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Implicit Nonholonomic Mechanics with Collisions ⋆

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Abstract: In this paper, variational techniques are used to analyze the dynamics of nonholonomic mechanical systems with impacts. Implicit nonholonomic smooth Lagrangian and Hamiltonian systems are extended to a nonsmooth context appropriate for collisions. In particular, we provide a variational formulation for implicit nonholonomic mechanical systems with collisions, for those collisions that preserve energy and momentum at the impact. Lastly, the theoretical results are illustrated by examining the example of a rolling disk hitting a wall.

Keywords: Implicit Lagrangian systems, Implicit Hamiltonian systems, Nonholonomic systems, Lagrange–d'Alembert–Pontryagin principle, Collision dynamics.

1. INTRODUCTION

A nonholonomic system is a mechanical system subject to constraint functions which are, roughly speaking, functions on the velocities that are not derivable from position constraints (see Neimark and Fufaev (1972)). They arise, for instance, in mechanical systems that have rolling or certain types of sliding contact. There are multiple applications in the context of wheeled motion, mobile robotics and robotic manipulation.

A geometrical formulation for mechanical systems with one-sided constraints was developed by Lacomba and Tulczyjew (see Lacomba and Tulczyjew (1990)). Ibort *et al.* studied the geometric aspects of Lagrangian systems subject to impulsive and one-sided constraints (Ibort et al. (1998)). This was extended to the Hamiltonian formalism by Cortés and Vinogradov (2006).

Mechanical systems subject to collisions are confined within a region of space with a boundary. Collision with the boundary for elastic impacts activates a constraint on the momentum and on the energy after and before the collision. The problem of collisions has been extensively treated in the literature since the early days of mechanics (see, for example, Ibort et al. (1997) or Brogliato (1999) and references therein for a comprenshive review). More recently, much work has been done on the rigorous mathematical foundation of impact problems (see Haddad et al. (2006) and Westervelt et al. (2018)). Nonholonomic systems subject to impacts or impulse effects have been previously studied in Clark and Bloch (2019) and Colombo et al. (2022).

In mechanics, implicit Lagrangian and Hamiltonian systems appear in controlled mechanical systems. An important class of implicit mechanical systems studied in Yoshimura and Marsden (2006b) are those with nonholonomic constraints. The aim of this paper is to take one step further and consider implicit mechanical systems subject to nonholonomic constraints and elastic collisions, which occurs when the nonholonomic system impacts the boundary of the configuration space under some suitable conditions. The goal of this paper is to provide a variational formulation for nonholonomic implicit mechanical systems with collisions. In particular, those collisions that preserve energy and momentum at the impact.

The remainder of the paper is structured as follows. Section 2 introduces nonholonomic systems from an explicit point of view via the Lagrange–d'Alembert principle and from an implicit point of view via the Lagrange–d'Alembert–Pontryagin principle. In Section 3 we define the configuration space and the phase space with the objective of introducing the action functional in Section 4, where we derive variationally, via the Hamilton–d'Alembert–Pontryagin principle, the equations of motion for implicit nonholonomic Lagrangian systems subject to elastic collisions. We extend the framework to the Hamiltonian side in Section 5, where we derive nonholonomic implicit Hamiltonian systems subject to collisions from a variational perspective. Finally, we study the vertical rolling disk hitting a wall in Section 6.

2. NONHOLONOMIC SYSTEMS

Let Q be a smooth manifold with $\dim(Q) = n$. Throughout the text, TQ and T^*Q will denote the tangent and the cotangent bundles of Q, respectively. Let $\mathcal{U} \subset Q$ be an open set that trivializes the tangent bundle, i.e., there exits a vector space V and an isomorphism of vector bundles

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$TQ|_{\mathcal{U}} \simeq \mathcal{U} \times V$. In turn, this induces a trivialization of the cotangent bundle: $T^*Q|_{\mathcal{U}} \simeq \mathcal{U} \times V^*$. For brevity, henceforth we will denote $\mathcal{U} = Q$.

A *distribution* Δ_Q on Q is a vector subspace $\Delta_Q(q)$ of T_qQ for each $q \in Q$. In addition, Δ_Q is smooth if for each $q \in Q$ there exist a neighborhood $\mathcal{U} \subset Q$ of q and smooth vector fields $X_1, \ldots, X_k \in \mathfrak{X}(\mathcal{U})$ such that $\Delta_Q(q) = \operatorname{span}\{X_1(q), \ldots, X_k(q)\}$ for each $q \in \mathcal{U}$. The *rank* of Δ_Q at $q \in Q$ is the dimension of the subspace $\Delta_Q(q)$, i.e., $\varrho : Q \to \mathbb{R}$, $\varrho(q) = \dim \Delta_Q(q)$. If ϱ is a constant function, then Δ_Q is said to be *regular*.

A *smooth regular codistribution* $\widetilde{\Delta_Q}$ on Q is a (locally trivial) vector subbundle of T^*Q. Let $\Delta_Q \subset TQ$ be a distribution, the *annihilator* of Δ_Q is the codistribution defined as

$$\Delta_Q^\circ(q) = \{\alpha \in T_q^*Q \mid \alpha(v) = \langle \alpha, v \rangle = 0, \ \forall v \in \Delta_Q(q)\}$$

for every $q \in Q$.

From an intrinsic point of view, *linear constraints* on the velocities of a mechanical system are defined by a regular distribution Δ_Q on Q.

Here we will restrict ourselves to the case of linear nonholonomic constraints. In this case, the constraints are given by a nonintegrable distribution Δ_Q. In addition to these constraints, we need to specify the dynamical evolution of the system by fixing a Lagrangian function $L\colon TQ \to \mathbb{R}$. The central concepts permitting the extension of mechanics from the Newtonian point of view to the Lagrangian one are the notions of virtual displacements and virtual work. These concepts were formulated in the developments of mechanics and in their application to statics. In nonholonomic dynamics, the procedure is given by the *Lagrange–d'Alembert principle*:

$$\delta \int_{t_0}^{t_1} L(q(t), \dot{q}(t))\, dt = 0,$$

where $\dot{q}(t) \in \Delta_Q(q(t))$ and the variations satisfy $\delta q(t) \in \Delta_Q(q(t)$ and vanish at the endpoints. This principle allows us to determine the set of possible values of the constraint forces from the set Δ_Q of admissible kinematic states alone. The resulting equations of motion are known as the *nonholonomic Euler–Lagrange equations* and are given by

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right] \cdot \delta q = 0.$$

An alternative approach consists of enlarging the phase space from TQ to the Pontryagin bundle $TQ \oplus T^*Q$, and considering the *Lagrange–d'Alembert–Pontryagin principle*:

$$\delta \int_{t_0}^{t_1} \left(L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t) \rangle\right) dt = 0,$$

where $v(t) \in \Delta_Q(q(t))$ and the variations $(\delta q(t), \delta v(t), \delta p(t))$ are such that $\delta q(t) \in \Delta_Q(q(t))$ and vanish at the endpoints. Then dynical equations for a stationary curve $(q(t), v(t), p(t))$ yield the so-called *implicit Lagrange–d'Alembert equations* on $TQ \oplus T^*Q$ (see Yoshimura and Marsden (2006b)):

$$p = \frac{\partial L}{\partial v}, \quad \dot{q} = v \in \Delta_Q(q), \quad \dot{p} - \frac{\partial L}{\partial q} \in \Delta_Q^\circ(q).$$

The forthcoming sections contain the main results of the paper and are devoted to generalize this latter situation to account for implicit nonholonomic systems undergoing elastic collisions. We also consider the Hamiltonian counterpart.

3. CONFIGURATION SPACE AND PHASE SPACE

Let Q be a smooth manifold with boundary, denoted by ∂Q, $L : TQ \to \mathbb{R}$ be a (possibly degenerate) Lagrangian, and $\Delta_Q \subset TQ$ be a (possibly nonholonomic) constraint distribution. According to Section 2, the annihilator of Δ_Q is denoted by $\Delta_Q^\circ \subset T^*Q$.

Given $[\tau_0, \tau_1] \subset \mathbb{R}$ and $\tilde{\tau} \in [\tau_0, \tau_1]$, the *path space with a unique collision (at $\tau = \tilde{\tau}$)* is defined as $\Omega(Q, \tilde{\tau}) = \mathcal{T} \times \mathcal{Q}(\tilde{\tau})$, where

$$\mathcal{T} = \{\alpha_T \in C^\infty([\tau_0, \tau_1]) \mid \alpha_T'(\tau) > 0, \ \tau \in [\tau_0, \tau_1]\}$$

and

$$\mathcal{Q}(\tilde{\tau}) = \big\{\alpha_Q \in C^0([\tau_0, \tau_1], Q) \mid \alpha_Q(\tilde{\tau}) \in \partial Q, \\ \alpha_Q \text{ is piecewise } C^2 \text{ and has only one singularity at } \tilde{\tau}\big\}. \tag{1}$$

We only consider one singularity at $\tau = \tilde{\tau}$ for brevity, but similar results hold for a finite amount of singularities, $\{\tilde{\tau}_i \mid 1 \leq i \leq N\} \subset [\tau_0, \tau_1]$.

Remark 1. Systems with collisions are a particular instance of hybrid systems. For systems with elastic impacts, the *guard* is given by $S = \{v_q \in T_qQ \mid q \in \partial Q, \ g(v_q, n_q) > 0\}$, where g is a Riemannian metric on Q and n is the outward-pointing, unit, normal vector field on the boundary. Similarly, the *reset map* is given by $R(v_q) = v_q^\parallel - v_q^\perp$, where $v_q^\perp = g(v_q, n_q)\, n_q$ and $v_q^\parallel = v_q - v_q^\perp \in T_q\partial Q$. Recall that hybrid systems may experience Zeno behaviour if a trajectory undergoes infinitely many impacts in finite time. In order to avoid this situation, we ask the system to satisfy two conditions (cf. (Goodman and Colombo, 2020, Remark 2.1)):

(i) $S \cap \overline{R}(S) = \emptyset$, where $\overline{R}(S)$ is the closure of $R(S) \subset TQ$. This condition is clearly satisfied in our case. Indeed, for each $v_q \in S$ we have $v_q^\perp \neq 0$ and, thus, $\parallel R(v_q) - v_q \parallel_g = 2 \parallel v_q^\perp \parallel_g > 0$, being $\parallel \cdot \parallel_g$ the norm induced by the metric g.

(ii) The set of collision times is closed and discrete. This condition, which depends on the topology of the configuration manifold, prevents the existence of an accumulation point and will be assumed in the following.

Under these assumptions, our development is valid in a neighborhood of each collision. Therefore, we can assume that the colission takes place in the interior of the interval, i.e., $\tilde{\tau} \in (\tau_0, \tau_1)$.

Lemma 1. (Fetecau et al., 2003, Corollary 2.3) $\Omega(Q, \tilde{\tau}) = \mathcal{T} \times \mathcal{Q}(\tilde{\tau})$ is a smooth manifold.

Remark 2. Given $\alpha_T \in \mathcal{T}$, we denote $[t_0, t_1] = \alpha_T([\tau_0, \tau_1])$ and, in order to distinguish between τ-derivatives and t-derivatives, we use different symbols; namely, $\alpha_T' = d\alpha_T/d\tau$ and $\dot{\alpha}_T^{-1} = d\alpha_T^{-1}/dt$, where $\alpha_T^{-1} : [t_0, t_1] \to [\tau_0, \tau_1]$ is the inverse of α_T. Analogously, we denote $\tilde{t} = \alpha_T(\tilde{\tau})$.

The tangent space of $\mathcal{Q}(\tilde{\tau})$ at $\alpha_Q \in \mathcal{Q}(\tilde{\tau})$ is given by

$$\begin{aligned} T_{\alpha_Q}\mathcal{Q}(\tilde{\tau}) = \Big\{ & \nu_{\alpha_Q} \in C^0([\tau_0,\tau_1], TQ) \mid \\ & \alpha_Q = \pi_{TQ}\circ\nu_{\alpha_Q},\ \nu_{\alpha_Q}(\tilde{\tau}) \in T_{\alpha_Q(\tilde{\tau})}\partial Q, \\ & \nu_{\alpha_Q} \text{ is piecewise } C^1 \text{ and has only one singularity at } \tilde{\tau}\Big\}, \end{aligned} \tag{2}$$

where $\pi_{TQ} : TQ \to Q$ is the natural projection. In order to incorporate the constraint distribution, we define the following subspace at each $\alpha_Q \in \mathcal{Q}(\tilde{\tau})$,

$$\Delta_{\mathcal{Q}(\tilde{\tau})}(\alpha_Q) = \left\{\nu_{\alpha_Q} \in T_{\alpha_Q}\mathcal{Q}(\tilde{\tau}) \mid \nu_{\alpha_Q} : [\tau_0,\tau_1] \to \Delta_Q\right\}.$$

As usual, we denote $T\mathcal{Q}(\tilde{\tau}) = \bigsqcup_{\alpha_Q\in\mathcal{Q}(\tilde{\tau})} T_{\alpha_Q}\mathcal{Q}(\tilde{\tau})$ and $\Delta_{\mathcal{Q}(\tilde{\tau})} = \bigsqcup_{\alpha_Q\in\mathcal{Q}(\tilde{\tau})} \Delta_{\mathcal{Q}(\tilde{\tau})}(\alpha_Q)$.

Let

$$\begin{aligned} T'_{\alpha_Q}\mathcal{Q}(\tilde{\tau}) = \{ & \phi_{\alpha_Q} : T_{\alpha_Q}\mathcal{Q}(\tilde{\tau}) \to \mathbb{R} \mid \\ & \phi_{\alpha_Q} \text{ is linear and continuous}\} \end{aligned}$$

be the topological dual of $T_{\alpha_Q}\mathcal{Q}(\tilde{\tau})$. Since $\mathcal{Q}(\tilde{\tau})$ is an infinite dimensional manifold, its topological cotangent bundle is too large to formulate mechanics. For that reason, we will restrict ourselves to an appropriate vector subbundle; namely, for each $\alpha_Q \in \mathcal{Q}(\tilde{\tau})$ we define

$$\begin{aligned} T^{\star}_{\alpha_Q}\mathcal{Q}(\tilde{\tau}) = \big\{ & \pi_{\alpha_Q} \in C^0([\tau_0,\tau_1], T^*Q) \mid \\ & \alpha_Q = \pi_{T^*Q}\circ\pi_{\alpha_Q},\ \pi_{\alpha_Q}(\tilde{\tau}) \in T^*_{\alpha_Q(\tau)}\partial Q, \\ & \pi_{\alpha_Q} \text{ is piecewise } C^1 \text{ and has only one singularity at } \tilde{\tau}\big\}, \end{aligned} \tag{3}$$

where $\pi_{T^*Q} : T^*Q \to Q$ is the natural projection. By construction, the image of the Legendre transform of the Lagrangian lie in this space, i.e., $\mathbb{F}L\circ\nu_{\alpha_Q} \in T^{\star}\mathcal{Q}(\tilde{\tau})$ for each $\nu_{\alpha_Q} \in T_{\alpha_Q}\mathcal{Q}(\tilde{\tau})$, where $\mathbb{F}L : TQ \to T^*Q$ is the Legendre transform of L. Nevertheless, in general

$$\left\{\mathbb{F}L\circ\nu_{\alpha_Q} \mid \nu_{\alpha_Q} \in T_{\alpha_Q}\mathcal{Q}(\tilde{\tau})\right\} \subsetneq T^{\star}_{\alpha_Q}\mathcal{Q}(\tilde{\tau}),$$

as the Lagrangian is possibly degenerate.

Lemma 2. For each $\alpha_Q \in \mathcal{Q}(\tilde{\tau})$, the vector space $T^{\star}_{\alpha_Q}\mathcal{Q}(\tilde{\tau})$ is a vector subspace of the topological dual of $T_{\alpha_Q}\mathcal{Q}(\tilde{\tau})$ by means of the following L^2-dual pairing:

$$\langle\pi_{\alpha_Q},\nu_{\alpha_Q}\rangle = \int_{\tau_0}^{\tau_1} \pi_{\alpha_Q}(\tau)\cdot\nu_{\alpha_Q}(\tau)\,d\tau,$$

where $\cdot$ represents the pairing between T^*Q and TQ. Furthermore, this pairing is nondegenerate.

As a straightforward consequence of the previous lemma, the vector bundle

$$T^{\star}\mathcal{Q}(\tilde{\tau}) = \bigsqcup_{\alpha_Q\in\mathcal{Q}(\tilde{\tau})} T^{\star}_{\alpha_Q}\mathcal{Q}(\tilde{\tau}) \to \mathcal{Q}(\tilde{\tau}), \quad \pi_{\alpha_Q} \mapsto \alpha_Q,$$

is a vector subbundle of the topological cotangent bundle of $\mathcal{Q}(\tilde{\tau})$.

In the same vein, for each $\nu_{\alpha_Q} \in T_{\alpha_Q}\mathcal{Q}(\tilde{\tau})$ and $\pi_{\alpha_Q} \in T^{\star}_{\alpha_Q}\mathcal{Q}(\tilde{\tau})$, the iterated bundles are given by

$$\begin{aligned} T_{\nu_{\alpha_Q}}(T\mathcal{Q}(\tilde{\tau})) = \Big\{ & \delta\nu_{\alpha_Q} \in C^0([\tau_0,\tau_1], T(TQ)) \mid \\ & \nu_{\alpha_Q} = \pi_{T(TQ)}\circ\delta\nu_{\alpha_Q},\ \delta\nu_{\alpha_Q}(\tilde{\tau}) \in T_{\nu_{\alpha_Q}(\tilde{\tau})}(T\partial Q)\Big\}, \end{aligned}$$

where $\pi_{T(TQ)} : T(TQ) \to TQ$ is the natural projection, and

$$\begin{aligned} T_{\pi_{\alpha_Q}}(T^{\star}\mathcal{Q}(\tilde{\tau})) = \Big\{ & \delta\pi_{\alpha_Q} \in C^0([\tau_0,\tau_1], T(T^*Q)) \mid \\ & \pi_{\alpha_Q} = \pi_{T(T^*Q)}\circ\delta\pi_{\alpha_Q},\ \delta\pi_{\alpha_Q}(\tilde{\tau}) \in T_{\pi_{\alpha_Q}(\tilde{\tau})}(T^*\partial Q)\Big\}, \end{aligned}$$

where $\pi_{T(T^*Q)} : T(T^*Q) \to T^*Q$ is the natural projection. In particular, we consider the constrained iterated bundle,

$$\begin{aligned} \Delta_{T\mathcal{Q}(\tilde{\tau})}(\nu_{\alpha_Q}) = \Big\{ & \delta\nu_{\alpha_Q} \in T_{\nu_{\alpha_Q}}(T\mathcal{Q}(\tilde{\tau})) \mid \\ & d\pi_{TQ}\circ\delta\nu_{\alpha_Q} \in C^0([\tau_0,\tau_1], \Delta_Q)\Big\}. \end{aligned} \tag{4}$$

4. NONHOLONOMIC IMPLICIT LAGRANGIAN MECHANICS WITH COLLISIONS

Given a path $\alpha = (\alpha_T, \alpha_Q) \in \Omega(Q,\tilde{\tau})$, the *associated curve* is defined as

$$q_\alpha : [t_0,t_1] \to Q, \quad t \mapsto q_\alpha(t) = \left(\alpha_Q\circ\alpha_T^{-1}\right)(t). \tag{5}$$

Similarly, given $\nu_{\alpha_Q} \in T_{\alpha_Q}\mathcal{Q}(\tilde{\tau})$ and $\pi_{\alpha_Q} \in T'_{\alpha_Q}\mathcal{Q}(\tilde{\tau})$, we set

$$\begin{aligned} v_\alpha &: [t_0,t_1] \to TQ, & t &\mapsto v_\alpha(t) = \left(\nu_{\alpha_Q}\circ\alpha_T^{-1}\right)(t), \\ p_\alpha &: [t_0,t_1] \to T^*Q, & t &\mapsto p_\alpha(t) = \left(\pi_{\alpha_Q}\circ\alpha_T^{-1}\right)(t). \end{aligned}$$

It is clear that $\pi_{TQ}\circ v_\alpha = \pi_{T^*Q}\circ p_\alpha = q_\alpha$.

By regarding $\Omega(Q,\tilde{\tau})$ as a trivial vector bundle over $\mathcal{Q}(\tilde{\tau})$ with the projection onto the second factor, the *Lagrange–d'Alembert–Pontryagin action functional*,

$$\mathbb{S} : \Omega(Q,\tilde{\tau}) \times_{\mathcal{Q}(\tilde{\tau})} (T\mathcal{Q}(\tilde{\tau}) \oplus T^{\star}\mathcal{Q}(\tilde{\tau})) \to \mathbb{R},$$

where $\times_{\mathcal{Q}(\tilde{\tau})}$ denotes the fibered product over $\mathcal{Q}(\tilde{\tau})$, is defined as

$$\begin{aligned} \mathbb{S}\left(\alpha,\nu_{\alpha_Q},\pi_{\alpha_Q}\right) &= \int_{t_0}^{t_1} \left(L(v_\alpha(t)) + p_\alpha(t)\cdot(\dot{q}_\alpha(t) - v_\alpha(t))\right) dt \\ &= \int_{\tau_0}^{\tau_1} \left(L\left(\nu_{\alpha_Q}(\tau)\right) + \pi_{\alpha_Q}(\tau)\cdot\left(\frac{\alpha_Q'(\tau)}{\alpha_T'(\tau)} - \nu_{\alpha_Q}(\tau)\right)\right)\alpha_T'(\tau)\,d\tau. \end{aligned}$$

The equality between the first and the second expressions can be easily checked by considering the change of variable $t = \alpha_T(\tau)$. By recalling that the *energy* of the system, $E : TQ\oplus T^*Q \to \mathbb{R}$, is given by

$$E(v_q,p_q) = p_q\cdot v_q - L(v_q), \quad (v_q,p_q) \in TQ\oplus T^*Q, \tag{6}$$

the action functional may be rewritten as

$$\begin{aligned} \mathbb{S}\left(\alpha,\nu_{\alpha_Q},\pi_{\alpha_Q}\right) &= \int_{t_0}^{t_1} \left(p_\alpha(t)\cdot\dot{q}_\alpha(t) - E(v_\alpha(t),p_\alpha(t))\right) dt \\ &= \int_{\tau_0}^{\tau_1} \left(\pi_{\alpha_Q}(\tau)\cdot\frac{\alpha_Q'(\tau)}{\alpha_T'(\tau)} - E\left(\nu_{\alpha_Q}(\tau),\pi_{\alpha_Q}(\tau)\right)\right)\alpha_T'(\tau)\,d\tau. \end{aligned}$$

Definition 1. (Hamilton–d'Alembert–Pontryagin principle). A path

$$\mathsf{c} = ((\alpha_T,\alpha_Q),\nu_{\alpha_Q},\pi_{\alpha_Q}) \in \Omega(Q,\tilde{\tau})\times_{\mathcal{Q}(\tilde{\tau})}\left(\Delta_{\mathcal{Q}(\tilde{\tau})}\oplus T^{\star}\mathcal{Q}(\tilde{\tau})\right)$$

is *stationary* (or *critical*) for the action functional $\mathbb{S}$ if it satisfies

$$d\mathbb{S}(\mathsf{c})(\delta\mathsf{c}) = 0,$$

for every variation $\delta\mathsf{c} = \left((\delta\alpha_T,\delta\alpha_Q),\delta\nu_{\alpha_Q},\delta\pi_{\alpha_Q}\right) \in T_\alpha\Omega(Q,\tilde{\tau}) \times \Delta_{T\mathcal{Q}(\tilde{\tau})}(\nu_{\alpha_Q}) \times T_{\pi_{\alpha_Q}}(T^{\star}\mathcal{Q}(\tilde{\tau}))$ such that $\delta\alpha_T(\tau_0) = \delta\alpha_T(\tau_1) = 0$, $\delta\alpha_Q(\tau_0) = \delta\alpha_Q(\tau_1) = 0$ and

$$d\pi_{TQ}\circ\delta\nu_{\alpha_Q} = d\pi_{T^*Q}\circ\delta\pi_{\alpha_Q} = \delta\alpha_Q. \tag{7}$$

Theorem 1. A (local) curve

$$\begin{aligned} & (\alpha,\nu_{\alpha_Q},\pi_{\alpha_Q}) \simeq \\ & (\alpha_T,\alpha_Q,\nu_Q,\pi_Q) \in \Omega(Q,\tilde{\tau}) \times_{\mathcal{Q}(\tilde{\tau})} \left(\Delta_{\mathcal{Q}(\tilde{\tau})}\oplus T^*\mathcal{Q}(\tilde{\tau})\right) \end{aligned}$$

is critical for the action functional $\mathbb{S}$ if and only if it satisfies the *implicit Euler–Lagrange equations*,

$$\begin{cases} \pi_Q' - \alpha_T'\dfrac{\partial L}{\partial q}(\alpha_Q,\nu_Q) \in \Delta_Q^\circ(\alpha_Q), & E'(\alpha_Q,\nu_Q,\pi_Q) = 0, \\ \pi_Q = \dfrac{\partial L}{\partial v}(\alpha_Q,\nu_Q), & \nu_Q = \dfrac{\alpha_Q'}{\alpha_T'} \in \Delta_Q(\alpha_Q), \end{cases}$$

on $[\tau_0,\tilde{\tau})\cup(\tilde{\tau},\tau_1]$, together with the conditions for the *elastic impact*,

$$\pi_Q(\tilde{\tau}^+)-\pi_Q(\tilde{\tau}^-)\in(T\partial Q\cap\Delta_Q)^\circ=(T\partial Q)^\circ+\Delta_Q^\circ,$$
$$E(\alpha_Q(\tilde{\tau}^-),\nu_Q(\tilde{\tau}^-),\pi_Q(\tilde{\tau}^-))=E(\alpha_Q(\tilde{\tau}^+),\nu_Q(\tilde{\tau}^+),\pi_Q(\tilde{\tau}^+)),$$

where the annihilitaros are with respect to TQ.

Proof: Let $TQ\simeq Q\times V$ be a trivialization of the tangent bundle of Q, and consider the induced trivializations of the cotangent bundle of Q, $T^*Q\simeq Q\times V^*$, as well as of the iterated bundles $T(TQ)\simeq Q\times V\times V\times V$ and $T(T^*Q)\simeq Q\times V^*\times V\times V^*$. Locally, we may write $\alpha'\simeq(\alpha_T,\alpha_Q,\alpha_T',\alpha_Q')$, $\nu_{\alpha_Q}\simeq(\alpha_Q,\nu_Q)$ and $\pi_{\alpha_Q}\simeq(\alpha_Q,\pi_Q)$ for some $\alpha_Q',\nu_Q:[\tau_0,\tau_1]\to V$ and $\pi_Q:[\tau_0,\tau_1]\to V^*$. Moreover, the variations locally read $\delta\alpha\simeq(\alpha_T,\alpha_Q,\delta\alpha_T,\delta\alpha_Q)$, $\delta\nu_{\alpha_Q}\simeq(\alpha_Q,\nu_Q,\beta_Q,\delta\nu_Q)$ and $\delta\pi_{\alpha_Q}\simeq(\alpha_Q,\pi_Q,\gamma_Q,\delta\pi_Q)$ for some $\delta\alpha_Q,\beta_Q,\gamma_Q,\delta\nu_Q:[\tau_0,\tau_1]\to V$, $\delta\pi_Q:[\tau_0,\tau_1]\to V^*$. In fact, equation (7) ensures that $\delta\alpha_Q=\beta_Q=\gamma_Q$. Moreover, by locally regarding $\Delta_Q(q)\subset V$ for each $q\in Q$, the conditions $\nu_{\alpha_Q}\in\Delta_{\mathcal{Q}(\tilde{\tau})}$ and $\delta\nu_{\alpha_Q}\in\Delta_{T\mathcal{Q}(\tilde{\tau})}(\nu_{\alpha_Q})$ read $\nu_Q(\tau)\in\Delta_Q(\alpha_Q(\tau))$ and $\delta\alpha_Q(\tau)\in\Delta_Q(\alpha_Q(\tau))$ for each $\tau\in[\tau_0,\tau_1]$, respectively. At last, the condition $\delta\nu_{\alpha_Q}(\tilde{\tau})\in T_{\nu_{\alpha_Q}(\tilde{\tau})}(T\partial Q)$ yields the local condition $\delta\alpha_Q(\tilde{\tau})\in W$, where $W\subset V$ is a subspace of co-dimension one such that $T\partial Q\simeq\partial Q\times W$.

As a result, the variation of the action functional reads

$$\begin{aligned}
&d\mathbb{S}(\alpha,\nu_{\alpha_Q},\pi_{\alpha_Q})\left(\delta\alpha,\delta\nu_{\alpha_Q},\delta\pi_{\alpha_Q}\right)\simeq\\
&d\mathbb{S}(\alpha_T,\alpha_Q,\nu_Q,\pi_Q)(\delta\alpha_T,\delta\alpha_Q,\delta\nu_Q,\delta\pi_Q)=\\
&\int_{\tau_0}^{\tau_1}\left(\frac{\partial L}{\partial q}\cdot\delta\alpha_Q+\frac{\partial L}{\partial v}\cdot\delta\nu_Q+\delta\pi_Q\cdot\left(\frac{\alpha_Q'}{\alpha_T'}-\nu_Q\right)\right.\\
&\qquad\left.+\pi_Q\cdot\left(\frac{\delta\alpha_Q'}{\alpha_T'}-\frac{\alpha_Q'\delta\alpha_T'}{(\alpha_T')^2}-\delta\nu_Q\right)\right)\alpha_T'\,d\tau\\
&\quad+\int_{\tau_0}^{\tau_1}\left(L+\pi_Q\cdot\left(\frac{\alpha_Q'}{\alpha_T'}-\nu_Q\right)\right)\delta\alpha_T'\,d\tau,
\end{aligned}$$

where the Lagrangian, as well as its partial derivatives, are evaluated at (α_Q,ν_Q). After splitting the integration domain, $[\tau_0,\tau_1]-\{\tilde{\tau}\}=[\tau_0,\tilde{\tau})\cup(\tilde{\tau},\tau_1]$, as well as integrating by parts on each sub-interval, we may rewrite the previous expression as

$$\begin{aligned}
&d\mathbb{S}(\alpha,\nu_{\alpha_Q},\pi_{\alpha_Q})\left(\delta\alpha,\delta\nu_{\alpha_Q},\delta\pi_{\alpha_Q}\right)\simeq\\
&\int_{\tau_0}^{\tilde{\tau}}\left(\left(\alpha_T'\frac{\partial L}{\partial q}-\pi_Q'\right)\cdot\delta\alpha_Q-\frac{d}{d\tau}(L-\pi_Q\cdot\nu_Q)\delta\alpha_T\right.\\
&\qquad\left.+\alpha_T'\left(\frac{\partial L}{\partial v}-\pi_Q\right)\cdot\delta\nu_Q+\delta\pi_Q\cdot(\alpha_Q'-\alpha_T'\nu_Q)\right)d\tau\\
&+\int_{\tilde{\tau}}^{\tau_1}\left(\left(\alpha_T'\frac{\partial L}{\partial q}-\pi_Q'\right)\cdot\delta\alpha_Q-\frac{d}{d\tau}(L-\pi_Q\cdot\nu_Q)\delta\alpha_T\right.\\
&\qquad\left.+\alpha_T'\left(\frac{\partial L}{\partial v}-\pi_Q\right)\cdot\delta\nu_Q+\delta\pi_Q\cdot(\alpha_Q'-\alpha_T'\nu_Q)\right)d\tau\\
&+\Big[\pi_Q\cdot\delta\alpha_Q+(L-\pi_Q\cdot\nu_Q)\delta\alpha_T\Big]_{\tau=\tau_0}^{\tau=\tilde{\tau}^-}\\
&+\Big[\pi_Q\cdot\delta\alpha_Q+(L-\pi_Q\cdot\nu_Q)\delta\alpha_T\Big]_{\tau=\tilde{\tau}^+}^{\tau=\tau_1}.
\end{aligned}$$

Since the previous expression vanishes for free variations $(\delta\alpha,\delta\nu_Q,\delta\pi_Q)$ such that $\delta\alpha_Q\in\Delta_Q(\alpha_Q)$, $\delta\alpha_T(\tau_0)=\delta\alpha_T(\tau_1)=0$ and $\delta\alpha_Q(\tau_0)=\delta\alpha_Q(\tau_1)=0$, we obtain the desired equations and impact conditions. □

By using the change of variable $t=\alpha_T(\tau)$, we have $\dot{q}_\alpha=\alpha_Q'/\alpha_T'$ and $\dot{p}_\alpha=\pi_Q'/\alpha_T'$. Then, the implicit Euler–Lagrange equations for a (local) curve

$$(v_\alpha,p_\alpha)\simeq(q_\alpha,v,p):[t_0,t_1]\to TQ\oplus T^*Q$$

take the form

$$\begin{cases}\dot{p}-\dfrac{\partial L}{\partial q}(q_\alpha,v)\in\Delta_Q^\circ(q_\alpha), & \dot{E}(q_\alpha,v,p)=0,\\[2mm] p=\dfrac{\partial L}{\partial v}(q_\alpha,v), & v=\dot{q}_\alpha\in\Delta_Q(q_\alpha),\end{cases}\tag{8}$$

on $\left[t_0,\tilde{t}\right)\cup\left(\tilde{t},t_1\right]$. Similarly, the conditions for the elastic impact read

$$p\left(\tilde{t}^+\right)-p\left(\tilde{t}^-\right)\in(T\partial Q\cap\Delta_Q)^\circ=(T\partial Q)^\circ+\Delta_Q^\circ,\tag{9}$$

$$\begin{aligned}&E\left(q_\alpha\left(\tilde{t}^-\right),v\left(\tilde{t}^-\right),p\left(\tilde{t}^-\right)\right)=E\left(q_\alpha\left(\tilde{t}^+\right),v\left(\tilde{t}^+\right),p\left(\tilde{t}^+\right)\right),\\ &v\left(\tilde{t}^+\right)=\dot{q}_\alpha\left(\tilde{t}^+\right)\in\Delta_Q.\end{aligned}\tag{10}$$

Energy balance: It may be shown that the conservation of the energy along the solutions, $\dot{E}(q_\alpha,v,p)=0$, is redundant, as it may be obtained from the remaining equations.

For unconstrained systems, i.e., $\Delta_Q=TQ$, the Hamilton–d'Alembert–Pontryagin principle reduces to the Hamilton–Pontryagin principle, and the implicit Euler–Lagrange equations of motion read as

$$\dot{p}=\frac{\partial L}{\partial q}(q_\alpha,v),\qquad p=\frac{\partial L}{\partial v}(q_\alpha,v),\qquad v=\dot{q}_\alpha.$$

5. NONHOLONOMIC IMPLICIT HAMILTONIAN MECHANICS WITH COLLISIONS

The results in the previous section may be obtained in the Hamiltonian side as well. Namely, given a (possibly degenerate) Hamiltonian, $H:T^*Q\to\mathbb{R}$, the *Hamilton–d'Alembert–Pontryagin action functional*,

$$\mathfrak{S}:\Omega(Q,\tilde{\tau})\times_{\mathcal{Q}(\tilde{\tau})}T^\star\mathcal{Q}(\tilde{\tau})\to\mathbb{R},$$

is defined as

$$\begin{aligned}\mathfrak{S}\left(\alpha,\pi_{\alpha_Q}\right)&=\int_{t_0}^{t_1}\left(p_\alpha(t)\cdot\dot{q}_\alpha(t)-H(p_\alpha(t))\right)dt\\ &=\int_{\tau_0}^{\tau_1}\left(\pi_{\alpha_Q}(\tau)\cdot\frac{\alpha_Q'(\tau)}{\alpha_T'(\tau)}-H\left(\pi_{\alpha_Q}(\tau)\right)\right)\alpha_T'(\tau)\,d\tau.\end{aligned}$$

Recall that from this point of view the *energy* of the system is simply given by the Hamiltonian.

Definition 2. (Variational principle in the phase space). A path

$$\mathsf{c}=((\alpha_T,\alpha_Q),\pi_{\alpha_Q})\in\Omega(Q,\tilde{\tau})\times_{\mathcal{Q}(\tilde{\tau})}T^\star\mathcal{Q}(\tilde{\tau})$$

such that $\alpha_Q'\in\Delta_{\mathcal{Q}(\tilde{\tau})}(\alpha_Q)$ is *stationary* (or *critical*) for the action functional $\mathfrak{S}$ if it satisfies $d\mathfrak{S}(\mathsf{c})(\delta\mathsf{c})=0$ for every variation $\delta\mathsf{c}=\left((\delta\alpha_T,\delta\alpha_Q),\delta\pi_{\alpha_Q}\right)\in T_\alpha\Omega(Q,\tilde{\tau})\times T_{\pi_{\alpha_Q}}(T^\star\mathcal{Q}(\tilde{\tau}))$ such that $\delta\alpha_T(\tau_0)=\delta\alpha_T(\tau_1)=0$, $\delta\alpha_Q(\tau_0)=\delta\alpha_Q(\tau_1)=0$ and

$$d\pi_{T^*Q}\circ\delta\pi_{\alpha_Q}=\delta\alpha_Q\in\Delta_{\mathcal{Q}(\tilde{\tau})}(\alpha_Q).\tag{11}$$

Theorem 2. A (local) curve

$$(\alpha,\pi_{\alpha_Q})\simeq(\alpha_T,\alpha_Q,\pi_Q)\in\Omega(Q,\tilde{\tau})\times_{\mathcal{Q}(\tilde{\tau})}T^\star\mathcal{Q}(\tilde{\tau})$$

such that $\alpha_Q'\in\Delta_{\mathcal{Q}(\tilde{\tau})}(\alpha_Q)$ is critical for the action functional $\mathfrak{S}$ if and only if it satisfies the *implicit Hamilton equations*,

$$\pi_Q' + \alpha_T' \frac{\partial H}{\partial q}(\alpha_Q, \pi_Q) \in \Delta_Q^\circ(\alpha_Q), \qquad H'(\alpha_Q, \pi_Q) = 0,$$
$$\frac{\alpha_Q'}{\alpha_T'} = \frac{\partial H}{\partial p}(\alpha_Q, \pi_Q) \in \Delta_Q(\alpha_Q),$$

on $[\tau_0, \tilde{\tau}) \cup (\tilde{\tau}, \tau_1]$, together with the conditions for the *elastic impact*,

$$\begin{cases} \pi_Q(\tilde{\tau}^+) - \pi_Q(\tilde{\tau}^+) \in (T\partial Q \cap \Delta_Q)^\circ = (T\partial Q)^\circ + \Delta_Q^\circ, \\ H(\alpha_Q(\tilde{\tau}^-), \pi_Q(\tilde{\tau}^-)) = H(\alpha_Q(\tilde{\tau}^+), \pi_Q(\tilde{\tau}^+)). \end{cases}$$

Proof: Let $TQ \simeq Q \times V$ be a trivialization of the tangent bundle of Q, and consider the induced trivializations of the cotangent bundle of Q, $T^*Q \simeq Q \times V^*$, as well as of the iterated bundle $T(T^*Q) \simeq Q \times V^* \times V \times V^*$. Locally, we may write $\alpha' \simeq (\alpha_T, \alpha_Q, \alpha_T', \alpha_Q')$ and $\pi_{\alpha_Q} \simeq (\alpha_Q, \pi_Q)$ for some $\alpha_Q' : [\tau_0, \tau_1] \to V$ and $\pi_Q : [\tau_0, \tau_1] \to V^*$. Moreover, the variations locally read $\delta\alpha \simeq (\alpha_T, \alpha_Q, \delta\alpha_T, \delta\alpha_Q)$ and $\delta\pi_{\alpha_Q} \simeq (\alpha_Q, \pi_Q, \gamma_Q, \delta\pi_Q)$ for some $\delta\alpha_Q, \gamma_Q : [\tau_0, \tau_1] \to V$ and $\delta\pi_Q : [\tau_0, \tau_1] \to V^*$. In fact, equation (11) ensures that $\delta\alpha_Q = \gamma_Q \in \Delta_Q(\alpha_Q)$. Moreover, we have $\delta\alpha_Q(\tilde{\tau}) \in W$, where $W \subset V$ is a subspace of co-dimension one such that $T\partial Q \simeq \partial Q \times W$.

As a result, the variation of the action functional reads

$$\begin{aligned} & d\mathfrak{S}(\alpha, \pi_{\alpha_Q})\left(\delta\alpha, \delta\pi_{\alpha_Q}\right) \simeq \\ & d\mathfrak{S}(\alpha_T, \alpha_Q, \pi_Q)(\delta\alpha_T, \delta\alpha_Q, \delta\pi_Q) = \\ & \int_{\tau_0}^{\tau_1} \left(\delta\pi_Q \cdot \frac{\alpha_Q'}{\alpha_T'} + \pi_Q \cdot \left(\frac{\delta\alpha_Q'}{\alpha_T'} - \frac{\alpha_Q' \delta\alpha_T'}{(\alpha_T')^2} \right) \right. \\ & \qquad \left. - \frac{\partial H}{\partial q} \cdot \delta\alpha_Q - \frac{\partial H}{\partial p} \cdot \delta\pi_Q \right) \alpha_T' \, d\tau \\ & + \int_{\tau_0}^{\tau_1} \left(\pi_Q \cdot \frac{\alpha_Q'}{\alpha_T'} - H \right) \delta\alpha_T' \, d\tau, \end{aligned}$$

where the Hamiltonian, as well as its partial derivatives, are evaluated at (α_Q, π_Q). After splitting the integration domain, $[\tau_0, \tau_1] - \{\tilde{\tau}\} = [\tau_0, \tilde{\tau}) \cup (\tilde{\tau}, \tau_1]$, as well as integrating by parts on each sub-interval, we may rewrite the previous expression as

$$\begin{aligned} & d\mathfrak{S}(\alpha, \pi_{\alpha_Q})\left(\delta\alpha, \delta\pi_{\alpha_Q}\right) \simeq \\ & \int_{\tau_0}^{\tilde{\tau}} \left(\left(-\pi_Q' - \alpha_T' \frac{\partial H}{\partial q} \right) \cdot \delta\alpha_Q + H' \delta\alpha_T \right. \\ & \qquad \left. + \delta\pi_Q \cdot \left(\alpha_Q' - \alpha_T' \frac{\partial H}{\partial p} \right) \right) d\tau \\ & + \int_{\tilde{\tau}}^{\tau_1} \left(\left(-\pi_Q' - \alpha_T' \frac{\partial H}{\partial q} \right) \cdot \delta\alpha_Q + H' \delta\alpha_T \right. \\ & \qquad \left. + \delta\pi_Q \cdot \left(\alpha_Q' - \alpha_T' \frac{\partial H}{\partial p} \right) \right) d\tau \\ & + \Big[\pi_Q \cdot \delta\alpha_Q - H \delta\alpha_T \Big]_{\tau=\tau_0}^{\tau=\tilde{\tau}^-} + \Big[\pi_Q \cdot \delta\alpha_Q - H \delta\alpha_T \Big]_{\tau=\tilde{\tau}^+}^{\tau=\tau_1}. \end{aligned}$$

Since the previous expression vanishes for free variations $(\delta\alpha, \delta\pi_Q)$ such that $\delta\alpha_Q \in \Delta_Q(\alpha_Q)$, $\delta\alpha_T(\tau_0) = \delta\alpha_T(\tau_1) = 0$ and $\delta\alpha_Q(\tau_0) = \delta\alpha_Q(\tau_1) = 0$, we obtain the desired equations and impact conditions. □

As for the Lagrangian equations, by means of the change of variable $t = \alpha_T(\tau)$, the implicit Hamilton equations for a (local) curve

$$p_\alpha \simeq (q_\alpha, p) : [t_0, t_1] \to T^*Q$$

take the form

$$\dot{p} + \frac{\partial H}{\partial q}(q_\alpha, p) \in \Delta_Q^\circ(q_\alpha), \qquad \dot{H}(q_\alpha, p) = 0,$$
$$\dot{q}_\alpha = \frac{\partial H}{\partial p}(q_\alpha, p) \in \Delta_Q(q_\alpha),$$

on $\left[t_0, \tilde{t}\right) \cup \left(\tilde{t}, t_1\right]$. Similarly, the conditions for the elastic impact read

$$\begin{aligned} & p\left(\tilde{t}^+\right) - p\left(\tilde{t}^-\right) \in (T\partial Q \cap \Delta_Q)^\circ = (T\partial Q)^\circ + \Delta_Q^\circ, \\ & H\left(q_\alpha\left(\tilde{t}^-\right), p\left(\tilde{t}^-\right)\right) = H\left(q_\alpha\left(\tilde{t}^+\right), p\left(\tilde{t}^+\right)\right) \\ & \dot{q}_\alpha\left(\tilde{t}^+\right) \in \Delta_Q. \end{aligned}$$

Energy balance: It may be shown that the conservation of the energy along the solutions, $\dot{H}(q_\alpha, p) = 0$, is redundant, as it may be obtained from the remaining equations.

For unconstrained systems, i.e., $\Delta_Q = TQ$, the Hamilton–d'Alembert–Pontryagin principle in the phase space reduces to the Hamilton–Pontryagin principle in the phase space, and the implicit Hamilton equations of motion read as

$$\dot{p} = -\frac{\partial H}{\partial q}(q_\alpha, p), \qquad \dot{q}_\alpha = \frac{\partial H}{\partial p}(q_\alpha, p).$$

Hyperregular Lagrangians: When $L : TQ \to \mathbb{R}$ is a hyperregular Lagrangian, i.e., when the Legendre transform $\mathbb{F}L : TQ \to T^*Q$ defines an isomorphism, then both the Lagrangian and the Hamiltonian approaches are equivalent. Namely, the Lagrangian L induces the Hamiltomian $H : T^*Q \to \mathbb{R}$ given by

$$H(p_q) = E\left((\mathbb{F}L)^{-1}(p_q), p_q\right), \quad p_q \in T^*Q.$$

By recalling the local expression of the Legendre transform,

$$\mathbb{F}L(q, v) = \left(q, \frac{\partial L}{\partial v}(q, v)\right),$$

it is easy to check that the implicit Euler–Lagrange equations together with the conditions for the elastic impact hold for a (local) curve $(v_\alpha, p_\alpha) : [t_0, t_1] \to TQ \oplus T^*Q$ if and only if $p_\alpha = \mathbb{F}L(v_\alpha)$ and the implicit Hamilton equations together with the conditions for the elastic impact hold for the (local) curve $p_\alpha = \mathbb{F}L(v_\alpha) : [t_0, t_q] \to T^*Q$.

6. ROLLING DISK HITTING A WALL

Let us consider a disk rolling without slipping, as in (Yoshimura and Marsden, 2006a, Section 7.1). However, here we assume that there is a wall that the disk may hit (see Anahory Simoes and Colombo (2023)). The configuration space is thus given by

$$Q = \{(x, y, \theta, \varphi) \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1 \mid y + R \sin\varphi \leq 10\},$$

where (x, y) denotes the contact point of the disk with the ground, θ denotes the angle of rotation and φ denotes the heading angle of the disk with respect to the x-axis. The Lagrangian $L : TQ \to \mathbb{R}$ is given by

$$L(x, y, \theta, \varphi; v_x, v_y, v_\theta, v_\varphi) = \frac{1}{2} m\left(v_x^2 + v_y^2\right) + \frac{1}{2}\left(I\, v_\theta^2 + J\, v_\varphi^2\right),$$

where $m, I, J \in \mathbb{R}^+$ are the mass and the moments of inertia of the disk, respectively. For each $(v_q, p_q) = (x, y, \theta, \varphi; v_x, v_y, v_\theta, v_\varphi; p_x, p_y, p_\theta, p_\varphi) \in TQ \oplus T^*Q$, the energy reads

$$E(v_q,p_q) = p_x\, v_x + p_y\, v_y + p_\theta\, v_\theta + p_\varphi\, v_\varphi - \frac{1}{2} m\left(v_x^2+v_y^2\right) - \frac{1}{2}\left(I\, v_\theta^2 + J v_\varphi^2\right).$$

Non-holonomic constraint: The non-slipping condition reads $v_x = R\, v_\theta \cos\varphi$, $v_y = R\, v_\theta \sin\varphi$, where $R \in \mathbb{R}^+$ is the radius of the disk, thus yielding following non-holonomic constraint:

$$\Delta_Q = \operatorname{span}\{\partial_\theta + R\cos\varphi\,\partial_x + R\sin\varphi\,\partial_y, \partial_\varphi\}.$$

The annihilator is easily seen to be

$$\Delta_Q^\circ = \operatorname{span}\{dx - R\cos\varphi\, d\theta, dy - R\sin\varphi\, d\theta\}.$$

On the other hand, the boundary of the configuration manifold is given by

$$\partial Q = \{(x,y,\theta,\varphi) \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1 \mid y + R\sin\varphi = 10\},$$

whose tangent bundle reads

$$T\partial Q = \operatorname{span}\{\partial_x, \partial_\theta, \partial_\varphi - R\cos\varphi\,\partial_y\}.$$

Therefore, its annihilator reads

$$(T\partial Q)^\circ = \operatorname{span}\{dy + R\cos\varphi\, d\varphi\}.$$

Dynamical equations: The implicit Euler–Lagrange equations with collisions for a curve

$(x,y,\theta,\varphi; v_x,v_y,v_\theta,v_\varphi; p_x,p_y,p_\theta,p_\varphi) : [t_0,t_1] \to TQ \oplus T^*Q$

given in (8) read

$$\begin{cases} R\,\dot p_x \cos\varphi + R\,\dot p_y \sin\varphi + \dot p_\theta = 0, & \dot p_\varphi = 0, \\ v_x = R\, v_\theta \cos\varphi, & v_y = R\, v_\theta \sin\varphi, \\ p_x = m\, v_x, & p_y = m\, v_y, \\ p_\theta = I\, v_\theta, & p_\varphi = J\, v_\varphi, \\ v_x = \dot x, & v_y = \dot y, \\ v_\theta = \dot\theta, & v_\varphi = \dot\varphi, \end{cases}$$

on $[t_0,t_1] - \{\tilde t\}$.

Conditions for the impact: The impact condition at $t = \tilde t$ given in (9) reads

$$\begin{cases} p_x^+ - p_x^- = \lambda^1, \\ p_y^+ - p_y^- = \lambda^0 + \lambda^2, \\ p_\theta^+ - p_\theta^- = -\lambda^1\, R\cos\varphi - \lambda^2\, R\sin\varphi, \\ p_\varphi^+ - p_\varphi^- = \lambda^0\, R\cos\varphi, \end{cases}$$

where we denote $p_x^+ = p_x\left(\tilde t^+\right)$, etc., and $\lambda^0, \lambda^1, \lambda^2 \in \mathbb{R}$ are the Lagrange multipliers. Similarly, the condition (10) reads

$$\begin{cases} v_x^+ = \lambda^3\, R\cos\varphi, & v_\theta^+ = \lambda^3, \\ v_y^+ = \lambda^3\, R\sin\varphi, & v_\varphi^+ = \lambda^4 \end{cases}$$

where $v_x^+ = v_x\left(\tilde t^+\right)$, etc., and $\lambda^3, \lambda^4 \in \mathbb{R}$ are the Lagrange multipliers.

For instance, when the disk hits the wall orthogonally, i.e., when $\varphi\left(\tilde t\right) = \pi/2$, the only admissible solution of the impact equations is

$$p_x^+ = p_x^- = 0, \quad p_y^+ = -p_y^-, \quad p_\theta^+ = -p_\theta^-, \quad p_\varphi^+ = p_\varphi^-.$$

For future work, we will consider other examples with degenerate Lagrangian, such as LC circuits.

REFERENCES

Anahory Simoes, A. and Colombo, L. (2023). Hamel equations and quasivelocities for nonholonomic systems with inequality constraints. *arXiv preprint arXiv:2303.17920.*

Brogliato, B. (1999). *Nonsmooth mechanics*, volume 3. Springer.

Clark, W. and Bloch, A. (2019). The bouncing penny and nonholonomic impacts. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, 2114–2119. IEEE.

Colombo, L., de León, M., Irazú, M.E.E., and López-Gordón, A. (2022). Geometric hamilton-jacobi theory and integrability for nonholonomic and forced hybrid systems. *arXiv preprint arXiv:2211.06252.*

Cortés, J. and Vinogradov, A.M. (2006). Hamiltonian theory of constrained impulsive motion. *Journal of mathematical physics*, 47(4).

Fetecau, R.C., Marsden, J.E., Ortiz, M., and West, M. (2003). Nonsmooth lagrangian mechanics and variational collision integrators. *SIAM Journal on Applied Dynamical Systems*, 2(3), 381–416. doi:10.1137/S1111111102406038. URL `https://doi.org/10.1137/S1111111102406038`.

Goodman, J.R. and Colombo, L.J. (2020). On the existence and uniqueness of poincaré maps for systems with impulse effects. *IEEE Transactions on Automatic Control*, 65(4), 1815–1821. doi:10.1109/TAC.2019.2941446.

Haddad, W.M., Chellaboina, V., and Nersesov, S.G. (2006). *Impulsive and hybrid dynamical systems: stability, dissipativity, and control.* Princeton University Press.

Ibort, A., de León, M., Lacomba, E.A., Marrero, J.C., de Diego, D.M., and Pitanga, P. (1998). Geometric formulation of mechanical systems subjected to time-dependent one-sided constraints. *Journal of Physics A: Mathematical and General*, 31(11), 2655.

Ibort, A., de León, M., Lacomba, E.A., de Diego, D.M., and Pitanga, P. (1997). Mechanical systems subjected to impulsive constraints. *Journal of Physics A: Mathematical and General*, 30(16), 5835. doi:10.1088/0305-4470/30/16/024. URL `https://dx.doi.org/10.1088/0305-4470/30/16/024`.

Lacomba, E. and Tulczyjew, W. (1990). Geometric formulation of mechanical systems with one-sided constraints. *Journal of Physics A: Mathematical and General*, 23(13), 2801.

Neimark, J. and Fufaev, R.G. (1972). *Dynamics of Nonholonomic Systems*, volume 33 of *Translations of Mathematical Monographs.* American Mathematical Society.

Westervelt, E.R., Grizzle, J.W., Chevallereau, C., Choi, J.H., and Morris, B. (2018). *Feedback control of dynamic bipedal robot locomotion.* CRC press.

Yoshimura, H. and Marsden, J. (2006a). Dirac structures in Lagrangian mechanics Part I: Implicit Lagrangian systems. *J. Geom. Phys.*, 57(1), 133–156.

Yoshimura, H. and Marsden, J.E. (2006b). Dirac structures in lagrangian mechanics part ii: Variational structures. *Journal of Geometry and Physics*, 57(1), 209–250.