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Infinite Dimensional Lagrange–Dirac Mechanics with Boundary Conditions*

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Abstract. The Lagrange–Dirac theory is extended to systems defined on the family of smooth functions on a manifold with boundary, which provides an instance of systems with a Fréchet space as a configuration space. To that end, we introduce the restricted cotangent bundle, a vector subbundle of the topological cotangent bundle which contains the partial derivatives of Lagrangian functions defined through a density. The main achievement of our proposal is that the Lagrange–Dirac equations on the restricted cotangent bundle properly account for the boundary value problem, i.e., the boundary conditions do not need to be imposed *ad hoc*, but they arise naturally from the Lagrange–Dirac formulation. After giving the main theoretical results, and showing how boundary forces can be naturally included in the Lagrange–Dirac formulation, we illustrate our framework with the dynamical equations of a vibrating membrane.

Keywords: Boundary problem · Lagrange–Dirac mechanics · Fréchet space.

1 Introduction

Lagrange–Dirac dynamical systems, which were introduced by Yoshimura and Marsden in [15], are based on the use of Dirac structures [2] in conjunction

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with the Lagrangian formalism. The main advantage of the Lagrange–Dirac approach to dynamical systems is that it provides a unified geometric formulation of systems that are both degenerate and nonholonomic, and that it admits an associated variational formulation. More recently, the Lagrange–Dirac formulation has been extended to a number of situations, including reduction by symmetries [16,6], interconnection [7] and nonequilibrium thermodynamics [4,5]. In addition, different discretizations have been proposed, as in [1] or [9]. Nevertheless, there exists a considerable gap in the literature about infinite dimensional systems from the Lagrange–Dirac point of view.

The aim of this contribution is to propose a Lagrange–Dirac theory for systems defined on the family of smooth functions on a domain with boundary. This family is an instance of an infinite dimensional Fréchet space [11,13]. When the Lagrangian is defined through a density, its partial derivatives lie in a vector subspace of the topological dual space, which will be called the restricted dual. The restricted dual gathers information about the behaviour of the system in both the interior and the boundary of the domain. As a result, the Lagrange–Dirac equations thus obtained incorporate the boundary conditions of the dynamical system, as well as boundary and interior forces, in a natural way. This is illustrated by computing the Lagrange–Dirac equations of a vibrating membrane with free boundary, which leads to the Neumann boundary conditions.

This work constitutes a first step towards a comprehensive Lagrange–Dirac theory for infinite dimensional systems and, more specifically, for systems with a Fréchet manifold as a configuration space. Such a theory would be highly desirable, since it will encompass many physical situations, such as waves, fluid dynamics or continuum mechanics [10].

2 Configuration space and induced Dirac structure

Let $\mathcal{B} \subset \mathbb{R}^m$ be a closed, bounded domain with smooth boundary. In this work, we consider continuum systems with configuration space given by the Fréchet space of smooth functions on $\mathcal{B}$, i.e.,

$$V = C^\infty(\mathcal{B}).$$

2.1 Restricted cotangent bundle

As we will see in the next section, the fiber derivatives of Lagrangians defined through a density lie in some subspace of the topological dual V' of V. For this reason, it is convenient to introduce the *restricted dual space*

$$V^\star = C^\infty(\mathcal{B}) \times C^\infty(\partial\mathcal{B}).$$

We regard $V^\star$ as a vector subspace of V' by means of the L^2-dual pairing given by

$$\langle(\alpha, \alpha_\partial), \varphi\rangle = \int_{\mathcal{B}} \alpha\varphi \,\mathrm{d}x + \int_{\partial\mathcal{B}} \alpha_\partial \varphi \,\mathrm{d}s, \qquad (\alpha, \alpha_\partial) \in V^\star,\ \varphi \in V,$$

where $\mathrm{d}x$ and $\mathrm{d}s$ denote the volume form on $\mathcal{B}$ and the boundary element on $\partial\mathcal{B}$, respectively. Note that this pairing is weakly non-degenerate, i.e., $\langle(\alpha,\alpha_\partial),\varphi\rangle = 0$ for all $\varphi \in V$ implies $(\alpha,\alpha_\partial) = 0$, and $\langle(\alpha,\alpha_\partial),\varphi\rangle = 0$ for all $(\alpha,\alpha_\partial) \in V^\star$ implies $\varphi = 0$. Such a pair $(V, V^\star)$ is referred to as a dual system [11, Chapter 23].

Definition 1. *The* ***restricted cotangent bundle*** *of* V *is given by* $T^\star V = V \times V^\star$.

As a consequence of $V^\star \subset V'$, we observe that the restricted cotangent bundle, $T^\star V$, is naturally included in the topological cotangent bundle, $T'V = V \times V'$. By following the same idea, we define the restricted iterated bundles:

$$\begin{aligned} T^\star(TV) = V \times V \times V^\star \times V^\star \subset T'(TV) &= V \times V \times V' \times V',\\ T(T^\star V) = V \times V^\star \times V \times V^\star \subset T(T'V) &= V \times V' \times V \times V',\\ T^\star(T^\star V) = V \times V^\star \times V^\star \times V \subset T'(T^\star V) &= V \times V^\star \times V' \times (V^\star)'. \end{aligned}$$

In the last equation, $(V^\star)'$ denotes the topological dual of $V^\star$ and the inclusion $V \subset (V^\star)'$ is understood again by means of the L^2-pairing (cf. [11, Chapter 23]). From the last equation, we also note that we choose to define $(V^\star)^\star = V$. While a more general setting is possible, this choice is enough for our purpose.

Definition 2. *The* ***restricted Pontryagin bundle*** *of* $T^\star V$ *is given by*

$$T(T^\star V) \oplus T^\star(T^\star V) = V \times V^\star \times (V \times V^\star \times V^\star \times V).$$

Of course, the restricted Pontryagin bundle is naturally included in the topological Pontryagin bundle $T(T^\star V) \oplus T'(T^\star V)$ of $T^\star V$.

2.2 Restricted Tulczyjew triple

Following its general definition on cotangent bundles, the *canonical one-form* $\Theta_{T^\star V} \in \Omega^1(T^\star V)$ is given by

$$\Theta_{T^\star V}(z) \cdot \delta z = \langle z, T_z\pi(\delta z)\rangle, \qquad z \in T^\star V,\ \delta z \in T_z(T^\star V),$$

where $\pi : T^\star V \to V$ is the (restricted) cotangent bundle projection and $T\pi : T(T^\star V) \to TV$ is the tangent map. In our case, by using $z = (\varphi, \alpha, \alpha_\partial)$ and $\delta z = (\delta\varphi, \delta\alpha, \delta\alpha_\partial)$ one gets the expression

$$\Theta_{T^\star V}(\varphi, \alpha, \alpha_\partial) \cdot (\delta\varphi, \delta\alpha, \delta\alpha_\partial) = \int_{\mathcal{B}} \alpha\delta\varphi\,\mathrm{d}x + \int_{\partial\mathcal{B}} \alpha_\partial\delta\varphi\,\mathrm{d}s.$$

The *canonical symplectic form* on the restricted cotangent bundle is the two-form

$$\Omega_{T^\star V} = -\mathrm{d}\Theta_{T^\star V} \in \Omega^2(T^\star V),$$

giving

$$\begin{aligned}&\Omega_{T^\star V}(\varphi,\alpha,\alpha_\partial)((\dot\varphi,\dot\alpha,\dot\alpha_\partial),(\delta\varphi,\delta\alpha,\delta\alpha_\partial))\\&=\langle(\delta\alpha,\delta\alpha_\partial),\dot\varphi\rangle-\langle(\dot\alpha,\dot\alpha_\partial),\delta\varphi\rangle\\&=\int_{\mathcal{B}}\delta\alpha\dot\varphi\,\mathrm{d}x+\int_{\partial\mathcal{B}}\delta\alpha_\partial\dot\varphi\,\mathrm{d}s-\int_{\mathcal{B}}\dot\alpha\delta\varphi\,\mathrm{d}x-\int_{\partial\mathcal{B}}\dot\alpha_\partial\delta\varphi\,\mathrm{d}s\end{aligned}$$

for each $(\varphi,\alpha,\alpha_\partial)\in T^\star V$ and $(\dot\varphi,\dot\alpha,\dot\alpha_\partial),(\delta\varphi,\delta\alpha,\delta\alpha_\partial)\in T_{(\varphi,\alpha,\alpha_\partial)}(T^\star V)$. A straightforward computation leads to the following result.

Proposition 1. *The flat map of the canonical symplectic form,*

$$\Omega^\flat_{T^\star V}:T(T^\star V)\to T'(T^\star V),\ \ (\varphi,\alpha,\alpha_\partial;\dot\varphi,\dot\alpha,\dot\alpha_\partial)\mapsto(\varphi,\alpha,\alpha_\partial;-\dot\alpha,-\dot\alpha_\partial,\dot\varphi)$$

takes values in the restricted iterated bundle, $T^\star(T^\star V)$. Furthermore, it defines a vector bundle isomorphism over the identity, $\mathrm{id}_{T^\star V}$, between $T(T^\star V)$ and the restricted iterated bundle, $T^\star(T^\star V)$.

Remark 1. Observe that the inclusion $T^\star(T^\star V)\subset T'(T^\star V)$ is strict. Therefore, the canonical symplectic form, $\Omega_{T^\star V}$, is weak, since it does not define an isomorphism between $T(T^\star V)$ and $T'(T^\star V)$. If we confine ourselves to the restricted iterated bundle, then it becomes a strong form. In the following, we focus on the latter situation without further mention.

By mimicking the finite dimensional case, see [15], the following isomorphism over the identity, id_V, is defined

$$\kappa_{T^\star V}:T(T^\star V)\to T^\star(TV),\ \ (\varphi,\alpha,\alpha_\partial;\dot\varphi,\dot\alpha,\dot\alpha_\partial)\mapsto(\varphi,\dot\varphi;\dot\alpha,\dot\alpha_\partial,\alpha,\alpha_\partial).$$

In the same vein, we set

$$\gamma_{T^\star V}=\Omega^\flat_{T^\star V}\circ\kappa^{-1}_{T^\star V}:T^\star(TV)\to T^\star(T^\star V).$$

By gathering the previous isomorphisms, we obtain the *restricted Tulczyjew triple* in the family of smooth functions as follows:

$$\begin{tikzcd}
T^\star(TV) \arrow[rr, bend left, "\gamma_{T^\star V}"] & T(T^\star V) \arrow[l, "\kappa_{T^\star V}"'] \arrow[r, "\Omega^\flat_{T^\star V}"] & T^\star(T^\star V)\\
(\varphi,\dot\varphi;\dot\alpha,\dot\alpha_\partial,\alpha,\alpha_\partial) & (\varphi,\alpha,\alpha_\partial;\dot\varphi,\dot\alpha,\dot\alpha_\partial) \arrow[l, maps to] \arrow[r, maps to] & (\varphi,\alpha,\alpha_\partial;-\dot\alpha,-\dot\alpha_\partial,\dot\varphi).
\end{tikzcd}$$

The following result can be shown by using the non-degeneracy of the L^2-dual pairing.

Proposition 2. *The subbundle*

$$D_{T^\star V}=\operatorname{graph}\Omega^\flat_{T^\star V}\subset T(T^\star V)\oplus T^\star(T^\star V)$$

is a Dirac structure on the restricted Pontryagin bundle of $T^\star V$.

By using the explicit expression of $\Omega^\flat_{T^\star V}$ provided in Proposition 1, we obtain

$$\begin{aligned} D_{T^\star V} = \big\{ & (\varphi, \alpha, \alpha_\partial; \dot\varphi, \dot\alpha, \dot\alpha_\partial; \delta\alpha, \delta\alpha_\partial, \delta\varphi) \in T(T^\star V) \oplus T^\star(T^\star V) \mid \\ & -\dot\alpha = \delta\alpha,\ -\dot\alpha_\partial = \delta\alpha_\partial,\ \dot\varphi = \delta\varphi \big\}. \end{aligned} \tag{1}$$

In the above expression, note that the relations $\dot\varphi = \delta\varphi$ and $-\dot\alpha = \delta\alpha$ hold in the interior of $\mathcal{B}$, while $-\dot\alpha_\partial = \delta\alpha_\partial$ holds on $\partial\mathcal{B}$.

3 Lagrange–Dirac equations

Let $L : TV \to \mathbb{R}$ be a Lagrangian given by

$$L(\varphi, \dot\varphi) = \int_{\mathcal{B}} \mathfrak{L}(\varphi(x), \dot\varphi(x), \nabla\varphi(x))\,\mathrm{d}x, \qquad (\varphi, \dot\varphi) \in TV, \tag{2}$$

where $\mathfrak{L} : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^m \to \mathbb{R}$ is the Lagrangian density. In the previous expression,

$$\nabla : C^\infty(\mathcal{B}) \to \Omega^1(\mathcal{B}) \simeq C^\infty(\mathcal{B}, \mathbb{R}^m), \quad \varphi \mapsto \left(\frac{\partial\varphi}{\partial x^1}, \dots, \frac{\partial\varphi}{\partial x^m} \right),$$

denotes the gradient, being $(x^1, \dots, x^m)$ the standard (global) coordinates of $\mathbb{R}^m$. In the same vein, the divergence is denoted by

$$\operatorname{div} : \mathfrak{X}(\mathcal{B}) \simeq C^\infty(\mathcal{B}, \mathbb{R}^m) \to C^\infty(\mathcal{B}), \quad \omega = \big(\omega^1, \dots, \omega^m\big) \mapsto \sum_{i=1}^m \frac{\partial\omega^i}{\partial x^i}.$$

The partial derivatives of the Lagrangian are the maps

$$\frac{\delta L}{\delta\varphi} : TV \to V', \qquad \frac{\delta L}{\delta\varphi}(\varphi, \dot\varphi)(\delta\varphi) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L(\varphi + \epsilon\,\delta\varphi, \dot\varphi)$$

and

$$\frac{\delta L}{\delta\dot\varphi} : TV \to V', \qquad \frac{\delta L}{\delta\dot\varphi}(\varphi, \dot\varphi)(\delta\dot\varphi) = \left.\frac{d}{d\epsilon}\right|_{t=0} L(\varphi, \dot\varphi + \epsilon\,\delta\dot\varphi),$$

for each $(\varphi, \dot\varphi) \in TV$ and $\delta\varphi, \delta\dot\varphi \in V$.

Remark 2. The contraction between $\omega = \big(\omega^1, \dots, \omega^m\big) \in \mathfrak{X}(\mathcal{B}) \simeq C^\infty(\mathcal{B}, \mathbb{R}^m)$ and $\zeta = (\zeta_1, \dots, \zeta_m) \in \Omega^1(\mathcal{B}) \simeq C^\infty(\mathcal{B}, \mathbb{R}^m)$ is given by the standard inner product on $\mathbb{R}^m$ and is denoted by $\omega \cdot \zeta = \sum_{i=1}^m \omega^i \zeta_i \in C^\infty(\mathcal{B})$.

The following result ensures that the partial derivatives introduced above lie in the restricted dual, $V^\star \subset V'$, when the Lagrangian is defined through a density.

Lemma 1. *Let $L : TV \to \mathbb{R}$ be a Lagrangian defined through a density, as in* (2)*. Then the partial derivatives of L lie in $V^\star$. Furthermore, they are given by*

$$\begin{aligned} \frac{\delta L}{\delta\varphi}(\varphi, \dot\varphi) &= \left(\frac{\partial\mathfrak{L}}{\partial\varphi}(\varphi, \dot\varphi, \nabla\varphi) - \operatorname{div}\frac{\partial\mathfrak{L}}{\partial\nabla\varphi}(\varphi, \dot\varphi, \nabla\varphi),\ \left.\frac{\partial\mathfrak{L}}{\partial\nabla\varphi}(\varphi, \dot\varphi, \nabla\varphi)\right|_{\partial\mathcal{B}} \cdot n \right), \\ \frac{\delta L}{\delta\dot\varphi}(\varphi, \dot\varphi) &= \left(\frac{\partial\mathfrak{L}}{\partial\dot\varphi}(\varphi, \dot\varphi, \nabla\varphi), 0 \right), \end{aligned}$$

for each $(\varphi, \dot\varphi) \in TV$*, where* $n \in C^\infty(\partial\mathcal{B}, \mathbb{R}^m)$ *is the outward-pointing, unit, normal vector field on the boundary.*

Proof. Let $\delta\varphi \in V$. By definition, we have

$$\begin{aligned}\frac{\delta L}{\delta \varphi}(\varphi,\dot\varphi)(\delta\varphi) &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0}\int_{\mathcal{B}} \mathfrak{L}(\varphi+\epsilon\,\delta\varphi,\dot\varphi,\nabla(\varphi+\epsilon\,\delta\varphi))\,\mathrm{d}x\\ &= \int_{\mathcal{B}}\left(\frac{\partial\mathfrak{L}}{\partial\varphi}(\varphi,\dot\varphi,\nabla\varphi)\,\delta\varphi+\frac{\partial\mathfrak{L}}{\partial\nabla\varphi}(\varphi,\dot\varphi,\nabla\varphi)\cdot\nabla(\delta\varphi)\right)\mathrm{d}x\\ &= \int_{\mathcal{B}}\left(\frac{\partial\mathfrak{L}}{\partial\varphi}(\varphi,\dot\varphi,\nabla\varphi)-\operatorname{div}\frac{\partial\mathfrak{L}}{\partial\nabla\varphi}(\varphi,\dot\varphi,\nabla\varphi)\right)\delta\varphi\,\mathrm{d}x\\ &\quad+\int_{\partial\mathcal{B}}\left(\frac{\partial\mathfrak{L}}{\partial\nabla\varphi}(\varphi,\dot\varphi,\nabla\varphi)\cdot n\right)\delta\varphi\,\mathrm{d}s,\end{aligned}$$

where we have used the standard integration by parts formula. Analogously,

$$\frac{\delta L}{\delta\dot\varphi}(\varphi,\dot\varphi)(\delta\dot\varphi)=\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\int_{\mathcal{B}}\mathfrak{L}(\varphi,\dot\varphi+\epsilon\,\delta\dot\varphi,\nabla\varphi)\,\mathrm{d}x=\int_{\mathcal{B}}\frac{\partial\mathfrak{L}}{\partial\dot\varphi}(\varphi,\dot\varphi,\nabla\varphi)\,\delta\dot\varphi\,\mathrm{d}x.$$

Since the previous expressions hold for every $\delta\varphi, \delta\dot\varphi \in V$, we conclude. □

The differential of L is given by

$$\mathrm{d}L : TV \to T'(TV), \quad (\varphi,\dot\varphi)\mapsto\left(\varphi,\dot\varphi,\frac{\delta L}{\delta\varphi}(\varphi,\dot\varphi),\frac{\delta L}{\delta\dot\varphi}(\varphi,\dot\varphi)\right).$$

The previous lemma ensures that it takes values in $T^\star(TV)$, which enables us to define the *Dirac differential* of L as

$$\mathrm{d}_D L = \gamma_{T^\star V}\circ\mathrm{d}L : TV \to T^\star(T^\star V). \tag{3}$$

By using the explicit expression of $\gamma_{T^\star V}$, it can be checked that

$$\mathrm{d}_D L(\varphi,\dot\varphi)=\left(\varphi,\frac{\delta L}{\delta\dot\varphi}(\varphi,\dot\varphi),-\frac{\delta L}{\delta\varphi}(\varphi,\dot\varphi),\dot\varphi\right)$$

for each $(\varphi,\dot\varphi)\in TV$.

Definition 3. *The* ***Lagrange–Dirac equations*** *for a curve*

$$(\varphi,\nu,\alpha,\alpha_\partial) : [t_0,t_1]\to TV\oplus T^\star V = V\times(V\times V^\star)$$

are given by

$$\big((\varphi,\alpha,\alpha_\partial;\dot\varphi,\dot\alpha,\dot\alpha_\partial),\mathrm{d}_D L(\varphi,\nu)\big)\in D_{T^\star V}(\varphi,\alpha,\alpha_\partial). \tag{4}$$

A straightforward computation from (1) and (3), as well as Lemma 1, gives the explicit expression of the equations.

Theorem 1. *Let $L : TV \to \mathbb{R}$ be a Lagrangian, as in* (2)*. The Lagrange–Dirac equations* (4) *are equivalent to the following system:*

$$\begin{cases} \dot{\varphi} = \nu, & \\ \alpha = \dfrac{\partial \mathfrak{L}}{\partial \dot{\varphi}}(\varphi, \nu, \nabla\varphi), & \dot{\alpha} = \dfrac{\partial \mathfrak{L}}{\partial \varphi}(\varphi, \nu, \nabla\varphi) - \operatorname{div} \dfrac{\partial \mathfrak{L}}{\partial \nabla\varphi}(\varphi, \nu, \nabla\varphi), \\ \alpha_\partial = 0, & \dot{\alpha}_\partial = \dfrac{\partial \mathfrak{L}}{\partial \nabla\varphi}(\varphi, \nu, \nabla\varphi)\Big|_{\partial\mathcal{B}} \cdot n. \end{cases}$$

Observe that the two conditions

$$\alpha = \frac{\partial \mathfrak{L}}{\partial \dot{\varphi}}(\varphi, \nu, \nabla\varphi), \qquad \alpha_\partial = 0,$$

above arise from Lagrange–Dirac equations (4) by noting that $(\varphi, \alpha, \alpha_\partial; \dot{\varphi}, \dot{\alpha}, \dot{\alpha}_\partial)$ and $\mathrm{d}_D L(\varphi, \nu)$ must have the same footpoint. These conditions can be written as $(\varphi, \alpha, \alpha_\partial) = \mathbb{F}L(\varphi, \nu)$, where

$$\mathbb{F}L : TV \to T^\star V, \quad (\varphi, \nu) \mapsto \left(\varphi, \frac{\delta L}{\delta \dot{\varphi}}(\varphi, \nu)\right) = \left(\varphi, \frac{\partial \mathfrak{L}}{\partial \dot{\varphi}}(\varphi, \nu, \nabla\varphi), 0\right)$$

is the Legendre transform of L. Note also that the boundary condition is part of the Lagrange–Dirac equations.

Example 1 (External forces). The above theory may be easily extended to systems with external forces. More specifically, given a Lagrangian, as in (2), and an external force with values on the restricted cotangent bundle,

$$F : TV \to T^\star V,$$

the *forced Lagrange–Dirac equations* for a curve $(\varphi, \nu, \alpha, \alpha_\partial) : [t_0, t_1] \to TV \oplus T^\star V$ are given by

$$\left((\varphi, \alpha, \alpha_\partial; \dot{\varphi}, \dot{\alpha}, \dot{\alpha}_\partial), \mathrm{d}_D L(\varphi, \nu) - \widetilde{F}_L(\varphi, \nu)\right) \in D_{T^\star V}(\varphi, \alpha, \alpha_\partial),$$

where $\widetilde{F}_L : TV \to T^\star(T^\star V)$ is defined as

$$(T\pi)^\star_{\mathbb{F}L(\varphi,\nu)}(F(\varphi, \nu)) \in T^\star_{\mathbb{F}L(\varphi,\nu)}(T^\star V), \qquad (\varphi, \nu) \in TV,$$

being $\pi : T^\star V = V \times V^\star \to V$ the projection onto the first component, and $(T\pi)_{\mathbb{F}(\varphi,\nu)} : T_{\mathbb{F}L(\varphi,\nu)}(T^\star V) \to T_\varphi V$ its tangent map at $\mathbb{F}L(\varphi, \nu)$, i.e.,

$$\left\langle \widetilde{F}_L(\varphi, \nu), \delta z \right\rangle = \left\langle F(\varphi, \nu), (T\pi)_{\mathbb{F}L(\varphi,\nu)}(\delta z) \right\rangle, \qquad \delta z \in T_{\mathbb{F}L(\varphi,\nu)}(T^\star V).$$

Since $V^\star = C^\infty(\mathcal{B}) \times C^\infty(\partial\mathcal{B})$, we may write

$$F(\varphi, \nu) = \left(\varphi, \mathcal{F}(\varphi, \nu), \mathcal{F}_\partial(\varphi, \nu)\right), \qquad (\varphi, \nu) \in TV,$$

for some $\mathcal{F} : TV \to C^\infty(\mathcal{B})$ and $\mathcal{F}_\partial : TV \to C^\infty(\partial\mathcal{B})$. Subsequently, the forced Lagrange–Dirac equations are equivalent to the following system:

$$\begin{cases} \dot{\varphi} = \nu, & \\ \alpha = \dfrac{\partial \mathfrak{L}}{\partial \dot{\varphi}}(\varphi, \nu, \nabla\varphi), & \dot{\alpha} = \dfrac{\partial \mathfrak{L}}{\partial \varphi}(\varphi, \nu, \nabla\varphi) - \operatorname{div} \dfrac{\partial \mathfrak{L}}{\partial \nabla\varphi}(\varphi, \nu, \nabla\varphi) + \mathcal{F}(\varphi, \nu), \\ \alpha_\partial = 0, & \dot{\alpha}_\partial = \left.\dfrac{\partial \mathfrak{L}}{\partial \nabla\varphi}(\varphi, \nu, \nabla\varphi)\right|_{\partial\mathcal{B}} \cdot n + \mathcal{F}_\partial(\varphi, \nu). \end{cases}$$

Example 2 (Vibrating membrane with free boundary). The Lagrangian of a vibrating membrane is given by (2) with the following Lagrangian density,

$$\mathfrak{L}(\varphi, \dot{\varphi}, \nabla\varphi) = \frac{1}{2}\rho_0 \dot{\varphi}^2 - \frac{1}{2}\tau |\nabla\varphi|^2, \qquad (\varphi, \dot{\varphi}, \nabla\varphi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m,$$

where $\rho_0 \in \mathbb{R}^+$ is the *density* and $\tau \in \mathbb{R}^+$ is the *tension*. Since the partial derivatives of $\mathfrak{L}$ are given by

$$\frac{\partial \mathfrak{L}}{\partial \varphi}(\varphi, \dot{\varphi}, \nabla\varphi) = 0, \qquad \frac{\partial \mathfrak{L}}{\partial \dot{\varphi}}(\varphi, \dot{\varphi}, \nabla\varphi) = \rho_0 \dot{\varphi}, \qquad \frac{\partial \mathfrak{L}}{\partial \nabla\varphi}(\varphi, \dot{\varphi}, \nabla\varphi) = -\tau \nabla\varphi,$$

for each $(\varphi, \dot{\varphi}, \nabla\varphi) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^m$, then the dynamical equations given in Theorem 1 for a curve $(\varphi, \nu, \alpha, \alpha_\partial) : [t_0, t_1] \to TV \oplus T^\star V$ read

$$\begin{cases} \dot{\varphi} = \nu, & \\ \alpha = \rho_0 \dot{\varphi}, & \dot{\alpha} = \operatorname{div}(\tau \nabla\varphi), \\ \alpha_\partial = 0, & \dot{\alpha}_\partial = -\tau \nabla\varphi|_{\partial\mathcal{B}} \cdot n. \end{cases}$$

Observe that we obtain the *wave equation*,

$$\ddot{\varphi} = (\tau/\rho_0)\nabla^2\varphi,$$

where ∇^2 denotes the Laplacian, together with the *Neumann boundary condition*,

$$\nabla\varphi|_{\partial\mathcal{B}} \cdot n = 0,$$

as expected for a vibrating membrane with free boundary. The forced case can be treated similarly, following Example 1.

4 Conclusions and prospective work

We have made a proposal for treating Lagrangian systems defined on $C^\infty(\mathcal{B})$ from the Dirac point of view. As we have seen, the main advantage is the incorporation of the boundary conditions in the Dirac structure. We have chosen to carry out the development on a closed, bounded domain, $\mathcal{B} \subset \mathbb{R}^m$, to keep the exposition simple, but our setting can be extended to m-dimensional, (pseudo-)Riemannian manifolds with boundary, (M, g).

On the other hand, we are currently working on the generalization to more general Fréchet spaces, such as the family of k-forms, $V = \Omega^k(\mathcal{B})$, for $0 \leq k \leq m$ (note that we have presented here the case $k = 0$), as well as on the relation between our approach and that using Stokes–Dirac structures [14]. The next step would be to consider the nonlinear case, i.e., dealing with Fréchet manifolds of smooth maps (cf., for example, [8, Chapter IX]) instead of Fréchet spaces. This will allow for treating more general physical systems, such as fluid dynamics in a fixed domain, where the configuration manifold is the family of diffeomorphisms of $\mathcal{B}$, $\mathrm{Diff}(\mathcal{B})$, or continuum mechanics with moving boundary, where the configuration manifold is the family of embeddings of $\mathcal{B}$ in $\mathbb{R}^m$, $\mathrm{Emb}(\mathcal{B}, \mathbb{R}^m)$. In addition, reduction by symmetries could be performed thanks to the relabeling symmetry and the material frame indifference, yielding the so-called spatial and convective representations.

In addition, it would be desirable to construct a discrete counterpart of this theory in order to obtain numerical integrators that preserve the geometric structures at the discrete level. In [9] and, more recently, in [1], variational discretizations in time for Lagrange–Dirac and Hamilton–Dirac systems have been proposed. We aim to extend them to account for infinite-dimensional systems by following for instance the variational approach to spatial discretization, such as considered in [3,12].

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