Semantic analysis of fuzzy models, application to Yager models

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Abstract: This paper proposes a methodology to analyse the semantic behaviour of a fuzzy rule model, that is, a pair of fuzzy implication and modus ponens generating function used for inference. The proposed methodology is applied to Yager models which are obtained from Yager implication function. It is shown that, for example, Yager implicative implication is midway between the usual residual and strong implications generated from the product t-norm, and that in fact Yager implication belongs to a more general family of implications that can also be generated from the t-norms.

Key-Words: fuzzy inference, possibility distributions, compositional rule of inference, fuzzy implications.

1 Introduction

This section describes the basic concepts used along the paper.

A fuzzy implication can be defined as a function $I(x,y)$:[0,1]×[0,1]→[0,1] verifying the following properties [1], [2]:

$$(1)\quad \begin{cases} I:[0,1]\times[0,1]\rightarrow[0,1] \\ I(1,1)=1 \\ I(1,0)=0 \\ I(x,y) \text{ non decreasing with y} \end{cases}$$

Given a rule $P\rightarrow Q$ ("if u is P then v is Q") where P and Q are fuzzy sets defined over U and V respectively, (we will also denote them by $P(u)$ and $Q(v)$), the fuzzy conditionals R that can be used to model the rule are usually generated from a fuzzy implication function as $R(u,v)=I(P(u),Q(v))$. Given an observation $P'(u)$, inference is performed using the compositional rule of inference (CRI) [1], [2]:

$$(2)\quad Q'(v)=\sup_u M(P'(u),R(u,v))$$

where a modus ponens generating function (MPGF for short) M must be used to combine the conditional and the observation [2]. The MPGF is a conjunctive operator verifying:

$$M(0,1)=M(1,0)=M(0,0)=0 \quad\text{and}\quad M(1,1)=1$$
$$M(x,y) \text{ non decreasing with } x \text{ and with } y.$$

If in addition it is:

$$M(x,R(x,y)) \leq y \ (\textit{modus ponens inequality})$$
$$M(1,y)=y \quad\text{and}\quad R(1,y)=y.$$

then the CRI verifies the generalised modus ponens (if $P'=P$ then $Q'=Q$). A pair (M,R) verifying all the above properties has been called a rule model for $P\rightarrow Q$ ((M,I) if the conditional R is replaced by the implication function I).

Two main types of implications, [3], [4], can be distinguished: n-implications $I_n(x,y)$, verifying $I_n(0,y)=1$ and non increasing with x (classical material conditional generalisation), and p-implications $I_p(x,y)$, verifying $I_n(0,y)=0$ and non decreasing with x (classical cartesian product generalisation). The models obtained from n-implications have been called n-models, and those obtained from p-implications p-models. Some typical n-models are for example (M_R^T,I_R^T), (M_S^T,I_S^T), and p-models (M_R^T,M_R^T) and (M_S^T,M_S^T), where the following operators have been defined:

(3)

$I_R^T(x,y)=\sup\{c\in[0,1]/\ T(x,c)\leq y\}$ (residual implication)
$I_S^T(x,y)=S(1\text{-}x,y)=1\text{-}T(x,1\text{-}y)$ (strong implication)
$M_R^T(x,y)=T(x,y)$ (t-norm)
$M_S^T(x,y)=\inf\{c\in[0,1]\ /S(1\text{-}x,c)\geq y\}$ (pseudo-conjunction)

where T and S are a t-norm and its dual t-conorm. Some properties of these models have already been studied in precedent papers [3], [4]. However, there are some particular implications that are not included in the above families. A typical example is the Yager n-implication, given by:

$$(4) \quad I_Y(x,y) = \begin{cases} y^x \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

and its conjunctive counterpart (Yager p-implication), its maximum MPGF $M_Y(x,y) = \inf\{c \in [0,1] / I_Y(x,c) \geq y\}$, [2], given by:

$$(5) \quad M_Y(x,y) = \begin{cases} y^{\frac{1}{x}} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

As far as we know, no way of generating both implications has been reported yet. The models (M_Y, I_Y) and (M_Y, M_Y) will be called Yager n and p-models respectively.

This paper proposes a procedure to analyse the behaviour of fuzzy rule models. An example of Yager models analysis is also provided.

In the sequel the core and the support of a fuzzy set P will be denoted respectively by $[P]$ and $]P[$. Only normalised fuzzy sets will be considered.

2 Semantic analysis of a rule model

The following steps are proposed.

2.1 Order relation between conclusion and consequent

Given a rule $P(u) \rightarrow Q(v)$, a n-model (M, I_n) and an observation $P'(u)$, the conclusion $Q_n'(v)$ is given by:

$$Q_n'(v) = sup_u M(P'(u), I_n(P(u), Q(v)))$$

Since $I_n(x,y)$ is non decreasing with x and $I(1,y)=y$, then $I_n(x,y) \geq y$ and the conclusion of the model verifies $Q_n'(v) = sup_u M(P'(u), I_n(P(u),Q(v))) \geq sup_u M(P'(u), Q(v))$. Since $M(x,y)$ is non decreasing with x then $sup_u M(P'(u),Q(v)) = M(sup_u P'(u), Q(v)) = Q(v)$ and thus it is:

$$(6) \quad Q'_n(v) \geq Q(v)$$

Similarly $I_p(x,y)$ is non decreasing with x and $I_p(1,y)=y$ which means that $I_p(x,y) \leq y$, that is $Q_p'(v) = sup_u M(P'(u), I_p(P(u),Q(v))) \leq sup_u M(P'(u),Q(v)) = Q(v)$, and thus it is:

$$(7) \quad Q'_p(v) \leq Q(v)$$

This means that n-models always generate a conclusion greater or equal than the consequent, while p-models generate conclusions lesser or equal than the consequent. In the same sense see [5] where similar concepts (expansion/reduction type implication) were introduced.

2.2 Consequent modification with singleton input

The conclusion obtained from the model, assuming a precise observation $P'(u)=\{u_0\}$, is given by $Q'(v)=I(\alpha, Q(v))$, where I is the implication of the model and $\alpha=P(u_0)$ [3].

At this stage it is possible to compare how different implications behave by comparing their conclusion for same $\alpha=P(u_0)$. Similar approaches have already been performed for example in [6]. Typical examples of n-models are shown in Fig. 1.

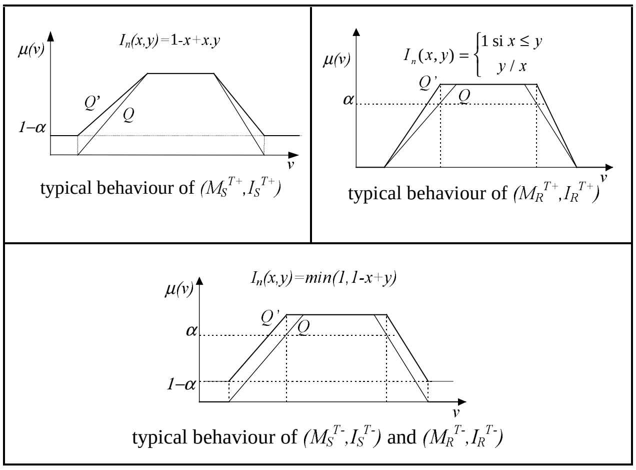

Fig. 1 Consequent modification with singleton input for some typical n-models

Three different behaviours are identified [7], [9]:

- residual implications, that generate outputs vaguer than the consequent, by means of an enlarged core (gradual models)
- strong implications, that generate more uncertain outputs, with same core but a non null level of indetermination (certainty models)
- Lukasiewicz implication, that shows both behaviours simultaneously

Similar results can be observed for p-models, where uncertainty level should be read as height, and core enlargement as support reduction.

2.3 Equivalence CRI-CM

The previous analysis can be generalised by determining when the equivalence between the compositional rule of inference -CRI- and the compatibility modification inference –CM [8]- holds. In these cases the conclusion is again given by directly applying the implication function to the consequent, but using a more complex compatibility index $\alpha=Comp(P',P)$. This index is highly informative since it reflects which features of the hypothesis and input are really being compared and used by the rule model to modify the consequent.

The equation to be solved is $Q'(v)=I(\alpha,Q(v))$, where $Q'(v)$ is supposed to be calculated applying the CRI. This equation can be formulated in terms of fuzzy truth values, that is $\tau_{QQ'}(y)=I(\alpha,y)$, and has been solved for some of the most common rule models in [3]. The main results are summarised in Table 1:

Model	Conditions	Comp(P',P)
(M_R^{T+},I_R^{T+})	$\mu_{\{\alpha\}} \le \tau(x) \le I_R^{T+}(\alpha,x) \Rightarrow\ ]P'[\subset]P[$	$\alpha = \inf_{u/P'(u)=1} P(u)$
(M_R^{T-},I_R^{T-})	$\mu_{\{\alpha\}} \le \tau(x) \le I_R^{T-}(\alpha,x) \Rightarrow [P']\subset]P[$	$\alpha = \inf_{u/P'(u)=1} P(u)$
(M_S^{T-},I_S^{T-})	$\mu_{\{\alpha\}} \le \tau(x) \le I_R^{T-}(\alpha,x) \Rightarrow [P']\subset]P[$	$\alpha = \inf_{u/P'(u)=1} P(u)$
(M_S^{T+},I_S^{T+})	$\mu_{\{\alpha\}} \le \tau(x) \le I_R^{T+}(\alpha,x) \Rightarrow\ ]P'[\subset]P[$	$\alpha = \inf_{u/P'(u)=1} P(u)$
(M_R^{T+},M_R^{T+})	$\tau(x) \le I_R^{T+}(x,\alpha) \Leftrightarrow\ ]P'[\cap]P[\neq 0$	$\alpha=sup_u M_R^{T+}(P'(u),P(u))$
(M_R^{T-},M_R^{T-})	$\tau(x) \le I_R^{T-}(x,\alpha) \Leftrightarrow f(P'(u))+f(P(u)) < f(0)$	$\alpha=sup_u M_R^{T-}(P'(u),P(u))$
(M_S^{T-},M_S^{T-})	$\tau(x) \le I_R^{T-}(x,\alpha) \Leftrightarrow f(P'(u))+f(P(u)) < f(0)$	$\alpha=sup_u M_R^{T-}(P'(u),P(u))$
(M_S^{T+},M_S^{T+})	$\tau(x) \le I_R^{T+}(x,\alpha) \Leftrightarrow\ ]P'[\cap]P[\neq 0$	$\alpha=sup_u M_R^{T+}(P'(u),P(u))$

Table 1: Equivalence between CRI and CM for some rule models

$\tau(x)$ is the truth value of the observation P' given the hypothesis P, $\tau_{PP'}(x)$, and f is the additive generator of the arquimedean t-norm that generates the model. The definition of $\tau_{PP'}(x)$ combined with its resulting upper and/or lower bounds allows to obtain some additional information in terms of support and core inclusions or intersections, reflected in the above table. It should be noted that the compatibility index of the n-models is a measure of the inclusion of the core of the observation P' into the hypothesis P of the rule (a pessimistic measure). On the other hand, for the p-models it is a measure of the intersection of the observation and the hypothesis (the optimistic version).

2.4 Firing condition of rule models

The third step proposed consists in determining the condition the hypothesis and the input of the rule must verify to infer a non trivial conclusion. In [4] it was proved that the trivial conclusion obtained from a n-model when the input P' is sufficiently different from the hypothesis P is $Q'(v)=1$, and that its firing condition is given by:

$$(7)\quad \forall u\in U,\ \sup_u M(P'(u),I(P(u),0))<1$$

where $\sup_u M(P'(u),I(P(u),0))$ is the level of indetermination or uncertainty of its conclusion. Similarly, the trivial conclusion of a p-model is $Q'(v)=0$ and its firing condition is given by:

$$(8)\quad \exists u\in U,\ \sup_u M(P'(u),M(P(u),1))>0$$

where $\sup_u M(P'(u),M(P(u),1))$ is the height or possibilistic uncertainty of its conclusion (see 2.5).

By solving the precedent expressions the level of uncertainty of the conclusions of n-models, or the height of the conclusions of p-models are obtained. Table 2 shows some results obtained in [4].

Model	Firing condition	indet. or height of Q'
(M_R^{T+},I_R^{T+})	$[P']\subset]P[$	$Q'(v)\ge 1\text{-}N(]P[/P')$
(M_R^{T-},I_R^{T-})	$[P']\subset]P[$	$Q'(v)\ge f^{[-1]}(f(0)\text{-}f(N(P/[P'])))$ and $Q'(v)\ge 1\text{-}N(]P[/P')$
(M_S^{T-},I_S^{T-})	$[P']\subset]P[$	$Q'(v)\ge 1\text{-}N(P/[P'])$ and $Q'(v)\ge 1\text{-}N(]P[/P')$
(M_S^{T+},I_S^{T+})	$]P'[\subset]P[$	$Q'(v)\ge 1\text{-}N(P/[P'])$
(M_R^{T+},M_R^{T+})	$]P'[\cap]P[\neq 0$	$Q'(v)\le \Pi_R^T(P/P')$
(M_R^{T-},M_R^{T-})	$f(P'(u))+f(P(u)) < f(0)$	$Q'(v)\le \Pi_R^T(P/P')$
(M_S^{T-},M_S^{T-})	$f(P'(u))+f(P(u)) < f(0)$	$Q'(v)\le 1\text{-}f^{[-1]}(f(0)\text{-}\Pi_R^T(P/P'))$
(M_S^{T+},M_S^{T+})	$]P'[\cap]P[\neq 0$	$Q'(v)=1$, or 0 if $]P'[\cap]P[=0$

Table 2: Firing condition of rule models

2.5 Some preliminary conclusions

The three precedent steps can be used to interpret the meaning of the conclusions obtained from both types of models [4], and thus the models themselves: n-models (necessity models) produce certain/necessary conclusions since:

- Their conclusion is always less restrictive (greater) than the consequent
- Their trivial conclusion is the whole universe of discourse: no value is excluded
- Their firing condition is based on a measure of the inclusion of the observation into the hypothesis.

Their conclusions, called in [4] necessary possibility distributions, are possibility distributions that exclude only impossible values, given the available information.

Similarly p-models (possibility models) produce possible (non contradictory) conclusions since:

- Their conclusion is always more restrictive (lesser) than the consequent
- Their trivial conclusion is the empty set: no value is included
- Their firing condition is based on a measure of the intersection of the observation and the hypothesis.

Their conclusions, called possible possibility distributions in [4], are possibility distributions that include only possible values, or equivalently, values that are not in total contradiction with the available information.

2.6 Core or support of the conclusion

As a forth step it is suggested the exact or approximate computation of the core of necessary possibility distributions and the support of possible possibility distributions. This information combined with the information obtained from the precedent points allows a more precise characterisation of the model under study. In [8] several cases are analysed.

3 Analysis of Yager implications

3.1 Consequent modification

For singleton input $P'(u)=\{u_0\}$, the conclusions obtained from the n-model (M_Y,I_Y) and the p-model (M_Y,M_Y) are:

$$Q'_n(v)=I_n(\alpha,Q(v))=(Q(v))^{\alpha}$$
$$Q'_p(v)=I_p(\alpha,Q(v))=(Q(v))^{1/\alpha}$$

Their approximate shapes are shown in Fig. 2.

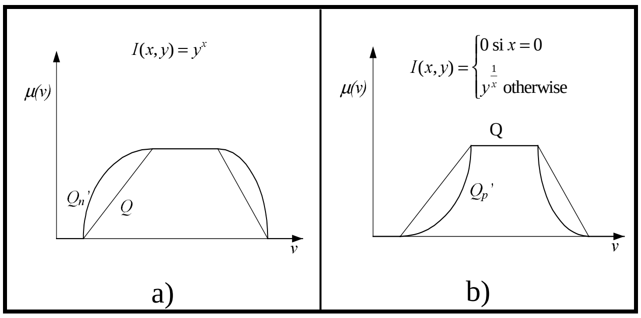

Fig. 2: Consequent modification with Yager implications

As it was expected Q'_n is always greater or equal than $Q(v)$ while $Q_p'(v)$ is always lesser or equal than $Q(v)$. In addition, the trivial conclusion of the n-model (when $P(u_0)=0$) is $Q'(v)=V$, and that of the p-model is $Q'(v)=0$. However, for both implications, and in contrast with the behaviour of other usual implications (see [4]), no modification of the core and support are performed. This means that the uncertainty of the conclusion (level of uncertainty for necessary possibility distributions, or height for possible possibility distributions) does not increase, unless the trivial conclusion is reached (in a non continuos way).

The imprecision or vagueness of the resulting possibility distributions increases in a different way than the above typical models, and something between. See for example the case of the n-implications, Fig. 1 and Fig. 2 case a).

3.2 Equivalence CRI-MS

The equation to be solved is $Q'(y)=I_Y(\alpha,Q(v))$ which is equivalent to:

$$sup_x\, M_Y(\tau(x),I_Y(x,y)) \leq I_Y(\alpha,y)$$

Solving the modus ponens inequality for $\tau(x)$ it is:

$$I_Y(x,y)^{1/\tau(x)} \leq I(\alpha,y) \Rightarrow y^{x/\tau(x)} \leq y^{\alpha} \Rightarrow \tau(x) \leq x/\alpha$$

It can be checked that if $\mu_{\{\alpha\}} \leq \tau(x) \leq x/\alpha$, being $\mu_{\{\alpha\}}$ the membership function of the singleton $\{\alpha\}$, the supreme is reached for $x=\alpha$ and the CRI reduces to the CM with compatibility index given by $\alpha = inf_{x/P'(u)=1}\, P(u)$. The condition for the equivalence is then $\mu_{\{\alpha\}} \leq \tau(x) \leq x/\alpha$. The compatibility index and the condition for equivalence are identical to those obtained for the n-model generated from the product t-norm $T(x,y)=xy$.

For the p-model the equation to be solved is $Q'(y)=M_Y(\alpha,Q(y))$, which is equivalent to:

$$sup_x\, M_Y(\tau(x),M_Y(x,y)) \leq M_Y(\alpha,y)$$

Solving the modus ponens inequality for $\tau(x)$ it is:

$$M_Y(x,y)^{1/\tau(x)} \leq M_Y(\alpha,y) \Rightarrow y^{1/(x.\,\tau(x))} \leq y^{1/\alpha} \Rightarrow \tau(x) \leq \alpha/x$$

If in addition $sup_x\, x.\,\tau(x)=\alpha$, that is $sup_x T(x,\tau(x)) \geq \alpha$ then the supreme is reached and the equivalence holds. Thus if $\tau(x) \leq I_R^T(x,\alpha)$, being T the product t-norm, the equivalence CRI-CM holds with compatibility index $\alpha=sup_x T(\tau(x),x)=sup_x\, x.\,\tau(x)$. Again the same results obtained for the p-model generated from $T(x,y)=x.y$ are recovered [3]. This suggest some mathematical relationship between Yager implications and the product t-norm that will be pursued in section 3.4. Fig. 3 shows the conditions for the equivalence CRI-CM for both types of models.

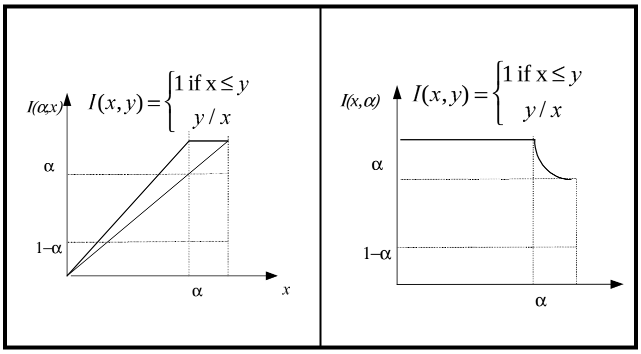

Fig. 3: Condition for CRI-CM equivalence of Yager models

3.3 Firing condition of Yager models

As done in [4] it is possible to determine the condition P and P' must verify to infer non trivial conclusions with both types of models. When using the n-model the following equation must be solved:

$$\forall u\in U,\ \sup_u M_Y(P'(u),I_Y(P(u),0))<1$$

Since $\sup_u M_Y(P'(u),1))=1^{1/P'(u)}$ and $1^x=1$ unless x=∞, the model fires if when $P(u)=0$ (that is $I_Y(0,0)=1$) then $P'(u)=0$ (that is $1^{1/P'(u)}\neq 1$):

$$]P'[\subset]P[$$

This condition is the same obtained for n-models generated from a positive t-norm, and it is the most restrictive one of those obtained for the n-models.

When inferring with the p-model the firing condition is obtained solving:

$$\exists u\in U,\ \sup_u M_Y(P'(u),M_Y(P(u),1))>0$$

Since $M_Y(x,0)=0$ and $M(0,y)=0$ the model fires if $P(u)\neq 0$ and $M(P'(u),1)\neq 0$, that is $P(u)\neq 0$ and $P'(u)\neq 0$, which is equivalent to:

$$]P'[\cap]P[\neq 0$$

This condition is the condition obtained for p-models generated from a positive t-norm.

As it was expected the Yager n-model is a necessity model since it fires when there is some kind of non null inclusion of the input into the hypothesis, and thus its conclusions can be interpreted as necessary possibility distributions. In the same way the p-model fires when there is a non null intersection between the input and the hypothesis, and thus it is a possibility model and its conclusions can be interpreted as possible possibility distributions.

3.4 Generating Yager implication from the product t-norm

There is an interesting relationship between Yager implications and the usual implications generated from the product t-norm. On the same way the additive generator of the product t-norm seems also to be related with Yager implication, since $f(x)=-ln(x)$ with pseudo-inverse $f^{[-1]}(x)=e^{-x}$, and it can be checked that the following holds:

$$(9)\quad M_Y(x,y)=f^{[-1]}(f(y)/x)$$
$$I_Y(x,y)=f^{[-1]}(x.f(y))$$

This seems to point out a new way of generating implications from a t-norm, or more specifically from the additive generator of a t-norm, being Yager implications a particular case.

4 Generating new implications from t-norms

Let's define the operator:

$$(10)\quad H(x,y)=\begin{cases}f^{[-1]}\left(\dfrac{f(y)}{x}\right) & x\neq 0\\ 0 \text{ otherwise}\end{cases}$$

with f being the additive generator of an arquimedean t-norm, and $f^{[-1]}$ its pseudo-inverse. Then the operator $H(x,y)$ is a conjunctive operator, associative but non commutative:

$$H(1,1)=f^{[-1]}(f(1)/1)=f^{[-1]}(0)=1$$
$$H(0,y)=0$$
$$H(x,0)=f^{[-1]}(f(0)/x)=0 \text{ (since } f(0)/x>f(0))$$
$$H(x,y) \text{ is non decreasing with } x \text{ and } y$$

$H(x,y)$ is non decreasing with both arguments since: $f(y)/x$ decreases when x increases, $f^{[-1]}$ is a decreasing function and thus $H(x,y)$ increases with x; in the same way if y increases then $f(y)$ decreases and thus $x.f(y)$ decreases which means that $H(x,y)$ is increasing with y. It can also be proved that $H(x,y)$ is associative:

$$H(x,H(y,z))=f^{[-1]}(f(H(y,z))/x)=f^{[-1]}((f(y)/z)/x) = f^{[-1]}((f(y)/x)/z)=H(H(x,y),z)$$

In addition H is a valid MPGF since $H(1,y)=y$ ($H(1,y)=f^{[-1]}(1.f(y))=y$). This means that it is possible to generate new pseudo-conjunctions (or p-implications) $H(x,y)$ from the additive generator of an arquimedean t-norm. Taking into account that from each new pseudo-conjunction $H(x,y)$ it is possible to obtain its corresponding n-implications by residuation [2], then the following n and p-models denoted by (M_H^T,I_H^T) and (M_H^T,M_H^T) can be obtained (being f the additive generator of an arquimedean t-norm, and $f^{[-1]}$ its pseudo-inverse):

(11) $I_R^H(x,y)= \sup\{c\in[0,1]/ H(x,c) \le y\}$

(12) $$M_H^T(x,y)=H(x,y)=\begin{cases} f^{[-1]}\left(\dfrac{f(y)}{x}\right) & x\neq 0 \\ 0 & \text{otherwise}\end{cases}$$

5 Conclusions

This paper proposes a methodology for analysing the mathematical and semantic behaviour of rule models. This methodology has been extensively developed in [9]. Two main types of models with clearly defined properties have been considered: n-models that conclude certain or necessary conclusions (called necessary possibility distributions), and p-models, that conclude non contradictory or plausible/possible conclusions (called possible possibility distributions). This interpretation logically explains why possible models do not produce an output when the input does "not match" (intersects) the hypothesis. The proposed methodology has been applied to two rule models generated from Yager implication. Several aspects of its behaviour have been pointed out and this analysis has shown up the close relationship existing between Yager operators and the product t-norm. This relationship has been used to propose a new method to generate implications from t-norms.

References

[1] A. Zadeh, Fuzzy Sets, *Information and Control*, vol8, pp338-353, 1965

[2] E. Trillas y L. Valverde, On modus ponens in fuzzy logic, *Proceedings of the fifteenth international symposium of multiple valued logics*, 0195-623X/85/0000/0294$01.00 IEEE, 1985.

[3] J. Villar, Conditions for Equivalence between the Compositional Rule of Inference and the Compatibility Modification Inference, *Fifth IEEE International conference on Fuzzy Systems*, New Orleans, September, 1996.

[4] J. Villar, Firing conditions of fuzzy rules, generalised possibility and necessity measures, *6th IEEE International conference on Fuzzy Systems*, Barcelone, July, 1997.

[5] I.B. Turksen, Y. Tian, Combination of rules or their consequences in fuzzy expert systems, *Fuzzy Sets and Systems* 58 (1993) 3-40

[6] J.F. Baldwin, B.W. Pilsworth, Axiomatic approach to implication for approximate reasoning with fuzzy logic, *Fuzzy Sets and Systems* 3 (1980) 193-219.

[7] Didier Dubois, Henri Prade, Fuzzy rules in knowledge based systems, modelling gradeness, uncertainty and preference, *An Introduction to Fuzzy Logic Applications in Intelligent Systems*, Edited by R.R. Yager and L.A. Zadeh, Kluwer Academic Publishers, 1993

[8] Valerie Cross, Thomas Sudkamp, Fuzzy implication and compatibility modification, *0-7803-0614-7/93$03.00, IEEE 1993*

[9] J. Villar, Aplicación de la teoría de conjuntos borrosos al diagnostico de procesos industriales, *Phd Thesis (in spanish), ICAI-UPCO 1997*