3rd *BYMAT Conference* (2020)

Covariant reduction by fiberwise actions in classical field theory

Álvaro Rodríguez Abella
Instituto de Ciencias Matemáticas
Universidad Complutense de Madrid
alvrod06@ucm.es

Abstract: Symmetries represent a central tool in the geometric analysis of mechanical systems. When a group of symmetries acts on the configuration space of a Lagrangian system, the quotient by this action of both the space and the variational principle is known as reduction. In the case of mechanics, this produces the well-known Lagrange-Poincaré equations, which have many applications in the literature.

In the realm of field theories, similar results have also been obtained. However, the typical nature of symmetries involved in the most relevant classical field theories is local and has not been addressed so far. In this case, symmetries are given by fiberwise actions of Lie group fiber bundles. The main instance of this situation are gauge theories.

The goal of this contribution is to determine the reduction procedure when a first order Lagrangian is invariant by a certain type of gauge group.

Resumen: El estudio de las simetrías constituye una herramienta fundamental en el análisis geométrico de los sistemas mecánicos. Cuando un grupo de simetrías actúa en el espacio de configuración de un sistema lagrangiano, el cociente por esta acción tanto del espacio, como del principio variacional es conocido como reducción. En el caso de la mecánica, esto da lugar a las ecuaciones de Lagrange-Poincaré, de las que se pueden encontrar muchas aplicaciones en la literatura.

En el contexto de las teorías de campos, se han obtenido resultados similares. Sin embargo, las simetrías involucradas en las teorías de campos más importantes son locales y no se han tratado todavía. En este caso, las simetrías están dadas por acciones fibradas de fibrados de grupos de Lie. El principal ejemplo de esta situación son las teorías gauge.

El objetivo de esta contribución es determinar el procedimiento de reducción cuando un lagrangiano de primer orden es invariante por un cierto tipo de grupo gauge.

Keywords: field theory, gauge group, Lagrangian, local symmetries, reduction.

MSC2010: 37J37, 53C05, 58D19.

Acknowledgements: The author was supported by an FPU grant from the Spanish Ministry of Science, Innovation and Universities (MICIU).

Reference: Rodríguez Abella, Álvaro. "Covariant reduction by fiberwise actions in classical field theory". In: *TEMat monográficos*, 2 (2021): *Proceedings of the 3rd BYMAT Conference*, pp. 211-214. ISSN: 2660-6003. URL: https://temat.es/monograficos/article/view/vol2-p211.

1. Introduction

In the context of Lagrangian mechanics, continuous (global) symmetries of physical systems emerge mathematically as actions of Lie groups on the configuration spaces. The reduction method consists of transferring the variational principle to the quotient space by the symmetry group, yielding the reduced equations when applied to specific Lagrangians. This has been thoroughly treated in the literature [2]. More recently, these ideas have been extended to field theories [1, 3].

The goal of the present contribution is to consider local symmetries of Lagrangian systems instead of global ones. In such case, the Lie group is replaced by an appropriate Lie group fiber bundle. Namely, we focus our attention on a particular type of symmetries whose derivatives are constrained to a prescribed Lie subalgebra. The ideas outlined here will be treated in detail and extended to more general symmetries in a forthcoming paper.

In the following, every manifold or map is assumed to be smooth in the sense of C^∞ unless otherwise stated. Likewise, every fiber bundle is assumed to be locally trivial. The superscript * will denote the dual bundle of the corresponding vector bundle.

2. Geometric setting

2.1. Actions of Lie group bundles and generalized principal connections

A Lie group bundle is a fiber bundle $\pi_{\mathcal{G},X}: \mathcal{G} \to X$ with typical fiber a Lie group G such that for any point $x \in X$ the fiber $\mathcal{G}_x = \pi_{\mathcal{G},X}^{-1}(\{x\})$ is equipped with a Lie group structure and there exist a neighborhood $\mathcal{U} \subset X$ and a diffeomorphism $\mathcal{U} \times G \to \pi_{\mathcal{G},X}^{-1}(\mathcal{U})$ preserving the Lie group structure fiberwisely. We denote by 1_x the identity element of $\mathcal{G}_x$ for each $x \in X$. Any Lie group bundle defines a Lie algebra bundle $\pi_{\mathfrak{g},X}: \mathfrak{g} \to X$ as the vector bundle whose fiber at each $x \in X$ is $\mathfrak{g}_x = T_{1_x}\mathcal{G}_x$, the Lie algebra of $\mathcal{G}_x$.

Definition 1. A *(right) fibered action* of a Lie group bundle $\mathcal{G} \to X$ on a fiber bundle $\pi_{Y,X}: Y \to X$ is a bundle morphism $\Phi: Y \times_X \mathcal{G} \to Y$ covering the identity id_X such that $\Phi(y, hg) = \Phi(\Phi(y,h), g)$ and $\Phi(y, 1_x) = y$, for all $(y,g), (y,h) \in Y \times_X \mathcal{G}$, $\pi_{Y,X}(y) = x$, where $\times_X$ denotes the fibered product. ◂

We denote the fibered action by $\Phi(y,g) = y \cdot g$ and the corresponding quotient by $Y/\mathcal{G}$. The action is said to be *free* if $y \cdot g = y$ for some $(y,g) \in Y \times_X \mathcal{G}$ implies that $g = 1_x$, $x = \pi_{Y,X}(y)$. In the same way, it is said to be *proper* if the bundle morphism $Y \times_X \mathcal{G} \ni (y,g) \mapsto (y, y\cdot g) \in Y \times_X Y$ is proper.

Proposition 2. *If $\mathcal{G} \to X$ acts on $Y \to X$ freely and properly, then $Y/\mathcal{G}$ admits a unique smooth structure such that $Y \to Y/\mathcal{G}$ is a fiber bundle with typical fiber G and $Y/\mathcal{G} \to X$ is a fibered manifold, i.e., a surjective submersion.*

Note that an Ehresmann connection (see [7]) on a Lie group bundle $\mathcal{G} \to X$ may be regarded as a map $\nu: T\mathcal{G} \to \mathfrak{g}$. It is natural to impose a compatibility of ν with the algebraic structure of $\mathcal{G}$.

Definition 3. A *Lie group bundle connection* on $\pi_{\mathcal{G},X}$ is an Ehresmann connection ν on $\pi_{\mathcal{G},X}$ satisfying:

(i) $\ker \nu|_{T_{1_x}\mathcal{G}} = (d1)_x(T_xX)$ for each $x \in X$, where $1: X \to \mathcal{G}$ is the unit section.

(ii) For every $(g,h) \in \mathcal{G} \times_X \mathcal{G}$ and $(U_g, U_h) \in T_g\mathcal{G} \times_{T_xX} T_h\mathcal{G}$, $x = \pi_{\mathcal{G},X}(g)$, then

$$\nu\left((dM)_{(g,h)}(U_g, U_h)\right) = \nu(U_g) + Ad_g\left(\nu(U_h)\right),$$

where $M: \mathcal{G} \times_X \mathcal{G} \to \mathcal{G}$ is the multiplication map and $Ad_g: \mathfrak{g}_x \to \mathfrak{g}_x$ is the adjoint representation. ◂

In the same vein, we ask connections on $Y \to Y/\mathcal{G}$ to be equivariant. Note the analogy with principal connections (see [6]).

Definition 4. Let ν be an Lie group bundle connection on $\mathcal{G} \to X$. A *generalized principal connection* on $Y \to Y/\mathcal{G}$ associated to ν is a 1-form $\omega \in \Omega^1(Y, \mathfrak{g})$ satisfying:

(i) (Complementarity) $\omega_y(\xi_y^*) = \xi$ for every $(y,\xi) \in Y \times_X \mathfrak{g}$, where $\xi_y^* = \left.\frac{d}{dt}\right|_{t=0} y \cdot \exp(t\xi)$ is the *infinitesimal generator* of ξ at y.

(ii) (Ad-equivariance) For each $(y,g) \in Y \times_X \mathcal{G}$ and $(U_y, U_g) \in T_yY \times_{T_xX} T_g\mathcal{G}$, $x = \pi_{Y,X}(x)$, then

$$\omega_{y\cdot g}\left((d\Phi)_{(y,g)}(U_y,U_g)\right) = Ad_{g^{-1}}\left(\omega_y(U_y) + \nu(U_g)\right). \qquad ◂$$

Recall that the *curvature* of ω (see for example [7, §9.4]) is the 2-form $\Omega \in \Omega^2(Y,\mathfrak{g})$ defined as

$$\Omega(U_1,U_2) = -\omega\left([U_1 - \omega(U_1)^*, U_2 - \omega(U_2)^*]\right), \qquad U_1, U_2 \in \mathfrak{X}(Y).$$

2.2. Geometry of the reduced configuration space

Let $\mathcal{G} \to X$ be a Lie group bundle. Then, $\pi_{J^1\mathcal{G},X} : J^1\mathcal{G} \to X$ is also a Lie group bundle (see [4, §3, Th. 1]). We take a *Lie group subbundle* of $\pi_{J^1\mathcal{G},X}$, that is, a Lie group bundle $\pi_{H,X} : H \to X$ such that

(i) H is a submanifold of $J^1\mathcal{G}$,

(ii) H_x is a Lie subgroup of $J^1_x\mathcal{G}$ for each $x \in X$.

We also assume that H_x is closed in $J^1_x\mathcal{G}$ for every $x \in X$, $\pi_{J^1\mathcal{G},\mathcal{G}}(H) = \mathcal{G}$ and $\pi_{H,\mathcal{G}}$ is an affine subbundle of $\pi_{J^1\mathcal{G},\mathcal{G}}$. A Lie group connection ν on $\mathcal{G} \to X$ gives an identification of the first jet bundle $J^1\mathcal{G}$ with the vector bundle modelling it, $J^1\mathcal{G} \simeq \mathcal{G} \times_X (T^*X \otimes \mathfrak{g})$. Under this identification, we suppose that $H = \mathcal{G} \times_X (T^*X \otimes \mathfrak{h})$, for certain Lie algebra subbundle $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h}_x$ is an ideal of $\mathfrak{g}_x$ for every $x \in X$.

It can be seen that $\mathcal{G} \times_X \mathfrak{h} \to X$ acts fiberwisely, freely and properly on the right on $Y \times_X \mathfrak{g} \to X$. Moreover, the corresponding quotient $ad_{\mathfrak{h}}(Y) = (Y \times_X \mathfrak{g})/(\mathcal{G} \times_X \mathfrak{h})$ is a vector bundle over $Y/\mathcal{G}$, whose elements are denoted by $[\![y,\xi]\!] \in ad_{\mathfrak{h}}(Y)$.

On the other hand, the first jet extension of the fibered action, i.e., $\Phi^1 : J^1Y \times_X J^1\mathcal{G} \to J^1Y$, turns out to be a right fibered action of $J^1\mathcal{G} \to X$ on $J^1Y \to X$ that can be restricted to H.

Theorem 5. *In the above conditions, let $\omega \in \Omega^1(Y,\mathfrak{g})$ be a generalized principal connection on $Y \to Y/\mathcal{G}$ associated to ν. Then, the following map is a fibered isomorphism over $Y/\mathcal{G}$:*

$$J^1Y/H \ni \left[j^1_x s\right]_H \longmapsto \left(j^1_x\left(\pi_{Y,Y/\mathcal{G}} \circ s\right), [\![s(x),(s^*\omega)_x]\!]\right) \in J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} \left(T^*X \otimes ad_{\mathfrak{h}}(Y)\right).$$

The connections ω and ν induce a linear connection $\nabla^{\mathfrak{h}}$ on $ad_{\mathfrak{h}}(Y) \to Y/\mathcal{G}$. Provided a linear connection ∇^X on TX, we get a linear connection ∇ on $T^*X \otimes ad_{\mathfrak{h}}(Y) \to Y/\mathcal{G}$. Likewise, a torsion free linear connection on $T(Y/\mathcal{G}) \to Y/\mathcal{G}$ projectable to ∇^X (see [5]) induces an affine connection on $J^1(Y/\mathcal{G}) \to Y/\mathcal{G}$. Hence, we obtain an affine connection on the reduced space.

3. Reduction of the variational principle

A *(first order) Lagrangian density* on a fiber bundle $Y \to X$ is a bundle morphism $\mathcal{L} : J^1Y \to \bigwedge^n T^*X$ covering the identity on X, where $n = \dim X$. Assuming that X is orientable and $v \in \Omega^n(X)$ is a volume form, we can write $\mathcal{L} = L\,v$ for certain $L : J^1Y \to \mathbb{R}$ called *Lagrangian*.

Let $\mathcal{G} \to X$ be a Lie group bundle acting freely and properly on $Y \to X$ and $H \subset J^1\mathcal{G}$ be a Lie subbundle as in Section 2.2. If the Lagrangian L is *H-invariant*, that is, $L\left(\Phi^1\left(j^1_x s, j^1_x\eta\right)\right) = L\left(j^1_x s\right)$ for each $\left(j^1_x s, j^1_x\eta\right) \in J^1Y \times_X H$, then the *reduced Lagrangian*, $l : J^1Y/H \to \mathbb{R}$, is well defined. Using a generalized principal connection, Theorem 5 enables us to regard l as defined on $J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} \left(T^*X \otimes ad_{\mathfrak{h}}(Y)\right)$.

The principle of stationary action used to obtain the Euler-Lagrange equations can be transferred to the reduced configuration space. When applied to l, the so-called reduced equations are obtained.

Theorem 6 (Reduced field equations). *Let $\mathcal{U} \subset X$ be an open set such that $\overline{\mathcal{U}}$ is compact. Let $s \in \Gamma\left(\overline{\mathcal{U}}, \pi_{Y,X}\right)$ and $\sigma_s = \pi_{Y,Y/\mathcal{G}} \circ s$, and consider the* reduced section $\overline{s} = [\![s, s^*\omega]\!]$. *Then, the following assertions are equivalent:*

(i) *The variational principle $\delta \int_{\mathcal{U}} L(j^1 s)\, v = 0$ holds for vertical variations of s such that $\delta s|_{\partial\mathcal{U}} = 0$.*

(ii) *The section s satisfies the Euler-Lagrange equations for L, i.e., $\mathcal{EL}(L)\left(j^1 s\right) = 0$ (see* [1, Section 2.4]*).*

(iii) *The variational principle $\delta \int_{\mathcal{U}} l(j^1\sigma_s, \overline{s})\, v = 0$ holds for variations of the form*

$$\left(\delta\overline{s}\right)^v = \overline{\nabla}^{\mathfrak{h}}\overline{\eta} - \left[\overline{s}, \overline{\eta}\right] + \sigma_s^* \widetilde{\Omega}\left(\delta\sigma_s, \cdot\right),$$

with $\overline{\eta} \in \Gamma\left(\overline{\mathcal{U}}, \pi_{ad_{\mathfrak{h}}(Y),X}\right)$ arbitrary such that $\pi_{ad_{\mathfrak{h}}(Y),Y/\mathcal{G}} \circ \overline{\eta} = \sigma_s$ and $\overline{\eta}|_{\partial\mathcal{U}} = 0$, and $\delta\sigma_s$ arbitrary vertical variation of σ_s such that $\delta\sigma_s|_{\partial\mathcal{U}} = 0$.

(iv) *The reduced section $\overline{s}$ satisfies the following* reduced field equations

$$\frac{\delta l}{\delta\sigma_s} - \operatorname{div}^{Y/\mathcal{G}}\left(\frac{\delta l}{\delta j^1\sigma_s}\right) = \left\langle \frac{\delta l}{\delta\overline{s}}, \widetilde{\Omega}\left(d\sigma_s, \cdot\right)\right\rangle, \qquad \operatorname{div}^{\mathfrak{h}}\left(\frac{\delta l}{\delta\overline{s}}\right) - \operatorname{ad}^*_{\overline{s}}\left(\frac{\delta l}{\delta\overline{s}}\right) = 0.$$

To conclude, let us define the objects that appear in the equations. First, $\widetilde{\Omega} \in \Omega^2(Y/\mathcal{G}, ad_{\mathfrak{h}}(Y))$ is the *reduced curvature* of ω, which is given by $\widetilde{\Omega}_{[y]_{\mathcal{G}}}(u_1, u_2) = [\![y, \Omega_y(U_1, U_2)]\!]$ for each $[y]_{\mathcal{G}} \in Y/\mathcal{G}$ and $u_1, u_2 \in T_{[y]_G}(Y/\mathcal{G})$, where $U_1, U_2 \in T_y Y$ project to u_1, u_2, respectively. On the other hand, $\operatorname{div}^{\mathfrak{h}}$ and $\operatorname{div}^{Y/\mathcal{G}}$ are the *divergence* of $\nabla^{\mathfrak{h}}$ and $\nabla^{Y/\mathcal{G}}$, respectively, that is, minus the adjoint of those linear connections. In the same manner, the coadjoint representation of $\mathfrak{g}$ is naturally extended to a map

$$\operatorname{ad}^* : \Gamma(T^*X \otimes ad_{\mathfrak{h}}(Y)) \times \Gamma(TX \otimes ad_{\mathfrak{h}}(Y)^*) \longrightarrow \Gamma(ad_{\mathfrak{h}}(Y)^*).$$

At last, the *partial derivatives* of the reduced Lagrangian are the sections

$$\frac{\delta l}{\delta\sigma_s} \in \Gamma\left(T^*(Y/\mathcal{G})\right), \quad \frac{\delta l}{\delta j^1\sigma_s} \in \Gamma\left(TX \otimes V^*(Y/\mathcal{G})\right), \quad \frac{\delta l}{\delta\overline{s}} \in \Gamma\left(TX \otimes ad_{\mathfrak{h}}(Y)^*\right).$$

The first one is the horizontal derivative using the affine connection on the reduced space, whereas the latter ones are fiber derivatives.

References

[1] Castrillón López, Marco and Ratiu, Tudor. "Reduction in Principal Bundles: Covariant Lagrange-Poincaré Equations". In: *Communications in Mathematical Physics* 236 (2003).

[2] Cendra, Hernan; Marsden, Jerrold, and Ratiu, Tudor. "Lagrangian Reduction by Stages". In: *Mem. Amer. Math. Soc. V* 152.722 (2001).

[3] Ellis, David C.P.; Gay-Balmaz, François; Holm, Darryl D., and Ratiu, Tudor S. "Lagrange-Poincaré field equations". In: *Journal of Geometry and Physics* 61.11 (2011), pp. 2120–2146. issn: 0393-0440.

[4] Forger, Michael and Soares, Bruno L. "Local symmetries in gauge theories in a finite-dimensional setting". In: *Journal of Geometry and Physics* 62.9 (2012), pp. 1925–1938. issn: 0393-0440.

[5] Janyška, Josef and Modugno, Marco. "Relations between linear connections on the tangent bundle and connections on the jet bundle of a fibred manifold". In: *Archivum Mathematicum* 032.4 (1996), pp. 281–288.

[6] Kobayashi, S. and Nomizu, K. *Foundations of Differential Geometry*. Foundations of Differential Geometry v. 1. Interscience Publishers, 1963. isbn: 9780470496480.

[7] Kolář, Ivan; Michor, Peter W., and Slovák, Jan. *Natural Operations in Differential Geometry*. 1st. Springer-Verlag Berlin Heidelberg, 1993. isbn: 978-3-662-02950-3.