some preliminary results in that direction were introduced in [29]. The core of the paper is in section 4, where the Lagrange–Poincaré reduction theory for implicit nonholonomic systems with collisions is developed. The theoretical results are illustrated by several examples, including the formulation of the original and reduced dynamics of a spherical pendulum hitting a cylindrical surface. Lastly, in section 5 we suggest some future lines of research.

2. Implicit nonholonomic systems. Let Q be a smooth manifold with $\dim(Q) = n$ and (q^i) be a particular choice of local coordinates on this manifold. In the following, TQ denotes the tangent bundle of Q , with T_qQ being the tangent space at each point $q \in Q$ and $v_q \in T_qQ$ being a vector. In addition, the coordinate chart (q^i) induces a natural coordinate chart on TQ denoted by $(q^i, \dot{q}^i)$. Similarly, let T^*Q be the cotangent bundle of Q , with T_q^*Q being the cotangent space at $q \in Q$. The coordinate chart (q^i) induces a coordinate chart (q^i, p_i) on T^*Q consisting of the positions and momenta of the system.

A k -dimensional distribution on Q is a vector subbundle $\Delta_Q \subset TQ$ with k -dimensional fiber; i.e., $\Delta_Q(q) \subset T_qQ$ is a k -dimensional subspace for each $q \in Q$. Moreover, Δ_Q is smooth if there exist a neighborhood U of each point $q \in Q$ and local vector fields $X_1, \dots, X_k \in \mathfrak{X}(U)$ such that $\Delta_Q(q) = \text{span}\{X_1(q), \dots, X_k(q)\}$ for all $q \in U$.

Analogously, a k -dimensional codistribution on Q is a vector subbundle $\widetilde{\Delta_Q} \subset T^*Q$ with k -dimensional fiber. Given the concept of codistribution, it is possible to define the *annihilator* of a distribution $\Delta_Q \subset TQ$; namely, it is the codistribution given by $\Delta_Q^\circ(q) = \{\alpha \in T_q^*Q \mid \langle \alpha, v \rangle = 0 \forall v \in \Delta_Q(q)\}$ for each $q \in Q$, where $\langle \cdot, \cdot \rangle$ denotes the dual pairing.

Linear constraints on the velocities are locally given by equations of the form $\phi^a(q^i, \dot{q}^i) = \mu_i^a(q) \dot{q}^i = 0$, $1 \leq a \leq m$, depending, in general, on the configuration coordinates and their velocities. From an intrinsic point of view, the linear constraints are defined by a distribution Δ_Q on Q of constant rank $n - m$ such that the annihilator of Δ_Q is locally given by $\Delta_Q^\circ(q) = \text{span} \{ \mu^a(q) = \mu_i^a dq^i \mid 1 \leq a \leq m \}$, where the 1-forms μ^a are linearly independent at each $q \in Q$. When the constraint distribution Δ_Q is nonintegrable, the linear constraints are said to be nonholonomic.

In addition to these constraints, the dynamical evolution of the system is specified by fixing a Lagrangian function $L: TQ \rightarrow \mathbb{R}$. The central concepts permitting the extension of mechanics from the Newtonian point of view to the Lagrangian one are the notions of virtual displacements and virtual work. These concepts were formulated in the developments of mechanics and in their application to statics. In nonholonomic dynamics, the procedure is given by the *Lagrange-d'Alembert principle*, which determines the set of possible values of the constraint forces from the set Δ_Q of admissible kinematic states alone. The resulting equations of motion are

$$\left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} \right] \delta q^i = 0, \quad 1 \leq i \leq n,$$

where (δq^i) denote the virtual displacements and verify $\mu_i^a \delta q^i = 0$, $1 \leq a \leq m$. By using Lagrange multipliers, we obtain

$$(2.1) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = \lambda_a \mu_i^a, \quad 1 \leq i \leq n.$$

The term on the right-hand side represents the constraint force, also known as reaction force, induced by the constraints. The functions λ_a are the so-called Lagrange multipliers, which, after being computed using the constraint equations, yield a set of second-order differential equations.

Alternatively to the use of Lagrange multipliers, the phase space may be enlarged to the Pontryagin bundle $TQ \oplus T^*Q$ and the Lagrange-d'Alembert-Pontryagin principle may be considered. This variational principle is given by

$$\delta \int_{t_0}^{t_1} (L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t) \rangle) dt = 0,$$

where $v(t) \in \Delta_Q(q(t))$ and the variations $(\delta q(t), \delta v(t), \delta p(t))$ are such that $\delta q(t) \in \Delta_Q(q(t))$ and vanish at the endpoints. The stationary condition for a curve $(q(t), v(t), p(t))$ yields the implicit Lagrange-d'Alembert equations on $TQ \oplus T^*Q$ (see [34]):

$$p = \frac{\partial L}{\partial v}, \quad \dot{q} = v \in \Delta_Q(q), \quad \dot{p} - \frac{\partial L}{\partial q} \in \Delta_Q^\circ(q).$$

2.1. Reduction of implicit nonholonomic systems. Let G be a (finite dimensional) Lie group and $\Phi: G \times Q \rightarrow Q$ be a left action of G on Q , i.e., $\Phi(e, q) = q$, where $e \in G$ denotes the identity element, and $\Phi(g, \Phi(h, q)) = \Phi(gh, q)$ for each $q \in Q$ and $g, h \in G$. In particular, for each $g \in G$, the map $\Phi_g: Q \rightarrow Q$ defined as $\Phi_g(q) := \Phi(g, q)$, is a diffeomorphism.

The infinitesimal generator corresponding to $\xi \in \mathfrak{g}$, where $\mathfrak{g} = T_e G$ is the Lie algebra of G , is the field $\xi_Q \in \mathfrak{X}(Q)$ defined as $\xi_Q(q) = d/dt|_{t=0} \Phi(\exp(t\xi), q)$, where $\exp : \mathfrak{g} \rightarrow G$ denotes the exponential map. In addition, the Lie bracket on $\mathfrak{g}$ is denoted by $[\cdot, \cdot]$. For each $\xi \in \mathfrak{g}$, the adjoint map $\text{ad}_\xi : \mathfrak{g} \rightarrow \mathfrak{g}$ is defined as $\text{ad}_\xi(\eta) = [\xi, \eta]$ for all $\eta \in \mathfrak{g}$. Similarly, the map $\text{ad}_\xi^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ denotes the co-adjoint operator and is defined as $\langle \text{ad}_\xi^*(\mu), \eta \rangle = \langle \mu, \text{ad}_\xi(\eta) \rangle$ for all $\eta \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$.

Euler–Poincaré reduction. Consider the particular case of the Lie group acting on itself by left multiplication, i.e., $\Phi : G \times G \rightarrow G$ defined as $\Phi(g, h) = L_g(h) = gh$ for all $g, h \in G$. For G -invariant nonholonomic (possibly degenerate) Lagrangian systems on Lie groups, the reduced Lagrangian is denoted by $\ell : \mathfrak{g} \rightarrow \mathbb{R}$, and the reduction of the Lagrange–d’Alembert equations (2.1) yields

$$(2.2) \quad \mu = \frac{\delta \ell}{\delta \eta}, \quad \xi = \eta \in \mathfrak{d}, \quad \dot{\mu} - \text{ad}_\xi^*(\mu) \in \mathfrak{d}^\circ,$$

where $\mathfrak{d} \subset \mathfrak{g}$ is the reduced constraint and $\mathfrak{d}^\circ \subset \mathfrak{g}^*$ denotes its annihilator. These equations are the implicit analogue of the Euler–Poincaré–Suslov equations [4]. For this reason, they are called the implicit Euler–Poincaré–Suslov equations [19, 35].

Lagrange–Poincaré reduction. For the general case of a free and proper action $\Phi : G \times Q \rightarrow Q$, the quotient projection defines a principal bundle, $\pi_{Q, \Sigma} : Q \rightarrow \Sigma$, where $\Sigma = Q/G$ is known as the shape space and is endowed with the unique manifold structure making $\pi_{Q, \Sigma}$ a submersion (see, for example, [24]). With the aid of a principal connection on $\pi_{Q, \Sigma}$ (recall (1.1)), the reduced Lagrangian corresponding to a G -invariant, (possibly degenerate) Lagrangian on Q , reads $\ell : T\Sigma \oplus \tilde{\mathfrak{g}} \rightarrow \mathbb{R}$. For the unconstrained case, the implicit Lagrange–Poincaré equations become [36]

$$\begin{aligned} \frac{\nabla^{\Sigma*} y}{dt} &= \frac{\delta \ell}{\delta \sigma} - \bar{\rho} \cdot \left(i_{\dot{\sigma}} \tilde{F}^\omega \right), & \dot{\sigma} &= u, \\ y &= \frac{\delta \ell}{\delta u}, & \bar{\xi} &= \bar{\eta}, \\ \frac{\nabla^{\omega*} \bar{\rho}}{dt} &= \text{ad}_\xi^*(\bar{\rho}), & \bar{\rho} &= \frac{\delta \ell}{\delta \bar{\eta}}. \end{aligned}$$

As explained in the introduction, the problem of implicit nonholonomic Lagrange–Poincaré reduction has not been considered in the previous literature. Specifically, [35] and [19] dealt with implicit Euler–Poincaré–Suslov reduction for the entire group and for semidirect products, respectively, whereas [36] explored implicit nonholonomic Lagrange–Poincaré reduction for systems without constraints.

3. Nonholonomic implicit Lagrangian mechanics with collisions. In the following, the dynamics with impacts is addressed from the nonsmooth variational approach presented in [17]. Let Q be a smooth manifold with boundary, denoted by ∂Q , $L : TQ \rightarrow \mathbb{R}$ be a (possibly degenerate) Lagrangian, and $\Delta_Q \subset TQ$ be a (possibly nonholonomic) constraint distribution. According to section 2, the annihilator of Δ_Q is denoted by $\Delta_Q^\circ \subset T^*Q$.

3.1. Configuration space and phase space. Given $[\tau_0, \tau_1] \subset \mathbb{R}$ and $\tilde{\tau} \in [\tau_0, \tau_1]$, the *path space with a unique collision (at $\tau = \tilde{\tau}$)* is defined as $\Omega(Q, \tilde{\tau}) = \mathcal{T} \times \mathcal{Q}(\tilde{\tau})$, where

$$\mathcal{T} = \{ \alpha_T \in C^\infty([\tau_0, \tau_1]) \mid \alpha'_T(\tau) > 0, \tau \in [\tau_0, \tau_1] \}$$

and

$$(3.1) \quad \mathcal{Q}(\tilde{\tau}) = \{\alpha_Q \in C^0([\tau_0, \tau_1], Q) \mid \alpha_Q(\tilde{\tau}) \in \partial Q, \\ \alpha_Q \text{ is piecewise } C^2 \text{ and has only one singularity at } \tilde{\tau}\}.$$

We only consider one singularity at $\tau = \tilde{\tau}$ for brevity, but similar results hold for a finite number of singularities, $\{\tilde{\tau}_i \mid 1 \leq i \leq N\} \subset [\tau_0, \tau_1]$.

Remark 3.1 (Zeno behavior). Systems with collisions are a particular instance of hybrid systems. For systems with elastic impacts, the *guard* is given by $S = \{v_q \in T_q Q \mid q \in \partial Q, g(v_q, n_q) > 0\}$, where g is a Riemannian metric on Q and n is the outward-pointing, unit, normal vector field on the boundary. Similarly, the *reset map* is given by $R(v_q) = v_q^\parallel - v_q^\perp$, where $v_q^\perp = g(v_q, n_q) n_q$ and $v_q^\parallel = v_q - v_q^\perp \in T_q \partial Q$. Recall that hybrid systems may experience Zeno behavior if a trajectory undergoes infinitely many impacts in finite time. In order to avoid this situation, we ask the system to satisfy two conditions (cf. [20, Remark 2.1]):

1. $S \cap \bar{R}(S) = \emptyset$, where $\bar{R}(S)$ is the closure of $R(S) \subset TQ$. This condition is clearly satisfied in our case. Indeed, for each $v_q \in S$ we have $v_q^\perp \neq 0$ and, thus, $\|R(v_q) - v_q\|_g = 2\|v_q^\perp\|_g > 0$, with $\|\cdot\|_g$ being the norm induced by the metric g .
2. The set of collision times is closed and discrete. This condition, which depends on the topology of the configuration manifold, prevents the existence of an accumulation point and will be assumed in the following.

Under these assumptions, our development is valid in a neighborhood of each collision.

Lemma 3.2. ([17, Corollary 2.3]). $\Omega(Q, \tilde{\tau}) = \mathcal{T} \times \mathcal{Q}(\tilde{\tau})$ is a smooth manifold.

Remark 3.3. Given $\alpha_T \in \mathcal{T}$, we denote $[t_0, t_1] = \alpha_T([\tau_0, \tau_1])$ and, in order to distinguish between τ -derivatives and t -derivatives, we use different symbols; namely, $\alpha'_T = d\alpha_T/d\tau$ and $\dot{\alpha}_T^{-1} = d\alpha_T^{-1}/dt$, where $\alpha_T^{-1} : [t_0, t_1] \rightarrow [\tau_0, \tau_1]$ is the inverse of α_T . Analogously, we denote $\tilde{t} = \alpha_T(\tilde{\tau})$.

The tangent space of $\mathcal{Q}(\tilde{\tau})$ at $\alpha_Q \in \mathcal{Q}(\tilde{\tau})$ is given by

$$(3.2) \quad T_{\alpha_Q} \mathcal{Q}(\tilde{\tau}) = \left\{ \nu_{\alpha_Q} \in C^0([\tau_0, \tau_1], TQ) \mid \alpha_Q = \pi_{TQ} \circ \nu_{\alpha_Q}, \nu_{\alpha_Q}(\tilde{\tau}) \in T_{\alpha_Q(\tilde{\tau})} \partial Q, \right. \\ \left. \nu_{\alpha_Q} \text{ is piecewise } C^2 \text{ and has only one singularity at } \tilde{\tau} \right\},$$

where $\pi_{TQ} : TQ \rightarrow Q$ is the natural projection. In order to incorporate the constraint distribution, we define the following subspace at each $\alpha_Q \in \mathcal{Q}(\tilde{\tau})$:

$$\Delta_{\mathcal{Q}(\tilde{\tau})}(\alpha_Q) = \{ \nu_{\alpha_Q} \in T_{\alpha_Q} \mathcal{Q}(\tilde{\tau}) \mid \nu_{\alpha_Q} : [\tau_0, \tau_1] \rightarrow \Delta_Q \}.$$

As usual, we denote $T\mathcal{Q}(\tilde{\tau}) = \bigsqcup_{\alpha_Q \in \mathcal{Q}(\tilde{\tau})} T_{\alpha_Q} \mathcal{Q}(\tilde{\tau})$ and $\Delta_{\mathcal{Q}(\tilde{\tau})} = \bigsqcup_{\alpha_Q \in \mathcal{Q}(\tilde{\tau})} \Delta_{\mathcal{Q}(\tilde{\tau})}(\alpha_Q)$.

Let $T'_{\alpha_Q} \mathcal{Q}(\tilde{\tau}) = \{\phi_{\alpha_Q} : T_{\alpha_Q} \mathcal{Q}(\tilde{\tau}) \rightarrow \mathbb{R} \mid \phi_{\alpha_Q} \text{ is linear and continuous}\}$ be the topological dual of $T_{\alpha_Q} \mathcal{Q}(\tilde{\tau})$. Since $\mathcal{Q}(\tilde{\tau})$ is an infinite dimensional manifold, its topological cotangent bundle is too large to formulate mechanics. For that reason, we will restrict ourselves to the vector subbundle where the Legendre transform of the Lagrangian lies; i.e., we consider a vector subbundle $T^* \mathcal{Q}(\tilde{\tau}) \subset T' \mathcal{Q}(\tilde{\tau})$ such that $\mathbb{F}L \circ \nu_{\alpha_Q} \in T^* \mathcal{Q}(\tilde{\tau})$ for each $\nu_{\alpha_Q} \in T_{\alpha_Q} \mathcal{Q}(\tilde{\tau})$, where $\mathbb{F}L : TQ \rightarrow T^*Q$ is the Legendre transform of L .

Lemma 3.4. For each $\alpha_Q \in \mathcal{Q}(\tilde{\tau})$, the vector space

$$(3.3) \quad T_{\alpha_Q}^* \mathcal{Q}(\tilde{\tau}) = \left\{ \pi_{\alpha_Q} \in C^0([\tau_0, \tau_1], T^*Q) \mid \alpha_Q = \pi_{T^*Q} \circ \pi_{\alpha_Q}, \pi_{\alpha_Q}(\tilde{\tau}) \in T_{\alpha_Q(\tilde{\tau})}^* \partial Q, \right. \\ \left. \pi_{\alpha_Q} \text{ is piecewise } C^2 \text{ and has only one singularity at } \tilde{\tau} \right\},$$

where $\pi_{T^*Q} : T^*Q \rightarrow Q$ is the natural projection, is a vector subspace of the topological dual of $T_{\alpha_Q} \mathcal{Q}(\tilde{\tau})$ by means of the following L^2 -dual pairing:

$$\langle \pi_{\alpha_Q}, \nu_{\alpha_Q} \rangle = \int_{\tau_0}^{\tau_1} \pi_{\alpha_Q}(\tau) \cdot \nu_{\alpha_Q}(\tau) d\tau,$$

where $\cdot$ represents the pairing between T^*Q and TQ . Furthermore, this pairing is nondegenerate.

Note that, in general, $\{\mathbb{F}L \circ \nu_{\alpha_Q} \in T_{\alpha_Q}^* \mathcal{Q}(\tilde{\tau}) \mid \nu_{\alpha_Q} \in T_{\alpha_Q} \mathcal{Q}(\tilde{\tau})\} \subsetneq T_{\alpha_Q}^* \mathcal{Q}(\tilde{\tau})$, as the Lagrangian is possibly degenerate. As a straightforward consequence of the previous lemma, the vector bundle

$$T^* \mathcal{Q}(\tilde{\tau}) = \bigsqcup_{\alpha_Q \in \mathcal{Q}(\tilde{\tau})} T_{\alpha_Q}^* \mathcal{Q}(\tilde{\tau}) \rightarrow \mathcal{Q}(\tilde{\tau}), \quad \pi_{\alpha_Q} \mapsto \alpha_Q,$$

is a vector subbundle of the topological cotangent bundle of $\mathcal{Q}(\tilde{\tau})$.

In the same vein, for each $\nu_{\alpha_Q} \in T_{\alpha_Q} \mathcal{Q}(\tilde{\tau})$ and $\pi_{\alpha_Q} \in T_{\alpha_Q}^* \mathcal{Q}(\tilde{\tau})$, the iterated bundles are given by

$$T_{\nu_{\alpha_Q}}(T \mathcal{Q}(\tilde{\tau})) = \left\{ \delta \nu_{\alpha_Q} \in C^0([\tau_0, \tau_1], T(TQ)) \mid \nu_{\alpha_Q} = \pi_{T(TQ)} \circ \delta \nu_{\alpha_Q}, \right. \\ \delta \nu_{\alpha_Q}(\tilde{\tau}) \in T_{\nu_{\alpha_Q}(\tilde{\tau})}(T \partial Q), \\ \left. \delta \nu_{\alpha_Q} \text{ is piecewise } C^2 \text{ and has only one singularity at } \tilde{\tau} \right\},$$

where $\pi_{T(TQ)} : T(TQ) \rightarrow TQ$ is the natural projection, and

$$T_{\pi_{\alpha_Q}}(T^* \mathcal{Q}(\tilde{\tau})) = \left\{ \delta \pi_{\alpha_Q} \in C^0([\tau_0, \tau_1], T(T^*Q)) \mid \pi_{\alpha_Q} = \pi_{T(T^*Q)} \circ \delta \pi_{\alpha_Q}, \right. \\ \delta \pi_{\alpha_Q}(\tilde{\tau}) \in T_{\pi_{\alpha_Q}(\tilde{\tau})}(T^* \partial Q), \\ \left. \delta \pi_{\alpha_Q} \text{ is piecewise } C^2 \text{ and has only one singularity at } \tilde{\tau} \right\},$$

where $\pi_{T(T^*Q)} : T(T^*Q) \rightarrow T^*Q$ is the natural projection. In particular, we consider the constrained iterated bundle,

$$(3.4) \quad \Delta_{T \mathcal{Q}(\tilde{\tau})}(\nu_{\alpha_Q}) = \left\{ \delta \nu_{\alpha_Q} \in T_{\nu_{\alpha_Q}}(T \mathcal{Q}(\tilde{\tau})) \mid d\pi_{TQ} \circ \delta \nu_{\alpha_Q} \in C^0([\tau_0, \tau_1], \Delta_Q) \right\}.$$

3.2. Nonholonomic implicit Euler–Lagrange equations with collisions. Given a path $\alpha = (\alpha_T, \alpha_Q) \in \Omega(Q, \tilde{\tau})$, the associated curve is defined as

$$(3.5) \quad q_\alpha : [t_0, t_1] \rightarrow Q, \quad t \mapsto q_\alpha(t) = (\alpha_Q \circ \alpha_T^{-1})(t).$$

Similarly, given $\nu_{\alpha_Q} \in T_{\alpha_Q} \mathcal{Q}(\tilde{\tau})$ and $\pi_{\alpha_Q} \in T'_{\alpha_Q} \mathcal{Q}(\tilde{\tau})$, we set

$$\begin{aligned} v_\alpha : [t_0, t_1] &\rightarrow TQ, & t &\mapsto v_\alpha(t) = (\nu_{\alpha_Q} \circ \alpha_T^{-1})(t), \\ p_\alpha : [t_0, t_1] &\rightarrow T^*Q, & t &\mapsto p_\alpha(t) = (\pi_{\alpha_Q} \circ \alpha_T^{-1})(t). \end{aligned}$$

It is clear that $\pi_{TQ} \circ v_\alpha = \pi_{T^*Q} \circ p_\alpha = q_\alpha$.

By regarding $\Omega(Q, \tilde{\tau})$ as a trivial vector bundle over $\mathcal{Q}(\tilde{\tau})$ with the projection onto the second factor, the *Lagrange–d'Alembert–Pontryagin action functional*,

$$\mathbb{S} : \Omega(Q, \tilde{\tau}) \times_{\mathcal{Q}(\tilde{\tau})} (T\mathcal{Q}(\tilde{\tau}) \oplus T^*\mathcal{Q}(\tilde{\tau})) \rightarrow \mathbb{R},$$

where $\times_{\mathcal{Q}(\tilde{\tau})}$ denotes the fibered product over $\mathcal{Q}(\tilde{\tau})$, is defined as

$$\begin{aligned} \mathbb{S}(\alpha, \nu_{\alpha_Q}, \pi_{\alpha_Q}) &= \int_{t_0}^{t_1} (L(v_\alpha(t)) + p_\alpha(t) \cdot (\dot{q}_\alpha(t) - v_\alpha(t))) dt \\ &= \int_{\tau_0}^{\tau_1} \left(L(\nu_{\alpha_Q}(\tau)) + \pi_{\alpha_Q}(\tau) \cdot \left(\frac{\alpha'_Q(\tau)}{\alpha'_T(\tau)} - \nu_{\alpha_Q}(\tau) \right) \right) \alpha'_T(\tau) d\tau. \end{aligned}$$

The equality between the first and the second expressions can be easily checked by considering the change of variable $t = \alpha_T(\tau)$. By recalling that the *energy* of the system is given by

$$(3.6) \quad E : TQ \oplus T^*Q \rightarrow \mathbb{R}, \quad (v_q, p_q) \mapsto E(v_q, p_q) = p_q \cdot v_q - L(v_q),$$

the action functional may be rewritten as

$$\begin{aligned} \mathbb{S}(\alpha, \nu_{\alpha_Q}, \pi_{\alpha_Q}) &= \int_{t_0}^{t_1} (p_\alpha(t) \cdot \dot{q}_\alpha(t) - E(v_\alpha(t), p_\alpha(t))) dt \\ &= \int_{\tau_0}^{\tau_1} \left(\pi_{\alpha_Q}(\tau) \cdot \frac{\alpha'_Q(\tau)}{\alpha'_T(\tau)} - E(\nu_{\alpha_Q}(\tau), \pi_{\alpha_Q}(\tau)) \right) \alpha'_T(\tau) d\tau. \end{aligned}$$

Definition 3.5 (Hamilton–d'Alembert–Pontryagin principle). A path

$$c = ((\alpha_T, \alpha_Q), \nu_{\alpha_Q}, \pi_{\alpha_Q}) \in \Omega(Q, \tilde{\tau}) \times_{\mathcal{Q}(\tilde{\tau})} (\Delta_{\mathcal{Q}(\tilde{\tau})} \oplus T^*\mathcal{Q}(\tilde{\tau}))$$

is stationary (or critical) for the action functional $\mathbb{S}$ if it satisfies

$$d\mathbb{S}(c)(\delta c) = 0$$

for every variation $\delta c = ((\delta\alpha_T, \delta\alpha_Q), \delta\nu_{\alpha_Q}, \delta\pi_{\alpha_Q}) \in T_\alpha \Omega(Q, \tilde{\tau}) \times \Delta_{T\mathcal{Q}(\tilde{\tau})}(\nu_{\alpha_Q}) \times T_{\pi_{\alpha_Q}}(T^*\mathcal{Q}(\tilde{\tau}))$ such that $\delta\alpha_T(\tau_0) = \delta\alpha_T(\tau_1) = 0$, $\delta\alpha_Q(\tau_0) = \delta\alpha_Q(\tau_1) = 0$, and

$$(3.7) \quad d\pi_{TQ} \circ \delta\nu_{\alpha_Q} = d\pi_{T^*Q} \circ \delta\pi_{\alpha_Q} = \delta\alpha_Q.$$

In order to give an intrinsic expression for the dynamical equations, we fix a linear connection on the tangent bundle of Q , $\nabla^Q : \mathfrak{X}(Q) \rightarrow \Omega^1(Q, TQ)$, as well as its dual, which is a linear connection on the cotangent bundle of Q , $\nabla^{Q*} : \Omega^1(Q) \rightarrow \Omega^1(Q, T^*Q)$. These connections enable us to compute the vertical part of the variations. For instance, given $\alpha_Q \in \mathcal{Q}(\tilde{\tau})$,

$\nu_{\alpha_Q} \in T_{\alpha_Q} \mathcal{Q}(\tilde{\tau})$, and $\delta\nu_{\alpha_Q} \in T_{\nu_{\alpha_Q}}(T\mathcal{Q}(\tilde{\tau}))$, we write $\delta^Q \nu_{\alpha_Q} = (\delta\nu_{\alpha_Q})^v \in T_{\alpha_Q} \mathcal{Q}(\tilde{\tau})$, where the superscript v denotes the vertical part given by ∇^Q , and analogously for the dual connection, δ^{Q*} . Additionally, the covariant derivatives induced by these connections are denoted by $\nabla^Q/d\tau$ and $\nabla^{Q*}/d\tau$, respectively.

The partial functional derivatives of the Lagrangian are denoted by

$$\begin{aligned} \frac{\delta L}{\delta v_q} : TQ \rightarrow T^*Q, \quad \frac{\delta L}{\delta v_q}(v_q) \cdot w_q &= \left. \frac{d}{ds} \right|_{s=0} L(v_q + s w_q), \\ \frac{\delta L}{\delta q} : TQ \rightarrow T^*Q, \quad \frac{\delta \ell}{\delta q}(v_q) \cdot w_q &= \left. \frac{d}{ds} \right|_{s=0} \left(L \circ \gamma_{v_q}^h \right)(s), \end{aligned}$$

for each $v_q, w_q \in T_q Q$, $q \in Q$, where $\gamma : (-\epsilon, \epsilon) \rightarrow Q$ is a curve such that $d/ds|_{s=0} \gamma = w_q$ and $\gamma_{v_q}^h : (-\epsilon, \epsilon) \rightarrow TQ$ is the horizontal lift at v_q given by ∇^Q . Observe that the first one is a vertical (fiber) derivative, whereas the second one is a horizontal derivative and depends on the choice of the connection.

Theorem 3.6 (nonholonomic implicit Euler–Lagrange equations with collisions). *A path*

$$((\alpha_T, \alpha_Q), \nu_{\alpha_Q}, \pi_{\alpha_Q}) \in \Omega(Q, \tilde{\tau}) \times_{\mathcal{Q}(\tilde{\tau})} (\Delta_{\mathcal{Q}(\tilde{\tau})} \oplus T^*\mathcal{Q}(\tilde{\tau}))$$

is critical for the action functional $\mathbb{S}$ if and only if it satisfies the implicit Euler–Lagrange equations, or Lagrange–d’Alembert–Pontryagin equations

$$\begin{cases} \frac{\nabla^{Q*} \pi_{\alpha_Q}}{d\tau} - \alpha'_T \frac{\delta L}{\delta q}(\nu_{\alpha_Q}) \in \Delta_Q^\circ(\alpha_Q), & E'(\nu_{\alpha_Q}, \pi_{\alpha_Q}) = 0, \\ \pi_{\alpha_Q} = \frac{\delta L}{\delta v_q}(\nu_{\alpha_Q}), & \nu_{\alpha_Q} = \frac{\alpha'_Q}{\alpha'_T} \in \Delta_Q(\alpha_Q), \end{cases}$$

on $[\tau_0, \tilde{\tau}) \cup (\tilde{\tau}, \tau_1]$, together with the conditions for the elastic impact,

$$\begin{cases} \pi_{\alpha_Q}^+ - \pi_{\alpha_Q}^- \in (T_{\alpha_Q(\tilde{\tau})} \partial Q \cap \Delta_Q(\alpha_Q(\tilde{\tau})))^\circ = (T_{\alpha_Q(\tilde{\tau})} \partial Q)^\circ + \Delta_Q^\circ(\alpha_Q(\tilde{\tau})), \\ E(\nu_{\alpha_Q}^+, \pi_{\alpha_Q}^+) = E(\nu_{\alpha_Q}^-, \pi_{\alpha_Q}^-), \end{cases}$$

where the annihilators are with respect to TQ and we denote $\pi_{\alpha_Q}(\tilde{\tau}^+) = \pi_{\alpha_Q}^+$, etc.

Proof. First, (3.7) ensures that the horizontal part of the variations is $\delta\alpha_Q \in \Delta_{\mathcal{Q}(\tilde{\tau})}(\alpha_Q)$. As a result, the variation of the action functional reads

$$\begin{aligned} d\mathbb{S}(\alpha, \nu_{\alpha_Q}, \pi_{\alpha_Q})(\delta\alpha, \delta\nu_{\alpha_Q}, \delta\pi_{\alpha_Q}) &= \int_{\tau_0}^{\tau_1} \left(\frac{\partial L}{\partial q} \cdot \delta\alpha_Q + \frac{\partial L}{\partial v_q} \cdot \delta^Q \nu_{\alpha_Q} + \delta^{Q*} \pi_{\alpha_Q} \cdot \left(\frac{\alpha'_Q}{\alpha'_T} - \nu_{\alpha_Q} \right) \right. \\ &\quad \left. + \pi_{\alpha_Q} \cdot \left(\frac{\delta^Q \alpha'_Q}{\alpha'_T} - \frac{\alpha'_Q \delta \alpha'_T}{(\alpha'_T)^2} - \delta^Q \nu_{\alpha_Q} \right) \right) \alpha'_T d\tau \\ &\quad + \int_{\tau_0}^{\tau_1} \left(L + \pi_{\alpha_Q} \cdot \left(\frac{\alpha'_Q}{\alpha'_T} - \nu_{\alpha_Q} \right) \right) \delta \alpha'_T d\tau, \end{aligned}$$

where the Lagrangian and its partial derivatives are evaluated at $\nu_{\alpha_Q}(\tau)$. Note that

$$\delta^Q \alpha'_Q = \frac{\nabla^Q \alpha_Q}{d\tau}, \quad (\pi_{\alpha_Q} \cdot \delta\alpha_Q)' = \frac{\nabla^{Q*} \pi_{\alpha_Q}}{d\tau} \cdot \delta\alpha_Q + \pi_{\alpha_Q} \cdot \frac{\nabla^Q \delta\alpha_Q}{d\tau}.$$

By using this, splitting the integration domain, $[\tau_0, \tau_1] - \{\tilde{\tau}\} = [\tau_0, \tilde{\tau}] \cup (\tilde{\tau}, \tau_1]$, integrating by parts on each subinterval, and regrouping terms, we may rewrite the previous expression as

$$d\mathbb{S}(\alpha, \nu_{\alpha_Q}, \pi_{\alpha_Q}) (\delta\alpha, \delta\nu_{\alpha_Q}, \delta\pi_{\alpha_Q}) = \mathcal{I}(\tau_0, \tilde{\tau}) + \mathcal{I}(\tilde{\tau}, \tau_1) + \mathcal{B}(\tau_0, \tilde{\tau}^-) + \mathcal{B}(\tilde{\tau}^+, \tau_1),$$

where for each $a, b \in \mathbb{R}$, $a < b$, we set

$$\begin{aligned} \mathcal{I}(a, b) &= \int_a^b \left(\left(\alpha'_T \frac{\delta L}{\delta q} - \frac{\nabla^{Q*} \pi_{\alpha_Q}}{d\tau} \right) \cdot \delta\alpha_Q + \alpha'_T \left(\frac{\delta L}{\delta v_q} - \pi_{\alpha_Q} \right) \cdot \delta^Q \nu_{\alpha_Q} \right. \\ &\quad \left. + \delta^{Q*} \pi_{\alpha_Q} \left(\frac{\alpha'_Q}{\alpha'_T} - \nu_{\alpha_Q} \right) \alpha'_T + E' \delta\alpha_T \right) d\tau, \\ \mathcal{B}(a, b) &= \left[\pi_{\alpha_Q}(\tau) \cdot \delta\alpha_Q(\tau) - E \delta\alpha_T(\tau) \right]_{\tau=a}^{\tau=b}, \end{aligned}$$

where the energy is evaluated at $(\nu_{\alpha_Q}(\tau), \pi_{\alpha_Q}(\tau))$. Since the previous expression vanishes for arbitrary variations $\delta\alpha_T \in T_{\alpha_T} \mathcal{T}$ and $\delta\alpha_Q \in \Delta_{\mathcal{Q}(\tilde{\tau})}(\alpha_Q)$ vanishing at the endpoints, as well as arbitrary variations $\delta^Q \nu_{\alpha_Q} \in T_{\alpha_Q}^* \mathcal{Q}(\tilde{\tau})$, $\delta^{Q*} \pi_{\alpha_Q} \in T_{\alpha_Q}^* \mathcal{Q}(\tilde{\tau})$, we obtain the desired equations, together with the impact conditions. ■

By using the change of variable $t = \alpha_T(\tau)$, we have $\dot{q}_\alpha = \alpha'_Q / \alpha'_T$ and $\dot{p}_\alpha = \pi'_{\alpha_Q} / \alpha'_T$. Then, the implicit Euler–Lagrange equations for a curve $(v_\alpha, p_\alpha) : [t_0, t_1] \rightarrow TQ \oplus T^*Q$ take the form

$$(3.8) \quad \begin{cases} \frac{\nabla^{Q*} p_\alpha}{dt} - \frac{\delta L}{\delta q}(v_\alpha) \in \Delta_Q^\circ(q_\alpha), & \dot{E}(v_\alpha, p_\alpha) = 0, \\ p_\alpha = \frac{\delta L}{\delta v_\alpha}(v_\alpha), & v_\alpha = \dot{q}_\alpha \in \Delta_Q(q_\alpha), \end{cases}$$

on $[t_0, \tilde{t}) \cup (\tilde{t}, t_1]$. Similarly, the conditions for the elastic impact read

$$(3.9) \quad p_\alpha^+ - p_\alpha^- \in \left(T_{q_\alpha(\tilde{t})} \partial Q \cap \Delta_Q(q_\alpha(\tilde{t})) \right)^\circ = \left(T_{q_\alpha(\tilde{t})} \partial Q \right)^\circ + \Delta_Q^\circ(q_\alpha(\tilde{t})),$$

$$(3.10) \quad \begin{aligned} E(v_\alpha^+, p_\alpha^+) &= E(v_\alpha^-, p_\alpha^-), \\ v_\alpha^+ &= \dot{q}_\alpha^+ \in \Delta_Q(q_\alpha(\tilde{t})), \end{aligned}$$

where we denote $p_\alpha(\tilde{t}^+) = p_\alpha^+$, etc.

For unconstrained systems, i.e., $\Delta_Q = TQ$, the Hamilton–d’Alembert–Pontryagin principle reduces to the *Hamilton–Pontryagin principle*, and the implicit Euler–Lagrange equations of motion read as

$$\begin{cases} \frac{\nabla^{Q*} p_\alpha}{dt} = \frac{\delta L}{\delta q}(v_\alpha), & \dot{E}(v_\alpha, p_\alpha) = 0, \\ p_\alpha = \frac{\delta L}{\delta v_q}(v_\alpha), & v_\alpha = \dot{q}_\alpha. \end{cases}$$

Remark 3.7 (energy balance). The conservation of the energy, $\dot{E}(v_\alpha, p_\alpha) = 0$, is redundant, as it may be obtained from the remaining equations:

$$\begin{aligned}\dot{E}(v_\alpha, p_\alpha) &= \frac{\nabla^{Q*} p_\alpha}{dt} \cdot v_\alpha + p_\alpha \cdot \frac{\nabla^Q v_\alpha}{dt} - \frac{\delta L}{\delta q} \cdot \dot{q}_\alpha - \frac{\delta L}{\delta v_q} \cdot \frac{\nabla^Q v_\alpha}{dt} \\ &= \left(\frac{\nabla^{Q*} p_\alpha}{dt} - \frac{\delta L}{\delta q} \right) \cdot v_\alpha = 0,\end{aligned}$$

where the last equality follows from the fact that $v_\alpha \in \Delta_Q(q_\alpha)$ while $\nabla^{Q*} p_\alpha/dt - \delta L/\delta q \in \Delta_Q^\circ(q_\alpha)$. Consequently, this equation may be omitted.

3.3. Example: The rolling disk hitting a circular surface. Let us consider a disk rolling without slipping, as in [33, section 7.1] or [3, section VI], that is confined to move in a solid circle. The configuration space is thus given by

$$Q = \{(x, y, \theta, \varphi) \in \mathbb{R}^2 \times S^1 \times S^1 \mid (x + R \cos \varphi)^2 + (y + R \sin \varphi)^2 \leq 1\},$$

where (x, y) denotes the contact point of the disk with the ground, θ denotes the angle of rotation, and φ denotes the heading angle of the disk with respect to the x -axis. The Lagrangian $L : TQ \rightarrow \mathbb{R}$ is given by

$$L(x, y, \theta, \varphi; v_x, v_y, v_\theta, v_\varphi) = \frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}(I v_\theta^2 + J v_\varphi^2),$$

where $m, I, J \in \mathbb{R}^+$ are the mass and the moments of inertia of the disk, respectively. For each $(v_q, p_q) = (x, y, \theta, \varphi; v_x, v_y, v_\theta, v_\varphi; p_x, p_y, p_\theta, p_\varphi) \in TQ \oplus T^*Q$, the energy reads

$$E(v_q, p_q) = p_x v_x + p_y v_y + p_\theta v_\theta + p_\varphi v_\varphi - \frac{1}{2}m(v_x^2 + v_y^2) - \frac{1}{2}(I v_\theta^2 + J v_\varphi^2).$$

In addition, the nonslipping condition reads $v_x = R v_\theta \cos \varphi$, $v_y = R v_\theta \sin \varphi$, where $R \in \mathbb{R}^+$ is the radius of the disk, thus yielding the following nonholonomic constraint:

$$\Delta_Q = \text{span}\{\partial_\theta + R \cos \varphi \partial_x + R \sin \varphi \partial_y, \partial_\varphi\}, \quad \Delta_Q^\circ = \text{span}\{dx - R \cos \varphi d\theta, dy - R \sin \varphi d\theta\}.$$

On the other hand, the boundary of the configuration manifold is given by

$$\partial Q = \{(x, y, \theta, \varphi) \in \mathbb{R}^2 \times S^1 \times S^1 \mid (x + R \cos \varphi)^2 + (y + R \sin \varphi)^2 = 1\}.$$

Hence, the tangent bundle of the boundary reads

$$T\partial Q$$

$$= \text{span}\{R(x \sin \varphi - y \cos \varphi) \partial_x + (x + R \cos \varphi) \partial_\varphi, R(x \sin \varphi - y \cos \varphi) \partial_y + (y + R \sin \varphi) \partial_\varphi, \partial_\theta\},$$

and its annihilator is given by

$$(T\partial Q)^\circ = \text{span}\{(x + R \cos \varphi) dx + (y + R \sin \varphi) dy + R(-x \sin \varphi + y \cos \varphi) d\varphi\}.$$

Lastly, since S^1 is a Lie group, its tangent bundle is trivial, and, thus, the canonical flat connection on the tangent bundle of Q may be chosen, i.e., $\nabla^Q/dt = d/dt$. By gathering all, the implicit Euler–Lagrange equations (3.8) for a curve

$$(x, y, \theta, \varphi; v_x, v_y, v_\theta, v_\varphi; p_x, p_y, p_\theta, p_\varphi) : [t_0, t_1] \rightarrow TQ \oplus T^*Q$$

read

$$\begin{cases} \dot{p}_x = \mu_1, & \dot{p}_y = \mu_2, \\ \dot{p}_\theta = -\mu_1 R \cos \varphi - \mu_2 R \sin \varphi, & \dot{p}_\varphi = 0, \\ p_x = m v_x, & p_y = m v_y, \\ p_\theta = I v_\theta, & p_\varphi = J v_\varphi, \\ v_x = \dot{x} = \mu_3 R \cos \varphi, & v_y = \dot{y} = \mu_3 R \sin \varphi, \\ v_\theta = \dot{\theta} = \mu_3, & v_\varphi = \dot{\varphi} = \mu_4, \end{cases}$$

on $[t_0, t_1] - \{\tilde{t}\}$, where $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{R}$ are the Lagrange multipliers. The impact condition at $t = \tilde{t}$ given in (3.9) reads

$$\begin{cases} p_x^+ - p_x^- = \lambda_0(x + R \cos \varphi) + \lambda_1, & p_\theta^+ - p_\theta^- = -\lambda_1 R \cos \varphi - \lambda_2 R \sin \varphi, \\ p_y^+ - p_y^- = \lambda_0(y + R \sin \varphi) + \lambda_2, & p_\varphi^+ - p_\varphi^- = \lambda_0 R(-x \sin \varphi + y \cos \varphi), \end{cases}$$

where we denote $p_x^+ = p_x(\tilde{t}^+)$, etc., and $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$ are the Lagrange multipliers. Similarly, the condition (3.10) reads

$$v_x^+ = \lambda_3 R \cos \varphi, \quad v_y^+ = \lambda_3 R \sin \varphi, \quad v_\theta^+ = \lambda_3, \quad v_\varphi^+ = \lambda_4,$$

where $v_x^+ = v_x(\tilde{t}^+)$, etc., and $\lambda_3, \lambda_4 \in \mathbb{R}$ are the Lagrange multipliers.

3.4. Example: The spherical pendulum hitting a cylindrical surface. Let us consider a spherical pendulum (cf. [16, section 5.1]) hitting a cylindrical surface. The configuration space of the system and its boundary are given by

$$Q = \{(\theta, \varphi) \in S^2 \mid L \sin \theta \leq 1\}, \quad \partial Q = \{(\theta, \varphi) \in S^2 \mid L \sin \theta = 1\},$$

where $0 < L < 1$ is the length of the pendulum, and the Lagrangian reads

$$(3.11) \quad L(\theta, \varphi; v_\theta, v_\varphi) = \frac{1}{2} m L^2 (v_\theta^2 + v_\varphi^2 \sin^2 \theta) - m g L \cos \theta,$$

where $m, g \in \mathbb{R}^+$ are the mass of the pendulum and the gravitational acceleration, respectively. Let us suppose that the polar and azimuthal velocities are proportionally related by a function depending only on the polar angle, i.e., $v_\varphi = f(\theta) v_\theta$ with $f(\theta) > 0$ for each $\theta \in \mathbb{R}$. This results in the following nonholonomic constraint:

$$(3.12) \quad \Delta_Q = \text{span}\{\partial_\theta + f(\theta) \partial_\varphi\}, \quad \Delta_Q^\circ = \text{span}\{f(\theta) d\theta - d\varphi\}.$$

In addition, the tangent bundle of the reduced space and the corresponding annihilator read

$$T\partial Q = \text{span}\{\partial_\varphi\}, \quad (T\partial Q)^\circ = \text{span}\{d\theta\}.$$

By considering the flat connection in these coordinates, the implicit Euler–Lagrange equations (3.8) for a curve

$$(\theta, \varphi; v_\theta, v_\varphi; p_\theta, p_\varphi) : [t_0, t_1] \rightarrow TQ \oplus T^*Q$$

read

$$\begin{cases} \dot{p}_\theta - mL \sin \theta (Lv_\varphi^2 \cos \theta + g) = \mu_0 f(\theta), & \dot{p}_\varphi = -\mu_0 \\ p_\theta = mL^2 v_\theta, & p_\varphi = mL^2 v_\varphi \sin^2 \theta, \\ v_\theta = \dot{\theta} = \mu_1, & v_\varphi = \dot{\varphi} = f(\theta) \mu_1, \end{cases}$$

on $[t_0, t_1] - \{\tilde{t}\}$, where $\mu_0, \mu_1 \in \mathbb{R}$ are the Lagrange multipliers. The impact condition at $t = \tilde{t}$ given in (3.9) reads

$$p_\theta^+ - p_\theta^- = \lambda_0 + f(\theta) \lambda_1, \quad p_\varphi^+ - p_\varphi^- = -\lambda_1,$$

where we denote $p_\theta^+ = p_\theta(\tilde{t}^+)$, etc., and $\lambda_0, \lambda_1 \in \mathbb{R}$ are the Lagrange multipliers. Similarly, the condition (3.10) reads

$$v_\theta^+ = \dot{\theta}^+ = \lambda_3, \quad v_\varphi^+ = \dot{\varphi}^+ = f(\theta) \lambda_3,$$

where $v_\theta^+ = v_\theta(\tilde{t}^+)$, etc., and $\lambda_3 \in \mathbb{R}$ is the Lagrange multiplier.

4. Nonholonomic implicit Lagrange–Poincaré reduction with collisions. Let G be a Lie group and $\Phi : G \times Q \rightarrow Q$ be a free and proper left action. The corresponding quotient, which is a smooth manifold, is denoted by $\Sigma = Q/G$. Furthermore, the quotient projection, $\pi_{Q,\Sigma} : Q \rightarrow \Sigma$, is a principal G -bundle. As usual, we denote $\Phi(g, q) = \Phi_g(q) = g \cdot q$ for each $(g, q) \in G \times Q$. Henceforth, the equivalence classes induced by the G -action on the different spaces are denoted by square brackets $[\cdot]_G$, e.g., $[q]_G \in \Sigma$ for each $q \in Q$ and $[v_q]_G \in (TQ)/G$ for each $v_q \in TQ$.

Remark 4.1. Since $\Phi_g : Q \rightarrow Q$ is a diffeomorphism for each $g \in G$, it leaves the boundary invariant, i.e., $\Phi_g(\partial Q) = \partial Q$. Therefore, Φ induces a left G -action on the boundary, $\Phi_\partial = \Phi|_{G \times \partial Q} : G \times \partial Q \rightarrow \partial Q$. This action is again free and proper, and, thus, the quotient is a smooth manifold, $(\partial Q)/G$. In fact, this quotient is nothing but the boundary of $\Sigma = Q/G$, i.e., $\partial \Sigma = (\partial Q)/G$. Observe that, in particular, we have that $\dim G < \dim Q$.

The moment map induced by this action, $\mathbf{J} : T^*Q \rightarrow \mathfrak{g}^*$, is denoted by the condition

$$\mathbf{J}(p_q) \cdot \xi = p_q \cdot \xi_q^*, \quad q \in Q, p_q \in T^*Q, \xi \in \mathfrak{g},$$

where $\xi^* \in \mathfrak{X}(Q)$ denotes the infinitesimal generator (or fundamental vector field) of ξ , i.e., $\xi_q^* = d/dt|_{t=0} \exp(t\xi) \cdot q$, with $\exp : \mathfrak{g} \rightarrow G$ being the exponential map. In addition, the G -action may be lifted to the tangent and the cotangent bundles of Q ,

$$(4.1) \quad \begin{cases} G \times TQ \rightarrow TQ, & (g, v_q) \mapsto (d\Phi_g)_q(v_q), \\ G \times T^*Q \rightarrow T^*Q, & (g, p_q) \mapsto (d\Phi_{g^{-1}})_{g \cdot q}^*(p_q), \end{cases}$$

where $(d\Phi_{g^{-1}})_{g \cdot q}^* : T_{g \cdot q}^*Q \rightarrow T_{g \cdot q}^*Q$ is the adjoint map of $(d\Phi_{g^{-1}})_{g \cdot q} : T_{g \cdot q}Q \rightarrow T_{g \cdot q}Q$. In turn, these actions may be lifted to the iterated bundles, yielding $G \times T(TQ) \rightarrow T(TQ)$ and $G \times T(T^*Q) \rightarrow T(T^*Q)$.

4.1. Reduced configuration space and reduced constraint distribution. Let $\omega \in \Omega^1(Q, \mathfrak{g})$ be a principal connection on $\pi_{Q,\Sigma} : Q \rightarrow \Sigma$, where $\mathfrak{g}$ is the Lie algebra of G . Let us denote by $\tilde{Q} = \pi_{Q,\Sigma}^*(T\Sigma)$ the pullback of the tangent bundle of Σ by the quotient projection. In the same fashion, we denote $\tilde{Q}^* = \pi_{Q,\Sigma}^*(T^*\Sigma)$. The principal connection thus induces the following trivializations of the tangent and the cotangent bundles of Q :

$$(4.2) \quad \begin{cases} TQ \simeq \tilde{Q} \times \mathfrak{g}, & v_q \mapsto ((q, (d\pi_{Q,\Sigma})_q(v_q)), \omega_q(v_q)), \\ T^*Q \simeq \tilde{Q}^* \times \mathfrak{g}^*, & p_q \mapsto ((q, \mathbb{H}_q^*(p_q)), \mathbf{J}(p_q)), \end{cases}$$

where $\mathbb{H}_q^* : T_q^*Q \rightarrow T_\sigma^*\Sigma$ is the dual map of the horizontal lift induced by ω , $\mathbb{H}_q : T_\sigma\Sigma \rightarrow T_qQ$, where $\sigma = \pi_{Q,\Sigma}(q)$. Under these identifications, it is easy to check that the G -actions (4.1) read as

$$(4.3) \quad \begin{cases} g \cdot ((q, v_\sigma), \xi) = ((g \cdot q, v_\sigma), \text{Ad}_g(\xi)), & (q, v_\sigma) \in \tilde{Q}, \xi \in \mathfrak{g}, \\ g \cdot ((q, p_\sigma), \rho) = ((g \cdot q, p_\sigma), \text{Ad}_{g^{-1}}^*(\rho)), & (q, p_\sigma) \in \tilde{Q}^*, \rho \in \mathfrak{g}^*, \end{cases}$$

for each $g \in G$, where $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$ and $\text{Ad}^* : G \rightarrow \text{Aut}(\mathfrak{g}^*)$ denote the adjoint and the coadjoint representations of G , respectively. As a result, we have the following identifications for the quotient spaces:

$$(4.4) \quad \begin{cases} (TQ)/G \simeq T\Sigma \oplus \tilde{\mathfrak{g}}, & [v_q] \mapsto ((d\pi_{Q,\Sigma})_q(v_q), [q, \omega_q(v_q)]_G), \\ (T^*Q)/G \simeq T^*\Sigma \oplus \tilde{\mathfrak{g}}^*, & [p_q]_G \mapsto (\mathbb{H}_q^*(p_q), [q, \mathbf{J}(p_q)]_G), \end{cases}$$

where $\tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G$ and $\tilde{\mathfrak{g}}^* = (Q \times \mathfrak{g}^*)/G$ are the adjoint and coadjoint bundles, respectively.

Given $[\tau_0, \tau_1] \subset \mathbb{R}$ and $\tilde{\tau} \in [\tau_0, \tau_1]$, path spaces analogous to (3.1), (3.2), and (3.3) may be defined by exchanging Q by Σ . Such spaces are denoted by $\Sigma(\tilde{\tau})$, $T_{\alpha_\Sigma}\Sigma(\tilde{\tau})$, and $T_{\alpha_\Sigma}^*\Sigma(\tilde{\tau})$, respectively, where $\alpha_\Sigma \in \Sigma(\tilde{\tau})$. Hence, the *reduced path space* is given by $\Omega(\Sigma, \tilde{\tau}) = \mathcal{T} \times \Sigma(\tilde{\tau})$. In the same vein, we define

$$\begin{aligned} \tilde{\mathfrak{g}}(\tilde{\tau})_{\alpha_\Sigma} &= \{\xi_{\alpha_\Sigma} \in C^0([\tau_0, \tau_1], \tilde{\mathfrak{g}}) \mid \pi_{\tilde{\mathfrak{g}}} \circ \xi_{\alpha_\Sigma} = \alpha_\Sigma, \\ &\quad \xi_{\alpha_\Sigma} \text{ is piecewise } C^2 \text{ and has only one singularity at } \tilde{\tau}\}, \end{aligned}$$

where $\pi_{\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}} \rightarrow \Sigma$ is the natural projection, and analogously for $\tilde{\mathfrak{g}}^*(\tilde{\tau})_{\alpha_\Sigma}$. As a result, we obtain two vector bundles over $\Sigma(\tilde{\tau})$; namely, $\tilde{\mathfrak{g}}(\tilde{\tau}) = \bigsqcup_{\alpha_\Sigma \in \Sigma(\tilde{\tau})} \tilde{\mathfrak{g}}(\tilde{\tau})_{\alpha_\Sigma} \rightarrow \Sigma(\tilde{\tau})$ and $\tilde{\mathfrak{g}}^*(\tilde{\tau}) = \bigsqcup_{\alpha_\Sigma \in \Sigma(\tilde{\tau})} \tilde{\mathfrak{g}}^*(\tilde{\tau})_{\alpha_\Sigma} \rightarrow \Sigma(\tilde{\tau})$.

Lastly, let $\Delta_Q \subset TQ$ be a G -equivariant constraint distribution, i.e.,

$$\Delta_Q(g \cdot q) = (d\Phi_g)_q(\Delta_Q(q)), \quad g \in G, q \in Q.$$

Hence, it may be dropped to a distribution on the reduced bundle, $[\Delta_Q]_G \subset (TQ)/G$. By means of (4.4), we may write

$$(4.5) \quad [\Delta_Q]_G \simeq \Delta_\Sigma \oplus \Delta_{\tilde{\mathfrak{g}}} \subset T\Sigma \oplus \tilde{\mathfrak{g}},$$

where Δ_Σ and $\Delta_{\tilde{\mathfrak{g}}}$ are the *horizontal* and the *vertical reduced constraint distributions*. This way, constrained paths may be considered:

$$\begin{aligned} \Delta_{\Sigma(\tilde{\tau})}(\alpha_\Sigma) &= \{\nu_{\alpha_\Sigma} \in T_{\alpha_\Sigma}\Sigma(\tilde{\tau}) \mid \nu_{\alpha_\Sigma} : [\tau_0, \tau_1] \rightarrow \Delta_\Sigma\}, \\ \Delta_{\tilde{\mathfrak{g}}(\tilde{\tau})}(\alpha_\Sigma) &= \{\xi_{\alpha_\Sigma} \in \tilde{\mathfrak{g}}(\tilde{\tau})_{\alpha_\Sigma} \mid \xi_{\alpha_\Sigma} : [\tau_0, \tau_1] \rightarrow \Delta_{\tilde{\mathfrak{g}}}\} \end{aligned}$$

for each $\alpha_\Sigma \in \Sigma(\tilde{\tau})$. Note that we obtain vector subbundles $\Delta_{\Sigma(\tilde{\tau})} \subset T\Sigma(\tilde{\tau})$ and $\Delta_{\tilde{\mathfrak{g}}(\tilde{\tau})} \subset \tilde{\mathfrak{g}}(\tilde{\tau})$. The following lemma is now straightforward.

Lemma 4.2. *In the above conditions, let*

$$c = ((\alpha_T, \alpha_Q), \nu_{\alpha_Q}, \pi_{\alpha_Q}) \in \Omega(Q, \tilde{\tau}) \times_{\mathcal{Q}(\tilde{\tau})} (\Delta_{\mathcal{Q}(\tilde{\tau})} \oplus T^*\mathcal{Q}(\tilde{\tau})).$$

Then the reduced path is given by

$$[c]_G \simeq ((\alpha_T, \alpha_\Sigma), (\nu_{\alpha_\Sigma}, \xi_{\alpha_\Sigma}), (\pi_{\alpha_\Sigma}, \rho_{\alpha_\Sigma})) \in \Omega(\Sigma, \tilde{\tau}) \times_{\Sigma(\tilde{\tau})} ((\Delta_{\Sigma(\tilde{\tau})} \oplus \Delta_{\tilde{\mathbf{g}}(\tilde{\tau})}) \oplus (T^*\Sigma(\tilde{\tau}) \oplus \tilde{\mathbf{g}}^*(\tilde{\tau}))),$$

where $\alpha_\Sigma = [\alpha_Q]_G$, $(\nu_{\alpha_\Sigma}, \xi_{\alpha_\Sigma}) \simeq [\nu_{\alpha_Q}]_G$ and $(\pi_{\alpha_\Sigma}, \rho_{\alpha_\Sigma}) \simeq [\pi_{\alpha_Q}]_G$.

Note that the reduced path may be split into the *horizontal reduced path*,

$$[c]_G^h \simeq ((\alpha_T, \alpha_\Sigma), \nu_{\alpha_\Sigma}, \pi_{\alpha_\Sigma},) \in \Omega(\Sigma, \tilde{\tau}) \times_{\Sigma(\tilde{\tau})} (\Delta_{\Sigma(\tilde{\tau})} \oplus T^*\Sigma(\tilde{\tau})),$$

and the *vertical reduced path*,

$$[c]_G^v \simeq ((\alpha_T, \alpha_\Sigma), \xi_{\alpha_\Sigma}, \rho_{\alpha_\Sigma}) \in \Omega(\Sigma, \tilde{\tau}) \times_{\Sigma(\tilde{\tau})} (\Delta_{\tilde{\mathbf{g}}(\tilde{\tau})} \oplus \tilde{\mathbf{g}}^*(\tilde{\tau})).$$

At last, given $\alpha_\Sigma \in \Sigma(\tilde{\tau})$, $\nu_{\alpha_\Sigma} \in T_{\alpha_\Sigma}\Sigma(\tilde{\tau})$, and $\pi_{\alpha_\Sigma} \in T_{\alpha_\Sigma}^*\Sigma(\tilde{\tau})$, the spaces $T_{\nu_{\alpha_\Sigma}}(T\Sigma(\tilde{\tau}))$ and $T_{\pi_{\alpha_\Sigma}}(T^*\Sigma(\tilde{\tau}))$ are defined in the same vein as $T_{\nu_{\alpha_Q}}(T\mathcal{Q}(\tilde{\tau}))$ and $T_{\pi_{\alpha_Q}}(T^*\mathcal{Q}(\tilde{\tau}))$ by exchanging Q with Σ . In addition, for each $\xi_{\alpha_\Sigma} \in \tilde{\mathbf{g}}(\tilde{\tau})_{\alpha_\Sigma}$, we denote

$$T_{\xi_{\alpha_\Sigma}}\tilde{\mathbf{g}}(\tilde{\tau}) = \left\{ \delta\xi_{\alpha_\Sigma} \in C^0([\tau_0, \tau_1], T\tilde{\mathbf{g}}) \mid \xi_{\alpha_\Sigma} = \pi_{T\tilde{\mathbf{g}}} \circ \delta\xi_{\alpha_\Sigma}, \right. \\ \left. \delta\xi_{\alpha_\Sigma} \text{ is piecewise } C^2 \text{ and has only one singularity at } \tilde{\tau} \right\},$$

where $\pi_{T\tilde{\mathbf{g}}} : T\tilde{\mathbf{g}} \rightarrow \tilde{\mathbf{g}}$ is the natural projection. Analogously, we denote $T_{\rho_{\alpha_\Sigma}}\tilde{\mathbf{g}}^*(\tilde{\tau})$ for each $\rho_{\alpha_\Sigma} \in \tilde{\mathbf{g}}^*(\tilde{\tau})_{\alpha_\Sigma}$. In order to incorporate the constraint distribution, we set $\Delta_{T\Sigma(\tilde{\tau})}(\nu_{\alpha_\Sigma})$ as in (3.4) by exchanging Q by Σ .

4.2. Nonholonomic implicit Lagrange–Poincaré equations with collisions. Let $L : TQ \rightarrow \mathbb{R}$ be a (possibly degenerate) G -invariant Lagrangian, i.e.,

$$L(v_q) = L((d\Phi_g)_h(v_q)), \quad g \in G, q \in Q, v_q \in T_qQ.$$

The *reduced* or *dropped Lagrangian* is defined as

$$\ell : (TQ)/G \simeq T\Sigma \oplus \tilde{\mathbf{g}} \rightarrow \mathbb{R}, \quad [v_q]_G \simeq (v_\sigma, \xi_\sigma) \mapsto \ell(v_\sigma, \xi_\sigma) = L(v_q),$$

where identification (4.4) has been used. As the energy of the system (3.6) is also G -invariant, the *reduced energy* may be defined as

$$e : (T\Sigma \times \tilde{\mathbf{g}}) \oplus (T^*\Sigma \times \tilde{\mathbf{g}}^*) \rightarrow \mathbb{R}, \quad ((v_\sigma, \xi_\sigma), (p_\sigma, \rho_\sigma)) \mapsto p_\sigma \cdot v_\sigma + \rho_\sigma \cdot \xi_\sigma - \ell(v_\sigma, \xi_\sigma),$$

where the identification $(TQ \oplus T^*Q)/G \simeq (T\Sigma \times \tilde{\mathbf{g}}) \oplus (T^*\Sigma \times \tilde{\mathbf{g}}^*)$ has been used.

4.2.1. Reduced variational principle. Given a path $\tilde{\alpha} = (\alpha_T, \alpha_\Sigma) \in \Omega(\Sigma, \tilde{\tau})$, its *associated curve* is defined as

$$\sigma_{\tilde{\alpha}} = \alpha_\Sigma \circ \alpha_T^{-1} : [t_0, t_1] \rightarrow \Sigma.$$

Additionally, for the paths $(\nu_{\alpha_\Sigma}, \xi_{\alpha_\Sigma}) \in T_{\alpha_\Sigma} \Sigma(\tilde{\tau}) \oplus \tilde{\mathbf{g}}(\tilde{\tau})_{\alpha_\Sigma}$ and $(\pi_{\alpha_\Sigma}, \rho_{\alpha_\Sigma}) \in T_{\alpha_\Sigma}^* \Sigma(\tilde{\tau}) \oplus \tilde{\mathbf{g}}^*(\tilde{\tau})_{\alpha_\Sigma}$, we set

$$\begin{aligned} u_{\tilde{\alpha}} &= \nu_{\alpha_\Sigma} \circ \alpha_T^{-1} : [t_0, t_1] \rightarrow T\Sigma, & y_{\tilde{\alpha}} &= \pi_{\alpha_\Sigma} \circ \alpha_T^{-1} : [t_0, t_1] \rightarrow T^*\Sigma, \\ \bar{\xi}_{\tilde{\alpha}} &= \xi_{\alpha_\Sigma} \circ \alpha_T^{-1} : [t_0, t_1] \rightarrow \tilde{\mathbf{g}}, & \bar{\rho}_{\tilde{\alpha}} &= \rho_{\alpha_\Sigma} \circ \alpha_T^{-1} : [t_0, t_1] \rightarrow \tilde{\mathbf{g}}^*. \end{aligned}$$

Observe that $\pi_{T\Sigma} \circ u_{\tilde{\alpha}} = \pi_{\tilde{\mathbf{g}}} \circ \bar{\xi}_{\tilde{\alpha}} = \pi_{T^*\Sigma} \circ y_{\tilde{\alpha}} = \pi_{\tilde{\mathbf{g}}^*} \circ \bar{\rho}_{\tilde{\alpha}} = \sigma_{\tilde{\alpha}}$.

Remark 4.3. As we will see below, given $\alpha = (\alpha_T, \alpha_Q) \in \Omega(Q, \tilde{\tau})$ and the corresponding reduced path $\tilde{\alpha} = (\alpha_T, \alpha_\Sigma = [\alpha_Q]_G) \in \Omega(\Sigma, \tilde{\tau})$, we set $\eta_{\alpha_\Sigma} = [\alpha_Q, \omega \circ \alpha'_Q]_G$. The change of variable $t = \alpha_T(\tau)$ yields $\dot{q}_\alpha = \alpha'_Q / \alpha'_T$, which together with (3.5) yields

$$\bar{\eta}_{\tilde{\alpha}} = \eta_{\alpha_\Sigma} \circ \alpha_T^{-1} = (\alpha'_T \circ \alpha_T^{-1}) [q_\alpha, \omega \circ \dot{q}_\alpha]_G.$$

By using the reduced Lagrangian, we may define the *reduced action functional*,

$$\mathfrak{S} : \Omega(\Sigma, \tilde{\tau}) \times_{\Sigma(\tilde{\tau})} \left((\Delta_{\Sigma(\tilde{\tau})} \oplus \Delta_{\tilde{\mathbf{g}}(\tilde{\tau})}) \oplus (T^*\Sigma(\tilde{\tau}) \oplus \tilde{\mathbf{g}}^*(\tilde{\tau})) \right) \times_{\Sigma(\tilde{\tau})} \tilde{\mathbf{g}}(\tilde{\tau}) \rightarrow \mathbb{R},$$

as follows:

$$\begin{aligned} &\mathfrak{S}((\alpha_T, \alpha_\Sigma), (\nu_{\alpha_\Sigma}, \xi_{\alpha_\Sigma}), (\pi_{\alpha_\Sigma}, \rho_{\alpha_\Sigma}), \eta_{\alpha_\Sigma}) \\ &= \int_{t_0}^{t_1} \left(\ell(u_{\tilde{\alpha}}(t), \bar{\xi}_{\tilde{\alpha}}(t)) + y_{\tilde{\alpha}}(t) \cdot (\dot{\sigma}_{\tilde{\alpha}}(t) - u_{\tilde{\alpha}}(t)) + \bar{\rho}_{\tilde{\alpha}}(t) \cdot \left(\frac{\bar{\eta}_{\tilde{\alpha}}(t)}{(\alpha'_T \circ \alpha_T^{-1})(t)} - \bar{\xi}_{\tilde{\alpha}}(t) \right) \right) dt \\ &= \int_{\tau_0}^{\tau_1} \left(\ell(\nu_{\alpha_\Sigma}(\tau), \xi_{\alpha_\Sigma}(\tau)) + \pi_{\alpha_\Sigma}(\tau) \cdot \left(\frac{\alpha'_\Sigma(\tau)}{\alpha'_T(\tau)} - \nu_{\alpha_\Sigma}(\tau) \right) \right. \\ &\quad \left. + \rho_{\alpha_\Sigma}(\tau) \cdot \left(\frac{\eta_{\alpha_\Sigma}(\tau)}{\alpha'_T(\tau)} - \xi_{\alpha_\Sigma}(\tau) \right) \right) \alpha'_T(\tau) d\tau. \end{aligned}$$

As for the unreduced case, the equality for the first and the second expressions comes from the change of variables $t = \alpha_T(\tau)$. Furthermore, the reduced action functional may be written in terms of the reduced energy,

$$\begin{aligned} &\mathfrak{S}((\alpha_T, \alpha_\Sigma), (\nu_{\alpha_\Sigma}, \xi_{\alpha_\Sigma}), (\pi_{\alpha_\Sigma}, \rho_{\alpha_\Sigma}), \eta_{\alpha_\Sigma}) \\ &= \int_{t_0}^{t_1} \left(y_{\tilde{\alpha}}(t) \cdot \dot{\sigma}_{\tilde{\alpha}}(t) + \bar{\rho}_{\tilde{\alpha}}(t) \cdot \frac{\bar{\eta}_{\tilde{\alpha}}(t)}{(\alpha'_T \circ \alpha_T^{-1})(t)} - e((u_{\tilde{\alpha}}(t), \bar{\xi}_{\tilde{\alpha}}(t)), (y_{\tilde{\alpha}}(t), \bar{\rho}_{\tilde{\alpha}}(t))) \right) dt \\ &= \int_{\tau_0}^{\tau_1} \left(\pi_{\alpha_\Sigma}(\tau) \cdot \frac{\alpha'_\Sigma(\tau)}{\alpha'_T(\tau)} + \rho_{\alpha_\Sigma}(\tau) \cdot \frac{\eta_{\alpha_\Sigma}(\tau)}{\alpha'_T(\tau)} - e((\nu_{\alpha_\Sigma}(\tau), \xi_{\alpha_\Sigma}(\tau)), (\pi_{\alpha_\Sigma}(\tau), \rho_{\alpha_\Sigma}(\tau))) \right) \alpha'_T(\tau) d\tau. \end{aligned}$$

In order to reduce the variational principle and obtain the reduced equations, we need to introduce linear connections on the reduced spaces. More specifically, recall that ω induces

a linear connection on the adjoint bundle, $\nabla^\omega : \Gamma(\tilde{\mathfrak{g}}) \rightarrow \Omega^1(\Sigma, \tilde{\mathfrak{g}})$. The dual of ∇^ω , which is a linear connection on the coadjoint bundle, is denoted by $\nabla^{\omega*} : \Gamma(\tilde{\mathfrak{g}}^*) \rightarrow \Omega^1(\Sigma, \tilde{\mathfrak{g}}^*)$. We also consider a linear connection on the tangent bundle of Σ , $\nabla^\Sigma : \mathfrak{X}(\Sigma) \rightarrow \Omega^1(\Sigma, T\Sigma)$, as well as its dual, which is a linear connection on the cotangent bundle of Σ , $\nabla^{\Sigma*} : \Omega^1(\Sigma) \rightarrow \Omega^1(\Sigma, T^*\Sigma)$. These connections allow for computing the vertical parts of the variations. For instance, given $\alpha_\Sigma \in \Sigma(\tilde{\tau})$, $\xi_{\alpha_\Sigma} \in \tilde{\mathfrak{g}}(\tilde{\tau})_{\alpha_\Sigma}$ and $\delta\xi_{\alpha_\Sigma} \in T_{\xi_{\alpha_\Sigma}}\tilde{\mathfrak{g}}(\tilde{\tau})$, we denote

$$\delta^\omega \xi_{\alpha_\Sigma} = (\delta\xi_{\alpha_\Sigma})^v \in \tilde{\mathfrak{g}}(\tilde{\tau})_{\alpha_\Sigma},$$

where the superscript v denotes the vertical part computed with ∇^ω , and analogously for the other connections: δ^Σ , $\delta^{\omega*}$, and $\delta^{\Sigma*}$. The covariant derivatives induced by these connections are denoted by $\nabla^\omega/d\tau$, $\nabla^\Sigma/d\tau$, $\nabla^{\omega*}/d\tau$, and $\nabla^{\Sigma*}/d\tau$, respectively.

In the following, the (fiberwise) adjoint representation of $\tilde{\mathfrak{g}}$ is denoted by $\text{ad}_{\xi_\sigma} : \tilde{\mathfrak{g}}_\sigma \rightarrow \tilde{\mathfrak{g}}_\sigma$ for each $\xi_\sigma \in \tilde{\mathfrak{g}}_\sigma$, $\sigma \in \Sigma$. Likewise, the (fiberwise) coadjoint representation, i.e., minus the adjoint of the adjoint representation, is denoted by $\text{ad}_{\xi_\sigma}^* : \tilde{\mathfrak{g}}_\sigma^* \rightarrow \tilde{\mathfrak{g}}_\sigma^*$. In addition, the reduced curvature of the principal connection $\omega \in \Omega^1(Q, \mathfrak{g})$ is denoted by $\tilde{F}^\omega \in \Omega^2(\Sigma, \tilde{\mathfrak{g}})$.

Theorem 4.4 (reduced Hamilton–d’Alembert–Pontryagin principle). *Let*

$$\mathbf{c} = ((\alpha_T, \alpha_Q), \nu_{\alpha_Q}, \pi_{\alpha_Q}) \in \Omega(Q, \tilde{\tau}) \times_{\mathcal{Q}(\tilde{\tau})} (\Delta_{\mathcal{Q}(\tilde{\tau})} \oplus T^*\mathcal{Q}(\tilde{\tau}))$$

be a path,

$$[\mathbf{c}]_G \simeq ((\alpha_T, \alpha_\Sigma), (\nu_{\alpha_\Sigma}, \xi_{\alpha_\Sigma}), (\pi_{\alpha_\Sigma}, \rho_{\alpha_\Sigma})) \in \Omega(\Sigma, \tilde{\tau}) \times_{\Sigma(\tilde{\tau})} ((\Delta_{\Sigma(\tilde{\tau})} \oplus \Delta_{\tilde{\mathfrak{g}}(\tilde{\tau})}) \oplus (T^*\Sigma(\tilde{\tau}) \oplus \tilde{\mathfrak{g}}^*(\tilde{\tau})))$$

be the corresponding reduced path, as in Lemma 4.2, and

$$\eta_{\alpha_\Sigma} = [\alpha_Q, \omega \circ \alpha'_Q]_G \in \tilde{\mathfrak{g}}(\tilde{\tau})_{\alpha_\Sigma}.$$

Then $\mathbf{c}$ is critical for the action functional $\mathbb{S}$ (recall Definition 3.5) if and only if

$$d\mathbb{S}([\mathbf{c}]_G, \eta_{\alpha_\Sigma})(\delta[\mathbf{c}]_G, \delta\eta_{\alpha_\Sigma}) = 0,$$

for every variation

$$\begin{aligned} \delta[\mathbf{c}]_G &= ((\delta\alpha_T, \delta\alpha_\Sigma), (\delta\nu_{\alpha_\Sigma}, \delta\xi_{\alpha_\Sigma}), (\delta\pi_{\alpha_\Sigma}, \delta\rho_{\alpha_\Sigma})) \\ &\in T_{(\alpha_T, \alpha_\Sigma)}\Omega(\Sigma, \tilde{\tau}) \times (\Delta_{T\Sigma(\tilde{\tau})}(\nu_{\alpha_\Sigma}) \oplus T_{\xi_{\alpha_\Sigma}}\tilde{\mathfrak{g}}(\tilde{\tau})) \times (T_{\pi_{\alpha_\Sigma}}(T^*\Sigma(\tilde{\tau})) \oplus T_{\rho_{\alpha_\Sigma}}\tilde{\mathfrak{g}}^*(\tilde{\tau})), \end{aligned}$$

such that $\delta\alpha_T(\tau_0) = \delta\alpha_T(\tau_1) = 0$, $\delta\alpha_\Sigma(\tau_0) = \delta\alpha_\Sigma(\tau_1) = 0$, and

$$(4.6) \quad d\pi_{T\Sigma} \circ \delta\nu_{\alpha_\Sigma} = d\pi_{\tilde{\mathfrak{g}}} \circ \delta\xi_{\alpha_\Sigma} = d\pi_{T^*\Sigma} \circ \delta\pi_{\alpha_\Sigma} = d\pi_{\tilde{\mathfrak{g}}^*} \circ \delta\rho_{\alpha_\Sigma} = \delta\alpha_\Sigma,$$

and for every variation $\delta\eta_{\alpha_\Sigma} \in T_{\eta_{\alpha_\Sigma}}\tilde{\mathfrak{g}}(\tilde{\tau})$ such that

$$(4.7) \quad \delta^\omega \eta_{\alpha_\Sigma} = \frac{\nabla^\omega \hat{\eta}_{\alpha_\Sigma}}{d\tau} + \text{ad}_{\eta_{\alpha_\Sigma}}(\hat{\eta}_{\alpha_\Sigma}) + \tilde{F}^\omega(\delta\alpha_\Sigma, \alpha'_\Sigma) \in \tilde{\mathfrak{g}}(\tilde{\tau})_{\alpha_\Sigma},$$

where $\nabla^\omega/d\tau$ denotes the covariant derivative induced by ∇^ω , for arbitrary paths $\hat{\eta}_{\alpha_\Sigma} \in \tilde{\mathfrak{g}}(\tilde{\tau})_{\alpha_\Sigma}$ such that $\hat{\eta}_{\alpha_\Sigma}(\tau_0) = \hat{\eta}_{\alpha_\Sigma}(\tau_1) = 0$.

Proof. To begin with, note that $\mathbb{S}(\mathbf{c}) = \mathfrak{S}([\mathbf{c}]_G, \eta_{\alpha_\Sigma})$ by construction. Now let

$$\left\{ \mathbf{c}^s = \left((\alpha_T^s, \alpha_Q^s), \nu_{\alpha_Q}^s, \pi_{\alpha_Q}^s \right) \in \Omega(Q, \tilde{\tau}) \times_{\mathcal{Q}(\tilde{\tau})} (\Delta_{\mathcal{Q}(\tilde{\tau})} \oplus T^*\mathcal{Q}(\tilde{\tau})) \mid s \in (-\epsilon, \epsilon) \right\}$$

be a variation of $\mathbf{c}$ such that $\delta \mathbf{c} = d/ds|_{s=0} \mathbf{c}^s = ((\delta \alpha_T, \delta \alpha_Q), \delta \nu_{\alpha_Q}, \delta \pi_{\alpha_Q})$. Then the reduced variation is computed from (4.4), leading to

$$\begin{aligned} \delta[\mathbf{c}]_G &= \frac{d}{ds} \Big|_{s=0} [\mathbf{c}^s]_G \\ &= \frac{d}{ds} \Big|_{s=0} ((\alpha_T^s, \alpha_\Sigma^s), (\nu_{\alpha_\Sigma}^s, \xi_{\alpha_\Sigma}^s), (\pi_{\alpha_\Sigma}^s, \rho_{\alpha_\Sigma}^s)) \\ &= \frac{d}{ds} \Big|_{s=0} ((\alpha_T^s, \pi_{Q,\Sigma} \circ \alpha_Q^s), (d\pi_{Q,\Sigma} \circ \nu_{\alpha_Q}^s, [\alpha_Q^s, \omega \circ \nu_{\alpha_Q}^s]_G), (\mathbf{H}^* \circ \pi_{\alpha_Q}^s, [\alpha_Q^s, \mathbf{J} \circ \pi_{\alpha_Q}^s]_G)) \\ &= ((\delta \alpha_T, \delta \alpha_\Sigma), (\delta \nu_{\alpha_\Sigma}, \delta \xi_{\alpha_\Sigma}), (\delta \pi_{\alpha_\Sigma}, \delta \rho_{\alpha_\Sigma})), \end{aligned}$$

where we have denoted $\delta \alpha_\Sigma = d\pi_{Q,\Sigma} \circ \delta \alpha_Q$, $\delta \nu_{\alpha_\Sigma} = d(d\pi_{Q,\Sigma}) \circ \delta \nu_{\alpha_Q}$, $\delta \pi_{\alpha_\Sigma} = d\mathbf{H}^* \circ \delta \pi_{\alpha_Q}$, and

$$\delta \xi_{\alpha_\Sigma} = \frac{d}{ds} \Big|_{s=0} [\alpha_Q^s, \omega \circ \nu_{\alpha_Q}^s]_G, \quad \delta \rho_{\alpha_\Sigma} = \frac{d}{ds} \Big|_{s=0} [\alpha_Q^s, \mathbf{J} \circ \pi_{\alpha_Q}^s]_G.$$

It is now clear that free variations $\delta \alpha_Q$, $\delta \nu_{\alpha_Q}$, and $\delta \pi_{\alpha_Q}$ induce free variations $\delta \alpha_\Sigma$, $\delta \xi_{\alpha_\Sigma}$, $\delta \pi_{\alpha_\Sigma}$, and $\delta \rho_{\alpha_\Sigma}$. Note that, although we have the restriction $\delta \nu_{\alpha_Q} \in \Delta_{T\mathcal{Q}}(\nu_{\alpha_Q})$, the condition (3.4) does not restrict $\delta \xi_{\alpha_\Sigma}$. This may be checked locally, where $\delta \nu_{\alpha_Q} = (\alpha, \nu, \delta \alpha, \delta \nu)$ and the restriction reads $\delta \alpha \in \Delta_{\mathcal{Q}(\tilde{\tau})}(\alpha_Q)$, but $\delta \xi_{\alpha_Q}$ only depends on $\delta \nu$, which is free.

Next, we have

$$\left\{ \begin{array}{ll} d\pi_{T\Sigma} \circ \delta \nu_{\alpha_\Sigma} &= \frac{d}{ds} \Big|_{s=0} \pi_{T\Sigma} \circ d\pi_{Q,\Sigma} \circ \nu_{\alpha_Q}^s = d\pi_{Q,\Sigma} \circ d\pi_{TQ} \circ \delta \nu_{\alpha_Q}, \\ d\pi_{\tilde{\mathfrak{g}}} \circ \delta \xi_{\alpha_\Sigma} &= \frac{d}{ds} \Big|_{s=0} \pi_{\tilde{\mathfrak{g}}} \circ [\alpha_Q^s, \omega \circ \nu_{\alpha_Q}^s]_G = d\pi_{Q,\Sigma} \circ \delta \alpha_Q = \delta \alpha_\Sigma, \\ d\pi_{T^*\Sigma} \circ \delta \pi_{\alpha_\Sigma} &= \frac{d}{ds} \Big|_{s=0} \pi_{T^*\Sigma} \circ \mathbf{H}^* \circ \pi_{\alpha_Q}^s = d\pi_{Q,\Sigma} \circ d\pi_{T^*Q} \circ \delta \pi_{\alpha_Q}, \\ d\pi_{\tilde{\mathfrak{g}}^*} \circ \rho_{\alpha_\Sigma} &= \frac{d}{ds} \Big|_{s=0} \pi_{\tilde{\mathfrak{g}}^*} \circ [\alpha_Q^s, \mathbf{J} \circ \pi_{\alpha_Q}^s]_G = d\pi_{Q,\Sigma} \circ \delta \alpha_Q = \delta \alpha_\Sigma, \end{array} \right.$$

where we have used that $\pi_{T\Sigma} \circ d\pi_{Q,\Sigma} = \pi_{Q,\Sigma} \circ \pi_{TQ}$ and $\pi_{T^*\Sigma} \circ \mathbf{H}^* = \pi_{Q,\Sigma} \circ \pi_{T^*Q}$. From the first equality, we conclude that the condition $\delta \nu_{\alpha_Q} \in \Delta_{T\mathcal{Q}}(\nu_{\alpha_Q})$ yields $\delta \nu_{\alpha_\Sigma} \in \Delta_{T\Sigma}(\nu_{\alpha_\Sigma})$, where we have used (4.4) and (4.5). Moreover, a straightforward check shows that the condition (3.7) reduces to (4.6).

The second part is a straightforward consequence of [10, Theorem 3.3.1], which can be applied independently on $[\tau_0, \tilde{\tau})$ and $(\tilde{\tau}, \tau_1]$. Recall that in our case the parameter of the paths is τ instead of t . ■

4.2.2. Reduced equations. The fiber derivatives of the reduced Lagrangian are denoted by

$$\begin{aligned} \frac{\delta \ell}{\delta v_\sigma} : T\Sigma \oplus \tilde{\mathfrak{g}} &\rightarrow T^*\Sigma, & \frac{\delta \ell}{\delta v_\sigma}(v_\sigma, \xi_\sigma) \cdot w_\sigma &= \left. \frac{d}{ds} \right|_{s=0} \ell(v_\sigma + s w_\sigma, \xi_\sigma), \\ \frac{\delta \ell}{\delta \xi_\sigma} : T\Sigma \oplus \tilde{\mathfrak{g}} &\rightarrow \tilde{\mathfrak{g}}^*, & \frac{\delta \ell}{\delta \xi_\sigma}(v_\sigma, \xi_\sigma) \cdot \eta_\sigma &= \left. \frac{d}{ds} \right|_{s=0} \ell(v_\sigma, \xi_\sigma + s \eta_\sigma) \end{aligned}$$

for each $(v_\sigma, \xi_\sigma), (w_\sigma, \eta_\sigma) \in T_\sigma \Sigma \oplus \tilde{\mathfrak{g}}_\sigma$, $\sigma \in \Sigma$. Moreover, the horizontal derivative of the reduced Lagrangian is defined by using the linear connection $\nabla^\Sigma \oplus \nabla^\omega$ on $T\Sigma \oplus \tilde{\mathfrak{g}}$ as

$$\frac{\delta \ell}{\delta \sigma} : T\Sigma \oplus \tilde{\mathfrak{g}} \rightarrow T^*\Sigma, \quad \frac{\delta \ell}{\delta \sigma}(v_\sigma, \xi_\sigma) \cdot w_\sigma = \left. \frac{d}{ds} \right|_{s=0} \left(\ell \circ \gamma_{(v_\sigma, \xi_\sigma)}^h \right)(s)$$

for each $(v_\sigma, \xi_\sigma) \in T_\sigma \Sigma \oplus \tilde{\mathfrak{g}}_\sigma$ and $w_\sigma \in T_\sigma \Sigma$, where $\gamma : (-\epsilon, \epsilon) \rightarrow \Sigma$ is a curve such that $d/ds|_{s=0} \gamma(s) = w_\sigma$ and $\gamma_{(v_\sigma, \xi_\sigma)}^h : (-\epsilon, \epsilon) \rightarrow T\Sigma \oplus \tilde{\mathfrak{g}}$ is its horizontal lift at (v_σ, ξ_σ) given by the linear connection $\nabla^\Sigma \oplus \nabla^\omega$.

Theorem 4.5 (reduced nonholonomic implicit Euler–Lagrange equations with collisions). *Let $\mathbf{c} = ((\alpha_T, \alpha_Q), (\nu_{\alpha_Q}, \pi_{\alpha_Q})) \in \Omega(Q, \tilde{\tau}) \times_{\mathcal{Q}(\tilde{\tau})} (\Delta_{\mathcal{Q}(\tilde{\tau})} \oplus T^*\mathcal{Q}(\tilde{\tau}))$ be a path and*

$$[\mathbf{c}]_G = ((\alpha_T, \alpha_\Sigma), (\nu_{\alpha_\Sigma}, \xi_{\alpha_\Sigma}), (\pi_{\alpha_\Sigma}, \rho_{\alpha_\Sigma})) \in \Omega(\Sigma, \tilde{\tau}) \times_{\Sigma(\tilde{\tau})} ((\Delta_{\Sigma(\tilde{\tau})} \oplus \Delta_{\tilde{\mathfrak{g}}(\tilde{\tau})}) \oplus (T^*\Sigma(\tilde{\tau}) \oplus \tilde{\mathfrak{g}}^*(\tilde{\tau})))$$

be the reduced path (recall Lemma 4.2). Then the following statements are equivalent:

1. $\mathbf{c}$ is stationary for the Hamilton–d’Alembert–Pontryagin variational principle introduced in Definition 3.5.
2. $\mathbf{c}$ satisfies the nonholonomic implicit Euler–Lagrange equations with collisions given in Theorem 3.6.
3. $[\mathbf{c}]_G$ and $\eta_{\alpha_\Sigma} = [\alpha_Q, \omega \circ \alpha'_Q]_G$ satisfy the reduced Hamilton–d’Alembert–Pontryagin variational principle given in Theorem 4.4.
4. $[\mathbf{c}]_G$ and $\eta_{\alpha_\Sigma} = [\alpha_Q, \omega \circ \alpha'_Q]_G$ satisfy the nonholonomic implicit Lagrange–Poincaré equations with collisions, which consists of the horizontal equations on $[\tau_0, \tilde{\tau}) \cup (\tilde{\tau}, \tau_1]$,

$$\begin{cases} \frac{\nabla^{\Sigma*} \pi_{\alpha_\Sigma}}{d\tau} - \frac{\delta \ell}{\delta \sigma}(\nu_{\alpha_\Sigma}, \xi_{\alpha_\Sigma}) \alpha'_T + \rho_{\alpha_\Sigma} \cdot (i_{\alpha'_\Sigma} \tilde{F}^\omega) \in \Delta_\Sigma^\circ(\alpha_\Sigma), \\ \frac{\alpha'_\Sigma}{\alpha'_T} = \nu_{\alpha_\Sigma} \in \Delta_\Sigma(\alpha_\Sigma), & \frac{\delta \ell}{\delta v_\sigma}(\nu_{\alpha_\Sigma}, \xi_{\alpha_\Sigma}) = \pi_{\alpha_\Sigma}, \end{cases}$$

where $i_U : \Omega^2(\Sigma, \tilde{\mathfrak{g}}) \rightarrow \Omega^1(\Sigma, \tilde{\mathfrak{g}})$ denotes the left interior product by $U \in \mathfrak{X}(\Sigma)$, the vertical equations on $[\tau_0, \tilde{\tau}) \cup (\tilde{\tau}, \tau_1]$,

$$\begin{cases} \frac{\nabla^{\omega*} \rho_{\alpha_\Sigma}}{d\tau} = ad_{\eta_{\alpha_\Sigma}}^*(\rho_{\alpha_\Sigma}), \\ \frac{\eta_{\alpha_\Sigma}}{\alpha'_T} = \xi_{\alpha_\Sigma} \in \Delta_{\tilde{\mathfrak{g}}}(\alpha_\Sigma), & \frac{\delta \ell}{\delta \xi_\sigma}(\nu_{\alpha_\Sigma}, \xi_{\alpha_\Sigma}) = \rho_{\alpha_\Sigma}, \end{cases}$$

and the reduced energy conservation on $[\tau_0, \tilde{\tau}) \cup (\tilde{\tau}, \tau_1]$,

$$e'((\nu_{\alpha_\Sigma}, \xi_{\alpha_\Sigma}), (\pi_{\alpha_\Sigma}, \rho_{\alpha_\Sigma})) = 0,$$

together with the reduced conditions for the elastic impact,

$$\begin{cases} \pi_{\alpha_\Sigma}^+ - \pi_{\alpha_\Sigma}^- \in (T_{\alpha_\Sigma(\tilde{\tau})} \partial \Sigma \cap \Delta_\Sigma(\alpha_\Sigma(\tilde{\tau})))^\circ = (T_{\alpha_\Sigma(\tilde{\tau})} \partial \Sigma)^\circ + \Delta_\Sigma^\circ(\alpha_\Sigma(\tilde{\tau})), \\ e((\nu_{\alpha_\Sigma}^+, \xi_{\alpha_\Sigma}^+), (\pi_{\alpha_\Sigma}^+, \rho_{\alpha_\Sigma}^+)) = e((\nu_{\alpha_\Sigma}^-, \xi_{\alpha_\Sigma}^-), (\pi_{\alpha_\Sigma}^-, \rho_{\alpha_\Sigma}^-)), \end{cases}$$

where we have denoted $\pi_{\alpha_\Sigma}(\tilde{\tau}^+) = \pi_{\alpha_\Sigma}^+$, etc.

Proof. The equivalences between 1 and 2, and between 1 and 3 were established in Theorems 3.6 and 4.4. To conclude, let us show the equivalence between 3 and 4. To that end, Theorem 4.4 is used, and the reduced variations are decomposed into its horizontal and vertical parts by means of the linear connections ∇^ω , ∇^Σ , $\nabla^{\omega*}$, and $\nabla^{\Sigma*}$. Note that (4.6) ensures that the horizontal part of all the variations is $\delta\alpha_\Sigma \in \Delta_{\Sigma(\tilde{\tau})}(\alpha_\Sigma)$. Therefore,

$$\begin{aligned} d\mathfrak{S}([c]_G, \eta_{\alpha_\Sigma}) (\delta[c]_G, \delta\eta_{\alpha_\Sigma}) &= \int_{\tau_0}^{\tau_1} \left(\frac{\delta\ell}{\delta\sigma} \cdot \delta\alpha_\Sigma + \frac{\delta\ell}{\delta v_\sigma} \cdot \delta^\Sigma \nu_{\alpha_\Sigma} + \frac{\delta\ell}{\delta \xi_\sigma} \cdot \delta^\omega \xi_{\alpha_\Sigma} + \delta^{\Sigma*} \pi_{\alpha_\Sigma} \cdot \left(\frac{\alpha'_\Sigma}{\alpha'_T} - \nu_{\alpha_\Sigma} \right) \right. \\ &\quad + \pi_{\alpha_\Sigma} \cdot \left(\frac{\delta^\Sigma \alpha'_\Sigma}{\alpha'_T} - \frac{\alpha'_\Sigma \delta \alpha'_T}{(\alpha'_T)^2} - \delta^\Sigma \nu_{\alpha_\Sigma} \right) + \delta^{\Sigma*} \rho_{\alpha_\Sigma} \cdot \left(\frac{\eta_{\alpha_\Sigma}}{\alpha'_T} - \xi_{\alpha_\Sigma} \right) \\ &\quad + \rho_{\alpha_\Sigma} \cdot \left(\frac{\delta^\omega \eta_{\alpha_\Sigma}}{\alpha'_T} - \frac{\eta_{\alpha_\Sigma} \delta \alpha'_T}{(\alpha'_T)^2} - \delta^\omega \xi_{\alpha_\Sigma} \right) \left. \right) \alpha'_T d\tau \\ &\quad + \int_{\tau_0}^{\tau_1} \left(\ell + \pi_{\alpha_\Sigma} \cdot \left(\frac{\alpha'_\Sigma}{\alpha'_T} - \nu_{\alpha_\Sigma} \right) + \rho_{\alpha_\Sigma} \cdot \left(\frac{\eta_{\alpha_\Sigma}}{\alpha'_T} - \xi_{\alpha_\Sigma} \right) \right) \delta \alpha'_T d\tau, \end{aligned}$$

where the reduced Lagrangian and its partial derivatives are evaluated at $(\nu_{\alpha_\Sigma}(\tau), \xi_{\alpha_\Sigma}(\tau))$.

Note that

$$\delta^\Sigma \alpha'_\Sigma = \frac{\nabla^\Sigma \delta \alpha_\Sigma}{d\tau}, \quad (\pi_{\alpha_\Sigma} \cdot \delta \alpha_\Sigma)' = \left(\frac{\nabla^{\Sigma*} \pi_{\alpha_\Sigma}}{d\tau} \right) \cdot \delta \alpha_\Sigma + \pi_{\alpha_\Sigma} \cdot \frac{\nabla^\Sigma \delta \alpha_\Sigma}{d\tau}.$$

By using this, splitting the integration domain, $[\tau_1, \tilde{\tau}) \cup (\tilde{\tau}, \tau_1]$, integrating by parts on each subinterval, and regrouping terms, the previous expression leads to

$$d\mathfrak{S}([c]_G, \eta_{\alpha_\Sigma}) (\delta[c]_G, \delta\eta_{\alpha_\Sigma}) = \mathcal{I}(\tau_0, \tilde{\tau}) + \mathcal{I}(\tilde{\tau}, \tau_1) + \mathcal{B}(\tau_0, \tilde{\tau}^-) + \mathcal{B}(\tilde{\tau}^+, \tau_1),$$

where for each $a, b \in \mathbb{R}$, $a < b$, we set

$$\begin{aligned} \mathcal{I}(a, b) &= \int_a^b \left(\left(\frac{\delta\ell}{\delta\sigma} \alpha'_T - \frac{\nabla^{\Sigma*} \pi_{\alpha_\Sigma}}{d\tau} \right) \cdot \delta \alpha_\Sigma + \alpha'_T \left(\frac{\delta\ell}{\delta v_\sigma} - \pi_{\alpha_\Sigma} \right) \cdot \delta^\Sigma \nu_{\alpha_\Sigma} \right. \\ &\quad + \alpha'_T \left(\frac{\delta\ell}{\delta \xi_\sigma} - \rho_{\alpha_\Sigma} \right) \cdot \delta^\omega \xi_{\alpha_\Sigma} + \delta^{\Sigma*} \pi_{\alpha_\Sigma} \cdot \left(\frac{\alpha'_\Sigma}{\alpha'_T} - \nu_{\alpha_\Sigma} \right) \alpha'_T \\ &\quad + \delta^{\omega*} \rho_{\alpha_\Sigma} \cdot \left(\frac{\eta_{\alpha_\Sigma}}{\alpha'_T} - \xi_{\alpha_\Sigma} \right) \alpha'_T + \rho_{\alpha_\Sigma} \cdot \delta^\omega \eta_{\alpha_\Sigma} + e' \delta \alpha_T \left. \right) d\tau, \\ \mathcal{B}(a, b) &= \left[\pi_{\alpha_\Sigma}(\tau) \cdot \delta \alpha_\Sigma(\tau) - e \delta \alpha_T(\tau) \right]_{\tau=a}^{\tau=b}, \end{aligned}$$

with the reduced energy evaluated at $((\nu_{\alpha_\Sigma}(\tau), \xi_{\alpha_\Sigma}(\tau)), (\pi_{\alpha_\Sigma}(\tau), \rho_{\alpha_\Sigma}(\tau)))$. The reduced conditions for the elastic impact are straightforwardly obtained by recalling that $\delta\alpha_T$ is free, $\delta\alpha_\Sigma \in \Delta_{\Sigma(\tilde{\tau})}(\alpha_\Sigma)$, and both of them vanish at the endpoints.

Next, we use (4.7), as well as integration by parts, to write

$$\begin{aligned} \mathcal{I}(a, b) = & \int_a^b \left(\left(\frac{\delta\ell}{\delta\sigma} \alpha'_T - \frac{\nabla^{\Sigma*} \pi_{\alpha_\Sigma}}{d\tau} - \rho_{\alpha_\Sigma} \cdot (i_{\alpha'_\Sigma} \tilde{F}^\omega) \right) \cdot \delta\alpha_\Sigma + \alpha'_T \left(\frac{\delta\ell}{\delta v_\sigma} - \pi_{\alpha_\Sigma} \right) \cdot \delta^\Sigma \nu_{\alpha_\Sigma} \right. \\ & + \alpha'_T \left(\frac{\delta\ell}{\delta \xi_\sigma} - \rho_{\alpha_\Sigma} \right) \cdot \delta^\omega \xi_{\alpha_\Sigma} + \delta^{\Sigma*} \pi_{\alpha_\Sigma} \cdot \left(\frac{\alpha'_\Sigma}{\alpha'_T} - \nu_{\alpha_\Sigma} \right) \alpha'_T + \delta^{\omega*} \rho_{\alpha_\Sigma} \cdot \left(\frac{\eta_{\alpha_\Sigma}}{\alpha'_T} - \xi_{\alpha_\Sigma} \right) \alpha'_T \\ & \left. + \left(\frac{\nabla^{\omega*} \rho_{\alpha_\Sigma}}{d\tau} - \text{ad}_{\eta_{\alpha_\Sigma}}^* (\rho_{\alpha_\Sigma}) \right) \cdot \hat{\eta}_{\alpha_\Sigma} + e' \delta\alpha_T \right) d\tau \end{aligned}$$

The reduced equations are now straightforward by using that $\delta\alpha_\Sigma \in \Delta_{\Sigma(\tilde{\tau})}(\alpha_\Sigma)$, $\delta^\Sigma \nu_{\alpha_\Sigma} \in T_{\alpha_\Sigma} \Sigma(\tilde{\tau})$, $\delta^\omega \xi_{\alpha_\Sigma} \in \tilde{\mathfrak{g}}(\tilde{\tau})_{\alpha_\Sigma}$, $\delta^{\Sigma*} \pi_{\alpha_\Sigma} \in T_{\alpha_\Sigma}^* \Sigma(\tilde{\tau})$, $\delta^{\omega*} \rho_{\alpha_\Sigma} \in \tilde{\mathfrak{g}}^*(\tilde{\tau})_{\alpha_\Sigma}$, $\hat{\eta}_{\alpha_\Sigma} \in \tilde{\mathfrak{g}}(\tilde{\tau})$, and $\delta\alpha_T \in T_{\alpha_T} \mathcal{T}$ are free variations. ■

As for the original (unreduced) equations, by using the change of variable $t = \alpha_T(\tau)$, the nonholonomic implicit Lagrange–Poincaré equations for a curve

$$((u_{\tilde{\alpha}}, \bar{\xi}_{\tilde{\alpha}}), (y_{\tilde{\alpha}}, \bar{\rho}_{\tilde{\alpha}}), \bar{\eta}_{\tilde{\alpha}}) : [t_0, t_1] \rightarrow (T\Sigma \oplus \tilde{\mathfrak{g}}) \oplus (T^*\Sigma \oplus \tilde{\mathfrak{g}}^*) \oplus \tilde{\mathfrak{g}}$$

read

$$(4.8) \quad \begin{cases} \frac{\nabla^{\Sigma*} y_{\tilde{\alpha}}}{dt} - \frac{\delta\ell}{\delta\sigma} (u_{\tilde{\alpha}}, \bar{\xi}_{\tilde{\alpha}}) + \bar{\rho}_{\tilde{\alpha}} \cdot (i_{\sigma_{\tilde{\alpha}}} \tilde{F}^\omega) \in \Delta_{\Sigma}^\circ(\sigma_{\tilde{\alpha}}), \\ \dot{\sigma}_{\tilde{\alpha}} = u_{\tilde{\alpha}} \in \Delta_{\Sigma}(\sigma_{\tilde{\alpha}}), & \frac{\delta\ell}{\delta v_\sigma} (u_{\tilde{\alpha}}, \bar{\xi}_{\tilde{\alpha}}) = y_{\tilde{\alpha}}, \\ \frac{\nabla^{\omega*} \bar{\rho}_{\tilde{\alpha}}}{dt} = \text{ad}_{\bar{\xi}_{\tilde{\alpha}}}^* (\bar{\rho}_{\tilde{\alpha}}), & \frac{\delta\ell}{\delta \xi_\sigma} (u_{\tilde{\alpha}}, \bar{\xi}_{\tilde{\alpha}}) = \bar{\rho}_{\tilde{\alpha}}, \\ \bar{\zeta}_{\tilde{\alpha}} = \bar{\xi}_{\tilde{\alpha}} \in \Delta_{\tilde{\mathfrak{g}}}(\sigma_{\tilde{\alpha}}), & e'((u_{\tilde{\alpha}}, \bar{\xi}_{\tilde{\alpha}}), (y_{\tilde{\alpha}}, \bar{\rho}_{\tilde{\alpha}})) = 0, \end{cases}$$

where we have used Remark 4.3 and denoted $\bar{\zeta}_{\tilde{\alpha}} = \bar{\eta}_{\tilde{\alpha}}/(\alpha'_T \circ \alpha_T^{-1})$. Moreover, the reduced conditions for the elastic impact read

$$(4.9) \quad y_{\tilde{\alpha}}^+ - y_{\tilde{\alpha}}^- \in \left(T_{\sigma_{\tilde{\alpha}}(\tilde{t})} \partial\Sigma \cap \Delta_{\Sigma}(\sigma_{\tilde{\alpha}}(\tilde{t})) \right)^\circ = \left(T_{\sigma_{\tilde{\alpha}}(\tilde{t})} \partial\Sigma \right)^\circ + \Delta_{\Sigma}^\circ(\sigma_{\tilde{\alpha}}(\tilde{t})),$$

$$(4.10) \quad e\left((u_{\tilde{\alpha}}^+, \bar{\xi}_{\tilde{\alpha}}^+), (y_{\tilde{\alpha}}^+, \bar{\rho}_{\tilde{\alpha}}^+)\right) = e\left((u_{\tilde{\alpha}}^-, \bar{\xi}_{\tilde{\alpha}}^-), (y_{\tilde{\alpha}}^-, \bar{\rho}_{\tilde{\alpha}}^-)\right),$$

$$(4.10) \quad \dot{\sigma}_{\tilde{\alpha}}^+ = u_{\tilde{\alpha}}^+ \in \Delta_{\Sigma}(\sigma_{\tilde{\alpha}}(\tilde{t})), \quad \bar{\zeta}_{\tilde{\alpha}}^+ = \bar{\xi}_{\tilde{\alpha}}^+ \in \Delta_{\tilde{\mathfrak{g}}}(\sigma_{\tilde{\alpha}}(\tilde{t})),$$

where we have denoted $y_{\tilde{\alpha}}(\tilde{t}^+) = y_{\tilde{\alpha}}^+$, etc.

Remark 4.6. For the unconstrained case, i.e., $\Delta_{\Sigma} = T\Sigma$ and $\Delta_{\tilde{\mathfrak{g}}} = \tilde{\mathfrak{g}}$, we recover the *implicit Lagrange–Poincaré equations* presented in [36].

4.3. Example: The spherical pendulum hitting a cylindrical surface. Let us continue with the example of section 3.4. This system is invariant by rotations about the vertical axis; that is, the Lagrangian (3.11) and the constraint distribution (3.12) are invariant by the (tangent lift of) following action: $\mathrm{SO}(2) \times Q \rightarrow Q$, $(\psi, (\theta, \varphi)) \mapsto (\theta, \varphi + \psi)$.

Note that this action is free and proper except when $\theta = k\pi$ for some $k \in \mathbb{Z}$. Hence, the following is only valid for trajectories not passing through those configurations. The reduced configuration space and its boundary are given by

$$\Sigma = Q/\mathrm{SO}(2) \simeq \{\theta \in S^1 \mid L \sin \theta \leq 1\}, \quad \partial\Sigma = \{\theta \in S^1 \mid L \sin \theta = 1\}.$$

Hence, the tangent bundle and its annihilator are given by $T\partial\Sigma = 0$, $(T\partial\Sigma)^\circ = \mathrm{span}\{d\theta\} = T^*\partial\Sigma$. Since we are working locally, $\tilde{\mathfrak{so}}(2) = \Sigma \times \mathfrak{so}(2) = \Sigma \times \mathbb{R}$, and we may choose the trivial principal connection on $Q \rightarrow \Sigma$, whence

$$(TQ)/\mathrm{SO}(2) \simeq T\Sigma \oplus \tilde{\mathfrak{so}}(2) = T\Sigma \times \mathbb{R}, \quad [\theta, \varphi; v_\theta, v_\varphi]_{\mathrm{SO}(2)} \mapsto (\theta, v_\theta; \xi = v_\varphi).$$

The reduced Lagrangian thus reads

$$\ell(\theta, v_\theta, \xi = v_\varphi) = \frac{1}{2}mL^2(v_\theta^2 + \xi^2 \sin^2 \theta) - mgL \cos \theta,$$

and analogously for the reduced constraint distributions,

$$\Delta_\Sigma = \mathrm{span}\{\partial_\theta\} = T\Sigma, \quad \Delta_{\tilde{\mathfrak{so}}(2)} = \mathrm{span}\{f(\theta)\partial_\varphi\}.$$

The corresponding annihilators vanish $\Delta_\Sigma^\circ = 0$ and $\Delta_{\tilde{\mathfrak{so}}(2)}^\circ = 0$.

By gathering all, the implicit Lagrange–Poincaré equations (4.8) for a curve

$$(\theta, v_\theta, p_\theta, \xi = v_\varphi, \rho = p_\varphi, \zeta) : [t_0, t_1] \rightarrow (T\Sigma \oplus T^*\Sigma) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

read

$$\begin{cases} \dot{p}_\theta = mL \sin \theta (L\xi^2 \cos \theta + g), \\ \dot{\theta} = v_\theta, & p_\theta = mL^2 v_\theta, \\ \dot{\rho} = 0, \\ \zeta = \xi = \mu_0 f(\theta), & \rho = mL^2 \xi \sin^2 \theta, \end{cases}$$

on $[t_0, t_1] - \{\tilde{t}\}$, where $\mu_0 \in \mathbb{R}$ is the Lagrange multiplier. The impact condition at $t = \tilde{t}$ given in (4.9) reads $p_\theta^+ - p_\theta^- = \lambda_0$, where we denote $p_\theta^+ = p_\theta(\tilde{t}^+)$, etc., and $\lambda_0 \in \mathbb{R}$ is the Lagrange multiplier. Similarly, the condition (4.10) reads $\dot{\theta}^+ = v_\theta^+$, $\zeta^+ = \xi^+ = \lambda_0 f(\theta)$, where $\dot{\theta}^+ = \dot{\theta}(\tilde{t}^+)$, etc., and λ_0 is the Lagrange multiplier.

5. Conclusions and future work. In this paper, the Lagrange–d’Alembert–Pontryagin action functional has been extended to the nonsmooth setting to account for nonholonomic systems undergoing elastic collisions. The configuration space of these systems is a smooth manifold with boundary and the impact takes place when the trajectory of the system reaches the boundary. The dynamical equations thus obtained are known as the implicit Euler–Lagrange equations with collisions and naturally include the energy conservation, as well as

the appropriate conditions for the impact. Furthermore, for systems with symmetries, the geometric structures describing the system are reduced, yielding the nonholonomic implicit Lagrange–Poincaré equations with collisions, which consist of the horizontal equations, the vertical equations, and the conditions for the impact. Both in the original and reduced formulations, the equations are first obtained by applying the variational principle with an auxiliary parameter τ , and then they are reparametrized so that they may be rewritten in terms of the time t . Lastly, the theory is illustrated through some examples.

For future work, we would like to explore the following lines:

1. *Symmetry breaking* appears when the system is not invariant by the action of the entire Lie group, but only by its restriction to a subgroup. This situation, which is particularly relevant in fluid dynamics [18], is related to systems whose symmetry group is a semidirect product and leads to advected parameters in the reduced equations [19].
2. *Variational integrators*. A discrete counterpart of this theory would be highly desirable in order to obtain numerical schemes that preserve the geometric structures underlying the dynamical equations, thus leading to well-behaved simulations even for long times [17, 30, 31].
3. *Interconnection* for Lagrange–Dirac systems [23, 28] may be extended to the non-smooth setting. The collisions are expected to be transferred between subsystems due to the coupling, thus yielding impacts on systems that originally have no collisions.
4. *Nonelastic impacts* may be modeled by introducing a restitution coefficient in the reset map [6], which will allow for treating a wider number of physical systems.

REFERENCES

- [1] A. AMES, R. GREGG, E. WENDEL, AND S. SASTRY, *On the geometric reduction of controlled three-dimensional bipedal robotic walkers*, in Lagrangian and Hamiltonian Methods for Nonlinear Control 2006: Proceedings from the 3rd IFAC Workshop, Springer, Nagoya, Japan, 2007, pp. 183–196.
- [2] A. AMES AND S. SASTRY, *Hybrid cotangent bundle reduction of simple hybrid mechanical systems with symmetry*, in 2006 American Control Conference, IEEE, 2006.
- [3] A. A. SIMOES AND L. COLOMBO, *Hamel equations and quasivelocities for nonholonomic systems with inequality constraints*, in 2023 62nd IEEE Conference on Decision and Control (CDC), IEEE, 2023, pp. 5741–5746.
- [4] A. BLOCH, *Nonholonomic mechanics*, in Nonholonomic Mechanics and Control, Springer, 2003, pp. 207–276.
- [5] A. BLOCH, P. KRISHNAPRASAD, J. MARSDEN, AND T. RATIU, *The Euler–Poincaré equations and double bracket dissipation*, Commun. Math. Phys., 175 (1996), pp. 1–42, <https://doi.org/10.1007/BF02101622>.
- [6] B. BROGLIATO, *Nonsmooth Mechanics: Models, Dynamics, and Control*, 2nd ed., Springer, 1999.
- [7] H. CENDRA AND V. DIAZ, *The Lagrange–d’Alembert–Poincaré equations and integrability for the rolling disk*, Regul. Chaotic Dyn., 11 (2006), pp. 67–81, <https://doi.org/10.1070/RD2006v011n01ABEH000335>.
- [8] H. CENDRA, S. FERRARO, AND S. GRILLO, *Lagrangian reduction of generalized nonholonomic systems*, J. Geom. Phys., 58 (2008), pp. 1271–1290, <https://doi.org/10.1016/j.geomphys.2008.05.002>.
- [9] H. CENDRA, J. MARSDEN, AND T. RATIU, *Geometric mechanics, Lagrangian reduction, and nonholonomic systems*, in Mathematics Unlimited—2001 and Beyond, Springer, 2001, pp. 221–273.
- [10] H. CENDRA, J. MARSDEN, AND T. RATIU, *Lagrangian reduction by stages*, Mem. Amer. Math. Soc., 152 (2001), <https://doi.org/10.1090/memo/0722>.
- [11] W. CLARK AND A. BLOCH, *The bouncing penny and nonholonomic impacts*, in 2019 IEEE 58th Conference on Decision and Control (CDC), IEEE, 2019, pp. 2114–2119.

- [12] W. CLARK, L. COLOMBO, AND A. BLOCH, *The interplay between symmetries and impact effects on hybrid mechanical systems*, in 2024 European Control Conference (ECC 24), 2024, pp. 2805–2810.
- [13] L. COLOMBO, M. DE LEÓN, M. IRAZÚ, AND A. LÓPEZ-GORDÓN, *Hamilton–Jacobi theory for nonholonomic and forced hybrid mechanical systems*, *Geom. Mech.*, 1 (2024), pp. 123–152.
- [14] L. COLOMBO, I. EYREA, AND A. GARCÍA-TORAÑO, *A note on hybrid Routh reduction for time-dependent Lagrangian systems*, *J. Geom. Mech.*, 12 (2020), pp. 309–321, <https://doi.org/10.3934/jgm.2020014>.
- [15] L. COLOMBO AND M. IRAZÚ, *Symmetries and periodic orbits in simple hybrid Routhian systems*, *Nonlinear Anal. Hybrid Syst.*, 36 (2020), 100857, <https://doi.org/10.1016/j.nahs.2020.100857>.
- [16] M. EYREA IRAZÚ, L. COLOMBO, AND A. BLOCH, *Reduction by symmetries of simple hybrid mechanical systems*, *IFAC–PapersOnLine*, 54 (2021), pp. 94–99, <https://doi.org/10.1016/j.ifacol.2021.11.061>.
- [17] R. FETEAU, J. MARSDEN, M. ORTIZ, AND M. WEST, *Nonsmooth Lagrangian mechanics and variational collision integrators*, *SIAM J. Appl. Dyn. Syst.*, 2 (2003), pp. 381–416, <https://doi.org/10.1137/S1111111102406038>.
- [18] F. GAY-BALMAZ AND C. TRONCI, *Reduction theory for symmetry breaking with applications to nematic systems*, *Phys. D*, 239 (2010), pp. 1929–1947, <https://doi.org/10.1016/j.physd.2010.07.002>.
- [19] F. GAY-BALMAZ AND H. YOSHIMURA, *Dirac reduction for nonholonomic mechanical systems and semidirect products*, *Adv. Appl. Math.*, 63 (2015), pp. 131–213, <https://doi.org/10.1016/j.aam.2014.10.004>.
- [20] J. GOODMAN AND L. COLOMBO, *On the existence and uniqueness of Poincaré maps for systems with impulse effects*, *IEEE Trans. Automat. Control*, 65 (2020), pp. 1815–1821, <https://doi.org/10.1109/TAC.2019.2941446>.
- [21] W. HADDAD, V. CHELLABOINA, AND S. NERSESOV, *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control*, Princeton University Press, 2006.
- [22] D. HOLM, J. MARSDEN, AND T. RATIU, *The Euler–Poincaré equations and semidirect products with applications to continuum theories*, *Adv. Math.*, 137 (1998), pp. 1–81, <https://doi.org/10.1006/aima.1998.1721>.
- [23] H. JACOBS AND H. YOSHIMURA, *Tensor products of Dirac structures and interconnection in Lagrangian mechanics*, *J. Geom. Mech.*, 6 (2014), pp. 67–98, <https://doi.org/10.3934/jgm.2014.6.67>.
- [24] S. KOBAYASHI AND K. NOMIZU, *Foundations of Differential Geometry, Vol. II*, Wiley Classics Lib. 61, John Wiley & Sons, 1996.
- [25] J. KOILLER, *Reduction of some classical non-holonomic systems with symmetry*, *Arch. Ration. Mech. Anal.*, 118 (1992), pp. 113–148, <https://doi.org/10.1007/BF00375092>.
- [26] J. MARSDEN AND J. SCHEURLE, *The reduced Euler–Lagrange equations*, *Fields Inst. Commun.*, 1 (1993), pp. 139–164, <https://doi.org/10.1090/fic/001/07>.
- [27] J. NEĬMARK AND N. FUFÁEV, *Dynamics of Nonholonomic Systems*, Transl. Math. Monogr. 33, AMS, 2004.
- [28] A. RODRÍGUEZ ABELLA, *Interconnection of Lagrange–Dirac systems through nonstandard interaction structures*, *Int. J. Geom. Methods Mod. Phys.*, 20 (2023), 2350124, <https://doi.org/10.1142/S0219887823501244>.
- [29] A. RODRÍGUEZ ABELLA AND L. COLOMBO, *Implicit nonholonomic mechanics with collisions*, *IFAC–PapersOnLine*, 58 (2024), pp. 71–76, <https://doi.org/10.1016/j.ifacol.2024.08.259>.
- [30] A. RODRÍGUEZ ABELLA AND M. LEOK, *Discrete Dirac reduction of implicit Lagrangian systems with abelian symmetry groups*, *J. Geom. Mech.*, 15 (2023), pp. 319–356, <https://doi.org/10.3934/jgm.2023013>.
- [31] K. TRAN AND M. LEOK, *Lie group variational collision integrators for a class of hybrid systems*, *SIAM J. Appl. Dyn. Syst.*, 24 (2025), pp. 1622–1667.
- [32] E. WESTERVELT, J. GRIZZLE, C. CHEVALLEREAU, J. CHOI, AND B. MORRIS, *Feedback Control of Dynamic Bipedal Robot Locomotion*, CRC Press, 2018.
- [33] H. YOSHIMURA AND J. MARSDEN, *Dirac structures in Lagrangian mechanics part I: Implicit Lagrangian systems*, *J. Geom. Phys.*, 57 (2006), pp. 133–156, <https://doi.org/10.1016/j.geomphys.2006.02.009>.
- [34] H. YOSHIMURA AND J. MARSDEN, *Dirac structures in Lagrangian mechanics part II: Variational structures*, *J. Geom. Phys.*, 57 (2006), pp. 209–250, <https://doi.org/10.1016/j.geomphys.2006.02.012>.

- [35] H. YOSHIMURA AND J. MARSDEN, *Reduction of Dirac structures and the Hamilton–Pontryagin principle*, Rep. Math. Phys., 60 (2007), pp. 381–426, [https://doi.org/10.1016/S0034-4877\(08\)00004-9](https://doi.org/10.1016/S0034-4877(08)00004-9).
- [36] H. YOSHIMURA AND J. MARSDEN, *Dirac cotangent bundle reduction*, J. Geom. Mech., 1 (2009), pp. 87–158, <https://doi.org/10.3934/jgm.2009.1.87>.