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Dynamical System Governing a Viscoelastic Thermosyphon Model

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Abstract— Thermoyphons, in the engineering literature, is a device composed of a closed loop containing a fluid who motion is driven by several actions such as gravity and natural convection. In this work I prove some result about the asymptotic behaviour for solutions of a closed loop thermosyphon model with a viscoelastic fluid in the interior (A. Jiménez-Casas et. al [8]). In this model a viscoelastic fluid described by the Maxwell constitutive equation is considered, this kind of fluids present elastic-like behavior and memory effects.
Their dynamics are governing for a coupled differential nonlinear systems. In several previous work we show chaos in the fluid, even with this kind of viscoelastic fluid (A. Jiménez-Casas and Mario Castro[7], Justine Yasapan, A. Jiménez-Casas and M. Castro[9, 8, 10]).
In this model I consider a prescribed heat flux like Rodríguez-Bernal and Van Vleck[17], Jiménez-Casas and Ovejero[4] between others (all of them with Newtonian fluids).
This work is a generalization of some previous results on standard (Newtonnian) fluids obtained by Rodríguez-Bernal and Van Vleck[17], when considering a viscoelastic fluid.

Keywords:Thermosyphon, Viscoelastic fluid, Heat flux, Asymptotic behaviour, Inertial Manifold.

1 Introduction

In engineering literature a thermosyphon is a device composed of a closed loop *pipe* containing a fluid whose motion is driven by the effect of several actions such as gravity and natural convection.The flow inside the loop is driven by an energetic balance between thermal energy and mechanical energy.

Here, we consider a thermosyphon model in which the confined fluid is viscoelastic. This has some *a-priori* interesting peculiarities that could affect the dynamics with respect to the case of a Newtonian fluid. On the one hand, the dynamics has memory so its behavior depends on the whole past history and, on the second hand, at small perturbations the fluid behaves like an elastic solid and a characteristic resonance frequency could, eventually, be relevant (consider for instance the behavior of jelly or toothpaste).

The simplest approach to viscoelasticity comes from the so-called Maxwell model [13]. In this model, both Newton's law of viscosity and Hooke's law of elasticity are generalized and complemented through an evolution equation for the stress tensor, σ.

Viscoelastic behavior is common in polymeric and biological suspensions and, consequently, our results may provide useful information on the dynamics of this sort of systems inside a thermosyphon.

In a thermosyphon the equations of motion can be greatly sim-

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plified because of the quasi-one-dimensional geometry of the loop. Thus, we assume that the section of the loop is constant and small compared with the dimensions of the physical device, so that the arc length co-ordinate along the loop (x) gives the position in the circuit. The velocity of the fluid is assumed to be independent of the position in the circuit, i.e. it is assumed to be a scalar quantity depending only on time. This approximations comes from the fact that the fluid is assumed to be incompressible. On the contrary temperature is assumed to depend both on time and position along the loop.

The derivation of the thermosyphon equations of motion is similar to that in Ref. [11] and are obtained in [8]. Finally, after adimensionalizing the variables (to reduce the number of free parameters) we get the followingo ODE/PDE system (see [8] and [20]):

$$(1)\quad \begin{cases} \varepsilon \dfrac{d^2 v}{dt^2} + \dfrac{dv}{dt} + G(v)v &= \displaystyle\oint Tf, \\ \dfrac{\partial T}{\partial t} + v\dfrac{\partial T}{\partial x} &= h(x,v,T) + \nu \frac{\partial^2 T}{\partial x^2}, \end{cases}$$

with $v(0) = v_0$, $\frac{dv}{dt}(0) = w_0$ and $T(0,x) = T_0(x)$.

Here $v(t)$ is the velocity, $T(t,x)$ is the distributions of the temperature of the viscoelastic fluid into the loop, ν is the temperature diffusion coefficient, $G(v)$, is the friction law at the inner wall of the loop, the function f is the geometry of the loop and the distribution of gravitational forces, in this cases $h(x,v,T) = h(x)$ as in [4],[8],[18, 19] is the general heat flux and ε in Eq. (1) is the viscoelastic parameter, wich is the dimensionless version of the viscoelastic time. Roughly speaking, it gives the time scale in which the transition from elastic to fluid-like occurs in the fluid.

We assume that $G(v)$ is positive and bounded away from zero. This function has been usually taken to be $G(v) = G_0$, a positive constant for the linear friction case [11], or $G(v) = |v|$ for the quadratic law [3, 12], or even a rather general function given by $G(v) = g(Re)|v|$,where Re is a Reynolds-like number that is assumed to be large [16, 18] and proportional to $|v|$. The functions G, f, l and h incorporate relevant physical constants of the model, such as the cross sectional area, D, the length of the loop, L, the Prandtl, Rayleigh, or Reynolds numbers, etc see [18]. Usually G, h are given continuous functions, such that $G(v) \geq G_0 > 0$, and $h(v) \geq h_0 > 0$, for G_0 and h_0 positive constants.

Finally, for physical consistency, it is important to note that all functions considered must be 1-periodic with respect to the spatial variable, and $\oint = \int_0^1 dx$ denotes integration along the closed path of the circuit. Note that $\oint f = 0$.

The contribution in this paper (Section 3) is to prove that, under suitable conditions, any solution either converges to the rest state or the oscillations of velocity around $v = 0$ must be large enough. This result, generalizes the one proposed in Rodríguez-Bernal and Van Vleck[17] for a thermosyphon model with a one-component **viscoelastic** fluid.

2 Previous results about well posedness and global attractor

First, we note that in this section we consider the case in which all periodic functions in Eq. (1) have zero average, i.e. we work in $\mathcal{Y} = I\!R^2 \times \dot{H}^1_{per}(0,1)$, where

$$\dot{L}^2_{per}(0,1) = \{u \in L^2_{loc}(I\!R),\ u(x+1) = u(x) a.e.,\ \oint u = 0\},$$

$$\dot{H}^m_{per}(0,1) = H^m_{loc}(I\!R) \cap \dot{L}^2_{per}(0,1).$$

In effect, we observe that, for $\nu > 0$, if we integrate the equation for the temperature along the loop, we have that $\frac{d}{dt}(\oint T) = \oint h$ and then $\oint T(t) = \oint T_0 + t\oint h$. Therefore, the temperature is unbounded, as $t \to \infty$, unless $\oint h = 0$. However, taking $\theta = T - \oint T$ and $h^* = h - \oint h$ reduces to the case $\oint \theta = \oint \theta_0 = \oint h^* = 0$, since θ would satisfy:

$$(2)\quad \frac{\partial \theta}{\partial t} + v\frac{\partial \theta}{\partial x} = h(x) + \nu \frac{\partial^2 \theta}{\partial x^2}, \quad \theta(0) = \theta_0 = T_0 - \oint T_0$$

Therefore, if we consider now $\theta = T - \oint T$ then from the second equation of system Eq. (1), we obtain that θ verifies the equations (2).

Finally, since $\oint f = 0$, we have that $\oint Tf = \oint \theta f$, and the equation for v reads

$$(3)\quad \frac{d^2 v}{dt^2} + \frac{dv}{dt} + G(v)v = \oint \theta . f, \quad v(0) = v_0,\ \frac{dv}{dt}(0) = w_0.$$

Thus, from Eqs. (2) and (3) we have (v,θ) verifies system Eq.(1) with h^*, θ_0 replacing h, T_0 respectively and now $\oint \theta = \oint h = \oint \theta_0 = 0$.

Thus, we consider the system Eq. (1) with $\oint T_0 = \oint h = 0$ and $\oint T(t) = 0$ for every $t \geq 0$.

Also, if $\nu > 0$ the operator $-\nu \frac{\partial^2}{\partial x^2}$, together with periodic boundary conditions, is an unbounded, self-adjoint operator with compact resolvent in $L^2_{per}(0,1)$ that is positive when resctricted to the space of zero-average functions $\dot{L}^2_{per}(0,1)$. Hence, the second equation in Eq. (1) is of parabolic type for $\nu > 0$.

Hereafter we denote by $w = \frac{dv}{dt}$ and we write the system (1) as the following evolution system for the acceleration,velocity and temperature:

$$(4)\quad \frac{d}{dt}\begin{pmatrix} w \\ v \\ T \end{pmatrix} + \begin{pmatrix} \frac{1}{\varepsilon} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\nu\frac{\partial^2}{\partial x^2} \end{pmatrix}\begin{pmatrix} w \\ v \\ T \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

with $F_1(w,v,T) = -\frac{1}{\varepsilon}G(v)v + \frac{1}{\varepsilon}\oint Tf$, $F_2(w,v,T) = w$ and $F_3(w,v,T) = -v\frac{\partial T}{\partial x} + h(v)$ and initial data $(w,v,T)^{\perp}(0) = (w_0, v_{0,0}T)^{\perp}$.

The operator $B = \begin{pmatrix} \frac{1}{\varepsilon} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\nu\frac{\partial^2}{\partial x^2} \end{pmatrix}$ is a sectorial operator in $\mathcal{Y} = I\!R^2 \times \dot{H}^1_{per}(0,1)$ with domain $D(B) = I\!R^2 \times \dot{H}^3_{per}(0,1)$ and has compact resolvent, where

$$\dot{L}^2_{per}(0,1) = \{u \in L^2_{loc}(I\!R), u(x+1) = u(x) a.e., \oint u = 0\},$$

$$\dot{H}^m_{per}(0,1) = H^m_{loc}(I\!R) \cap \dot{L}^2_{per}(0,1).$$

Thus, we can apply the result about sectorial operator of [2] to prove the existence of solutions of system (1). Moreover, if we consider some aditionally hypothesis (H) to add for the friction G using in the tecnique Lemma 3.1 in [8], which are satisfied for all friction functions G consider in the previous works,i.e., the thermsyphon models where G is constant or linear or quadratic law, and also for $G(s) \equiv A|s|^n$, as $s \to \infty$. Then, we have the next result.

PROPOSITION 1 *We suppose that $H(r) = rG(r)$ is locally Lipschitz, $f, h \in \dot{L}^2_{per}$ $h(v) \geq h_0 > 0$. Then, given $(w_0, v_0, T_0) \in \mathcal{Y} = I\!R^2 \times \dot{H}^1_{per}$, there exists a unique solution of (1) satisfying*

$$(w,v,T) \in C([0,\infty], \mathcal{Y}) \cap C(0,\infty, I\!R^2 \times \dot{H}^3_{per}(0,1)),$$

$$(\dot{w}, w, \frac{\partial T}{\partial t}) \in C(0,\infty, I\!R^2 \times \dot{H}^{3-\delta}_{per}(0,1)),$$

where $w = \dot{v} = \frac{dv}{dt}$ and $\dot{w} = \frac{d^2v}{dt^2}$ for every $\delta > 0$. In particular, (1) defines a nonlinear semigroup, $S(t)$ in $\mathcal{Y}$, with $S(t)(w_0, v_0, T_0) = (w(t), v(t), T(t))$.

Moreover, from (H) (see [8]) Eq. (1) has a global compact and connected attractor, $\mathcal{A}$, in $\mathcal{Y}$. Also if $h \in \times\dot{H}^m_{per}(0,1)$ with $m \geq 1$, the global attractor $\mathcal{A} \subset I\!R^2 \times \dot{H}^{m+2}_{per}$ and is compact in this space.

Proof: This result has been proved in Theorem 2.1, Proposition 3.1 and Corollary 4.1 from A.Jiménez-Casas at al.[8].

3 Asymptotic behaviour of finite length solutions

In previous works, like A. Jiménez-Casas and Mario [7], A. Jiménez-Casas at al.[8, 10], the asymptotic behaviour of the system Eq.(1) for large enough time is studied.

In this sense the existence of a inertial manifold associated to the functions f (loop-geometry) and h (prescribed heat flux) have proved. The abstract operators theory (Henry[2], Foias et al. [1] and Rodríguez-Bernal[15, 14]) has been used for this purpose. In this section we consider the linear friction case [11] where $G(v) = G_0$, a positive constant and we prove in Proposition 3 the results which rise an important consequence: for large time the velocity reaches the equilibrium - null velocity -, or takes a value to make its integral diverge, which means that either it remains with a constant value without changing its sign or it will alternate an infinite number of times so the oscillations around zero become large enough to make the integral diverge.

3.1 Previous results and notations

In this section we assume also that $G^*(r) = rG(r)$ is locally Lipschitz satisfying (H) (see [8]), and $f, h \in \dot{L}^2_{per}$ are given by following Fourier expansions

$$(5) \quad h(x) = \sum_{k \in Z\!\!Z^*} b_k e^{2\pi kix}; \qquad f(x) = \sum_{k \in Z\!\!Z^*} c_k e^{2\pi kix};$$

where $Z\!\!Z^* = Z\!\!Z - \{0\}$, while $T_0 \in \dot{H}^2_{per}$ is given by $T_0(x) = \sum_{k \in Z\!\!Z^*} a_{k0} e^{2\pi kix}$. Finally assume that $T(t,x) \in \dot{H}^2_{per}$ is given by

$$(6) \qquad T(t,x) = \sum_{k \in Z\!\!Z^*} a_k(t) e^{2\pi kix}$$

where $Z\!\!Z^* = Z\!\!Z - \{0\}$, We note that $\bar{a_k} = -a_k$ since all functions consider are real and also $a_0 = 0$ since they have zero average.

Now we observe the dynamics of each Fourier mode and from Eq. (1), we get the following system for the new unknowns, v and the coefficients $a_k(t)$.

$$(7) \quad \begin{cases} \varepsilon\frac{d^2v}{dt^2} + \frac{dv}{dt} + G(v)v = \sum_{k \in Z\!\!Z^*} a_k(t) c_{-k} \\ \dot{a}_k(t) + \left[2\pi kiv(t) + 4\nu\pi^2k^2\right] a_k(t) = b_k \end{cases}$$

- Assume that the prescribed heat flux $h \in \dot{H}^m_{per}$, are given by
$$h(x) = \sum_{k \in K} b_k e^{2\pi kix},$$
and $b_k \neq 0$ for every $k \in K \subset Z\!\!Z$ with $0 \neq K$, since $\oint h = 0$. We denote by V_m the clousure of the subspace of $\dot{H}^m_{per}$ generated by $\{e^{2\pi kix}, k \in K\}$. Then we have from Theorem 4.2 in A. Jiménez-Casas et al.[8] the set $\mathcal{M} = I\!R^2 \times V_m$ is an **inertial manifold** for the flow of $S(t)(w_0, v_0, T_0) = (w(t), v(t), T(t))$ in the space $\mathcal{Y} = I\!R^2 \times \dot{H}^m_{per}$. By this, the dynamics of the flow is given by the flow in $\mathcal{M}$ associated to the prescribed heat flux h. This is

$$\begin{cases} \varepsilon\frac{d^2v}{dt^2} + \frac{dv}{dt} + G(v)v = \sum_{k \in K} a_k(t) c_{-k} \\ \dot{a}_k(t) + \left[2\pi kiv(t) + 4\nu\pi^2k^2\right] a_k(t) = b_k,\ k \in K \end{cases} \quad (8)$$

- Moreover, we assume that the function associated to the geometry of the loop f, are given by

$$f(x)=\sum_{k\in J} c_k e^{2\pi kix}$$

and $c_k \neq 0$ for every $k \in J \subset \mathbb{Z}$ with $0 \neq K$, since $\oint f = 0$.

We note also that on the inertial manifold $\oint Tf = \sum_{k\in K\cap J} a_k(t)c_{-k}$. Thus, the dynamics of the system depends only on the coefficients in $K \cap J$.

- Herafter, we consider de functions h and f are given by following Fourier expansions

(9) $$h(x)=\sum_{k\in K} b_k e^{2\pi kix}; \qquad f(x)=\sum_{k\in J} c_k e^{2\pi kix};$$

where

$K = \{k \in \mathbb{Z}^*/b_k \neq 0\}$, $J = \{k \in \mathbb{Z}^*/c_k \neq 0\}$ with $\mathbb{Z}^* = \mathbb{Z} - \{0\}$, First, from the equations Eq. (7) we can observe the velocity of the fluid is independent of the coefficients for temperature $a_k(t)$ for every $k \in \mathbb{Z}^* - (K \cap J)$. That is, the **relevant coefficients** for the velocity are only $a_k(t)$ with k belonging to the set $K \cap J$. This important result about the asymptotic behaviour has been proved in Corollary 4.2 from A. Jiménez-Casas at al.[8].

We also note that $0 \notin K \cap J$ and since $K = -K$ and $J = -J$ then the set $K \cap J$ has an even number of elements, wihch we denote by $2n_0$. Therefore the number of the positive elements of $K \cap J$, $(K \cap J)_+$ is n_0. Moreover the equations for a_{-k} are conjugates of the equations for a_k and therefore we have $\sum_{k\in K\cap J} a_{k(t)}c_{-k} = 2Re(\sum_{k\in(K\cap J)_+} a_k(t)c_{-k})$.

Thus,

(10) $$\oint Tf = 2Re(\sum_{k\in(K\cap J)_+} a_k(t)c_{-k}).$$

The aim is to prove the Proposition 3 which generalize the result of thermosyphon model for Newtoniann fluids of Rodríguez-Bernal and Van Vleck[17] in the case of a prescribed heat flux, i.e. $h = h(x)$. To do so we examine which are these steady-state solutions, also called *equilibrium points*.

We have to make the difference between equilibrium points (constants respect to the time) null velocity, called *rest equilibrium*, and equilibrium points with non-vanishing constant velocity.

Equilibrium conditions.

i) The system Eq. (7) presents the *rest equilibrium* $w = v = 0$, $a_k = \frac{b_k}{4\nu\pi^2k^2}$ $\forall k \in K \cap J$ under the assumption of the following orthogonality condition:

(11) $$I_0 = Re(\sum_{k\in(K\cap J)_+} \frac{b_k c_{-k}}{k^2}) = 0.$$

ii) Any other equilibrium position will have a non-vanishing velocity and the equilibrium is given by:

(12) $$\begin{cases} G(v)v = 2Re(\sum_{k\in(K\cap J)_+} a_k c_{-k}) \\ a_k = \frac{b_k}{4\nu\pi^2k^2+2\pi kiv} \end{cases}$$

3.2 Asymptotic behaviour

LEMMA 2 *If we assume that a solution of Eq.* (7) *satisfies* $\int_0^\infty |v(s)|ds < \infty$*, then for every* $\eta > 0$ *there exists* t_0 *such that*

(13) $$\int_{t_0}^t e^{-4\nu\pi^2k^2(t-r)}(e^{-\int_r^t 2\pi ikv} - 1)dr \leq \eta,$$

with $t \geq t_0$.

Proof: If $\int_0^\infty |v(s)|ds < \infty$, then for all δ there exits $t_0 > 0$ such that for every $t_0 \leq r \leq t$ we have $|\int_r^t v| \leq \delta$. Then, for any $\eta > 0$ we can take t_0 large enough such that

(14) $$|e^{-\int_r^t 2\pi ikv} - 1| \leq \eta \text{ for all } t_0 \leq r \leq t.$$

Therefore, we get

$$\int_{t_0}^t e^{-4\nu\pi^2k^2(t-r)}(e^{-\int_r^t 2\pi ikv} - 1)dr \leq$$

$\leq \frac{\eta}{4\nu\pi^2k^2}(1 - e^{-4\nu\pi^2k^2(t-t_0)}) \leq \eta$ with $t \geq t_0$ and taking into account that $\eta \to 0$ for $t \to \infty$ and $\nu > 0$, we get Eq. (13).

□

PROPOSITION 3 *i) if* $K \cap J = \emptyset$*, then the global attractor for system Eq.* (1) *in* $\mathbb{R}^2 \times \dot{H}^1_{per}$ *is reduced to a point* $\{(0, 0, \theta_\infty)\}$*, where* θ_∞ *is the unique solution in* $\dot{H}^2_{per}(0, 1)$ *of*

$$-\nu\frac{\partial^2\theta}{\partial x^2} = h(x).$$

ii) We assume that $I_0 = Re(\sum_{k\in(K\cap J)_+} \frac{b_k c_{-k}}{k^2}) = 0$*, with* $K \cap J$ *finite set, and that a solution of Eq.* (7) *satisfies* $\int_0^\infty |v(s)|ds < \infty$*. Then the system reaches the rest stationary solution, that:*

$$\begin{cases} v(t) \to 0, \text{ and } w(t) \to 0, \text{ as } t \to \infty \\ a_k(t) \to \frac{b_k}{4\nu\pi^2k^2}, \text{ as } t \to \infty \end{cases}$$

This is, he also have in this situation the global attractor for system Eq. (1) *in* $\mathbb{R}^2 \times \dot{H}^1_{per}$ *is reduced to a point* $\{(0, 0, \theta_\infty)\}$.

iii) Conversely, if $I_0 = Re(\sum_{k\in(K\cap J)_+} \frac{b_k c_{-k}}{k^2}) \neq 0$ *then for every solution* $\int_0^\infty |v(s)|ds = \infty$*, and* $v(t)$ *does not converge to zero.*

Proof:

First, we study the behaviour for large time of the coefficients $a_k(t)$.

The distance between the coefficients that represents the solution of the system, $a_k(t)$ to the values of those coefficients in the equilibrium, $\frac{b_k}{4\nu\pi^2k^2}$ is computed.

For t_0 enough large, we known that for every $t > t_0$ we have

$$a_k(t) = a_k(t_0)e^{-\int_{t_0}^t 2\pi ikv+4\nu\pi^2k^2} + b_k\int_{t_0}^t e^{-\int_r^t 2\pi ikv+4\nu\pi^2k^2}dr \quad (15)$$

and using $\int_{t_0}^t e^{-\int_r^t 4\nu\pi^2k^2} = \frac{1}{4\nu\pi^2k^2}(1-e^{-4\nu\pi^2k^2(t-t_0)})$ we have that

$$a_k(t) - \frac{b_k(1-e^{-4\nu\pi^2k^2(t-t_0)})}{4\nu\pi^2k^2} = a_k(t_0)e^{-\int_{t_0}^t 2\pi ikv+4\nu\pi^2k^2} +$$

$$+b_k\int_{t_0}^t e^{-\int_r^t 4\nu\pi^2k^2}(e^{-\int_r^t 2\pi ikv}-1)dr.$$

Taking limits when $t \to \infty$, we get

$a_k(t) - (1-e^{-4\nu\pi^2k^2(t-t_0)})\frac{b_k}{4\nu\pi^2k^2} \to 0$, since $a_k(t_0)e^{-\int_0^t 2\pi ikv+4\nu\pi^2k^2} \to 0$ and from Eq. (13) we have that $b_k\int_{t_0}^t e^{-\int_r^t 4\nu\pi^2k^2}(e^{-\int_r^t 2\pi kiv}-1) \to 0$. Now taking into account that $(1-e^{-4\nu\pi^2k^2(t-t_0)})\frac{b_k}{4\nu\pi^2k^2}$ converges to $\frac{b_k}{4\nu\pi^2k^2}$ for large time we conclude that:

$$(16)\begin{cases} a_k(t) \to \frac{b_k}{4\nu\pi^2k^2} \\ I(t) = 2Re(\sum_{k\in(K\cap J)_+} a_k(t)c_{-k}) \to \frac{I_0}{2\nu\pi^2} = I_0^* \end{cases}$$

with $I_0 = 2Re(\sum_{k\in(K\cap J)_+} \frac{b_k c_{-k}}{4\nu\pi^2k^2})$.

We note that $T(t,x) = \sum_k a_k(t)e^{2\pi kix} \to \theta_\infty = \sum_k \frac{b_k}{4\nu\pi^2k^2}e^{2\pi kix}$ and we also note $\frac{\partial^2\theta_\infty}{\partial x^2} = \frac{-1}{\nu}\sum_k b_k e^{2\pi kix} = \frac{-1}{\nu}h(x)$.

To conclude, we study now when the velocity $v(t)$ and the acceleration $w(t)$ go to zero. From (10) we can reading the equation for v, the first equation of system Eq. (7), as

$$\varepsilon\frac{d^2v}{dt^2} + \frac{dv}{dt} + G(v)v = I(t).$$

we consider now, $G(v) = G_0 > 0$ and then we note that:

I) First, if we denoted by $v_H(t)$ any solution of linear homogeneous equation given by:

$$\varepsilon\frac{d^2v}{dt^2} + \frac{dv}{dt} + G_0v = 0$$

then $v_H(t) \to 0$ as $t \to \infty$ since there exits a base of solutions given by exponential functions wihch converge to zero.

II) Second,using (16) for every $\delta > 0$ there exists t_0 such that $|I(t) - I_0^*| \le \delta$ for ever $t \ge t_0$, and

$$\varepsilon\frac{d^2v}{dt^2} + \frac{dv}{dt} + G_0v = I(t) = (I(t) - I_0^*)) + I_0^* \le \delta + I_0^*.$$

Now taking into accoun that

$$v_p(t) = \frac{\delta + I_0^*}{G_0} \text{ satisfies } \varepsilon\frac{d^2v}{dt^2} + \frac{dv}{dt} + G_0v = \delta + I_0^*$$

and any solution of

$$\varepsilon\frac{d^2v}{dt^2} + \frac{dv}{dt} + G_0v = \delta + I_0^*,$$

are given by $v(t) = v_H(t)+v_p(t)$ for some $v_H(t)$; we conclude that

$$limsup_{t\to\infty}|v(t)| \le \frac{\delta + I_0^*}{G_0}$$

for every $\delta > 0$ and

$$|v(t)| \to \frac{I_0^*}{G_0}.$$

i)-ii) In particular, when $K \cap J = \emptyset$ or $I_0 = 0, i.e. I_0^* = 0$, we get $v(t) \to 0$ and if we also prove that $w(t) \to 0$ then we conclude.

In effect, if $v(t) \to 0$, for every $\delta > 0$ there exists t_0 such that $|G_0v| \le \delta$ and $\varepsilon\frac{d|w|}{dt} + |w| \le \delta$ for every $t \ge t_0$, this is

$$|w(t)| \le |w(t_0)|e^{-\frac{1}{\varepsilon}(t-t_0)} + \delta[1 - e^{-\frac{1}{\varepsilon}(t-t_0)}] \le \delta$$

i.e $w(t) \to 0$ as $t \to \infty$.

iii) Finally, we also note that

$$liminf_{t\to\infty}|v(t)| \ge \frac{\delta + I_0^*}{G_0}$$

for every $\delta > 0$ and in the case of $I_0 \neq 0$, then $I_0^* \neq 0$ and we get $liminf_{t\to\infty}|v(t)| > 0$, which implies that $\int_0^\infty |v(s)|ds = \infty$. This result is in contradiction with the initial condition $\int_0^\infty |v(s)|ds < \infty$, what implies that it is not a valid hypothesis.

□

3.3 Concluding remarks

Recalling that functions associated to circuit geometry, f, and to prescribed heat flux, h, are given by $f(x) = \sum_{k\in J} c_k e^{2\pi kix}$ and $h(x) = \sum_{k\in K} b_k e^{2\pi kix}$, respectively. In Jiménez-Casas et. al [8], using the operator abstract theory, it is proved that if $K \cap J = \emptyset$, then the global attractor for system Eq. (1) in $I\!R^2 \times \dot{H}^1_{per}$ is reduced to a point $\{(0,0,\theta_\infty)\}$, where θ_∞ is the unique solution in $\dot{H}^2_{per}(0,1)$ of $-\nu\frac{\partial^2\theta}{\partial x^2} = h(x)$.

In this sense the Proposition 3 offers the possibility to obtain the same asymptotic behaviour for the dynamics, i.e., the attractor is also reduced to a point taking functions f and h without this condition, that is with $K \cap J \neq \emptyset$, its enough that the set $(K \cap J) \neq \emptyset$, but $Re(\sum_{k\in(K\cap J)_+} \frac{b_k c_{-k}}{k^2}) = 0$, when we consider the linear friction case $G = G_0$.

We note, the result about the inertial manifold (Jiménez-Casas et. al [8]) reduces the asymptotic behaviour of the initial system Eq. (1) to the dynamics of the reduced explicit system Eq. (8) with $k \in K \cap J$.

We observe also that from the analysis above, it is possible to design the geometry of circuit, f, and/or heat flux, h, so that the resulting system has an arbitrary number of equations of the form $N = 4n_0 + 1$ where n_0 is the number of elements of $(K \cap J)_+$ and we consider the real an imaginary parts of relevant coefficients for the temperature $a_k(t)$ and solute concentration $d_k(t)$ with $k \in (K \cap J)_+$.

Note that it may be the case that K and J are infinite sets, but their intersection is finite. Also, for a circular circuit we have $f(x) \sim asen(x) + bcos(x)$, i.e. $J = \{\pm 1\}$ and then $K \cap J$ is either $\{\pm 1\}$ or the empty set.

Recently, we have considered a thermosyphon model containing a viscoelastic fluid and we have shown chaos in some closed-loop thermosyphon model with one-component viscoelastic fluid not only in this model [8], also in other kind of transfer law (Jiménez-Casas and Castro [7], Yasappan and Jiménez-Casas et al. [9]), and even in some cases with a viscoelastic binary fluid (Yasappan and Jiménez-Casas et al. [10])

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