INVOLUTIONS OF THE MODULI SPACES OF G-HIGGS BUNDLES OVER ELLIPTIC CURVES

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Abstract. We present a systematic study of involutions on the moduli space of G-Higgs bundles over an elliptic curve X, where G is complex reductive affine algebraic group. The fixed point loci in the moduli space of G-Higgs bundles on X, and in the moduli space of representations of the fundamental group of X into G, are described. This leads to an explicit description of the moduli spaces of pseudo-real G-Higgs bundles over X.

Contents

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1. Introduction

Given a complex reductive affine algebraic group G and a compact Riemann surface X, a *G-Higgs bundle* over X is a pair $(E,\,\varphi)$, where E is a holomorphic principal G-bundle over X, and φ is a holomorphic section of $E(\mathfrak{g})\otimes K$, the adjoint vector bundle $E(\mathfrak{g})$ of E twisted by the canonical bundle K of X. These objects were introduced by Hitchin [27], while Simpson [36, 37, 38], and Nitsure [33] constructed the moduli space $\mathcal{M}(G)$ of G-Higgs bundles.

Let X be a complex elliptic curve. Let σ_+ and α_+ denote holomorphic involutions on G and X, respectively. Consider the involution (see Section 2.2 for details)

$$I(\alpha_+,\sigma_+,\pm)\,:\,\mathcal{M}(G)\,\longrightarrow\,\mathcal{M}(G)\,,\quad(E,\,\varphi)\,\longmapsto\,(\alpha_+^*\sigma_+(E),\,\pm\alpha_+^*\sigma_+(\varphi))\,.$$

Analogously, for anti-holomorphic involutions σ_- and α_-, on G and X, define

$$I(\alpha_-,\sigma_-,\pm)\,:\,\mathcal{M}(G)\,\longrightarrow\,\mathcal{M}(G)\,,\quad(E,\,\varphi)\,\longmapsto\,(\alpha_-^*\sigma_-(E),\,\pm\alpha_-^*\sigma_-(\varphi))\,.$$

The first goal of this paper is to classify these involutions and describe their fixed points.

Let Z denote the center of G and $Z_2^{\sigma_-}\,\subset\,Z^{\sigma_-}$ the group elements of order two of Z fixed pointwise by σ_-. Given any $z\,\in\,Z_2^{\sigma_-}$, we consider the pseudo-real $(G,\alpha_-,\sigma_-,\pm,z)$-Higgs bundles [6, 7] which are G-Higgs bundles equipped with a certain real structure (see Section 2.3). The second goal of this paper is to obtain a description of the moduli space $\mathcal{M}(G,\alpha_-,\sigma_-,\pm,z)$ of pseudo-real G-Higgs bundles; this is achieved in Section 4.4.

A major result of the theory of Higgs bundles is the non-abelian Hodge correspondence, proved by Hitchin [27] and Donaldson [15] for $\mathrm{SL}(2,\mathbb{C})$, and by Simpson [37, 38] and Corlette [14] for an arbitrary group (see [8] also for the general case). It relates the moduli of G-Higgs bundles with the moduli space of representations of the fundamental group

$$\mathcal{R}(G)\,:=\,\mathrm{Hom}(\pi_1(X,x_0),\,G)/\!\!/G \tag{1.1}$$

as follows:

Theorem 1.1. *There is a natural homeomorphism* $\mathcal{M}(G)\,\cong\,\mathcal{R}(G)\,.$

To treat simultaneously the holomorphic and anti-holomorphic cases, we will write σ_ϵ and α_ϵ (instead of σ_+, σ_- and α_+, α_-). Along with the study of the involutions $I(\alpha_\epsilon,\sigma_\epsilon,\pm)$, we also describe in this article their images $J(\alpha_\epsilon,\sigma_\epsilon,\pm)$ under the homeomorphism of Theorem 1.1, which are involutions on $\mathcal{R}(G)$, making the following diagram commutative,

$$\begin{tikzcd}
\mathcal{M}(G) \arrow[r, "\text{homeo.}"] \arrow[d, "I(\alpha_\epsilon,\sigma_\epsilon,\pm)"'] & \mathcal{R}(G) \arrow[d, "J(\alpha_\epsilon,\sigma_\epsilon,\pm)"] \\
\mathcal{M}(G) \arrow[r, "\text{homeo.}"] & \mathcal{R}(G).
\end{tikzcd}$$

The complex structure Γ_1 on $\mathcal{M}(G)$ comes from X, while the complex structure Γ_2 on $\mathcal{R}(G)$ is given by that of the group. In view of Theorem 1.1 one can identify the spaces $\mathcal{M}(G)$ and $\mathcal{R}(G)$. As we have seen, there are two different complex structures on it and, in fact, on can endow the smooth locus $\mathcal{M}(G)_{sm}$ with a hyper-Kähler structure $(\omega_1,\omega_2,\omega_3)$, where the complex structure Γ_3 is $\Gamma_1\Gamma_2$. Note also that one can define three corresponding (holomorphic) symplectic structures Ω_1, Ω_2 and Ω_3.

In the context of Mirror Symmetry, an A-brane is a Lagrangian submanifold together with a flat bundle supported on it, while a B-brane is a complex submanifold with a holomorphic bundle, although in this paper we will use the terms A-brane and B-brane to refer simply to their support, i.e., the special Lagrangian or complex submanifold. In their ground-breaking paper [30], Kapustin and Witten introduced the concept of $(*,*,*)$-brane as a submanifold which is either complex or special Lagrangian for each of the complex structures Γ_1, Γ_2 and Γ_3. This is a hyper-Kähler submanifold in the case of a (B,B,B)-brane, or a Ω_i-complex Lagrangian submanifold in the case of (B,A,A), (A,B,A) or (A,A,B)-branes.

In [3], Baraglia and Schaposnik identified the fixed points of $I(\alpha_-,\sigma_-,-)$ with an (A,B,A)-brane (see also [6]). The involutions $I(\mathrm{Id}_X,\sigma_+,+)$ have been studied by Hitchin [27, 28] and in [20, 21, 23]. It turns out that the fixed points of $I(\mathrm{Id}_X,\sigma_+,+)$ constitute a (B,B,B)-brane, while the fixed points of $I(\mathrm{Id}_X,\sigma_+,-)$ give a (B,A,A)-brane. See [23] for a detailed study of these involutions and their fixed points. For a general picture of anti-holomorphic involutions on the smooth locus $\mathcal{M}(G)_{sm}\subset\mathcal{M}(G)$ we refer to [4, 6]. The involution $I(\alpha_+,\mathrm{Id}_X,+)$ is a particular case of the more general situation studied in [24] (see also [26], which contains a detailed study of the case of α_+ being fix point free).

Over an elliptic curve one can achieve a greater level of explicitness regarding the description of the moduli spaces of Higgs bundles. In 1957 Atiyah [2] described the moduli space of rank n vector bundles on an elliptic curve X as the symmetric product of the curve $M(\mathrm{GL}(n,\mathbb{C}))\cong Sym^n(X)$ and some 30 years later, Laszlo [32] and Friedman, Morgan and Witten [18, 19], gave a description of the moduli space of principal G–bundles for a complex reductive Lie group G, in terms of the finite quotient $M(G)\cong(X\otimes_{\mathbb{Z}}\Lambda_T)/W$, where T is a Cartan subgroup of G, Λ_T is the cocharacter lattice and W the associated Weyl group. In [16] the third and fourth authors studied the case of G-Higgs bundles, showing

$$\mathcal{M}(G)\;\cong\;(T^*X\otimes_{\mathbb{Z}}\Lambda_T)/W.$$

One can find in [17] a detailed study of the involution $I(\mathrm{Id}_X,\sigma_+,-)$ on the moduli spaces of Higgs bundles for real forms of G, and in [10], a description of the fixed points of this involution $I(\alpha_-,\sigma_-,\pm)$ of the Atiyah's moduli space $M(\mathrm{GL}(n,\mathbb{C}))$ and its relation with the moduli spaces of (real) vector bundles over a real elliptic curve (see also [5] for the case of a Klein bottle).

This article is organized as follows. After a quick review of involutions on complex Lie groups in Section 2.1, we define in Section 2.2 the involutions $I(\alpha_\epsilon,\sigma_\epsilon,\pm)$ which are the objects of our study. In Section 2.3 we study these involution in the anti-holomorphic case and their relation with pseudo-real G-Higgs bundles.

In Section 3 we recall the case of elliptic curves. We study holomorphic involutions on elliptic curves in Section 3.1 and anti-holomorphic involutions in Section 3.2. We review the description of the moduli space of G-Higgs bundles over elliptic curves in Section 3.3.

The new results of this article are contained in Section 4. We first describe the involutions $I(\alpha_\epsilon,\sigma_\epsilon,\pm)$ on the moduli space of Higgs line bundles in Section 4.1 and using that, we obtain an explicit description of the involutions for the general case in Section 4.2. This explicitness, allow us to describe their fixed point loci in Section 4.3, and in Section 4.4 we apply this description to the moduli space of pseudo-real G-Higgs bundles.

2. Higgs bundles and involutions

2.1. **Involutions and conjugations of complex Lie groups.** Take G to be a complex reductive affine algebraic group. A conjugation $\sigma_- : G \longrightarrow G$ is an anti-holomorphic automorphism of G of order two. The set of all inner automorphisms of G will be denoted by $\mathrm{Conj}(G)$. A real form of a conjugation σ_- is the fixed point subgroup $G^{\sigma_-} \subset G$. There is a $\sigma_K \in \mathrm{Conj}(G)$ such that the real form $K = G^{\sigma_K}$ is a maximal compact subgroup of G whose complexification is G.

Let $\mathrm{Aut}(G)$ be the group of all holomorphic automorphisms of G; denote by $\mathrm{Int}(G)$ the subgroup of $\mathrm{Aut}(G)$ consisting of inner automorphisms of G. This is a normal subgroup of $\mathrm{Aut}(G)$ so one can consider the quotient group $\mathrm{Aut}(G)/\mathrm{Int}(G)$ which we denote by $\mathrm{Out}(G)$. This all fits in the short exact sequence

$$1 \longrightarrow \mathrm{Int}(G) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1. \tag{2.1}$$

Let $\mathrm{Aut}_2(G)$ the set of elements of order two in $\mathrm{Aut}(G)$. Two elements f, f' of $\mathrm{Aut}_2(G)$ will be said to be equivalent if there is some $\alpha \in \mathrm{Int}(G)$ such that

$$f' = \alpha f \alpha^{-1}.$$

Similarly, we define an equivalence relation $\sim$ on $\mathrm{Conj}(G)$; in other words, two anti-holomorphic automorphisms of G of order two are said to be equivalent if they differ by conjugation by an inner automorphism.

Theorem 2.1 (Cartan)**.** *There is a bijection:*

$$C : \mathrm{Conj}(G)/\sim \stackrel{1:1}{\longrightarrow} \mathrm{Aut}_2(G)/\sim, \quad \sigma_- \longmapsto \sigma_+ := \sigma_-\sigma_K,$$

where σ_K is the above conjugation.

2.2. **Involutions on $\mathcal{M}(G)$ and branes.** In this section X will be a compact connected Riemann surface. Let $\sigma_+ : G \longrightarrow G$ be a holomorphic involution of the complex Lie group G, and let $\sigma_- : G \longrightarrow G$ be an anti-holomorphic involution.

Given a principal G-bundle E on X, let $\sigma_+(E)$ (respectively, $\sigma_-(E)$) denote the principal G-bundle on X obtained by extending the structure group of E using σ_+ (respectively, σ_-). Note that the total spaces of both $\sigma_+(E)$ and $\sigma_-(E)$ have a natural holomorphic structure. In fact, $\sigma_+(E)$ is a holomorphic principal G-bundle, while $\sigma_-(E)$ is a holomorphic principal $\overline{G}$-bundle, where $\overline{G}$ is the Lie group G equipped with the almost complex structure $-J_G$ with J_G being the almost complex structure on G.

For any holomorphic involution $\alpha_+ : X \longrightarrow X$, define

$$I(\alpha_+, \sigma_+, +) : \mathcal{M}(G) \longrightarrow \mathcal{M}(G), \quad (E, \varphi) \longmapsto (\alpha_+^*\sigma_+(E), +\alpha_+^*\sigma_+(\varphi)),$$

$$I(\alpha_+, \sigma_+, -) : \mathcal{M}(G) \longrightarrow \mathcal{M}(G), \quad (E, \varphi) \longmapsto (\alpha_+^*\sigma_+(E), -\alpha_+^*\sigma_+(\varphi)).$$

For an anti-holomorphic involution $\alpha_- : X \longrightarrow X$, define

$$I(\alpha_-, \sigma_-, +) : \mathcal{M}(G) \longrightarrow \mathcal{M}(G), \quad (E, \varphi) \longmapsto (\alpha_-^*\sigma_-(E), +\alpha_-^*\sigma_-(\varphi)),$$

$$I(\alpha_-, \sigma_-, -) : \mathcal{M}(G) \longrightarrow \mathcal{M}(G), \quad (E, \varphi) \longmapsto (\alpha_-^*\sigma_-(E), -\alpha_-^*\sigma_-(\varphi)).$$

Sometimes we write $I(\alpha_\epsilon, \sigma_\epsilon, \pm)$ when we want to deal with these involutions simultaneously. Note that ϵ indicates whether it is the holomorphic ($\epsilon = +$) case or the anti-holomorphic ($\epsilon = -$) case.

See [6, 23] for a proof of the following proposition.

Proposition 2.2. *If* $\sigma_\epsilon \sim \sigma'_\epsilon$, *then* $I(\alpha_\epsilon, \sigma_\epsilon, \pm) = I(\alpha_\epsilon, \sigma'_\epsilon, \pm)$.

Let $\mathcal{M}(G)_{sm} \subset \mathcal{M}(G)$ be the smooth locus. This manifold $\mathcal{M}(G)_{sm}$ is hyper-Kähler with three complex structures $\Gamma_1, \Gamma_3, \Gamma_3$ and three associated Kähler forms $\omega_1, \omega_2, \omega_3$. Also recall that $\mathcal{M}(G)_{sm}$ has three (holomorphic) symplectic structures: $\Omega_1 = \omega_2 + \sqrt{-1}\omega_3$, $\Omega_2 = \omega_3 + \sqrt{-1}\omega_1$ and $\Omega_3 = \omega_1 + \sqrt{-1}\omega_2$. Now, we describe the geometric structure of the fixed points of the involutions in $\mathcal{M}(G)_{sm}$ defined above (see [6, 23] for instance):

- the fixed point locus of $I(\alpha_+, \sigma_+, +)$ is an hyper-Kähler submanifold,
- the fixed point locus of $I(\alpha_+, \sigma_+, -)$ is a Ω_1-Lagrangian submanifold,
- the fixed point locus of $I(\alpha_-, \sigma_-, -)$ is a Ω_2-Lagrangian submanifold,
- the fixed point locus of $I(\alpha_-, \sigma_-, +)$ is a Ω_3-Lagrangian submanifold.

Remark 2.3. In the context of Mirror Symmetry:

- the fixed point locus of $I(\alpha_+, \sigma_+, +)$ is a (B, B, B)-brane,
- the fixed point locus of $I(\alpha_+, \sigma_+, -)$ is a (B, A, A)-brane,
- the fixed point locus of $I(\alpha_-, \sigma_-, -)$ is a (A, B, A)-brane,
- the fixed point locus of $I(\alpha_-, \sigma_-, +)$ is a (A, A, B)-brane.

The above involutions induce involutions of the moduli space of representations defined in (1.1):

$$\begin{aligned} J(\alpha_+, \sigma_+, +) &: \mathcal{R}(G) \longrightarrow \mathcal{R}(G), \quad \rho \longmapsto \sigma_+ \circ \rho \circ (\alpha_+)_*, \\ J(\alpha_-, \sigma_-, +) &: \mathcal{R}(G) \longrightarrow \mathcal{R}(G), \quad \rho \longmapsto \sigma_- \circ \rho \circ (\alpha_-)_*, \\ J(\alpha_+, \sigma_+, -) &: \mathcal{R}(G) \longrightarrow \mathcal{R}(G), \quad \rho \longmapsto \sigma_- \circ \rho \circ (\alpha_+)_*, \\ J(\alpha_-, \sigma_-, -) &: \mathcal{R}(G) \longrightarrow \mathcal{R}(G), \quad \rho \longmapsto \sigma_+ \circ \rho \circ (\alpha_-)_*. \end{aligned}$$

Let $Z = Z(G)_0$ be the connected component of the center $Z(G) \subset G$ containing the identity element. Define the homomorphism

$$\mu : Z \times G \longrightarrow G, \quad (y, z) \longmapsto yz.$$

For any principal G-bundle E and any principal Z-bundle F, we have the principal G-bundle

$$F \otimes E := \mu_*(F \times_X E);$$

note that the fiber product $F \times_X E$ is a principal $(Z \times G)$-bundle and hence $\mu_*(F \times_X E)$ is a principal G-bundle. It is straight-forward to check that $F \otimes E$ is semistable, stable or polystable if and only if E is semistable, stable or polystable respectively. If F' is a principal Z-bundle, then using the multiplication operation $\mu' : Z \times Z \longrightarrow Z$ we get a principal Z-bundle

$$F \otimes F' := \mu'_*(F \times_X F').$$

This operation makes the moduli space $\mathcal{M}(Z)$ of topologically trivial principal Z–bundles on X a complex Lie group.

The Lie algebra of Z will be denoted by $\mathfrak{z}$. The adjoint bundle $F(\mathfrak{z})$ for the principal Z-bundle F is the trivial vector bundle $\mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{z}$ over X with fiber $\mathfrak{z}$. The Lie algebra of G will be denoted by $\mathfrak{g}$. For the adjoint action of G on $\mathfrak{g}$, each point of $\mathfrak{z}$ is fixed. Hence

$$H^0(X, F(\mathfrak{z})) = \mathfrak{z} \subset H^0(X, E(\mathfrak{g})).$$

The group $\mathcal{M}(Z)$ has the following holomorphic action on $\mathcal{M}(G)$:

$$\mathcal{M}(Z)\times\mathcal{M}(G) \longrightarrow \mathcal{M}(G)\,, \quad ((F,\,\phi),\,(E,\,\varphi)) \longmapsto (F,\,\phi)\otimes(E,\,\varphi) = (F\otimes E,\,\phi+\varphi)\,. \tag{2.2}$$

Analogously, given a homomorphism

$$\chi\,:\,\pi_1(X) \longrightarrow Z\,,$$

one gets for any homomorphism $\rho\,:\,\pi_1(X) \longrightarrow G$ a homomorphism $\chi\cdot\rho\,:\,\pi_1(X) \longrightarrow G$ given by the composition

$$\pi_1(X) \stackrel{\chi\times\rho}{\longrightarrow} Z\times G \stackrel{\mu}{\longrightarrow} G\,.$$

More precisely, $\mathcal{R}(Z)$ (defined as in (1.1)) is a complex Lie group that acts holomorphically on $\mathcal{R}(G)$ as follows

$$\mathcal{R}(Z)\times\mathcal{R}(G) \longrightarrow \mathcal{R}(G)\,, \quad (\chi,\,\rho) \longmapsto \chi\cdot\rho\,. \tag{2.3}$$

Note that if we set $G = Z$ in (1.1), then the GIT quotient becomes an ordinary quotient.

Since the anti-holomorphic involution σ_- of G preserves $Z \subset G$, we can combine (2.2) and (2.3) to obtain more involutions. For every $\mathcal{F} = (F,\,\phi) \in \mathcal{M}(Z)$, set

$$I(\alpha_\epsilon,\sigma_\epsilon,\pm,\mathcal{F})\,:\,\mathcal{M}(G) \longrightarrow \mathcal{M}(G)\,, \quad (E,\,\varphi) \longmapsto \mathcal{F}\otimes I(\alpha_\epsilon,\sigma_\epsilon,\pm)(E,\varphi)\,.$$

It is an involution whenever $\mathcal{F} = (F,\,\phi)$ satisfies

$$\mathcal{F}^{-1} = I(\alpha_\epsilon,\sigma_\epsilon,\pm)(F,\phi)\,, \tag{2.4}$$

where $\mathcal{F}^{-1}$ denotes $(F^{-1},\,-\phi)$. The inverse of F is denoted by F^{-1}; note that F^{-1} coincides with the principal Z–bundle obtained by extending the structure group of F using the automorphism $z \longmapsto z^{-1}$ of Z.

Analogously, for every $\chi \in \mathcal{R}(Z)$, define

$$J(\alpha_\epsilon,\sigma_\epsilon,\pm,\chi)\,:\,\mathcal{R}(G) \longrightarrow \mathcal{R}(G)\,, \quad \rho \longmapsto \chi\cdot J(\alpha_\epsilon,\sigma_\epsilon,\pm)(\rho)\,.$$

The condition for $J^{\sigma_\epsilon,\chi}_{\alpha_\epsilon}$ to be an involution is that

$$\chi^{-1} = J(\alpha_\epsilon,\sigma_\epsilon,\pm)(\chi)\,. \tag{2.5}$$

Theorem 2.4. *For all $\mathcal{F} \in \mathcal{M}(Z)$ and $\chi \in \mathcal{R}(Z)$ satisfying* (2.4) *and* (2.5)*, the above maps $I(\alpha_\epsilon,\sigma_\epsilon,\pm,\mathcal{F})$ and $J(\alpha_\epsilon,\sigma_\epsilon,\pm,\chi)$ are involutions.*

The fixed points of $I(\alpha_+,\sigma_+,+,\mathcal{F})$ restricted to $\mathcal{M}(G)_{sm}$ are (B,B,B)-branes, while the fixed points of $I(\alpha_+,\sigma_+,-,\mathcal{F})$ are (B,A,A)-branes. On the other hand, the fixed points of $I(\alpha_-,\sigma_-,-,\mathcal{F})$ are (A,B,A)-branes when we restrict to $\mathcal{M}(G)_{sm}$ and the fixed points of $I(\alpha_-,\sigma_-,+,\mathcal{F})$ are (A,A,B)-branes.

If $\mathcal{F} = (F,\phi) \in \mathcal{M}(Z)$ is the Z-Higgs bundle associated to the representation $\chi \in \mathcal{R}(Z)$, then the diagram

$$\begin{tikzcd} \mathcal{M}(G) \arrow[r, "\text{homeo.}"] \arrow[d, "I(\alpha_\epsilon,\sigma_\epsilon,\pm,\mathcal{F})"'] & \mathcal{R}(G) \arrow[d, "J(\alpha_\epsilon,\sigma_\epsilon,\pm,\chi)"] \\ \mathcal{M}(G) \arrow[r, "\text{homeo.}"] & \mathcal{R}(G) \end{tikzcd} \tag{2.6}$$

is commutative.

Proof. The statements for $I(\alpha_\epsilon, \sigma_\epsilon, \pm)$ follow from the works [10], [6] and [23]. One can prove that the diffeomorphism in Theorem 1.1 takes the map in (2.2) to the map in (2.3). Then $I(\alpha_\epsilon, \sigma_\epsilon, \pm, \mathcal{F})$ commutes or anticommutes with Γ_1 and Γ_2 whenever $I(\alpha_\epsilon, \sigma_\epsilon, \pm)$ does so. By the above references, the involutions $I(\alpha_\epsilon, \sigma_\epsilon, \pm)$ preserve the hyper-Kähler metric on $\mathcal{M}(G)_{sm}$. Note that the translation (2.2) preserves the hyper-Kähler metric as well, and so does $I(\alpha_\epsilon, \sigma_\epsilon, \pm, \mathcal{F})$. As a consequence, the statements proved for $I(\alpha_\epsilon, \sigma_\epsilon, \pm)$ are also valid for $I(\alpha_\epsilon, \sigma_\epsilon, \pm, \mathcal{F})$. □

Remark 2.5. When $\sigma_+ = \mathrm{Id}_G$, the involution $I(\alpha_+, \mathrm{Id}_G, +)$ is just the pull-back by α_+, while $J(\alpha_+, \mathrm{Id}_G, +)$ coincides with the composition of homomorphisms with $(\alpha_+)_*$. Denote by Id_X^{-1} and Id_G^{-1} the inversion given by the group structures on X and G, respectively. For any representation of the fundamental group, note that $\rho \circ (\mathrm{Id}_X^{-1})_* = \rho^{-1} = \mathrm{Id}_G^{-1} \circ \rho$. As a consequence of the commutativity of (2.6), the pull-back by Id_X^{-1} commutes with the extension of structure group associated to Id_G^{-1}.

2.3. **Pseudo-real Higgs bundles.** Let $\alpha_- : X \longrightarrow X$ be an anti-holomorphic involution of X and $\sigma_- : G \longrightarrow G$ an anti-holomorphic involution of G. Let $z \in Z_2^{\sigma_-} \subset Z^{\sigma_-}$ be an element of order 2 of the fixed point locus of $Z^{\sigma_-} \subset Z$ under σ_-. A pseudo-real $(G, \alpha_-, \sigma_-, \pm, z)$-Higgs bundle is a G-Higgs bundle (E, φ) over X equipped with a lift

$$\begin{tikzcd} E \arrow[r, "\widetilde{\alpha}_-"] \arrow[d] & E \arrow[d] \\ X \arrow[r, "\alpha_-"] & X, \end{tikzcd}$$

satisfying the following conditions:

- $\widetilde{\alpha}_-$ is anti-holomorphic;
- $\widetilde{\alpha}_-(eg) = \widetilde{\alpha}_-(e)\sigma_-(g)$, for all $e \in E$, $g \in G$;
- $\widetilde{\alpha}_-^2(e) = ez$, for $e \in E$;
- $\widetilde{\alpha}_-(\varphi) = \pm\varphi$.

The last condition needs an explanation. The canonical line bundle K of X has an anti-holomorphic involution induced by α_-. The bundle $E(\mathfrak{g})$ has also an anti-holomorphic involution given by $d\sigma_-$ and α_-. Using this two involutions together, we can define

$$\widetilde{\alpha}_- : E(\mathfrak{g}) \otimes K \longrightarrow E(\mathfrak{g}) \otimes K .$$

The $(G, \alpha_-, \sigma_-, \pm, z)$-Higgs bundles are also called pseudo-real G-bundles; these are called real G-Higgs bundles if further $z = \mathrm{Id}_G$. These are studied in [6] and [7]; the reader is referred to [6], [7] for the introduction of the moduli space

$$\mathcal{M}(G, \alpha_-, \sigma_-, \pm, z)$$

of isomorphism classes of polystable $(G, \alpha_-, \sigma_-, \pm, z)$-Higgs bundles.

Note that the holomorphic isomorphism $\widetilde{\alpha}_- : E \longrightarrow E$ and α_- produce a holomorphic isomorphism of G-Higgs bundles

$$\theta : (E, \varphi) \xrightarrow{\cong} (\alpha_-^*\sigma_-(E), \pm\alpha_-^*\sigma_-(\varphi)), \tag{2.7}$$

such that

$$\theta \circ \alpha_-^*\sigma_-(\theta) = z \cdot \mathrm{Id}_E . \tag{2.8}$$

We will refer to the triples $(E,\,\varphi,\,\theta)$ too as $(G,\alpha_-,\sigma_-,\pm)$-Higgs bundles. An isomorphism of $(G,\alpha_-,\sigma_-,\pm,z)$-Higgs bundles $(E_1,\varphi_1,\widetilde{\alpha}_1)$ and $(E_2,\,\varphi_2,\,\widetilde{\alpha}_2)$ is an isomorphism of Higgs bundles

$$f\,:\,(E_1,\,\varphi_1)\stackrel{\cong}{\longrightarrow}(E_2,\,\varphi_2)$$

such that $f^{-1}\circ\widetilde{\alpha}_2\,=\,\widetilde{\alpha}_1\circ f$, or, equivalently,

$$\theta_2\,=\,f\circ\theta_1\circ\alpha_-^*\sigma_-(f)^{-1}\,. \tag{2.9}$$

We consider the forgetful map $\mathcal{M}(G,\alpha_-,\sigma_-,\pm,z)\,\longrightarrow\,\mathcal{M}(G)$. Let $\widetilde{\mathcal{M}}(G,\alpha_-,\sigma_-,\pm,z)$ be the image of this forgetful map.

Theorem 2.6 ([6]). *The set*

$$\bigcup_{z\in Z_2^{\sigma_-}}\widetilde{\mathcal{M}}(G,\alpha_-,\sigma_-,\pm,z)$$

is contained in the fixed points of $I(\alpha_-,\sigma_-,\pm)$ in $\mathcal{M}(G)$.

Remark 2.7. In the case of holomorphic involutions on the curve, one could consider as well $(G,\alpha_+,\sigma_+,\pm)$-Higgs bundles and we should obtain a similar description of the fixed point locus as in Theorem 2.6. For the trivial involution $\sigma_+=\mathrm{Id}_X$, these objects would be equivalent to $(G_{\sigma_+},\pm)$-Higgs bundles as described in [23]. For the trivial involution $\sigma_+=\mathrm{Id}_G$, $(G,\alpha_+,\mathrm{Id}_G,+)$-Higgs bundles have been studied in [24].

See [17] for a concrete description of the fixed point loci of the involutions $I(\mathrm{Id}_X,\sigma_+,-)$ in the moduli space $\mathcal{M}(G)$ of G-Higgs bundles over elliptic curves.

Fix a point $x\,\in\,X$ such that $\alpha_-(x)\,\neq\,x$. Let

$$p\,:\,\widetilde{X}\,\longrightarrow\,X$$

be the universal cover of X associated to x. The orbifold fundamental group $\Gamma(X,x)$ of (X,α_-) is the set

$$\{f:\widetilde{X}\to\widetilde{X},p\circ f\,=\,q(f)\circ p\}\,,$$

where $q(f)$ can be σ_-. So, we have a short exact sequence

$$0\longrightarrow\pi_1(X,x)\xrightarrow{\;i\;}\Gamma(X,x)\xrightarrow{\;q\;}\mathbb{Z}/2\mathbb{Z}\longrightarrow 0\ .$$

Let $\widehat{G}$ be $G\times(\mathbb{Z}/2\mathbb{Z})$ as a set, with the group operation:

$$(g_1,\,e_1)(g_2,\,e_2)\,=\,(g_1^{e_1}g_2c^{e_1e_2},\,e_1+e_2)\,.$$

A z-homomorphism of the orbifold fundamental group is a group homomorphism

$$\rho\,:\,\Gamma(X,x)\,\longrightarrow\,\widehat{G}$$

that fits in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(X,x) \arrow[r, "i"] \arrow[d] & \Gamma(X,x) \arrow[r, "q"] \arrow[d, "\rho"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & G \arrow[r, "i"] & \widehat{G} \arrow[r, "q"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd} \tag{2.10}$$

Two c-homomorphism ρ, ρ' are called equivalent if there is an element $g \in G$ with $\rho' = g\rho g^{-1}$. Consider the moduli space of equivalence classes of reductive representations $\mathcal{R}(G, \alpha_-, \sigma_-, \pm,)$ [6]. Let $\mathcal{R}(G)_{sm}$ be the smooth locus corresponding to $\mathcal{M}(G)_{sm}$. Let $\widetilde{\mathcal{R}}(G, \alpha_-, \sigma_-, \pm, z)$ be the image of the forgetful map $\mathcal{R}(G, \alpha_-, \sigma_-, \pm, z) \longrightarrow \mathcal{R}(G)$.

Theorem 2.8. *The set*

$$\bigcup_{z \in Z_2^{\sigma_-}} \mathcal{R}(G, \alpha_-, \sigma_-, \pm, z)$$

is contained in the fixed points of $J(\alpha_-, \sigma_-, \pm)$ in $\mathcal{R}(G)$.

Proof. It is a consequence of Theorem 2.6 and [6, Theorems 4.5 and 4.8].

Remark 2.9. In [24], a similar description is worked out for the holomorphic case, defining $(G, \alpha_+, \sigma_+, \pm, z)$–Higgs bundles and their moduli spaces $\mathcal{M}(G, \alpha_+, \sigma_+, \pm, z)$, as well as the associated moduli spaces of representations $\mathcal{R}(G, \alpha_+, \sigma_+, \pm, z)$.

3. Elliptic curves and Higgs bundles

3.1. **Elliptic curves.** Let X be a compact Riemann surface of genus 1, and let x_0 be a distinguished point on it; the pair $(X,\, x_0)$ defines an *elliptic curve* (a connected compact complex Lie group of dimension one). However, by abuse of notation, we usually denote the elliptic curve simply by X. Every elliptic curve is isomorphic to some $X_\gamma = \mathbb{C}/\langle 1,\, \gamma\rangle_{\mathbb{Z}}$, where $\langle 1,\, \gamma\rangle_{\mathbb{Z}}$ is the lattice generated by 1 and $\gamma \in \mathbb{H}$, the subset of points of the complex plane $\mathbb{C}$ with positive imaginary part. One has $X_{\gamma_1} \cong X_{\gamma_2}$ if and only if $\gamma_1, \gamma_2 \in \mathbb{H}$ are related by a Möbius transformation given by an element of $\mathrm{SL}(2, \mathbb{Z})$.

The Picard group $\mathrm{Pic}^0(X)$ will be denoted by $\widehat{X}$. Let

$$p_{x_0} \,:\, X \xrightarrow{\cong} \widehat{X} \tag{3.1}$$

be the holomorphic isomorphism that sends any $x \in X$ to the holomorphic line bundle $\mathcal{O}_X(x - x_0)$. Note that the group structure on $\widehat{X}$ produces a group structure on $T^*\widehat{X}$. Recall that $T^*\widehat{X} \cong \widehat{X} \times H^0(X,\, \mathcal{O}_X)$; the group structure on the second factor is simply given by the linear structure. The subgroup of X defined by the points of order two will be denoted by $X[2]$.

The Abel-Jacobi morphism

$$\mathrm{aj}_1 \,:\, X \longrightarrow \mathrm{Pic}^1(X)\,, \quad x \longmapsto \mathcal{O}_X(x)$$

is an isomorphism. For every $y \in X$, let

$$t_y \,:\, X \longrightarrow X\,, \quad x \longmapsto x + y$$

be the translation automorphism.

To present a consistent notation with that of Section 3.2, we write $\alpha_{(+,1)}$ for the identity Id_X and $\alpha_{(+,-1)}$ for the inversion map $x \longmapsto -x$ of X. When they are simultaneously referred, we shall write $\alpha_{(+,a)}$. Composing $\alpha_{(+,a)}$ with the translations t_y one gets another involution provided $y = -\alpha_{(+,a)}(y)$. For an involution α of X, let $X^\alpha \subset X$ be the fixed point locus and X/α the quotient by α. Table 1 describes the holomorphic involutions on X.

For $\gamma \in \mathbb{H}$, consider the paths $\widetilde{\delta}_1,\, \widetilde{\delta}_2 \,:\, [0,\, 1] \longrightarrow \mathbb{C}$, where

$$\widetilde{\delta}_1(t) \,=\, t \tag{3.2}$$

and

$$\widetilde{\delta}_2(t) = \gamma t\,. \tag{3.3}$$

Let δ_1, δ_2 be the images of $\widetilde{\delta}_1$, $\widetilde{\delta}_2$ in $X_\gamma \,=\, \mathbb{C}/\langle 1,\, \gamma\rangle_{\mathbb{Z}}$. The homotopy classes of δ_1 and δ_2 generate the fundamental group $\pi_1(X_\gamma,\, 0)$. The involutions $\alpha_{(+,a)}$ induce an action on the fundamental group; this action is described in in Table 2.

Remark 3.1. Every semistable principal G-bundle $E \longrightarrow X$ of trivial topological type is homogeneous, meaning

$$t_y^* E \,\cong\, E$$

for every $y \in X$ [8, Theorem 4.1]. This implies that the pull-back by $\alpha_{(+,a)}$ coincides with the pull-back by $t_y \circ \alpha_{(+,a)}$; therefore, both involutions define the same involution on the moduli space of G–bundles of trivial topological type, meaning

$$I(t_y \circ \alpha_{(+,a)}, \sigma_+, \pm) = I(\alpha_{(+,a)}, \sigma_+, \pm).$$

Since the fundamental group of X is abelian, $\pi_1(X, x_0)$ is identified with $\pi_1(X, y)$ (in general they are identified uniquely up to an inner automorphism); with this identification, the action of $(t_y)_*$ on $\pi_1(X)$ is trivial. Therefore,

$$J(t_y \circ \alpha_{(+,a)}, \sigma_+, \pm) = J(\alpha_{(+,a)}, \sigma_+, \pm).$$

3.2. **Real elliptic curves.** A *real elliptic curve* is a triple of the form $(X,\, \alpha_-,\, x_0)$, where $(X,\, x_0)$ is an elliptic curve, and α_- is an anti-holomorphic involution of X. It should be emphasized that the point x_0 need not be fixed by α_-. Now, again, by abuse of notation, we will denote a real elliptic curve simply by $(X,\, \alpha_-)$. A morphism of real elliptic curves is a holomorphic group homomorphism $X \longrightarrow Y$ that intertwines the involutions.

A Klein surface is a pair consisting on compact Riemann surface and an anti-holomorphic involution. The topological classification of Klein surfaces was given by Klein [31]. In the particular case of real elliptic curves, the topological type of a real elliptic curve is given by a pair (n, b) where n is the number of connected components of the fix point locus α_-, and $b \in \mathbb{Z}/2\mathbb{Z}$ is the index of orientability defined by

$$b = \begin{cases} 0 & \text{if } X/\alpha_- \text{ is oriented} \\ 1 & \text{if } X/\alpha_- \text{ is not oriented.} \end{cases}$$

A theorem of Harnack says that $n \,\leq\, \mathrm{genus}(X) + 1$ [25].

There are three different topological types of real elliptic curves: a Klein bottle, a Möbius strip and a closed annulus; see Table 3.

Consider the elliptic curve $X_\gamma \,=\, \mathbb{C}/\langle 1, \gamma\rangle_{\mathbb{Z}}$ for some $\gamma \in \mathbb{H}$, and let α_- be an anti-holomorphic involution on it. Fix a lift

$$\widetilde{\alpha}_- \,:\, \mathbb{C} \longrightarrow \mathbb{C}\,, \quad z \longmapsto a\overline{z} + b$$

of α_-, where a, b are complex numbers. Table 4 gives the possible values of γ up to Möbius transformations.

Continuing with the notation of Section 3.1, an anti-holomorphic involution on $X \,=\, X_\gamma$ that fixes the identity element and lifts to the automorphism $z \longmapsto a\overline{z}$ of $\mathbb{C}$, where $a \in$

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, will be denoted by $\alpha_{(-,a)}$. The anti-holomorphic involution $t_y \circ \alpha_{(-,a)}$ lifts to $z \longmapsto a\overline{z} + b$, where $y \in X$ is the projection of b to X_γ. Consider the two involutions $\alpha_{(\epsilon_1,a_1)}$ and $\alpha_{(\epsilon_2,a_2)}$; note that

$$\alpha_{(\epsilon_1,a_1)} \circ \alpha_{(\epsilon_2,a_2)} = \alpha_{(\epsilon_1\epsilon_2,a_1a_2)}$$

if $\epsilon_1 = +$, and

$$\alpha_{(\epsilon_1,a_1)} \circ \alpha_{(\epsilon_2,a_2)} = \alpha_{(\epsilon_1\epsilon_2,a_1\overline{a}_2)}$$

when $\epsilon_1 = -$.

All possible anti-holomorphic involutions are contained in Table 5 (see [1, Section 9]).

Alling and Greenleef gave the following classification of real tori.

Proposition 3.2 ([1, Section 9]). *In the cases in Table 5, for any possible $y \in X$, the isomorphisms of real tori are as follows:*

- *In the region A, for any $y \in X^{\alpha_{(-,-1)}}$ such that $y \neq x_0$,*
$$\left(X_\gamma,\, t_y \circ \alpha_{(-,1)}\right) \cong \left(X_\gamma,\, t_{\frac{1}{2}} \circ \alpha_{(-,1)}\right)$$
and, for any $y \in X^{\alpha_{(-,1)}}$ with $y \neq x_0$,
$$\left(X_\gamma,\, t_y \circ \alpha_{(-,-1)}\right) \cong \left(X_\gamma,\, t_{\frac{\gamma}{2}} \circ \alpha_{(-,1)}\right).$$
- *In the region B,*
$$\left(X_\gamma,\, \alpha_{(-,1)}\right) \cong \left(X_\gamma,\, \alpha_{(-,-1)}\right),$$
$$\left(X_\gamma,\, \alpha_{(-,\mathbf{i})}\right) \cong \left(X_\gamma,\, \alpha_{(-,-\mathbf{i})}\right)$$
and for every $y \in X^{\alpha_{(-,\mp 1)}}$ such that $y \neq x_0$,
$$\left(X_\gamma,\, t_y \circ \alpha_{(-,1)}\right) \cong \left(X_\gamma,\, t_{\frac{1}{2}} \circ \alpha_{(-,1)}\right).$$
- *In the regions C, D and E, for all $y \in X^{\alpha_{(-,\mp a)}}$,*
$$\left(X_\gamma,\, t_y \circ \alpha_{(-,a)}\right) \cong \left(X_\gamma,\, \alpha_{(-,a)}\right).$$
- *In the region D,*
$$\left(X_\gamma,\, \alpha_{(-,\pm 1)}\right) \cong \left(X_\gamma,\, \alpha_{(-,\mp\gamma)}\right) \cong \left(X_\gamma, \alpha_{(-,\pm\gamma^2)}\right).$$

Recall that δ_1, δ_2, defined in (3.2) and (3.3) respectively, are generators of the fundamental group $\pi_1(X)$. Table 6 describes the action of $\alpha_{(\epsilon,a)}$ on $\pi_1(X)$.

Remark 3.3. As in Remark 3.1, the homogeneity of topologically trivial semistable principal G-bundles on elliptic curves implies that the pullbacks by $\alpha_{(-,a)}$ and $t_y \circ \alpha_{(-,a)}$ of such a bundle are isomorphic. We also have the identifications

$$I(t_y \circ \alpha_{(-,a)},\, \sigma_-, \pm) = I(\alpha_{(-,a)},\, \sigma_-, \pm)$$

and

$$J(t_y \circ \alpha_{(-,a)},\, \sigma_-, \pm) = J(\alpha_{(-,a)},\, \sigma_-, \pm)$$

in the anti-holomorphic case. The notation used here is guided by these identities.

3.3. **Higgs bundles over elliptic curves.** As above, let G be a connected complex reductive affine algebraic group. Let $T \subset G$ be a Cartan subgroup, and let $\Lambda_T := \mathrm{Hom}(\mathbb{C}^*, T)$ be the corresponding cocharacter lattice. We denote by $Z(G)$ the center of G, and by Z the connected component of it containing the identity element. Fix, once and for all, a basis

$$\{\lambda_1, \cdots, \lambda_\ell, \lambda_{\ell+1}, \cdots, \lambda_s\}$$

of Λ_T such that $\{\lambda_1, \cdots, \lambda_\ell\} \oplus \{\lambda_{\ell+1}, \cdots, \lambda_s\}$ is an orthogonal decomposition of it, with $\{\lambda_1, \cdots, \lambda_\ell\}$ being an orthogonal basis of $\Lambda_Z := \mathrm{Hom}(\mathbb{C}^*, Z)$. We consider the natural isomorphism

$$\eta : \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T \xrightarrow{\cong} T, \quad \sum z_i \otimes_{\mathbb{Z}} \lambda_i \longmapsto \Pi \lambda_i(z_i).$$

Let $\mu : T \times T \longrightarrow T$ be the multiplication map of the group T. For any two principal T-bundles E and E' of trivial topological type, consider the principal $(T \times T)$-bundle $E \times_X E'$, and define

$$E \otimes E' := \mu_*(E \times_X E'),$$

which is again a principal T-bundle of trivial topological type. Note that $E(\mathfrak{t}) \cong \mathfrak{t} \otimes \mathcal{O}_X$, so Higgs fields on a principal T-bundle are elements of $\mathfrak{t}$. One can express the extension of structure groups associated to η as follows

$$\eta_* : T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \xrightarrow{\cong} \mathcal{M}(T), \quad \sum (L_i, \psi_i) \otimes \lambda_i \longmapsto \left(\bigotimes (\lambda_i)_* L_i, \sum (d\lambda_i)_* \psi_i \right).$$

Consider also the extension of structure group associated to the injection $T \hookrightarrow G$, and compose it with η_*, to obtain the following map:

$$\dot{\xi} : T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \longrightarrow \mathcal{M}(G).$$

Analogously, one can define

$$\dot{\zeta} : \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T \longrightarrow \mathcal{R}(G), \quad \sum_i \rho_i \otimes \lambda_i \longmapsto \prod_i (\lambda_i \circ \rho_i).$$

The action of the Weyl group W associated to (T, G) on Λ_T can be extended to $A \otimes_{\mathbb{Z}} \Lambda_T$, where A is any abelian group, as follows:

$$\omega \cdot \left(\sum a_i \otimes \lambda_i \right) = \sum a_i \otimes \omega(\lambda_i), \quad \omega \in W, \quad a_i \in A.$$

If A is the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, this action commutes with the natural action of W on T,

$$\begin{tikzcd} \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\eta"] \arrow[d, "\omega\cdot"'] & T \arrow[d, "\omega\cdot"] \\ \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\eta"] & T. \end{tikzcd} \tag{3.4}$$

The commutativity of (3.4) implies that $\dot{\xi}$ and $\dot{\zeta}$ factor through the quotient by the action of W.

Theorem 3.4 ([16]). *The morphism $\dot{\xi}$ and $\dot{\zeta}$ induce isomorphisms*

$$\xi : (T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T)/W \xrightarrow{\cong} \mathcal{M}(G)$$

and

$$\zeta : (\mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T)/W \xrightarrow{\cong} \mathcal{R}(G).$$

Furthermore, the diffeomorphism between $T^*\widehat{X}$ *and* $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$ *induced by the Hodge theory induces the top row map of the commuting diagram*

$$\begin{tikzcd} T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\text{diffeo.}"] \arrow[d, "\dot{\xi}"'] & \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T \arrow[d, "\dot{\zeta}"] \\ \mathcal{M}(G) \arrow[r, "\text{diffeo.}"] & \mathcal{R}(G) \end{tikzcd} \tag{3.5}$$

4. Description of the involutions of the moduli space

4.1. **The rank 1 case.** The identity map of the multiplicative group $\mathbb{C}^*$ is associated to the anti-holomorphic involution of the compact real form $\mathrm{U}(1)$

$$\sigma_{\mathrm{U}(1)} \,:\, \mathbb{C}^* \longrightarrow \mathbb{C}^*\,, \quad z \longmapsto \overline{z}^{-1}\,. \tag{4.1}$$

For the group $G = \mathbb{C}^*$, set

$$i(\alpha_{(+,a)},\, \pm) \,:=\, I(\alpha_{(+,a)},\, \mathrm{Id}_{\mathbb{C}^*},\, \pm)$$

and

$$i(\alpha_{(-,a)},\, \pm) \,:=\, I(\alpha_{(-,a)},\, \sigma_{\mathrm{U}(1)},\, \pm).$$

Assume that $\mathcal{F} = (F,\, \phi)$ satisfies the involution condition in (2.4). Then we can define the following involutions on $T^*\widehat{X}$

$$i(\alpha_{(+,a)}, \pm, \mathcal{F}) \,:\, T^*\widehat{X} \longrightarrow T^*\widehat{X}\,, \quad (L,\, \psi) \longmapsto (F \otimes \alpha^*_{(+,a)}L,\, \phi \pm \alpha^*_{(+,a)}\psi)\,,$$

and

$$i(\alpha_{(-,a)}, \pm, \mathcal{F}) \,:\, T^*\widehat{X} \longrightarrow T^*\widehat{X}\,, \quad (L,\, \psi) \longmapsto (F \otimes \alpha^*_{(-,a)}\overline{L}^*,\, \phi \mp \alpha^*_{(-,a)}\overline{\psi})\,.$$

Accordingly, set $j(\alpha_{(+,a)}, \pm) = J(\alpha_{(+,a)}, \mathrm{Id}_{\mathbb{C}^*}, \pm)$ and $j(\alpha_{(-,a)}, \pm) = J(\alpha_{(-,a)}, \sigma_{\mathrm{U}(1)}, \pm)$. Similarly, if χ satisfies (2.5), consider the involutions on $\mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*)$

$$j(\alpha_{(\epsilon,a)}, +, \chi) \,:\, \mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*) \longrightarrow \mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*)\,, \quad \rho \longmapsto \chi\left(\cdot\rho \circ (\alpha_{(\epsilon,a)})_*\right)$$

and

$$j(\alpha_{(\epsilon,a)}, -, \chi) \,:\, \mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*) \longrightarrow \mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*)\,, \quad \rho \longmapsto \chi \cdot (\overline{\rho}^{-1} \circ \left(\alpha_{(\epsilon,a)})_*\right)\,.$$

Since $T^*\widehat{X} \cong \widehat{X} \times H^0(X,\, \mathcal{O}_X)$, the isomorphism p_{x_0} in (3.1) produces an isomorphism

$$p_{x_0} \,:\, X \times H^0(X,\, \mathcal{O}_X) \xrightarrow{\cong} T^*\widehat{X}$$

(it is a slight abuse of notation to use p_{x_0} for this map). Given an anti-holomorphic involution $\alpha \,:\, X \longrightarrow X$, let

$$\mathrm{conj} \,:\, H^0(X,\, \mathcal{O}_X) \longrightarrow H^0(X,\, \mathcal{O}_X)\,, \quad X \longmapsto \alpha^*\overline{X}$$

be the induced conjugate-linear homomorphism.

Lemma 4.1. *Take* $\mathcal{F} = (F,\, \phi)$ *satisfying* (2.4)*, and let* $y \in X$ *be such that* $F = p_{x_0}(y)$*. Then,*

$$i(\alpha_{(+,a)}, \pm, \mathcal{F}) \,=\, p \circ (t_y \circ \alpha_{(+,a)},\, \pm\mathrm{Id}) \circ p^{-1}\,,$$

and

$$i(\alpha_{(-,a)}, \pm, \mathcal{F}) \,=\, p \circ (t_y \circ \alpha_{(-,-a)},\, \mp\mathrm{conj}) \circ p^{-1}\,.$$

Proof. The first statement is straight-forward. The lemma follows from the fact that for all $x \in X_\gamma$,

$$\begin{aligned}\alpha^*_{(-,a)}\sigma_{\mathrm{U}(1)}\left(\mathcal{O}_X(x-x_0)\right) &= \alpha^*_{(-,a)}\overline{\mathcal{O}(x-x_0)}^{\,*}\\ &= \mathcal{O}_X(-\alpha_{(-,a)}(x)-x_0)\\ &= \mathcal{O}_X(\alpha_{(-,-a)}(x)-x_0)\,.\end{aligned}$$

□

Recall the generators δ_1, δ_2 of $\pi_1(X)$ defined in (3.2) and (3.3). One has the isomorphism

$$q \,:\, \mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*) \longrightarrow \mathbb{C}^* \times \mathbb{C}^*\,, \quad \rho \longmapsto (\rho(\delta_1), \rho(\delta_2))\,.$$

Note that the involution $j(\alpha_{(-,a)}, \pm, \chi)$ is not defined for every isomorphism class of elliptic curves; rather it is defined only for certain values specified in Table 4. For each region R of Table 4, or for the entire upper-half plane $\mathbb{H}$ when $\epsilon = 1$, let

$$f^{\pm}_{(\epsilon,a,R)} := q \circ j(\alpha_{(\epsilon,a)}, \pm) \circ q^{-1}$$

be the automorphism of $\mathbb{C}^* \times \mathbb{C}^*$.

Remark 4.2. Given $\chi \in \mathcal{R}(Z)$, set $b_i = \chi(\delta_i)$. Note that χ satisfies (2.5) if and only if

$$(b_1^{-1}, b_2^{-1}) = f^{\pm}_{(\epsilon,a,R)}(b_1, b_2)\,.$$

If this holds, then

$$f^{\pm,(b_1,b_2)}_{(\epsilon,a,R)} \,:\, \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathbb{C}^* \times \mathbb{C}^*\,, \quad (z_1,\, z_2) \longmapsto (b_1, b_2)\cdot f^{\pm}_{(\epsilon,a,R)}(z_1,\, z_2)$$

are involutions, and

$$j(\alpha_{(\epsilon,a)}, \pm, \chi) = q^{-1} \circ f^{\pm,(b_1,b_2)}_{(\epsilon,a,R)} \circ q\,.$$

For any involution $\alpha_{(\epsilon,a)}$ of the curve X, using Tables 2 and 6, we can describe $f^{\pm}_{(\epsilon,a,R)}$, and therefore we can describe $f^{\pm,(b_1,b_2)}_{(\epsilon,a,R)}$; see Table 7.

4.2. **The general case.** Let σ_+ and σ_- respectively be the Cartan (holomorphic) involution and the associated real form (anti-holomorphic involution) of G. Recall from Theorem 2.1 that they are related through the composition by the compact real form involution σ_K commuting with σ_-,

$$\sigma_+ = \sigma_- \sigma_K\,. \tag{4.2}$$

Given a Cartan subgroup T of G preserved by σ_+, σ_- and σ_K, we denote by the same symbols the involutions in the cocharacter lattice $\Lambda_T = \mathrm{Hom}(\mathbb{C}^*, T)$,

$$\sigma_+ \,:\, \Lambda_T \longrightarrow \Lambda_T\,, \quad \lambda \longmapsto \sigma_+ \circ \lambda \tag{4.3}$$

and

$$\sigma_- \,:\, \Lambda_T \longrightarrow \Lambda_T\,, \quad \lambda \longmapsto \sigma_- \circ \lambda \circ \sigma_{\mathrm{U}(1)}\,. \tag{4.4}$$

Define an action of σ_K on Λ_T by $\sigma_K(\lambda) = \sigma_K \circ \lambda \circ \sigma_{\mathrm{U}(1)}$; note that the equality (4.2), considered as an equality of involutions of Λ_T, is recovered.

Since the Cartan subgroup T is preserved by σ_+, σ_- and σ_K, these involutions induce involutions of the normalizer $N_G(T)$, and therefore produce involutions of the Weyl group $W = N_G(T)/Z_G(T)$; these involutions of the Weyl group are denoted by the same symbol.

Remark 4.3. Let $T_0 := T^{\sigma_K}$ be the compact torus fixed pointwise by σ_K. Since $T_0^{\mathbb{C}} = T$, it follows that $\mathrm{Hom}(\mathbb{C}^*, T) \cong \mathrm{Hom}(\mathrm{U}(1), T_0)$. The action of σ_K is trivial on $\Lambda_T \cong \mathrm{Hom}(\mathrm{U}(1), T_0)$. Also, we recall that σ_K acts trivially on W, because it is a compact real form. As a consequence, the actions of σ_+ and σ_- on Λ_T coincide. The same statement holds for the actions of σ_+ and σ_- on W.

Remark 4.4. The Vogan diagram is constructed with the root data of a maximally compact Cartan subalgebra $\mathfrak{t}_{\mathbb{R}}$ of a simple real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ and it consists on a triple $V = (D, \theta, S)$ where D is the Dynkin diagram of $\mathfrak{g} = (\mathfrak{g}_{\mathbb{R}})^{\mathbb{C}}$, θ is an automorphism of the diagram given by the Cartan involution σ_+ and S is a subset (possibly empty) of the vertices of D fixed by θ.

The Vogan diagram encodes the action of σ_ϵ on Λ_T when T is the complexification of a maximal compact subgroup of $G_{\mathbb{R}}$, the real subgroup fixed by σ_-.

Recall form Remark 4.3 that it suffices to describe the action of σ_+ on Λ_T. The automorphism θ is an involution on the set of simple roots, so we obtain a description on the entire set of roots. Translate this involution to the coroot lattice Δ by setting $\theta(\alpha^\vee) = (\theta(\alpha))^\vee$, one gets an involution on $\mathfrak{t} \cong \mathbb{C} \otimes_{\mathbb{Z}} \Delta$ which is precisely the Cartan involution σ_+. The restriction of σ_+ to $\Lambda \subset \mathfrak{t}$ coincides with (4.3).

As done in (4.3) and (4.4), define the holomorphic involution

$$\dot{\sigma}_+ \,:\, \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T \longrightarrow \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T\,, \quad \sum z_i \otimes_{\mathbb{Z}} \lambda_i \longmapsto \sum z_i \otimes_{\mathbb{Z}} \sigma_+(\lambda_i)$$

and the anti-holomorphic involution

$$\dot{\sigma}_- \,:\, \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T \longrightarrow \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T\,, \quad \sum z_i \otimes_{\mathbb{Z}} \lambda_i \longmapsto \sum \sigma_{\mathrm{U}(1)}(z_i) \otimes_{\mathbb{Z}} \sigma_-(\lambda_i)\,.$$

We use again $\dot{\sigma}_\epsilon$ to refer simultaneously $\dot{\sigma}_+$ and $\dot{\sigma}_-$. Now the following diagram is commutative

$$\begin{tikzcd} \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\eta"] \arrow[d, "\dot{\sigma}_\epsilon"'] & T \arrow[d, "\sigma_\epsilon"] \\ \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\eta"] & T. \end{tikzcd} \tag{4.5}$$

The action of W on Λ_T is σ_ϵ-equivariant, meaning for any $\omega \in W$ and any $\lambda \in \Lambda_T$,

$$\sigma_\epsilon(\omega \cdot \lambda) = \sigma_\epsilon(\omega) \cdot \sigma_\epsilon(\lambda)\,. \tag{4.6}$$

Therefore, we have

$$\dot{\sigma}_\epsilon \circ \omega = \sigma_\epsilon(\omega) \circ \dot{\sigma}_\epsilon\,. \tag{4.7}$$

Now define the involutions

$$\dot{I}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm) \,:\, T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \longrightarrow T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T\,,$$

$$\sum (L_i, \psi_i) \otimes \lambda_i \longmapsto \sum i(\alpha_{(\epsilon,a)}, \pm)(L_i, \psi_i) \otimes \sigma_\epsilon(\lambda_i)\,,$$

and

$$\dot{J}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm) \,:\, \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T \longrightarrow \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T\,,$$

$$\sum \rho_i \otimes \lambda_i \longmapsto \sum j(\alpha_{(\epsilon,a)}, \pm)(\rho_i) \otimes \sigma_\epsilon(\lambda_i)\,.$$

Fix any $\mathcal{F} = (F, \phi) \in \mathcal{M}(Z)$; let

$$\chi : \pi_1(X) \longrightarrow Z$$

be the corresponding representation of the fundamental group. We recall that

$$\mathcal{M}(Z) \cong T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_Z\,, \quad \mathcal{R}(Z) \cong \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_Z\,.$$

Let $[\mathcal{F}] = \dot{\xi}^{-1}(\mathcal{F})$ and $[\chi] = \dot{\zeta}^{-1}(\chi)$ be the corresponding elements of the groups $T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T$ and $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T$ respectively. Following the definitions of $I(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F})$ and $J(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F})$, define

$$\dot{I}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F}) : T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \longrightarrow T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T\,,$$
$$\sum (L_i, \psi_i) \otimes \lambda_i \longmapsto [\mathcal{F}] + \dot{I}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm) \left(\sum z_i \otimes_{\mathbb{Z}} \lambda_i \right)$$

and

$$\dot{J}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \chi) : \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T \longrightarrow \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T\,,$$
$$\sum \rho_i \otimes \lambda_i \longmapsto [\chi] + \dot{J}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm) \left(\sum \rho_i \otimes \lambda_i \right)\,.$$

Lemma 4.5. *The diagrams*

$$\begin{tikzcd} T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\dot{\xi}"] \arrow[d, "\dot{I}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})"'] & \mathcal{M}(G) \arrow[d, "I(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})"] \\ T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\dot{\xi}"] & \mathcal{M}(G) \end{tikzcd} \tag{4.8}$$

and

$$\begin{tikzcd} \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\dot{\zeta}"] \arrow[d, "\dot{J}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})"'] & \mathcal{R}(G) \arrow[d, "J(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})"] \\ \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\dot{\zeta}"] & \mathcal{R}(G) \end{tikzcd} \tag{4.9}$$

commute.

Proof. In view of (4.7) it follows that $\dot{I}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F})$ induces an involution on the quotient $T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T/W$. From (4.5) and the construction of ξ it is clear that this induced involution coincides with $I(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F})$.

The proof of the commutativity of (4.9) is analogous. □

In Section 3.3 we chose a basis of Λ_T with an orthogonal decomposition

$$\{\lambda_1, \cdots, \lambda_\ell\} \oplus \{\lambda_{\ell+1}, \cdots, \lambda_s\}\,,$$

where $\{\lambda_1, \cdots, \lambda_\ell\}$ is an orthogonal basis of $\Lambda_Z = \mathrm{Hom}(\mathbb{C}^*, Z)$. Take a Z-Higgs bundle $\mathcal{F} = (F, \phi)$, and set $\mathcal{F}_i^\epsilon = (F_i^\epsilon, \phi_i^\epsilon)$ to be the Higgs line bundles such that

$$[\mathcal{F}] = \sum [\mathcal{F}_i^\epsilon] \otimes \sigma_\epsilon(\lambda_i)\,. \tag{4.10}$$

Let us extend this by setting $\mathcal{F}_i^\epsilon := (\mathcal{O}_X, 0)$ for $\ell + 1 \leq i \leq s$. Analogously, we take a decomposition of the representation $\chi : \pi_1(X) \longrightarrow Z$

$$[\chi] = \sum [\chi_i^\epsilon] \otimes \sigma_\epsilon(\lambda_i) \tag{4.11}$$

where χ_i^ϵ are homomorphisms from $\pi_1(X)$ to $\mathbb{C}^*$. As before, set $\chi_i^\epsilon = \mathrm{Id}$ for $\ell+1 \leq i \leq s$.

Proposition 4.6. *For $\mathcal{F}$ and χ as above, construct $\mathcal{F}_i^\epsilon$ and χ_i^ϵ as in* (4.10) *and* (4.11) *respectively. Then*

$$\dot{I}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F}) \left(\sum (L_i, \psi_i) \otimes \lambda_i \right) = \sum i(\alpha_{(\epsilon,a)}, \pm, \mathcal{F}_i^\epsilon)(L_i, \psi_i) \otimes \sigma_\epsilon(\lambda_i)$$

and

$$\dot{J}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F}) \left(\sum \rho_i \otimes \lambda_i \right) = \sum j(\alpha_{(\epsilon,a)}, \pm, \chi_i^\epsilon)(\rho_i) \otimes \sigma_\epsilon(\lambda_i) \, .$$

Proof. Once we have the descriptions of $\dot{I}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm)$ and $\dot{J}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm)$, it is immediate to derive descriptions of $\dot{I}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F})$ and $\dot{J}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \chi)$. We have

$$\begin{aligned} \dot{I}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F}) \left(\sum (L_i, \psi_i) \otimes \lambda_i \right) &= \sum [\mathcal{F}_i^\epsilon] \otimes \sigma_\epsilon(\lambda_i) + \sum i(\alpha_{(\epsilon,a)}, \pm)(L_i, \psi_i) \otimes \sigma_\epsilon(\lambda_i) \\ &= \sum \left([\mathcal{F}_i^\pm] + i(\alpha_{(\epsilon,a)}, \pm)(L_i, \psi_i) \right) \otimes \sigma_\epsilon(\lambda_i) \\ &= \sum i(\alpha_{(\epsilon,a)}, \pm, \mathcal{F}_i^\epsilon)(L_i, \psi_i) \otimes \sigma_\epsilon(\lambda_i) \, . \end{aligned}$$

The case of $\dot{J}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \chi)$ is analogous. □

4.3. **Description of the fixed point locus.** Consider the cocharacter lattice Λ_T of a complex reductive Lie group G. Denote by σ_- the anti-holomorphic involution of a certain real form, and denote by σ_+ the Cartan involution associated to it. Recall from (4.6) that the Weyl group W acts σ_ϵ-equivariantly on Λ_T.

Let A be a complex abelian group; construct the tensor product

$$\dot{B} := A \otimes_{\mathbb{Z}} \Lambda_T \, .$$

Consider also the action of W on $\dot{B}$ induced by the action of W on Λ and take its quotient

$$B := \dot{B}/W \, .$$

Fix a basis $\{\lambda_1, \cdots, \lambda_s\}$ of Λ and suppose that A is equipped with a set of analytic group homomorphisms of order two $\{t_1, \cdots, t_s\}$

$$t_i : A \longrightarrow A$$

such that we can combine them with σ_ϵ to obtain the analytic involution

$$\dot{\tau} : \dot{B} = A \otimes_{\mathbb{Z}} \Lambda \longrightarrow \dot{B} \, , \quad \sum a_i \otimes \lambda_i \longmapsto \sum t_i(a_i) \otimes \sigma_\epsilon(\lambda_i) \, .$$

Since the action of the Weyl group is σ_ϵ-equivariant, it follows that $\dot{\tau}$ induces an involution on the quotient

$$\tau : B = (A \otimes_{\mathbb{Z}} \Lambda)/W \longrightarrow B \, , \quad \left[\sum a_i \otimes \lambda_i \right]_W \longmapsto \left[\sum t_i(a_i) \otimes \sigma_\epsilon(\lambda_i) \right]_W \, .$$

Evidently, the diagram

$$\begin{array}{ccc} A \otimes_{\mathbb{Z}} \Lambda_T & \xrightarrow{p} & A \otimes_{\mathbb{Z}} \Lambda_T/W \\ \downarrow{\scriptstyle \dot{\tau}} & & \downarrow{\scriptstyle \tau} \\ A \otimes_{\mathbb{Z}} \Lambda_T & \xrightarrow{p} & A \otimes_{\mathbb{Z}} \Lambda_T/W \end{array}$$

commutes, where p is the projection induced by the quotient map for the action of W. By construction, we have a similar commuting diagram for every element $\omega \in W$,

$$\begin{tikzcd} A \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "p"] \arrow[d, "\omega\dot{\tau}"'] & A \otimes_{\mathbb{Z}} \Lambda_T/W \arrow[d, "\tau"] \\ A \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "p"] & A \otimes_{\mathbb{Z}} \Lambda_T/W \end{tikzcd} \tag{4.12}$$

The aim of this section is to describe the fixed point set $(A \otimes_{\mathbb{Z}} \Lambda_T/W)^\tau$. To do so, we will make use of the commutativity of (4.12). Define, for any $\omega \in W$, the subgroup of $A \otimes_{\mathbb{Z}} \Lambda_T/W$

$$(A \otimes_{\mathbb{Z}} \Lambda_T/W)^\tau_\omega \,:=\, p((A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\dot{\tau}})\,.$$

One has that

$$(A \otimes_{\mathbb{Z}} \Lambda_T/W)^\tau \,=\, \bigcup_{\omega \in W} (A \otimes_{\mathbb{Z}} \Lambda_T/W)^\tau_\omega\,.$$

Remark 4.7. Note that each $\dot{B}^{\omega\dot{\tau}}$ is a closed subset, because it is a fixed point locus of an involution. The projection p is continuous, and therefore $B^\tau_\omega \,=\, p(\dot{B}^{\omega\dot{\tau}})$ is closed as well.

Lemma 4.8. *The equality*

$$(A \otimes_{\mathbb{Z}} \Lambda_T/W)^\tau_{\omega_1} \,=\, (A \otimes_{\mathbb{Z}} \Lambda_T/W)^\tau_{\omega_2}$$

holds if and only if there exists an element $\omega' \in W$ such that

$$\omega_2 \,=\, \omega'\omega_1\sigma_\epsilon(\omega')^{-1}\,. \tag{4.13}$$

Proof. We have that $x' \in (A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\dot{\tau}}$ if and only if $x' \,=\, \omega\dot{\tau}(x')$. Now $x \,=\, \omega' x'$ lies in $\omega' \cdot (A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\dot{\tau}}$ if and only if

$$(\omega')^{-1}x \,=\, \omega\dot{\tau}((\omega')^{-1}x)\,.$$

In that case,

$$x \,=\, \omega'\omega\sigma_\epsilon(\omega')^{-1}\dot{\tau}(x)\,.$$

We see that for $\omega, \omega' \in W$,

$$\omega' \cdot (A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\dot{\tau}} \,=\, (A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega'\omega\sigma_\epsilon(\omega')^{-1}\dot{\tau}}\,,$$

which implies the lemma. $\square$

Following (4.13), define the σ_ϵ-adjoint action

$$\mathrm{ad}_{\sigma_\epsilon} \,:\, W \times W \,\longrightarrow\, W\,, \quad (\omega',\, \omega) \,\longmapsto\, \omega'\omega\sigma_\epsilon(\omega')^{-1}\,.$$

From Lemma 4.8 it follows that the components are parametrized by

$$W/_{\sigma_\epsilon}W \,=\, W/\mathrm{ad}_{\sigma_\epsilon}(W)\,.$$

Therefore, we write

$$(A \otimes_{\mathbb{Z}} \Lambda_T/W)^\tau \,=\, \bigcup_{\overline{\omega} \in W/_{\sigma_\epsilon}W} (A \otimes_{\mathbb{Z}} \Lambda_T/W)^\tau_\omega\,,$$

where ω is a representative of the σ_ϵ-conjugacy class $\overline{\omega} \in W/_{\sigma_\epsilon}W$.

Proposition 4.9. *Denote by $T^{\omega\sigma_\epsilon}$ the subtorus fixed by $\omega\sigma_\epsilon$. Then*

$$(A \otimes_{\mathbb{Z}} \Lambda_T/W)^{\tau}_{\omega} \cong ((A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\dot{\tau}})/N_W(T^{\omega\sigma_\epsilon}). \tag{4.14}$$

Furthermore,

$$\dim(B^{\tau}_{\omega}) = (\dim(A \otimes_{\mathbb{Z}} \Lambda_T))/2 \cdot \mathrm{ord}(\omega\sigma_\epsilon(\omega)). \tag{4.15}$$

Proof. We claim that

$$N_W(T^{\omega\sigma_\epsilon}) = \{\omega' \in W \mid \omega = \omega'\omega\sigma_\epsilon(\omega')^{-1}\}. \tag{4.16}$$

To prove this claim, first recall that

$$N_W(T^{\omega\sigma_\epsilon}) = \{\omega' \in W \mid \omega'(T^{\omega\sigma_\epsilon}) = T^{\omega\sigma_\epsilon}\}.$$

Now note that

$$\omega'(T^{\omega\sigma_\epsilon}) = T^{\omega'\omega\circ\sigma_\epsilon\circ(\omega')^{-1}} = T^{\omega'\omega\sigma_\epsilon(\omega')^{-1}\circ\sigma_\epsilon}.$$

We have $T^{\omega'\omega\sigma_\epsilon(\omega')^{-1}\circ\sigma_\epsilon} = T^{\omega\sigma_\epsilon}$ if and only if $\omega'\omega\sigma_\epsilon(\omega')^{-1} = \omega$. This proved the claim.

The first statement in the proposition follows from the combination of Lemma 4.8 and the above claim.

To prove the second statement, since $\dot{\tau}$ is an analytic homomorphism, so is $\omega\dot{\tau}$. We assume that they are all nontrivial (so we exclude the case of $\dot{\tau}$ being an element of W). Since p is the quotient by a finite subgroup, one has

$$\dim((A \otimes_{\mathbb{Z}} \Lambda_T/W)^{\tau}_{\omega}) = \dim((A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\dot{\tau}}),$$

and therefore

$$\dim((A \otimes_{\mathbb{Z}} \Lambda_T/W)^{\tau}_{\omega}) = (\dim(A \otimes_{\mathbb{Z}} \Lambda_T))/\mathrm{ord}(\omega\dot{\tau}).$$

Note that

$$\omega\dot{\tau}\omega\dot{\tau} = \omega\sigma_\epsilon(\omega)\dot{\tau}^2 = \omega\sigma_\epsilon(\omega), \tag{4.17}$$

so $\mathrm{ord}(\omega\dot{\tau}) = 2 \cdot \mathrm{ord}(\omega\sigma_\epsilon(\omega))$. ☐

We now proceed to describe the fixed locus $\dot{B}^{\omega\sigma_\epsilon}$. Using the basis $\{\lambda_1, \cdots, \lambda_s\}$ of Λ one gets an isomorphism between $\dot{B}$ and $\overbrace{A \times \cdots \times A}^{s-\text{times}}$. Denote by $M \in \mathrm{GL}(s, \mathbb{Z})$ the matrix of $\omega\sigma_\epsilon$ in this base. We denote by $\overline{t} = (t_1, \cdots, t_s)$ the involution on A^s given by the t_i. We observe that $\omega\dot{\tau}$ corresponds with $M \circ \overline{t}$. It is then clear that

$$(A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\sigma_\epsilon} \cong \left(\overbrace{A \times \cdots \times A}^{s-\text{times}}\right)^{M\circ\overline{t}}. \tag{4.18}$$

The following is a consequence of Propositions 4.6 and 4.9, Lemma 4.1 and Remarks 3.1, 3.3 and 4.2.

Corollary 4.10. *Let $t_y \circ \alpha_{(\epsilon,a)}$ be a holomorphic (respectively, anti-holomorphic) involution on X and σ_ϵ a holomorphic (respectively, anti-holomorphic) involution of the complex reductive Lie group G. The fixed locus for the involution $I(t_y \circ \alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F})$ of $\mathcal{M}(G)$ is the union*

$$\mathcal{M}(G)^{I(t_y\circ\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})} = \bigcup_{\overline{\omega}\in W/_{\sigma_\epsilon}W} \left(T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T\right)^{\omega\left(\dot{I}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})\right)} / N_W(T^{\omega\sigma_\epsilon}),$$

where ω *is a representative of* $\overline{\omega} \in W/_{\sigma_\epsilon}W$*. Furthermore, if* $M_\omega \in \mathrm{GL}(s,\mathbb{Z})$ *is the automorphism* $\omega\sigma_\epsilon(\omega)$ *expressed in a certain basis of* Λ_T*, and if* $y_1, \cdots, y_s \in T^*\widehat{X}$ *are the coordinates of* $\mathcal{F} \in T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T$ *in this basis, then*

$$\left(T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T\right)^{\omega\left(\dot{I}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})\right)} \cong \left(\overbrace{T^*\widehat{X} \times \cdots \times T^*\widehat{X}}^{s-\text{times}}\right)^{M_\omega \circ \overline{t}},$$

where

$$\overline{t} = \left((\nu_{y_1} \circ \alpha_{(+,a)}, \pm\mathrm{Id}), \cdots, (\nu_{y_s} \circ \alpha_{(+,a)}, \pm\mathrm{Id})\right)$$

if $\epsilon = +$*, and*

$$\overline{t} = \left((\nu_{y_1} \circ \alpha_{(-,-a)}, \mp\mathrm{conj}), \cdots, (\nu_{y_s} \circ \alpha_{(-,-a)}, \mp\mathrm{conj})\right)$$

if $\epsilon = -$.

Analogously, the fixed locus for the involution $J(t_y \circ \alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \chi)$ *of* $\mathcal{R}(G)$ *is the union*

$$\mathcal{R}(G)^{I(t_y\circ\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\chi)} = \bigcup_{\overline{\omega}\in W/_{\sigma_\epsilon}W} (\mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\left(\dot{J}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\chi)\right)} / N_W(T^{\omega\sigma_+})\,.$$

If the homomorphisms $\chi_i \in \mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*)$ *are the coordinates of* $\chi \in \mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*)\otimes_{\mathbb{Z}} \Lambda_T$ *in the basis, and if* $b_{i,1} := \chi_i(\delta_1)$ *and* $b_{i,2} := \chi_i(\delta_2)$ *are the images of the generators* δ_1, δ_2 *of* $\pi_1(X)$*, then*

$$(\mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\left(\dot{J}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\chi)\right)} \cong \left((\mathbb{C}^* \times \mathbb{C}^*) \times \overset{s}{\cdots} \times (\mathbb{C}^* \times \mathbb{C}^*)\right)^{M_\omega \circ \overline{t}},$$

where

$$\overline{t} = \left(f_{(\epsilon,a,R)}^{\pm,(b_{1,1},b_{1,2})}, \cdots, f_{(\epsilon,a,R)}^{\pm,(b_{s,1},b_{s,2})}\right)$$

with R *being the region of* Ω *where* X *lies.*

It is straight-forward to check that the subset of W given by the elements of the form $\omega\sigma_\epsilon(\omega)$ is closed under the usual adjoint action of W. We define the quotient

$$\Upsilon_{\sigma_\epsilon} = \{\omega\sigma_\epsilon(\omega) \ \mid\ \omega \in W\}/\mathrm{ad}(W)\,.$$

Define the map

$$\delta \,:\, W/_{\sigma_\epsilon}W \longrightarrow \Upsilon_{\sigma_\epsilon}\,, \quad [\omega]_{\mathrm{ad}_{\sigma_\epsilon}} \longmapsto [\omega\sigma_\epsilon(\omega)]_{\mathrm{ad}}\,.$$

In view of (4.15), the dimension of B^τ_ω is determined by $\delta([\omega]_{\mathrm{ad}_{\sigma_\epsilon}})$.

Following (4.15), we say that a component B^τ_ω is *maximal* is the order of $\omega\dot{\tau}$ is 2. It is clear that the dimension of each maximal component is half the dimension of B. We see that a component is maximal when

$$\mathrm{Id}_W = \omega\sigma_\epsilon(\omega)\,. \tag{4.19}$$

We define $B^{\tau,\mathrm{Id}}$ to be the union of all maximal components.

Consider the non-abelian group cohomology associated to the action of $\mathbb{Z}/2\mathbb{Z}$ on W given by σ_ϵ. Note that the cocycle condition is precisely (4.19), that is, the condition for B^τ_ω to be maximal. Also, the coboundary condition is precisely (4.13). These tell us that maximal components are indexed by $H^1(\sigma_\epsilon,\, W)$,

$$(A \otimes_{\mathbb{Z}} \Lambda_T/W)^{\tau,\mathrm{Id}} = \bigcup_{\overline{\omega}\in H^1(\sigma_\epsilon,W)} (A \otimes_{\mathbb{Z}} \Lambda_T/W)^\tau_{\overline{\omega}}. \tag{4.20}$$

Analogously, for a fixed representative of each $\gamma \in \Upsilon_{\sigma_\epsilon}$ one can define a γ-shifted cocycle condition

$$\gamma = \omega\sigma_\epsilon(\omega)\,. \tag{4.21}$$

We also consider the union of all the components with a fixed $\gamma \in \Upsilon_{\sigma_\epsilon}$,

$$(A \otimes_{\mathbb{Z}} \Lambda_T/W)^{\tau,\gamma} := \bigcup_{\delta([\omega])=\gamma} (A \otimes_{\mathbb{Z}} \Lambda_T/W)^{\tau}_{\omega}\,;$$

it is clear that

$$(A \otimes_{\mathbb{Z}} \Lambda_T/W)^{\tau} = \bigcup_{\gamma\in\Upsilon_{\sigma_\epsilon}} (A \otimes_{\mathbb{Z}} \Lambda_T/W)^{\tau,\gamma}\,.$$

Taking the coboundary condition (4.13) as before, we can define the shifted non-abelian group cohomology $H^1_\gamma(\sigma_\epsilon, W)$. We see that

$$(A \otimes_{\mathbb{Z}} \Lambda_T/W)^{\tau,\gamma} = \bigcup_{\overline{\omega}\in H^1_\gamma(\sigma_\epsilon,W)} (A \otimes_{\mathbb{Z}} \Lambda_T/W)^{\tau}_{\overline{\omega}}\,. \tag{4.22}$$

One can easily see that every point $b \in \dot{B}$ such that

$$b = \omega\dot{\tau}(b)$$

actually satisfies the condition

$$b = \omega\dot{\tau}\omega\dot{\tau}(b)\,.$$

Recalling (4.17), we observe that b is fixed by $\omega\sigma_\epsilon(\omega)$, and this implies that

$$\dot{B}^{\omega\dot{\tau}} \subset \dot{B}^{\omega\sigma_\epsilon(\omega)}\,. \tag{4.23}$$

Remark 4.11. Note that if $\dim(A) \geq 2$, then the singular locus of $B = \dot{B}/W$ is

$$\mathrm{Sing}(B) = \bigcup_{\mathrm{Id}\neq\omega\in W} p(\dot{B}^{\omega})\,.$$

An antiholomorphic involution on a complex manifold has a half dimensional fixed point locus. In our case, non-maximal components $B^\tau_\omega \in B^{\tau,\gamma}$ have dimension less than $\frac{1}{2}\dim(B)$. Although, (4.23) implies that any non-maximal component satisfies $B^\tau_\omega \subset B^{\tau,\gamma}$, and therefore we have $B^\tau_\omega \subset \mathrm{Sing}(B)$.

4.4. **Moduli spaces of pseudo-real Higgs bundles.** Every element $z \in Z_2$ of order 2 of the center defines an element of the Weyl group ω_z in the following way (see for instance [18, 16]). Take an alcove $A \subset \mathfrak{t}$ containing the origin. We know (see for instance [13]) that there is a vertex a_z of the alcove A such that $z = \exp(a_z)$. We see that $A - a_z$ is another alcove containing the origin. Hence there is a unique element $\omega_z \in W$ such that

$$A - a_z = \omega_z(A)\,.$$

In the trivial case we obviously have $\omega_0 = \mathrm{Id}$.

Remark 4.12. Note that the action of ω_z on T coincides with the action of z on it.

Now we can provide a description of the moduli spaces of pseudo-real G-Higgs bundles.

Theorem 4.13. *Take the central element $z \in Z_2^{\sigma_-}$, the antiholomorphic involution*

$$t_y \circ \alpha_{(-,a)} \,:\, X \longrightarrow X\,,$$

and the pair of holomorphic and anti-holomorphic involutions $\sigma_+, \sigma_- \,:\, G \longrightarrow G$ such that $\sigma_+ = \sigma_-\sigma_K$. The image of the moduli space of $(G, \alpha_{(-,a)}, \sigma_-, \pm, z)$-Higgs bundles under the forgetful morphism is

$$\widetilde{\mathcal{M}}(G, t_y \circ \alpha_{(-,a)}, \sigma_-, \pm, z) \;=\; \bigcup_{\overline{\omega} \in H^1_{\omega_z}(\sigma_-, W)} \left(T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \right)^{\omega\left(\dot{I}(\alpha_{(-,a)}, \sigma_-, \pm)\right)} / N_W(T^{\omega\sigma_-})\,.$$

Furthermore, the forgetful morphism

$$\mathcal{M}(G, t_y \circ \alpha_{(-,a)}, \sigma_-, \pm, z) \longrightarrow \widetilde{\mathcal{M}}(G, t_y \circ \alpha_{(-,a)}, \sigma_-, \pm, z)$$

is bijective. Analogously, the image of the moduli space of representations is

$$\widetilde{\mathcal{R}}(G, t_y \circ \alpha_{(-,a)}, \sigma_-, \pm, z) \;=\; \bigcup_{\overline{\omega} \in H^1_{\omega_z}(\sigma_-, W)} \left(\mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T \right)^{\omega\left(\dot{J}(\alpha_{(-,a)}, \sigma_-, \pm)\right)} / N_W(T^{\omega\sigma_-})\,,$$

and the forgetful morphism

$$\mathcal{R}(G, t_y \circ \alpha_{(-,a)}, \sigma_-, \pm, z) \longrightarrow \widetilde{\mathcal{R}}(G, t_y \circ \alpha_{(-,a)}, \sigma_-, \pm, z)$$

is bijective as well.

Proof. In view of Remark 3.3, without any loss of generality, we can take $\alpha_{(\epsilon,a)}$ to be our anti-holomorphic involution. From Theorem 2.6 we have that $\widetilde{\mathcal{M}}(G, \alpha_{(-,a)}, \sigma_-, \pm, z)$ lies in the fixed-point locus of $I(\alpha_{(-,a)}, \sigma_-, \pm)$. In fact, recalling Remark 4.12 and the equivalent definition of a pseudo-real $(G, \alpha_{(-,a)}, \sigma_-, \pm, z)$-Higgs bundles given in (2.7) and (2.8), we know that the points lying in $\widetilde{\mathcal{M}}(G, \alpha_{(-,a)}, \sigma_-, \pm, z)$ are those lying in the components $(T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T/W)_{\omega}^{\dot{I}(\alpha_{(-,a)}, \sigma_-, \pm)}$ where

$$\omega\sigma_-(\omega) = \omega_c.$$

From this and (4.22), we see that $\widetilde{\mathcal{M}}(G, \alpha_{(-,a)}, \sigma_-, \pm, z)$ is given by the union of the components of $H^1_{\omega_z}(\sigma_-, W)$, and the first statement follows from Corollary 4.10.

Recall the definition of an isomorphism of pseudo-real Higgs bundles given in 2.9 and the description of the group $N_W(T^{\omega\sigma_-})$ given in (4.16). Then, every isomorphism between pseudo-real G-Higgs bundles parametrized by $T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T$ is given by the elements of $N_W(T^{\omega\sigma_-})$. The second statement follows from this observation and the description of $\widetilde{\mathcal{M}}(G, \alpha_{(-,a)}, \sigma_-, \pm, z)$ given above.

Finally, the third an fourth statements follow from the previous description and [6, Theorems 4.5 and 4.8]. ∎

Recall that for $z = \mathrm{Id}_G$, one has $\omega_z = \mathrm{Id}_W$, and therefore the moduli space of real $(G, \alpha_-, \sigma_-, \pm)$-Higgs bundles (corresponding to $z = \mathrm{Id}_G$) is the union of the components classified by $H^1(\sigma_-, W)$.

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Tables

Involution	Admissible translations	X^α	X/α
$\alpha_{(+,1)}$	$-$	X_γ	X_γ
$t_y \circ \alpha_{(+,1)}$	$y \in X[2], y \neq x_0$	$\emptyset$	$X_{(\gamma - \widetilde{y})}$
$\alpha_{(+,-1)}$	$-$	$X[2]$	$\mathbb{P}^1$
$t_y \circ \alpha_{(+,-1)}$	$y \in X$	$\frac{1}{2}y + X[2]$	$\mathbb{P}^1$

TABLE 1. Holomorphic involutions of X_γ.

Involution	$\big((\alpha_{\alpha_\epsilon})_*\delta_1, (\alpha_{(\epsilon,a)})_*\delta_2\big)$
$\alpha_{(+,1)}$	(δ_1, δ_2)
$\alpha_{(+,-1)}$	$(-\delta_1, -\delta_2)$

TABLE 2. Action of $\alpha_{(+,a)}$ on $\pi_1(X_\gamma)$.

Topological type	X^{α_-}	X/α_-
$(0,1)$	$\emptyset$	Klein bottle
$(1,1)$	S^1	Möbius strip
$(2,0)$	$S^1 \sqcup S^1$	Closed annulus

TABLE 3. Topological type of real elliptic curves.

Region of $\mathbb{H}$	Values of γ
A	$\{\Im(\gamma) > 1, \Re(\gamma) = 0\}$
B	$\{\Im(\gamma) = 1, \Re(\gamma) = 0\}$
C	$\{0 < \Re(\gamma) < \frac{1}{2}, \Im(\gamma) = \sqrt{1 - \Re(\gamma)}\}$
D	$\{\Im(\gamma) = \frac{\sqrt{3}}{2}, \Re(\gamma) = \frac{1}{2}\}$
E	$\{\Im(\gamma) > \frac{\sqrt{3}}{2}, \Re(\gamma) = \frac{1}{2}\}$

TABLE 4. X_γ admitting an antiholomorphic involution.

Region of $\mathbb{H}$	Involution	Admissible translations	Topological type
A	$\alpha_{(-,\pm 1)}$	$-$	(2,0)
	$t_y \circ \alpha_{(-,\pm 1)}$	$y \neq x_0, y \in X^{\alpha_{(-,\mp 1)}} \cong S^1 \sqcup S^1$	(0,1)
B	$\alpha_{(-,\pm 1)}$	$-$	(2,0)
	$\alpha_{(-,\pm \mathbf{i})}$	$-$	(1,1)
	$t_y \circ \alpha_{(-,\pm 1)}$	$y \neq x_0, y \in X^{\alpha_{(-,\mp 1)}} \cong S^1 \sqcup S^1$	(0,1)
C	$t_y \circ \alpha_{(-,\pm\gamma)}$	$y \in X^{\alpha_{(-,\mp\gamma)}} \cong S^1$	(1,1)
D	$t_y \circ \alpha_{(-,\pm 1)}$	$y \in X^{\alpha_{(-,\mp 1)}} \cong S^1$	(1,1)
	$t_y \circ \alpha_{(-,\pm\gamma)}$	$y \in X^{\alpha_{(-,\mp\gamma)}} \cong S^1$	(1,1)
	$t_y \circ \alpha_{(-,\pm\gamma^2)}$	$y \in X^{\alpha_{(-,\mp\gamma^2)}} \cong S^1$	(1,1)
E	$t_y \circ \alpha_{(-,\pm 1)}$	$y \in X^{\alpha_{(-,\mp 1)}} \cong S^1$	(1,1)

TABLE 5. Anti-holomorphic involutions on X_γ.

Region of $\mathbb{H}$	Involution	$(\alpha_{(\epsilon,a)})_*(\delta_1, \delta_2)$
A, B	$\alpha_{(-,1)}$	$(\delta_1, -\delta_2)$
	$\alpha_{(-,-1)}$	$(-\delta_1, \delta_2)$
C, D	$\alpha_{(-,1)}$	$(\delta_1, -\delta_2)$
	$\alpha_{(-,-1)}$	$(-\delta_1, \delta_2)$
	$\alpha_{(-,\gamma)}$	(δ_2, δ_1)
	$\alpha_{(-,-\gamma)}$	$(-\delta_2, -\delta_1)$
E	$\alpha_{(-,1)}$	$(\delta_1, -\delta_2 + \delta_1)$
	$\alpha_{(-,-1)}$	$(-\delta_1, \delta_2 - \delta_1)$

TABLE 6. Action of $\alpha_{(-,a)}$ on $\pi_1(X)$.

Region of $\mathbb{H}$	Involution	$f^{+}_{(\epsilon,a,R)}(z_1,z_2)$	$f^{-}_{(\epsilon,a,R)}(z_1,z_2)$
$\mathbb{H}$	$\alpha_{(+,1)}$	(z_1,z_2)	$(\overline{z}_1^{-1},\overline{z}_2^{-1})$
	$\alpha_{(+,-1)}$	(z_1^{-1},z_2^{-1})	$(\overline{z}_1,\overline{z}_2)$
A,B	$\alpha_{(-,1)}$	(z_1,z_2^{-1})	$(\overline{z}_1^{-1},\overline{z}_2)$
	$\alpha_{(-,-1)}$	(z_1^{-1},z_2)	$(\overline{z}_1,\overline{z}_2^{-1})$
C,D	$\alpha_{(-,1)}$	(z_1,z_2^{-1})	$(\overline{z}_1^{-1},\overline{z}_2)$
	$\alpha_{(-,-1)}$	(z_1^{-1},z_2)	$(\overline{z}_1,\overline{z}_2^{-1})$
	$\alpha_{(-,\gamma)}$	(z_2,z_1)	$(\overline{z}_2^{-1},\overline{z}_1^{-1})$
	$\alpha_{(-,-\gamma)}$	(z_2^{-1},z_1^{-1})	$(\overline{z}_2,\overline{z}_1)$
E	$\alpha_{(-,1)}$	$(z_1,z_2^{-1}z_1)$	$(\overline{z}_1^{-1},\overline{z}_2\overline{z}_1^{-1})$
	$\alpha_{(-,-1)}$	$(z_1^{-1},z_2z_1^{-1})$	$(\overline{z}_1^{-1},\overline{z}_2^{-1}\overline{z}_1)$

TABLE 7. Values of $f^{\pm}_{(\epsilon,a,R)}$.

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