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VALUATION MODELS FOR FINANCIAL ASSETS: ANALYSIS AND THEIR EVOLUTION

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All graphs and figures were created using Excel.

List of Abbreviations and Acronyms

APT	Arbitrage Pricing Theory
C-CAPM	Consumption Capital Asset Pricing Model
CAPM	Capital Asset Pricing Model
CMA	Conservative Minus Aggressive
CML	Capital Market Line
D-CAPM	Downside Capital Asset Pricing Model
DDM	Dividend Discount Model
ERP	Equity Risk Premium
ETF	Exchange-Traded Funds
GNP	Gross National Product
HML	High Minus Low
I-CAPM	Intertemporal Capital Asset Pricing Model
M&A	Mergers and Acquisitions
MOM	Momentum
MPT	Modern Portfolio Theory
MSB	Mean Semivariance Behavior
MVB	Mean Variance Behavior
NASDAQ	National Association of Securities Dealers Automated Quotation
NYSE	New York Stock Exchange
RMW	Robust Minus Weak
SMB	Small Minus Big

Resumen

El objetivo principal de este Trabajo de Fin de Grado es analizar y evaluar los métodos y teorías de valoración de activos financieros más importantes, desde un punto de vista teórico.

En este trabajo, inicialmente, se realizará un desarrollo del marco conceptual, explicando conceptos básicos de valoración, así como la evolución histórica de los diferentes modelos objeto de estudio. Después, pasaremos a estudiar en detalle cada uno de los siete modelos incluidos en este trabajo, entre ellos el *Modern Porfolio Theory* de Markowitz y el *Capital Asset Pricing Model* de Sharpe, describiendo sus características concretas, sus diferencias con otros modelos, así como sus aplicaciones y limitaciones.

Finalmente, se extraerán conclusiones respecto a la validez de cada uno de los modelos, y se introducirá el concepto de "Zoo de los Factores", que aparece en la literatura más reciente, como respuesta a las limitaciones de modelos anteriores.

Palabras Clave: Modelos, Valoración, Activos Financieros, Modern Portfolio Theory, Capital Asset Pricing Model, Arbitrage Pricing Theory, Fama and French, Multifactor Model, Momentum, Factor Zoo.

Abstract

The main objective of this Dissertation is to analyze and evaluate the most important financial asset valuation methods and theories, from a theoretical perspective.

In this paper, an analysis of the conceptual framework will be initially carried out, explaining the basic valuation concepts, as well as the historical evolution of the different models under study. Then, we will study in detail each of the seven models studied in this paper, including the *Modern Portfolio Theory* by Markowitz and the *Capital Asset Pricing Model* by Sharpe, describing their specific characteristics, their differences with other models, as well as their applications and limitations.

Finally, conclusions will be drawn regarding the validity of each of the models, and the concept of the "Factor Zoo", which appears in the most recent literature, will be introduced as it emerges as a response to the limitations of previous models.

Keywords: Models, Valuation, Financial Assets, Modern Portfolio Theory, Capital Asset Pricing Model, Arbitrage Pricing Theory, Fame and French, Multifactor Model, Momentum, Zoo Factor.

1. Introduction

In this project, different valuation models will be studied through the analysis of their literature, with objective of seeing how the valuation scenario and its models have been evolving through time, and determining which models are the most accurate or useful ones. The structure of this final project will consist of two main blocks:

First, an analysis of the conceptual framework will be made. For the reader to be able to fully grasp the contents of this study, basic valuation concepts will be explained, and the different financial assets will be defined. The historical evolution of the currently existing models will be briefly explained so that readers can get into context, before explaining each of them in more detail.

Secondly, a detailed study of each of the models will be carried out, describing their specific characteristics, and differences with the other models. For the purpose of this essay, seven distinct models and their later contributions will be examined, following a chronological order. The project will cover the most relevant models developed and studied from 1952 to 2015.

1.1 Objectives

The main objectives of this project are:

1. To analyse the historical evolution and changes of asset valuation methods.
2. Describing in depth each of the models' subject of study and comparing the accuracy and effectiveness of each one of them.
3. Analysis of the advantages and disadvantages of using them, as well as their applications and uses in real life situations.

Before going into the aforementioned points, some context regarding stock markets, and basic principles needed in order to understand the functioning of these models, will be given. Describing the financial assets to which these models are going to be applied is also key for the reader to be able to grasp how these valuation methods work, and for them to know which factors to take into account when choosing one method over another.

Based on the idea of valuation models, the focus will primarily be on the different models used by bankers and other professionals in the financial industry when valuing financial assets such as stock or bonds.

1.2 Methodology

Firstly, it is important to highlight that given the enormous relevance of the subject under study, there are several authors dedicated to its study, although the project will focus on those of greater relevance in the theoretical field. It is also of high importance to point out that the information obtained from these sources will constitute a great part of this project. Most of the information gathered for this project is formed by the original papers and articles released by the corresponding authors for each model but will be complemented with contributions or analysis made later on.

Regarding the type of research that will be carried out, it is necessary to point out that in no case would it be predictive since no predictions will be made regarding the future behaviour of variables. The aim is to establish and explain relationships between variables so that the best model can be chosen, through a descriptive analysis.

2. Conceptual Framework

To be able to grasp the contents of this thesis, there are several theoretical concepts that need to be understood by the reader. A conceptual framework will be provided, highlighting three main notions: basic valuation concepts, a historical evolution of the valuation methods, and finally, a description of the different financial assets.

2.1 Basic Valuation Concepts

Several papers have investigated the relationship between expected return and risk. It should not be surprising as this trade-off is one of the basic principles of valuation. What it shows is that the higher the amount of risk the investor assumes, the higher the return he can expect. In other words, there is a positive relationship between the two. One of the most common formulae used to represent this relationship is:

$$\boldsymbol{E(R_i) = R_f + \beta_i \left(R_m - R_f\right)},$$

where $E(R_i)$ is the rate of return for the company, R_f is the rate of return of a risk free asset, R_m is the rate of return on the market, and β is the beta of the company, which can be also expressed by $Cov(R_i, R_m)/Var(R_m)$. This is the CAPM formula, which will be explained in further detail in point 3. The β is used as a measure of the systematic risk associated to the company or asset under study.

To understand what the betas represent, the concept of risk must be explained. Risk refers to the likelihood that some unfavourable outcome occurs, differing from our expectations. Risk can be classified into two categories: systematic and unsystematic risk. Systematic risks are those that are inherent to the market and affect a large number of assets. It is also known as non-diversifiable risk as it is unpredictable, and in the majority of cases, it is not possible to avoid it completely. Unsystematic risk is that risk inherent to a company. This risk only affects a small number of assets, and it can be mitigated or eliminated.

When choosing between different investment opportunities, investors look at historical rates of returns. By looking at the CAPM formula, it can be deduced that investors ask for a risk premium on the assets, and as it can be observed, as the β increases (meaning systematic risk is higher), the expected return of the company rises as well. This explains the positive relationship between risk and return.

2.2 Description of the Different Financial Assets

An asset class is a group of assets with the same characteristics, which investors can include in their portfolios. There are seven main categories: equity, fixed income, commodities, currencies, derivatives, funds, and real state. For the purpose of this paper, only the two most common ones will be defined: equities and fixed income.

Equities can be defined as capital invested in a company by means of purchasing a proportion of its shares in the stock market. They usually carry a higher risk than fixed income securities which results in higher returns associated to this kind of investment. There are two ways by which investors can be retributed: dividends and share appreciation. These two ways usually tend to be associated to the type of company, which can be distinguished between growth and value. Growth companies normally remunerate their shareholders through share appreciation as they tend to be younger companies but with prospectus of very high future growth. On the contrary, value companies are usually more-established companies, with high net profits but with not much projections of high growth. These companies use dividends to distribute their profits among their shareholders.

Fixed income securities are debt instruments that provide the investor with fixed interest payments and repayments of the principal amount at maturity. This type of securities is divided into corporate bonds, which are issued by companies, and sovereign debt of T-bills, which are issued by governments. They normally carry a lower risk than other investments, but it is important to bear in mind that they will consequently give the investor a lower return.

For the objectives of this paper, only valuation models for equities will be studied.

2.3 Historical Evolution of the Models and their Applications

Harry Markowitz published in 1952 the Modern Portfolio Theory (also known as MPT), which has been the fundamental basis of portfolio management for more than 50 years. It was the first time in history where the problem of portfolio selection is articulated, and a solution is found. Previous contributions such as that of Keynes in 1936 or Marschak in 1938, only analyse investment decisions from a superficial approach, while Markowitz goes deeper.

In 1958, James Tobin expanded on what Markowitz developed, and included the possibility of investing on the risk-free asset to diversify the risk of the portfolio. After this, in 1964, William Sharpe developed the Capital Asset Pricing Model (most known as CAPM), which unlike Tobin, only took into account systematic risk in the equation. In 1965, John Lintner included some additions to the model developed by Sharpe a year earlier. Eight years later, Robert Merton introduced the Intertemporal CAPM, which differed from the initial CAPM in that it assumed that investors would hedge the possible risk of their positions, which is a more realistic approach.

Stephen Ross developed a new model called the Arbitrage Pricing Theory (APT) in 1976. It appeared as an alternative to the mean-variance CAPM introduced by Sharpe and Lintner, which was at the time (and still is) one of the most popular valuation models. The APT is a multi-factor model that, like the CAPM, includes the relationship between systematic risk and return, but as a multi factor model, includes several macroeconomic variables to capture that risk. Between 1978 and 1979, Douglas Breeden and Robert Lucas created the Consumption CAPM, which extends on the concept introduced by Sharpe and Lintner, and later, Merton, but simplifies his multi-beta into a single beta, which made it easier to use.

In 1992, Eugene Fama and Kenneth French, developed a three-factor model that elaborates on the CAPM, but includes size and value risk factors to the model. During 1993, Narasimhan Jegadeesh and Sheridan Titman developed a model, which was a pioneer in including momentum as a factor. In 1997, Mark Carhart introduced a Four-Factor Model that stemmed from Fama and French’s Three-Factor Model and the Momentum model. Javier Estrada developed in 2002 the Downside CAPM (D-CAPM), which differs from the CAPM in that he replaces the beta for the downside beta, a measure of systematic risk in a downside risk scenario. Finally, the last model that will be studied is the Fama-French five-factor model, which was published in 2015, and expands on the three-factor model by adding profitability and investment patterns.

3. Valuation Methods

In this section, a thorough analysis of the different financial-asset's valuation models will be conducted, highlighting the advantages and disadvantages of using each of them.

3.1 Markowitz's Modern Portfolio Theory

As mentioned before, Harry Markowitz developed the Modern Portfolio Theory in 1952. It was one of the first models for stock valuation to ever appear and has been used until this day.

3.1.1 Concept

In the Modern Portfolio Theory (MPT), Markowitz focuses on the second stage of the portfolio selection process. In this stage, the investor poses his ideas on what the expected future performances of available securities are and ends with the selection of his portfolio. He emphasizes the importance of evaluating securities as a group, and not in isolation, which is why he developed the MPT. Markowitz asserts that the rule stating that the goal of an investor is maximizing returns should not be considered as a maxim to explain and understand investment behaviours. This rule would imply that any investor who seeks to maximize his returns, would invest in the stock with the highest return, meaning that there could only be a sole investment asset for all investors. However, the problem with investment selection is that it is always done in a context of uncertainty that carries a risk, and therefore, it means recognizing that any financial investment has more than one possible outcome in terms of profitability. Moreover, it would also lead to the believe that it is only possible to know or to infer a probability distribution for the outcome; there is no strictly precise way to know the future return of the asset. This rule also debunks the idea of diversification of portfolios (Markowitz, 1952).

This is the reason why Markowitz believes that the investor sees return as a positive element, while the variance of returns is a negative element, and tries to find his desirable balance by either compromising returns for lower risk or vice versa. Here is where Markowitz introduces the concept of diversification, as it allows the investor to build a portfolio in which he can take a desired return for a desired amount of risk. It should be noted that diversification does not remove variance completely, but it can reduce it as it implies decorrelation between the assets held in the portfolio. If the assets within the portfolio are highly correlated, it means that they are affected by the same market forces, and in the case of a crisis, the risk would maximize and would not be offset by other (decorrelated) assets. In the end, what he concludes is that investors find their own optimal portfolio which gives them the highest expected return for a set standard deviation, or the lowest risk for a given expected return.

Markowitz then goes on to discuss the attainable combinations diagram, from which he extracts the efficient frontier. In Figure 1, the set of all attainable combinations is seen, as well as the efficient frontier. Under Markowitz model, the investor would choose a portfolio in the efficient frontier according to the level of risk he wants to sustain. Any point chosen which does not fall on the green line, would be inefficient for the investor as they could be obtaining a higher level of return for the amount of risk they are sustaining.

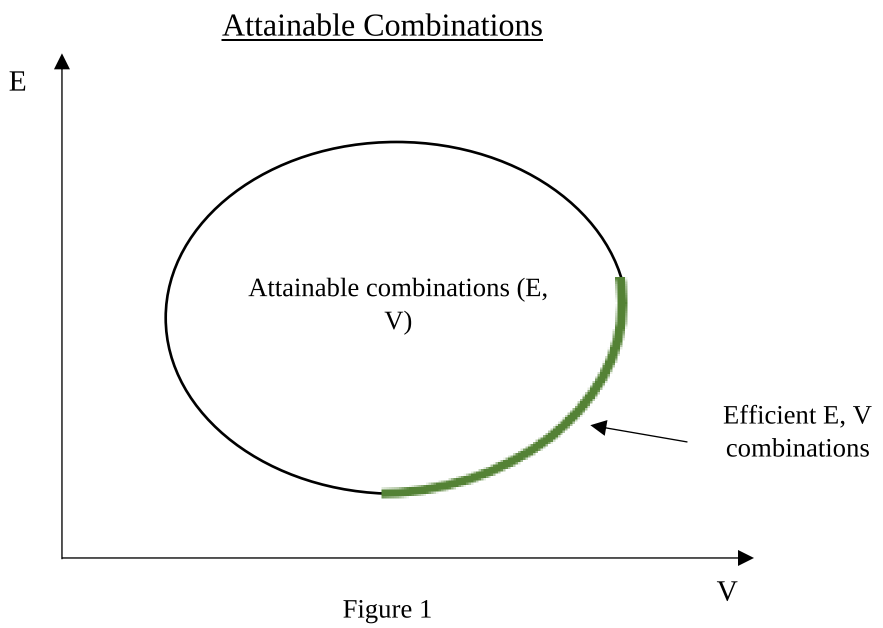

Figure 1

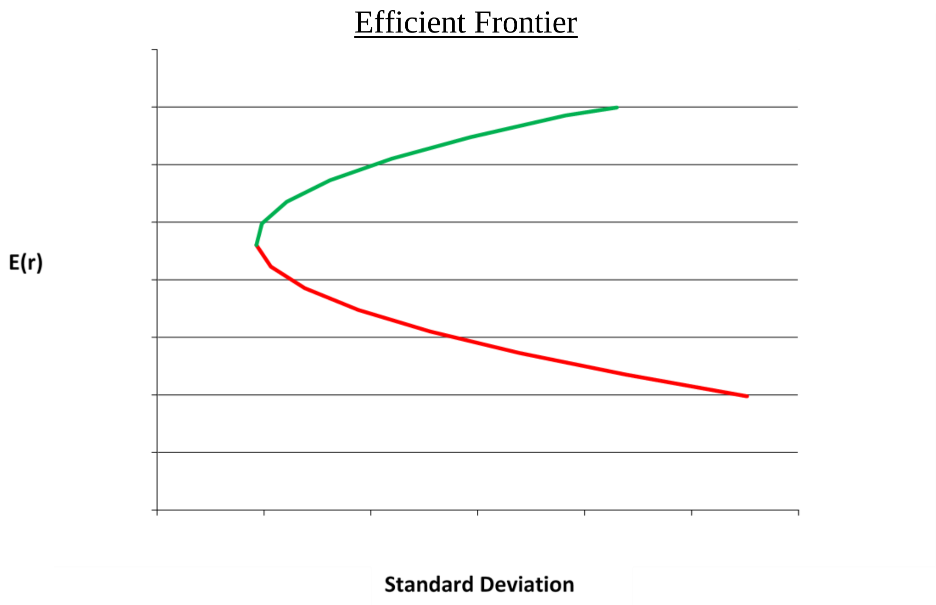

Figure 2

In his theory, Markowitz incorporates some hypothesis on the investor's behavior that relate to how they respond to the uncertainty associated with the outcome of their investments. These assumptions are the following:

1. Investors are risk averse and want, for every level of return expected, the least amount of risk possible.

2. The expected return on all assets is known to the public, as well as their risk (variances and covariances). It also assumes that investors understand this information and are able to make rational decisions. This is assumed to be sufficient for them to determine their optimal portfolios.

3. There are no transaction costs or taxes.

4. Investors maximize expected returns each period, and the earnings curves show decreasing marginal returns.

5. Portfolios include all the assets and liabilities of an investor.

Harry Markowitz developed a measure for the profitability and risk of a portfolio. In order to do this, he determined the expected return ($E(R_i)$) and risk (σ_i) of an individual asset, by using the statistical measures of mean and variance. He then establishes two formulae to calculate the expected return of the portfolio (E(Rp)) and the risk or variance associated to that portfolio (σ_p). These are:

$E(R_p) = \sum_i w_i E(R_i)$, where w_i represents the weight of each asset in the portfolio, and $E(R_i)$ the return of that specific asset.

$\sigma_p^2 = \sum_{i,j} w_i w_j \sigma_i \sigma_j \rho_{i,j}$, where w_i and w_j are the weights of two assets, σ_i and σ_j are the risk of two assets, and $\rho_{i,j}$ is the correlation between assets i and j.

These formulae can be simplified for a two-asset portfolio, and these would be:

$$\boldsymbol{E(R_p) = w_A E(R_A) + (1 - w_A) E(R_B)}$$

$$\boldsymbol{\sigma_p^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \sigma_A \sigma_B \rho_{A,B}}$$

In section 4 of this paper, these formulae would be put in place by creating a fictional portfolio of assets, in order to provide a practical view on each of the models for the purpose of comparing the results.

3.1.2 Evaluation of Markowitz's Model: Applications and Limitations

In this section, the advantages, and disadvantages of using the Modern Portfolio Theory will be discussed.

Markowitz presents the concept of diversification, which is widely used nowadays. It is a tool used to avoid financial ruin, as investors do not rely on a sole financial asset for financial stability, which may allow for the desired (or maximum) portfolio's expected return with reduced risk. This model also helps evaluating and managing returns and risks associated with the investments, which allows the investor to find over and underperforming assets by looking at the Efficient Frontier. Furthermore, this method's simplicity and ease of use, make it very popular.

On the other hand, this theory carries some disadvantages which is why many models have been developed after this one. The model takes under consideration past data regarding the performance of the company, in order to estimate the future return and risk. In terms of accuracy, this might not be the best estimation as future stock market situations may be completely different to past ones, and some risks which were not present in the past, may play an important role in future performances (for example technological advances). In addition, the MPT is based on the idea that the portfolio can be diversified by selecting stocks that are decorrelated, but market historians have proved that in times of market stress, originally independent assets can act as though they are correlated. Moreover, this theory assumes that returns follow a normal distribution which might not be the case as correlation between equities might change over time. Furthermore, the assumptions that this model makes do not represent reality as in the real world, there are taxes and transaction costs, investors sometimes do not make rational decisions or there is some information that might not be available to the general public. All these make it difficult to fully extrapolate this model to the real world. Finally, while this method helps you diversify, there is no precise way to know how many financial assets you should include in your portfolio in order to diversify the maximum amount of risk possible.

3.1.3 Tobin's contribution

In 1958, Tobin made some additions to the Modern Portfolio Theory. In his paper he included the possibility of including the risk-free asset (T-bills or sovereign bonds) to diversify the portfolio with lower risk. He also allows for the investor to take a short position in the risk-free asset, and long stocks. Shorting is a trading technique where an investor borrows a security and sells it, with the intention of buying it in the future. Going off of this, he introduced the Capital Market Line, which is used to find the tangency portfolio in the efficient frontier, or in other words, the most efficient portfolio for our investor's needs.

Capital Market Line

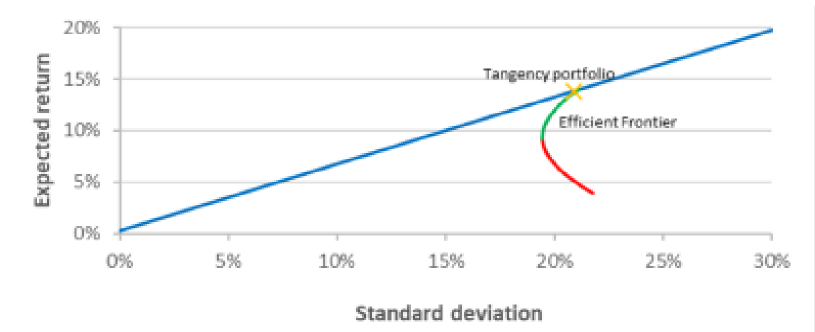

Figure 3

The line crosses the y-axis at the risk-free asset's return as it is the asset with a standard deviation of 0. To calculate the slope of this line, the most-commonly used method is the Sharpe ratio, developed in 1966. Its formula is:

$\boldsymbol{S = \frac{E(R_i)-E(R_f)}{\sigma_i}}$, where E(Ri) is the return of the stock held, E(Rf) is the return of the risk-free asset, and σ_i is the standard deviation/risk of the stock.

3.2 Sharpe and Lintner, and the CAPM (Capital Asset Pricing Model)

Between 1964 and 1965, Sharpe and Lintner developed the Capital Asset Pricing Model. Though Lintner made some additions on Sharpe's model, they are both considered its joint creators.

3.2.1 Concept

As mentioned in the previous section, Markowitz's model is built on the basis that investors are risk averse and are only interested in their one-period mean-variance relationship. This leads to them choosing their portfolio following two main criterions: minimizing risk and maximizing returns. Sharpe and Lintner add two key assumptions to Markowitz's model to identify a portfolio that must be mean-variance-efficient (Fama & French, 2004), though for this model to work, nine assumptions should be taken into account (Boďa & Kanderová, 2014). These are:

1. The goal of investors is to maximize the level of utility associated to their terminal wealth.
2. By assessing the risk-return trade-off, investors decide on the assets to include in their portfolios.
3. Investors are assumed to be rational and risk averse.
4. All investors hold homogenous expectations regarding risk and return.
5. Investors have the same time horizons for their investments.
6. Information is accessible and free to the general public, and assuming market efficiency, all price changes will be reflected immediately.
7. Investors can make use of the risk-free asset, which is the same to all investors, to diversify the risk in their portfolios. They can borrow and lend at that rate without restrictions.

8. Short-selling is a form of speculation, which occurs when an investor borrows a security and then sells it in the open market, with the expectations of buying it back in the future at a lower price, and benefiting from that price difference. In the CAPM, it is assumed that there do not exist any restrictions on short selling, nor taxes and transactions costs.
9. All securities are divisible and marketable. Divisibility means that an investor is not obliged to purchase a full share and can buy only a fraction of it. Moreover, when a security is marketable, it means that it can easily be bought or sold in the market.

The Capital Asset Pricing Model (CAPM) further expands on Markowitz and Tobin’s analysis on the optimal selection of a portfolio by introducing the concept of a capital market equilibrium model. Sharpe was driven to create his model due to the “absence of a body of positive microeconomic theory dealing with conditions of risk” (Sharpe, 1964). This issue has been in the minds of those trying to foresee the behavior of capital markets, as traditional models, even if they include the concept of uncertainty, do not directly asses or quantify the exact level of risk associated to that investment. To solve this matter, the CAPM uses the β, previously explained in this paper, to measure the asset’s (usually stock) sensitivity to systematic risk (Boďa & Kanderová, 2014). Moreover, it shows that in an efficient market the rate of return of any risky financial asset is a function of its covariance with the rate of return of the market portfolio. This market portfolio is a theoretical and diversified portfolio, consisting of every single asset available in the market, measuring each of them in proportion to its total market position. In other words, the risk premium of an asset is equal to its beta multiplied by the risk premium of the market portfolio. The CAPM formula is:

$$\boldsymbol{E(R_i) = R_f + \beta_i\left(R_m - R_f\right),}$$

where $E(R_i)$ is the rate of return for the company, R_f is the risk free asset’s rate, R_m is the rate of return on the market, and β represents the systematic risk of the company, which can be also expressed by $Cov(R_i, R_m)/Var(R_m)$. Under this model, the risk-return trade-off is still present, as when the beta associated to our stock increases, our return increases. This portrays a positive relationship between the two components.

To graphically represent the equilibrium relationship between risk and return, the Security Market Line (SML) was developed. This line represents the position of equilibrium portfolios chosen by investors. The SML is depicted as:

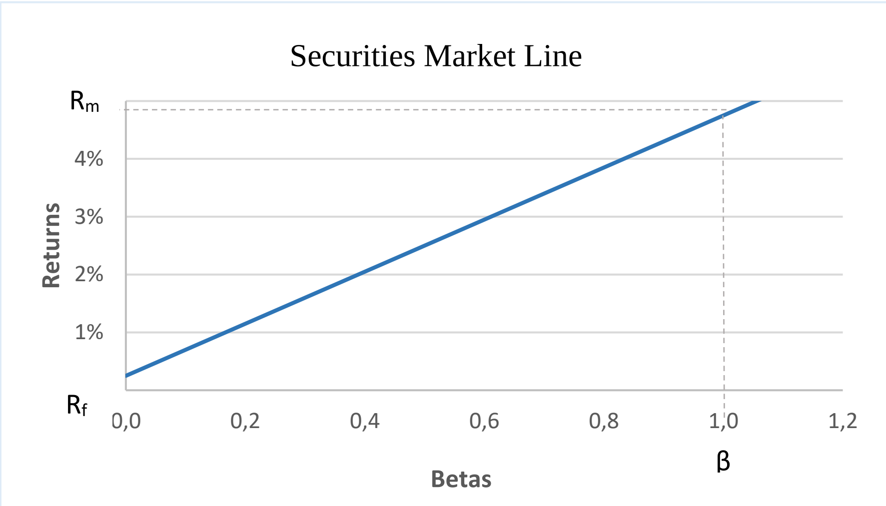

Figure 4

This model allows us to diagnose whether a financial asset is overvalued or undervalued in the stock market. Assets holding a higher systematic risk than the market (β=1) should offer a higher return than the R_m, and vice versa. Stocks with a beta higher than 1 are considered aggressive positions due to their associated risk, and stocks with beta lower than 1 are defensive stocks. In equilibrium, all assets and portfolios should be valued so that their required return is consistent with their levels of systematic risk and therefore graphically on the Security Market Line. Any assets above the SML should be considered undervalued as the estimated return of that asset is higher than the return it will receive according to its beta. On the other hand, assets that fall under the SML are overvalued as their expected return is lower than its actual return (using the CAPM formula). To calculate this numerically, we use a variation of the Sharpe ratio, called the Treynor ratio. Treynor's ratio relates excess profitability on the risk-free return earned by a portfolio with the beta of that portfolio. The formula is:

$$\boldsymbol{T = \frac{E(R_i) - E(R_f)}{\beta_i}}$$

It differs from the Sharpe ratio used in the CML, in that risk is measured with beta, meaning that only systematic risk is considered. Moreover, due to the fact that the risk of the portfolio is only systematic or undiversifiable risk, this ratio assumes that the portfolio is fully diversified. Therefore, the Sharpe ratio would give us the same result for perfectly diversified portfolios. This ratio can be used to find whether a portfolio or asset is undervalued or overvalued. Portfolios that have a higher than market ratio are those that are above the SML and are worth investing in, and vice versa. Thus, from this we conclude that Treynor's ratio also gives us the slope of the SML. On the other hand, there are some limitations of using Treynor's ratio for obtaining over and undervalued assets, even if is the correct formula for the slope. As mentioned before, the Sharpe ratio would give us the same result for diversified portfolios, but for poorly diversified ones, the results obtained from each ratio will vary, and if that is the case, it would be more precise to use the Sharpe ratio.

3.2.2 Evaluation of the CAPM: Applications and Limitations

In this section, the CAPM will be evaluated, and its applications and limitations will be described.

Over five decades later, the CAPM is still commonly used by financial managers for a variety of reasons, ranging from estimating the cost of equity, to valuing stock to invest in. There are several advantages of using the CAPM instead of other methods to calculate expected return. As a model, it is fairly simple and easy to use, which makes it an effective tool when presenting the asset pricing and portfolio theory concepts. This is one of the main reasons why it is still very popular as it takes little time and gives somewhat accurate representations of reality. Moreover, it only takes into account systematic risk, which is close to the reality of most investors, who own diversified portfolios with reduced or non-existent unsystematic risk. Other models, which quantify total risk, may give those investors a less accurate representation of the expected return to be obtained with their portfolios. Furthermore, in M&A (Mergers and Acquisitions) transactions, some models such as the Weighted Average Cost of Capital (WACC) cannot be used due to the variability in the business mix and financial risk between companies, but the CAPM can be used.

As all models, the CAPM is only an attempt to represent reality in the most accurate way possible, but as all models, it has its flaws. The CAPM started to spark criticism in the 1970's, when many researchers and economists started to question the use of the model for explaining the risk-return tradeoff and market risk. In 1977, economist Richard Roll tested the CAPM and draw some important conclusions (Boďa & Kanderová, 2014). Firstly, he criticized the idea of the market portfolio, for he believes that a fully diversified one is unattainable because it would comprise both publicly traded assets and non-traded assets, such as some real estate. Portfolios or ETFs (Exchange Traded Funds) composed of all the assets in a given stock exchange, such as de S&P 500, are only a proxy for a fully diversified portfolio. Another criticism of the model arises from the fact that some relevant factors, such as the size of the firm or other financial indicators, should be included for the purpose of increasing its adequacy and specificity. Moreover, in order to use the CAPM, the user needs to assign values to the risk-free rate, the market risk premium, and the beta, and we find issues in each of them. For the risk-free rate, the common substitute is the yield of government bonds. The issue with this approximation stems from the fact that the bond yield varies with the economic situation, and that volatility can alter the results of the CAPM from one month to another, without the expected return being actually higher or lower. However, the impact of these changes can be diminished through the use of averages. When estimating the equity risk premium (ERP), we encounter more difficulties. The expected return of the market is usually calculated by the addition of the average dividend yield, and the sum of the mean capital gains. If at any given point, the market return is adverse, due to falling share prices exceeding the dividend yield, some issues or inaccuracies may arise. This is why it is common to use a long-term average market return, though, it has been proven that the ERP is not constant over time; it fluctuates. Thus, the calculation of the CAPM will not be precise. Finally, the estimation for the Beta, can also generate additional errors. As with the ERP, we encounter that it is not stable over time and therefore, alters the result for our expected return.

Lastly, we can conclude, that as a model, it can be useful to use from a theoretical perspective, but also as a tool to complement other calculations for the expected return. However, it should not be used individually as it is not the most precise model, as it is built on many assumptions and estimations that may not resemble reality.

3.2.3 Later Contributions to the Model

Due to the high amount of criticism the model has sparked as a result of its inaccuracies, many economists and researchers have made contributions or improvements on the CAPM. In this paper, we will be discussing three of them: the Intertemporal CAPM, the Consumption CAPM, and the Downside CAPM.

3.2.3.1 Intertemporal CAPM

Robert Merton developed in 1973 the Intertemporal Capital Asset Pricing Model. With his model, he provides an alternative capital market equilibrium model, which is, in terms of simplicity, similar to Sharpe and Lintner's CAPM. This model also takes into consideration the principle of maximizing utility and limited liability of assets, and it also provides a description of the relationship between yields, which is more coherent with empirical evidence than the CAPM (Merton, 1973). It also states that the equilibrium of the beta is given by the utility function of the consumption of the investor, finding the optimal point when the marginal utility of the consumption is equal to the marginal utility of wealth. Moreover, the Intertemporal CAPM defines that the excess return on assets, will be given by a "multi-beta", that will be equal to the number of state variables used in the description of the opportunity set, plus one. It should be noted that this model has also been subject to some criticism, as in order to achieve the three points that have just been mentioned, some assumptions or expectations have to be made.

Some of the assumptions are similar to the CAPM as they are standard assumptions for the perfect market, and there is a new one added to this model. The ICAPM, unlike the CAPM, is an intertemporal model, meaning that it is not static over time. This acknowledges the fact that investors normally engage in investment activities for multiple years, and consequently, develop dynamic strategies that change with market conditions and risk. Therefore, the assumptions of this model are:

1. All assets have limited liability.
2. There are no taxes, nor transaction costs.
3. The market is always at equilibrium.
4. There are enough investors, with similar wealth levels. Each investor believes that he can purchase and sell as much of an asset as he desires, at the market price.

5. Short selling is permitted.
6. There is a stock market where all investors can borrow and lend at the same interest rate.
7. Trading of assets occurs repeatedly and continuously over time.

After describing the assumptions of this model that determine the structure of the capital market, we will go on to explain the dynamics of the assets being traded in the market, and their associated expected returns. According to Merton, for an investor to be able to make an informed decision on the assets to invest in, it is only necessary that he knows two pieces of information. Firstly, he needs to know the transition possibilities of all the assets' returns over the next trading interval (which Merton calls the opportunity set). Secondly, they need to be aware of the future periods' transition possibilities of the returns. With this model being more dynamic, but mostly, as it englobes several time periods (unlike the traditional static CAPM), investors need to take into account the relationship between returns in the present time, and the available returns in the future. The reasoning behind this is that if we find a negatively correlated asset with changes in capitalization rates (its return goes up as capitalization rates go down, and vice versa), then the investor can expect that if yields next period are lower than expected, his expected return will be higher.

This model introduces a new CAPM formula, which is:

$$\boldsymbol{\alpha_i = r_f + \beta_{im}(\alpha_m - r_f) + \beta_{ih}(\alpha_h - r_f),}$$

where α represents expected returns, m represents the market portfolio, and h is a portfolio used to hedge the risk. This new variable h includes all macroeconomic factors, such as the GDP or interest rates, that impact the performance of the portfolio depending on the state of the economy. The essence of the formula is the same as in the CAPM but provides the investor with the possibility of hedging his position.

In the end, this model is considered a consumption CAPM, as it attempts to contribute to the original CAPM by adding that investment behaviors remains over time (not just one period), and that investors hedge their risky positions against changes in the future opportunity set, or falls in consumption.

3.2.3.2 Consumption CAPM

Robert Lucas and Douglas Breeden developed between 1978 and 1979, a contribution to the CAPM, which they called the Consumption CAPM (C-CAPM). As we mentioned in the previous section, the Intertemporal CAPM is also considered a consumption CAPM, though its most relevant characteristic is its continuous-time economic framework. Lucas and Breeden's Consumption CAPM, also follows the intertemporal extension, but takes the "multi-beta" modeling of Merton's CAPM, and collapses it into a single beta. With this beta, the correlation between the asset's expected return and the beta is observed, which is proportional, with respect to aggregate consumption alone. Due to this characteristic (single-beta), this model is much easier to test and implement compared to others (Breeden, 1979).

In this model, it is assumed that there is only one good available that either individuals, or firms invest in. Under this model, the economy is presumed to be populated by many households that are identical in their investment preferences, as well as their level of wealth. Markets are frictionless and assumed to be perfect, but incomplete in some senses, and investors are price-takers under this scenario. It also assumes, as in the Intertemporal CAPM, that investors can trade continuously, and that short selling is allowed. Moreover, trading only occurs at equilibrium prices, and it is also believed that investors hold the same ideas on the probability of different states of the world. Finally, investors' wealth is represented by either an instantaneously riskless asset, or by shares of a risky asset.

This model relies on aggregate consumption, and risky assets create uncertainty in this consumption as they depend on investors' wealth, and how they decide to spend or invest it. The amount of risk this generates can be quantified using the risk premium over consumption growth, which is also called the "consumption risk". This "consumption risk" will represent the amount of uncertainty on the investor's consumption, generated by holding said asset. Thus, in reality, this model can help us understand how the stock market shifts, according to the movements in consumptions (growth or declines). The expected return under the C-CAPM, can be calculated with the following equation:

$$\boldsymbol{E(R_i) = R_f + \beta_{i,c}(R_m - R_f)},$$

where R_f is the risk-free rate, $\beta_{i,c}$ determines the volatility of the expected return in relation to possible changes in consumption needs, and $(R_m - R_f)$ is the market risk premium.

From a theoretical perspective, the C-CAPM is widely used, but it normally is not used empirically. Firms tend to use the traditional CAPM rather than the C-CAPM, despite the inaccuracies explained before, as the C-CAPM does not have a good performance on empirical terms. This is mainly due to the fact that, as the model assumes that all investors actively participate in the market, and are almost identical, which is not consistent with real life, the link between expected stock returns and consumption does not hold. On the other hand, this model is extensively used academically, as it helps to understand the change in returns over time and considers other types of wealth apart from stock market wealth. Finally, it is a very useful tool to showcase the relationship between consumption and wealth, which will later determine an investors risk aversion.

3.2.3.3 Estrada's Downside CAPM

Almost 40 years later after Sharpe and Lintner published their CAPM, Javier Estrada developed what he called the Downside CAPM. He was concerned about the traditional way of estimating the beta, as he argues that the measurement of a financial asset's risk by means of the variance of its returns, is inaccurate, given that these returns should follow a symmetrical and normal distribution, and this does not occur in real life with all relevant factors of the economy (Estrada, 2002). As a solution to this, Estrada proposes a new way of estimating the beta, which he calls the "Downside beta".

With the Downside beta, Estrada measures risk by using the semi variance of the financial securities' returns, as, like he explains, the semi variance considers investors' risk aversion when there is a negative variation on risk. This means that investors are not against the volatility of returns, but they dislike is when this volatility shows a downward tendency. In addition, the use of the semi variance, can be useful for both symmetric and asymmetric distributions, and it merges variance and deviation information from the data to be used. The use of the semi variance is also a simple method to calculate risk, leading to the creation of the D-CAPM, a model based on the mean-variance relationship.

In the traditional CAPM, utility was defined as the relationship between the mean and the variance (Mean Variance Behavior, or MVB), while in the D-CAPM, it is determined by the mean and semi variance (Mean Semi variance Behavior, or MSB). The semi variance is calculated as:

$$\boldsymbol{\Sigma_i = \sqrt{\mathrm{E}\{\mathrm{Min}[(R_i - \mu_i), 0]^2\}}} \ ,$$

where R_i is the expected return of the asset, and μ_i is the mean. The relationship between an asset and the behavior of the market, can be described with the cosemivariance, whose formula is:

$$\boldsymbol{\Sigma_{i,M} = \mathrm{E}\{\mathrm{Min}[(R_i - \mu_i)\mathrm{Min}[(R_M - \mu_M), 0]\}}$$

Both these formulae, attempt to build a CAPM, following the differences explained by Estrada. The next logical step, would therefore be, defining the formula for the correlation, that will be later used to obtain the beta. The formula is as follows:

$\boldsymbol{\theta_{i,M} = \frac{\Sigma_{i,M}}{\Sigma_i \Sigma_M}}$, which is used to calculate the downside beta as: $\boldsymbol{\beta_i^D = \frac{\Sigma_i}{\Sigma_M} \cdot \theta_{i,M}}$

Now, all the necessary data to build the CAPM is available, but replacing the traditional beta with the downside beta:

$$\boldsymbol{E(R_i) = R_f + \beta_i^D \cdot (R_M - R_f)}$$

There are several conclusions that can be extracted from Estrada's analysis, Firstly, he is able to portray that MSB is as plausible as an MVB model, and that, from an empirical perspective, the Downside beta has proven to be more accurate than the traditional beta. Why? Estrada put his model to practice with emerging markets, and based on the results obtained, he concluded that the Downside beta is the most significant risk measure, as it explains approximately 55% of the variability of expected returns in a cross-section analysis. Moreover, under the D-CAPM, required returns on equity, are on average, around 2,5% higher than with the CAPM. This difference, even if it seems small, can drastically determine whether an investment is made or not, and highly impacts the valuation of companies.

On the other hand, the D-CAPM has also faced some criticism. In 2009, Sergei Cheremushkin, published an article, "Why D-CAPM is a Big Mistake? The Incorrectness of the Cosemivariance Statistics", in which he criticizes Estrada's model. One of the critics is that the model is not consistent with portfolio theory, and that it does not hold the principle of portfolio diversification. Moreover, even if the semi variance calculations are correct, the measure of the cosemivariance is inaccurate, and is not useful to represent real life situations, and the actual dependence between two assets. Finally, this calculation also ignores the fact that investors can use the positive returns of one asset, to offset or hedge the falling returns of another asset in the portfolio, which is something that investors tend to do.

3.3 Arbitrage Pricing Theory (APT)

Economist Stephen Ross developed the Arbitrage Pricing Theory (APT) in 1976, which is considered to be an equilibrium model. In this section, the characteristics of the model will be explained, and the advantages and drawbacks of its use will be presented.

3.3.1 Concept

The APT model is a valuation model that was created in 1976 by Ross. It is a more complex model compared to the MPT or the CAPM, as it includes several variables (it is a multifactor model), and it requires a more tailored analysis for each specific case. This complexity is one of the main reasons why it is not as popular as other models. To understand the APT, the concept of arbitrage must be explained, which can be defined as the possibility of an investor of taking advantage of mispricing through the sale or purchase of an asset. Unlike the CAPM, the APT assumes that markets sometimes misprice securities. However, these mistakes generally get corrected through as investors create their portfolios in search for arbitrage profits. Once all opportunities are exhausted, an equilibrium is reached, and prices return to their fair value, which is why it is almost impossible to benefit from these price inefficiencies. Therefore, under the APT, price efficiency exists when all arbitrage opportunities have been eliminated by arbitragers.

The model attempts to determine the price of an asset, but instead of being a single-factor model, like the CAPM, it is a multifactorial one. Thus, it tries to explain the performance of an asset through several independent influencing factors. Moreover, it does not assume the efficiency of a portfolio, but calculates the return on an asset itself based on its linear relationship to several factors, which are mainly macroeconomic, and which largely impact the performance of the asset or company. This theory is based on the idea that the risk premium of an asset derives from the risk of the different factors that influence the company.

Like all models, the APT also makes some assumptions. The most relevant are:

1. The model assumes that there is perfect competition on capital markets.
2. A macroeconomic/multifactor model can describe the expected return of an asset, and its correlation with risk effectively.
3. Investors want to maximize their level of utility, by means of maximizing their returns and minimizing their risks. They are able to achieve this through portfolio diversification.
4. Assets carry limited liability.
5. All investors hold homogeneous expectations.

Mathematically, this model follows this formula:

$$\boldsymbol{E(r_j) = r_f + \beta_{j1}F_1 + \beta_{j2}F_2 + \ldots + \beta_{jn}F_n + \epsilon_j},$$

where $E(r_j)$ represents the expected return, r_f is the risk-free rate, β_{jn} is the asset's sensitivity to the n factor, F is the risk premium for each macroeconomic variable, and finally, ϵ_j is the asset's zero-mean error. From this formula, conclusions can be drawn regarding that, the expected return is given by the riskless return and the sum of the beta coefficients of each factor. Beta is usually estimated through regression, using either a multivariate or the least-squares model. The risk premium attributed to each macroeconomic factor, derives from the factor's return minus the risk-free rate.

If when applying this model, the investor finds an asset whose beta coefficients are 0, it is either a risk-free asset, or that the factors that have been chosen do not have any correlation with the asset (Çetin et al., 2003). Therefore, the risk-free rate of return should be attributed to this asset, and here is where arbitragers comes in, as the asset might be mispriced. If the return obtained with the APT formula is higher than the risk-free rate, then there is an arbitrage opportunity, that the investor can benefit from if he buys the asset; it will be undervalued. If the situation is the opposite, then the investor should sell the asset as it is overvalued.

As mentioned before, the model is based on different market factors, which are the determinants of risk. The APT assumes that portfolios are diversified, so that there is no unsystematic risk. Therefore, the risk that is measured by these factors is systematic risk. These macroeconomic factors vary from one asset to another, but the most reliable ones include: Gross National Product (GNP), inflation, shifts in the yield curve and corporate bond spreads. In addition, there are different indices that can be used, which are specific to the industry to which the company belongs to. For example, for a car manufacturer, the change in steel prices might be very relevant, but not for a clothing company.

3.3.2 Evaluation of the APT: Applications and Limitations

The APT is a model that sparked some criticism, though there are some advantages that favor its use. They will be explained in this part of the paper.

As it has been previously mentioned, the APT is a complex model, that builds on the CAPM. Some economists favor its use as it quantifies risk by taking into account many different macroeconomic factors, and therefore, the impact on returns of each specific variable can be measured. This can be useful for companies to realize if they are dependent on any asset (for example on oil) and can use this information towards developing strategies to reduce this dependency. Moreover, compared to other models, it has less assumptions, or at least, the assumptions are closer to reality. This makes it a more realistic model, from which more precise results can be obtained. Furthermore, the APT model includes into the formula the zero-mean error, which represents the existence of unanticipated changes. As unexpected risk is already considered in the calculations, it is easier for investors to anticipate which stocks have growth potential looking only at the opportunity itself. Lastly, the model is based on the concept of arbitrage, and thus allows investors to find these opportunities, from which they can highly benefit.

On the other hand, this model does not come without criticism. When including the different factors for modeling, it is difficult to know which factors to include, as the model does not provide any insights in this regard, and how many of them are relevant enough to be included. The model will only be useful if a sufficient number of relevant factors can be found and if each one of them can be determined and measured precisely, which in practice is difficult. In this sense, the APT only tells us that the expected return of the asset can be explained by several common factors, whose changes affect this profitability, but which are unknown. Moreover, the model assumes a linear relationship between factors and returns, when in reality, this is not necessarily the case, for example, when factors have a certain correlation. In terms of complexity, it is a more complete model, with regards to the number of factors, but it takes more time and expertise to apply.

3.4 Fama and French Three-Factor Model

Eugene Fama and Kenneth French developed between 1992 and 1993 the Three-Factor Model, which appeared as an extension of the CAPM to conduct an asset valuation that captures the returns on assets in a more accurate manner.

3.4.1 Concept

The Three-Factor Model expands on the CAPM by adding two additional factors: the size of firms and the book-to-market value. In order to do so, they include two new variables, which they call Small Minus Big (SMB) and High Minus Low (HML), and a third variable being the expected excess returns, calculated by subtracting the risk-free rate from the expected return of the asset (Fama & French, 1992).

In the development of their model, Fama and French considered many different variables such as the Price-to-Earnings Ratio (PER), the company's leverage ratio, its size, and the book-to-market ratio. The book-to-market ratio shows if an asset is over or undervalued, by comparing its book value to its market value. After analyzing each of them and their impact on expected returns, they reached the conclusion that the size of the company and book-to-market ratio were those that had the greatest explanatory power over the change in returns, when analyzed together with a third factor, the expected excess returns. Their study was conducted on companies listed in different American Stock Exchanges (NYSE and NASDAQ), from 1963 to 1990, where they studied their evolution and the impact of each of the variables mentioned.

Based on their findings, incorporated the two new variables that were previously mentioned: SMB and HML. These can be explained as:

- SMB (Small Minus Big): This is the difference between the return of the smaller asset and the return of the largest asset in the portfolio, which is sometimes called the size premium. Moreover, it measures the additional expected return that historically has been obtained by investors through investing in different assets belonging to companies with a relatively small market capitalization. This variable assumes that smaller companies are more sensitive to risk factors than larger companies because they have less capacity to absorb negative factors in the economy. Once the SMB has been identified, its beta coefficient (β) can be obtained through linear regression. This beta coefficient can take positive values, as well as negative ones. According to this model, in the long run, small companies tend to outperform compared to larger companies, and value companies tend to overperform growth companies.

- HML (High Minus Low): This is the difference between the return of the assets from the company with the highest book-to-market ratio and the return of assets of the company with the lowest book-to-market ratio. It can also be called the value premium. It measures the additional expected return that investors have historically acquired by investing in different assets of companies with a high book-to-market ratio. The assumption behind this variable is that those companies with a higher book-to-market ratio are more vulnerable to financial shocks than those companies with a lower book-to-market ratio.

This model can be calculated by using the following formula:

$$\mathbf{R}_i = \mathbf{R}_f + \beta_{\mathrm{i}}\,(\mathbf{R}_M - \mathbf{R}_f) + \beta_{SMB}\,SMB + \beta_{HML}\,HML\ ,$$

where R_i is the expected return of the asset, R_f is the risk-free rate, β_{i} is the risk associated to the asset, R_M is the return of the market, β_{SMB} and β_{HML} represent the sensitivity of the assets' returns to variations in the corresponding premiums (size and value), and finally, SMB and HML are the returns of the Small Minus Big and High Minus Low factors respectively.

Fama and French were able to prove that the model explains more than 90% of diversified portfolios' returns, which is a significant amount compared to the 70% that is explained by the CAPM. Moreover, they found that higher returns are correlated to a higher β, and a smaller size, like mentioned before.

3.4.2 Evaluation of the Fama & French Three-Factor Model: Applications and Limitations

Like all models, the Fama French Three-Factor Model is just an approximation to reality, and it is useful to some extent, but there are some limitations to its use, which will be now explained.

As an extension of the CAPM, this model has shown some improvements, compared to its predecessor. This model gives us a more accurate description of reality, even if obtaining the SMB and HML factors is complex. Moreover, the model is a useful tool that allows investors to understand stock market returns and helps them build portfolios suited to their desired level of risk. Investors can now create portfolios, bearing higher expected returns than the global portfolio, through the identification of the true priced risk sources that generate returns and managing their exposure to these factors. They can do so through the use of passive strategies, which can be done at low costs.

Contrarily, this model has also faced lots of criticism. Some argue that the estimation of the SMB and HML factors is complicated and time consuming, which is why analysts tend to favor the CAPM, even if it is slightly less precise. In addition, there is much controversy about the basis of the effects and whether they are a result of market efficiency, as both factors are derived from empirical studies and not from fundamental economic or financial theory. Critics argue that the reason why a company with a smaller market cap has a higher return is because the change has a greater relative effect (for example, an increase in price by 1 dollar on an asset quoting at 10, is higher than for an asset quoting at 100). Furthermore, it is not known whether the effect is due to an efficient or an inefficient market. On the one hand there are authors who defend that the reason for the higher return is due to the higher risk, while, on the other hand there are authors who suggest that the higher return is due to inefficient markets and an undervaluation of the companies that is corrected over time (Reilly & Brown, 2012). Finally, analyzing the past can be useful, but it is not wise to assume that what has occurred in the past, will happen again in the future, without taking into account other factors.

3.5 Jegadeesh and Titman's Momentum Strategies

In 1993, Narasimhan Jegadeesh and Sheridan Titman released their article "Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency". In this article, they investigate stock market efficiency and question whether it can be rejected or not.

3.5.1 Concept

In their paper, Jegadeesh and Titman first take a look at market efficiency, and do so by examining profitability, through the use of several strategies. They gathered data from the NYSE, specifically its daily returns, from 1965 to 1989, which they later used to create relative strength strategies. Relative strength strategies are based on the idea that stocks that outperform the contemporary market are winners, while those stocks who underperform are considered losers, and the specifics will be explained later.

From this, the concept of "Momentum" is obtained. The momentum effect is defined as the strategy were investors go long on securities that have shown a good performance in the past and going short on securities that have had poor performances in the past. Jegadeesh and Titman were able to prove that this strategy yields positive returns in 3 to 12 months holding periods. In order to do so, they select stocks based on their performance over the past 3, 6, 9 and 12 months. Investors are also assumed to hold these assets for 3, 6, 9 or 12 months, which leads us to a total of 16 possible strategies or combinations. There is the popular belief, by investors and journalists, that stock prices overreact to information, and therefore, contrarian strategies (selling past winners and buying past losers) can give abnormal returns. However, Jegadeesh and Titman, in their 1990 paper, prove that even if this works in the short term, it may be due to lack of liquidity and not an overreaction, and can therefore reverse. To avoid these short-term problems, along with price pressures, delayed reactions or bid-ask spreads, this model includes a second set of momentum strategies, that skip a week between the portfolio formation period and the retention period (Jegadeesh & Titman, 1993). This means, that there are overlapping periods, that allow the test to be more accurate and powerful. Securities are then rated according to their performance in the last few months J, which are then allocated in ten decile portfolios. All securities have the same weighting within each portfolio. The upper decile is attribute to the decile "losers" and the lower decile is given to the decile "winners". Each month, investors buy the winning portfolio and sell the losing portfolio and hold this position for K months. The strategy will then close the position in month t - K. The most successful zero-cost strategy selects stocks based on their return in the last 12 months and holds the portfolio for 3 months.

Through their research, Jegadeesh and Titman conclude that profits from these momentum or relative strength strategies, are not caused by their systematic risk. Moreover, seasonality appears as an important factor to take into consideration. They found out that winners always outperform losers, except in January, where is the other war around. With all this, the evidence given by this paper is in line with the idea that price reactions are delayed to firm-specific information. Finally, they conclude that, with a certain level of significance, the traditional efficient market hypothesis can be discarded.

3.5.2 Evaluation of Jegadeesh and Titman’s Model: Applications and Limitations

The Momentum model has sparked some academic debate in the past years. In this section, the possible applications and limitations of this model will be analyzed.

Currently, this model is not commonly used by professionals, and instead, they are favoring strategies like DCF (discounted cash flows), or fundamental factors. Jegadeesh and Titman’s model, is only tested on the US stock exchanges, which is why some are averse to its use, as this behavior may not extrapolate to other countries. Moreover, many financial analysts believe that using momentum strategies leads to the disregard of the company’s fundamental valuable aspects and is just based on psychological predisposition for humans to assume past events will repeat themselves in the future. Some professionals are not keen of its use, due to the fact that it contradicts the hypothesis of efficient markets. In addition, this type of strategy is considered to be active as investors need to buy or sell stocks according to whether they fall on the category of winners or losers. The tendency towards passive strategies, due to their lower costs, as well as evidence that, in the long term, investment managers have not been able to beat the market, have led to investors not using momentum strategies. Finally, critics of this strategies, often claim that this type of investment is not meritorious as instead of focusing on the actual value of the company, investors try to benefit from market inefficiencies.

On the other hand, supporters of this model argue that even if markets are efficient as a whole, there might be times were inefficiencies occur, and before the market reflects new information on prices, investors can benefit from this. Moreover, advocates of active management believe that it is important for human interaction and intuition to play a part in their investments. Furthermore, Jegadeesh and Titman's model has been proven in the US market, which means that in that scenario, it can be a useful model to be used.

3.6 Carhart's Four-Factor Model

In 1997, Mark Carhart developed an extension of Fama and French's Three-Factor Model, which included momentum as a factor. This model will be explained in more detail in the following section.

3.6.1 Concept

In his paper, "On Persistence in Mutual Fund Performance", Mark Carhart develops a model which combines characteristics from both Fama and French's Three-Factor Model (1993), and Jegadeesh and Titman's Momentum Model (1993). The 4-factor model is a multifactor model that includes the three factors from Fama and French's model, size of firms, book-to-market value, and excess return on the market, as well as the one-year momentum anomaly. This new factor was added as Carhart believed that the 3-factor model could not be used to explain cross-sectional variables in returns in momentum portfolios. This model is considered to be an equilibrium model (Carhart, 1997).

Momentum (MOM) can be described as the propensity for the securities price to keep increasing if it is going up and continue going down if it is falling. It can be calculated by calculating the equal weighted average of the best/highest performing firms and subtracting the equal weighted average of the lowest performing firms from it. The model's formula could then be defined as follows:

$$\mathbf{R}_i = \mathbf{R}_f + \beta_i (\mathbf{R}_M - \mathbf{R}_f) + \beta_{SMB}\, SMB + \beta_{HML}\, HML + \beta_{MOM}\, MOM + \epsilon_t \ ,$$

where R_i is the asset's expected return, R_f is the risk-free rate, β_i is the risk associated to the asset, R_M is the return of the market, β_{SMB}, β_{HML} and β_{MOM} represent the sensitivity of the assets' returns to variations in the corresponding premiums (size, value and momentum), ϵ_t represents the residuals of the regression model, and finally, SMB and HML are the returns of the Small Minus Big and High Minus Low factors respectively, and MOM (or WML, Winners Minus Losers) is the return of the momentum. Winner stocks will be in the top 30% of the data, while losers will be at the bottom 30%.

As it has already been mentioned, the momentum factor describes the tendency of growing prices to continue growing and falling prices to continue falling. Therefore, if an investor decides to buy stock whose price has been increasing lately, and other investors follow, the simple belief that the stock price will continue to increase, will lead to the actual price rising. This factor hints that investors who are able to predict which stocks will be praised in the market, will be capable of earning high profits. Moreover, the four-factor model has been later applied by Fama & French, and they were able to prove the momentum factor exists and in statistically relevant for stock-markets in Europe, North America, and the Asia Pacific Region (Fama & French, 2011).

3.6.2 Evaluation of Carhart's Four-Factor Model: Applications and Limitations

In this section, evaluations of the model will be carried, focusing specifically on its applications and limitations.

In Carhart's model, it is evident that the model is capable of explaining significant variations in yields. It indicates that it is a precise model in which the different factors have no correlation to each other. As the models increase the number of factors to be taken into account, the amount of error is reduced, being higher in the CAPM (0.35%), lower in the Three-Factor model (0.31%) and even lower in the Four-Factor model (0.14%), as explained in Carhart's report (Carhart, 1997). As seen before, with Fama & French's Three-Factor Model and Jegadeesh and Titman's Momentum, compared to previous models, Carhart's Model provides a deeper analysis and insights to real market factors, which makes the model more accurate. Taking into account that the factors represented are characteristics that are systematically valued, the analysis of the components of those factors allows us to find the characteristics that investors are concerned about.

Some limitations to the use of this model are also found. As already seen with its predecessor models, Carhart's Four-Factor Model is complex in practice, as it is difficult to estimate each of the variables used. Moreover, each extended model, like the Four-Factor Model, include more factors than previous models, but there are two issues that still persist. First of all, it is difficult to know how many factors should be included in the model to make it as accurate as possible, and secondly, it is difficult to determine which factors are "the winners", in terms of the relevance of each of them.

3.7 Fama and French Five-Factor Model

Eugene Fama and Kenneth French, released in 2015 their Five-Factor Model, which is an extension to their model in 1993.

3.7.1 Concept

In 1993, Fama and French introduced their Three-Factor model, which included the size and value factors to the CAPM. This model was an improvement on the CAPM, but it still failed to explain some abnormalities that arose from the effect of some variables that were not taken into account in the model. The Fama French Five-Factor Model added two factors, profitability and investment, which were not included in their previous model, but had proven to be relevant factors as they led to variations in average returns (Fama & French, 2015).

This model builds on the DDM (Dividend Discount Model), as this model is based on the idea that the value of a stock today, depends on future distribution of dividends. From this model, Fama and French extract the variables of profitability and investment. Therefore, the Five-Factor Model includes size, book-to-market ratio, expected excess returns, profitability, and investment. Fama and French, then go on to compare the performance of the Three-Factor model, with this model, to see if the anomalies found in the former, are solved by the latter.

The mathematical representation of this model is as follows:

$$\boldsymbol{R_{it} - R_{ft} = a_i + b_i(R_{Mt} - R_{ft}) + s_i SMB_t + h_i HML + r_i RMW_t + c_i CMA_t + e_{it}} ,$$

where R_{it} is the return of one portfolio in month t, R_{ft} is the risk-free rate, $(R_{Mt} - R_{ft})$ market risk premium or the spread between the return of the market and the risk-free return, SMB is the small minus big spread, HML is the high minus low spread, RMW is the spread of returns from most profitable firms minus the least profitable, and CMA is the return spread for firms that invest conservatively minus those who invest aggressively. Finally, a_i, b_i, s_i, h_i, r_i and c_i are the "betas" or risks that measure the sensitivity of the model to each of the five factors.

This formula or regression test, is used to see whether the 5-Factor Model is precise when capturing average returns, observe if there exists any correlation between variables, and if there exists any correlation, if it is negative or positive. Moreover, it is useful to identify how each of the components affects overall average stock returns. Fama and French perform in their 2015 some tests to prove the validity of their model and draw conclusions regarding the different variables included are relevant. In the case of the HML factor, it was concluded that in some situations, if the model includes profitability and investment factors, it is redundant to include this variable. The final results conclude that this model explains between 71% and 94% of the cross-section variance of expected returns for the examined size, profitability, value and investment portfolios (Fama & French, 2015). This proves that, compared to the Three-Factor Model, the Five-Factor Model performs better as fewer average returns are left unexplained. Furthermore, the new model reaches the conclusion that companies that are small, profitable and value firms, with no high growth prospects are those who obtain the highest expected returns.

3.7.2 Evaluation of the Fama & French Five-Factor Model: Applications and Limitations

This model, even if it is an improvement from Fama and French's previous multifactor model, still has some limitations.

The Five-Factor Model fails to explain or analyze whether the conclusions reached can be extrapolated and applied to other international stock markets besides the North American ones. In the case of the conclusion that the highest returns are obtained by small, high-value and profitable companies that do not embark on large growth initiatives, it is not clear whether it applies perfectly in the real world, and more qualitative methods should be used to see if investors follow this investment behavior. Secondly, this model discards the momentum and liquidity factors, that had started to be included in most models, which may weaken the results as these might be relevant variables. One of the main drawbacks of the five-factor model is its inability to capture the low average returns of small stocks, whose returns are similar to those of companies that invest aggressively despite their low returns, as well as the fact that the model's performance is indifferent to how its factors are defined (Fama and French, 2015). Thus, further research could be made to investigate whether this phenomenon is present in emerging countries. Finally, the profitability factor included in the model, has received some criticism in recent studies. This factor excludes the analysis of how much of the profitability is due to how the firm has registered accruals, and how much is due to actual return or value of the firm.

On the bright side, there are some applications and benefits of using this model. As it has already been shown, this model is more precise than its antecessor, by including two new and relevant factors. Fama and French's article does capture robust results between these factors and average returns, leading the reader to believe that these factors should be included in future models. In addition, the proposed model proves that the investment factor has a high correlation with the profitability and value variables. Overall, it can be concluded, that as a model, it is an improvement on previous ones, but still has some inaccuracies and limitations that should be solved.

4. **Conclusions**

In this section, some general conclusions regarding the models' subject of study, will be drawn, and discussions will be made on how this has led to the new concept of the "Factor Zoo".

4.1 Main Conclusions Derived from the Analysis of the Models

Asset valuation is a very important part of corporate finance and fundamental not only for professional portfolio managers, but for everyone who has some savings they seek to invest. All the models studied, which are analyzed in depth in this work, are relatively easy to understand though, some are more difficult to implement.

Throughout this project, an analysis and review of the literature on the most relevant portfolio management theories has been conducted. The paper has included different theories which are the "Modern Portfolio Theory" by Markowitz, "Capital Asset Pricing Model" by William Sharpe, John Lintner and its later contributions by Robert Merton, Lucas and Breeden, and Javier Estrada, "The Arbitrage Pricing Theory" by Stephen Ross, the "Three-Factor Model" by Fama and French, the "Momentum Model" by Jegadeesh and Titman, the "Four-Factor Model" by Carhart, and finally, the "Five-Factor Model" by Fama and French.

Markowitz's Modern Portfolio Theory has been a very important discovery for portfolio valuation, as it has helped set the foundations for the field, and all the following theories that are explained in this project, relate to it in some way. The efficiency frontier, which consists of optimal portfolio combinations that offer the highest expected return given a level of risk, has been a great revolution in portfolio management, emphasizing the importance of correlation between assets and looking at the portfolio as a whole and not each asset separately. This model has been subjected to much criticism, but it was the origin of other later theories such as Sharpe's CAPM, whose academic and practical relevance reaches our days.

As mentioned before, the following development was Sharpe and Lintner's CAPM, which is a simple and easy model to use, and explains the expected return and its correlation with the market portfolio. The main issue faced with the CAPM is that it is a model that describes expected returns and risk through the analysis of past data with statistical tools but does not explain how or why those returns occur. In this case, its lack of complexity, which may be seen as an advantage by some, is the reason why it is not very precise. Further contributions in this model like the Intertemporal CAPM, Consumption CAPM and Downside CAPM are analyzed, which improve the accuracy of the model, but still faced criticism.

The next model, developed by S. Ross in 1976, is the Arbitrage Pricing Theory, which uses several factors to explain expected performance. The systematic risk in this case stems from factors affecting the asset or portfolio. This model consists of the explanatory part that is missing from the CAPM and is therefore considered more accurate. However, its main weakness is that the explanatory factors are unknown, and it is more complex to use than the CAPM, which is why it is only used in a complementary way.

After the APT, the Fama-French Three-Factor Model appears between 1992 to 1993, which is a multifactor model, which eliminates some of the limitations of the CAPM and APT. This model does not come without criticism as many argue that the relevance of the two factors included, is not homogeneous among stock markets, and that these factors should be selected specifically for each of them.

That same year, Jegadeesh and Titman introduce their Momentum model, a model that explores the usefulness of momentum investment strategies. A few years later, in 1997, Carhart released his Four-Factor model, which combines the Three-Factor Model, with the momentum concept introduced by Jegadeesh and Titman. Then, in 2015, Fama and French develop their Three-Factor Model, by including two new factors to increase the preciseness of the model. These three models have also faced criticism, which leads us to believe, that as models, they attempt to represent reality in the most accurate way possible, but it is almost impossible for them to be perfectly exact.

In the end, all the models described have been disruptive in their time, and these authors have been changing the preconceived ideas on the financial and investment arenas ever since. Even if there are some limitations to their use, they are the most precise ways investors have to analyze the market and be able to compute their expected returns. These limitations have also led to a continuous improvement process, which is why there has been huge development in this field in the last 60 years. All of this has led to what now is called the Factor Zoo, which will be explained in detail in the following section.

4.2 Introduction to the Factor Zoo

The factor zoo is a concept that stems from the idea of multifactor models. It appears as there is a growing number of investment factors to consider, proposed by academics and financial experts. In their academic journals, Harvey, Liu and Zhu (2016), study this new concept.

In the 1980s and 1990s, factors such as size, value, and momentum were shown to generate returns that could not be explained by the Capital Asset Price Model (CAPM). Since then, the number of factors identified in the academic literature has skyrocketed, giving rise to what has come to be called the "zoo" of equity factors. This has led to a heated debate about how many different factors actually exist. While it is true that the small group of factors used in academic asset pricing models can be a very good starting point, the analysis cannot stop there. In recent studies, it has been shown that there are dozens of factors that are wrongly rejected, factors which are based on unconventional data sources or a "new generation" of factors based on machine learning, big data or artificial intelligence.

Some recent papers talk about grouping some of these factors into smaller strategic groups, that are: low risk factors, value factors, quality factors, momentum factor and short-term factors. In summary, from an academic zoo of hundreds of supposed factors, a few dozen of them would be taken, which are actually operational, and group them into a small number of composite factors. It should be noted that this is a very recent concept, and there is not much literature nor specific models implementing this. We will just have to wait and see what the future awaits.

5. **Bibliography:**

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