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Positive weak solutions for heterogeneous elliptic logistic BVPs with glued Dirichlet-Neumann mixed boundary conditions

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Abstract: This article concerns the existence of positive weak solutions of a heterogeneous elliptic boundary value problem of logistic type in a very general annulus. The novelty of this work lies in considering non-classical mixed glued boundary conditions. Namely, Dirichlet boundary conditions on a component of the boundary, and glued Dirichlet-Neumann boundary conditions on the other component of the boundary. In this paper we perform a complete analysis of the existence of positive weak solutions of the problem, giving a necessary condition on the λ parameter for the existence of them, and a sufficient condition for the existence of them, depending on the λ-parameter, the spatial dimension $N \geq 2$ and the exponent $q > 1$ of the reaction term. The main technical tools used to carry out the mathematical analysis of this work are variational and monotonicity techniques. The results obtained in this paper are pioners in the field, because up the knowledge of the autor, this is the first time where this kind of logistic problems have been analyzed.

Keywords: positive weak solutions; heterogeneous elliptic logistic BVPs; mixed boundary conditions; glued Dirichlet-Neumann boundary conditions; singular boundary conditions; spatial heterogeneities
Mathematics Subject Classification: 35B09, 35B35, 35B40, 35J25, 35J65

1. Introduction and previous results

This work is devoted to analyze the existence of positive weak solutions of the following heterogeneous elliptic logistic boundary value problem with mixed and glued Dirichlet-Neumann boundary conditions given by

$$\begin{cases} -\Delta u = \lambda u - a(x)u^q & \text{in} \quad \Omega, \quad q > 1, \\ u = 0 & \text{on} \quad \Gamma_0, \\ u = 0 & \text{on} \quad \Gamma_1^{\mathcal{D}}, \\ \partial u = 0 & \text{on} \quad \Gamma_1^{\mathcal{N}}, \end{cases} \tag{1.1}$$

where the following assumptions are assumed:

i) The domain Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$ of class C^2, with boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are two disjoint components of $\partial\Omega$ and $\Gamma_1 = \Gamma_1^{\mathcal{D}} \cup \Gamma_1^{\mathcal{N}}$, being $\Gamma_1^{\mathcal{D}}$ and $\Gamma_1^{\mathcal{N}}$ two connected pieces, open and closed respectively as $N-1$ dimensional manifolds, such that $\partial\Gamma_1^{\mathcal{D}} = \partial\Gamma_1^{\mathcal{N}} \subset \Gamma_1^{\mathcal{N}}$;
ii) $-\Delta$ stands for the minus Laplacian operator in $\mathbb{R}^N$ and $\lambda \in \mathbb{R}$;
iii) The potential $a \in C(\bar{\Omega})$, with $a > 0$, measures the spatial heterogeneities in Ω and satisfies that

$$\Omega_0 := int\,\{x \in \Omega \ : \ a(x) = 0\} \neq 0\,, \qquad \Omega_0 \in C^2\,,$$

$$\partial\Omega_0 = \Gamma_1 \cup \Gamma_0^0\,, \qquad \Gamma_0^0 := \partial\Omega_0 \cap \Omega\,, \qquad \text{dist}(\Gamma_0^0, \Gamma_1) > 0\,.$$

Set $\Omega^+ := \Omega \setminus \bar{\Omega}_0$.
iv) $\partial u = \nabla u \circ \bar{n}$, where $\bar{n}$ is the outward normal vector field to $\partial\Omega$.

Figure 1 shows a possible configuration of the domain Ω, its boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1^{\mathcal{D}} \cup \Gamma_1^{\mathcal{N}}$ and the boundary conditions in each piece of the boundary.

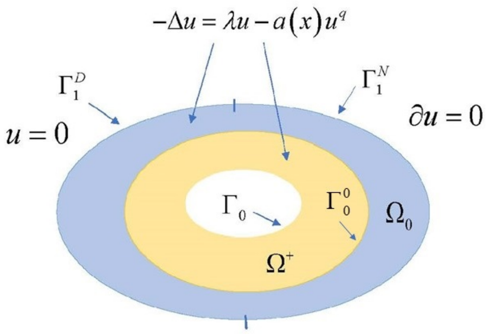

Figure 1. Configuration of Ω and $\partial\Omega = \Gamma_0 \cup \Gamma_1^{\mathcal{D}} \cup \Gamma_1^{\mathcal{N}}$.

The positive solutions of (1.1) are the positive steady-states of the associated evolutionary problem given by

$$\begin{cases} \partial_t v(x,t) - \Delta v(x,t) = \lambda v(x,t) - a(x)v(x,t)^q & \text{in} \quad \Omega \times \mathbb{R}, \quad q > 1, \\ v(x,t) = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}, \\ v(x,t) = 0 & \text{on} \quad \Gamma_1^{\mathcal{D}} \times \mathbb{R}, \\ \partial v(x,t) = 0 & \text{on} \quad \Gamma_1^{\mathcal{N}} \times \mathbb{R}, \\ v(x,0) = v_0(x) > 0 & \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

which describes the dynamics of the positive solutions of many reaction-diffusion problems appearing in the applied sciences and engineering. In population dynamics, (1.2) describes the dynamics of a population inhabiting a heterogeneous environment Ω, growing accordingly with a generalized logistic law. From this point of view, $v(x,t)$ stands for the population density, $-\Delta v(x,t)$ is the diffusion term, λ is the growth rate of the population and $a(x)$ measures the saturation effect responses to the population stress in Ω^+. As for the boundary conditions, the homogeneous Dirichlet boundary condition on $\Gamma_0 \cup \Gamma_1^{\mathcal{D}}$ means that $\Gamma_0 \cup \Gamma_1^{\mathcal{D}}$ are hostile regions, and the homogeneous Neumann boundary condition on $\Gamma_1^{\mathcal{N}}$ guarantees no migration or null flux of population through $\Gamma_1^{\mathcal{N}}$. The different boundary conditions considered in (1.1) and (1.2) may be due to a heterogeneous distribution of the natural resources through the boundary or close to the boundary. The analysis of the positive solutions of (1.1) is crucial to have a complete understanding of the long time behavior of the positive solutions of (1.2). Also, the analysis of the existence of positive weak solutions of (1.1) is pivotal in the study of the asymptotic behavior as $\gamma \uparrow \infty$ of the strong positive solutions of heterogeneous logistic elliptic boundary value problems with nonlinear mixed boundary conditions like the following

$$\begin{cases} -\Delta u = \lambda u - a(x)u^q & \text{in} \quad \Omega, \quad q > 1, \\ u = 0 & \text{on} \quad \Gamma_0, \\ \partial u = -\gamma b\, u^r & \text{on} \quad \Gamma_1, \quad r > 1, \end{cases} \tag{1.3}$$

where $b \in \mathcal{C}(\Gamma_1)$ with $b > 0$ on Γ_1 and

$$\Gamma_1^{\mathcal{N}} = b^{-1}(0), \qquad \Gamma_1^{\mathcal{D}} = b^{-1}((0, \|b\|_{L_\infty(\Gamma_1)}]),$$

which stand for again the positve steady-states of the associated parabolic problem with nonlinear flux on Γ_1. In this kind of problems, when λ belongs to a suitable interval, the limiting profile of the strong positive solutions when $\gamma \uparrow \infty$, is a positive weak solution of (1.1), as it will be proved elsewhere. The proof of this fact is out the scope of this work.

Although we have assumed throughout this paper that Γ_1 splits in two connected pieces $\Gamma_1^{\mathcal{D}}$ and $\Gamma_1^{\mathcal{N}}$, the results of this work may be generalized in a natural way to cover the case when Γ_1 splits in $2k$ connected pieces $\left\{\Gamma_{1i}^{\mathcal{D}}, \Gamma_{1,i}^{\mathcal{N}}\right\}_{i=1}^{k}$, where now

$$\Gamma_1 = \Gamma_1^{\mathcal{D}} \cup \Gamma_1^{\mathcal{N}}, \qquad \Gamma_1^{\mathcal{N}} = \cup_{i=1}^{k} \Gamma_{1i}^{\mathcal{N}}, \qquad \Gamma_1^{\mathcal{D}} = \cup_{i=1}^{k} \Gamma_{1i}^{\mathcal{D}},$$

with $\Gamma_1^{\mathcal{D}}$ and $\Gamma_1^{\mathcal{N}}$ unconnected and where each piece $\Gamma_{1i}^{\mathcal{D}}$, $i = 1, \ldots, k$ is between two consecutive pieces of the family $\left\{\Gamma_{1j}^{\mathcal{N}}\right\}_{j=1}^{k}$ and viceversa.

On the other hand, owing to [5], the results into this work also may be generalized to cover the case when, instead of imposing a Neumann boundary condition on $\Gamma_1^{\mathcal{N}}$, it is imposed a boundary condition of Robin type like $\partial u + b(x)u = 0$, where $b \in \mathcal{C}(\Gamma_1^{\mathcal{N}})$ with arbitrary sign satisfies adequate technical conditions. The novelty of the results in this work is considering glued Dirichlet and Neumann boundary conditions on a same component of the boundary. These results are pioners in the field, because up the knowledge of the author, this is the first time where this kind of logistic problems have been analyzed.

Before stating our main findings, we introduce some notations and previous results. Let us denote

$$\mathcal{C}^\infty_{\Gamma_0 \cup \Gamma_1^{\mathcal{D}}}(\Omega) := \left\{ \phi : \bar{\Omega} \to \mathbb{R} \,:\, \phi \in \mathcal{C}^\infty(\Omega) \cap \mathcal{C}(\bar{\Omega}) \,\wedge\, supp\, \phi \subset \bar{\Omega} \setminus (\Gamma_0 \cup \Gamma_1^{\mathcal{D}}) \right\},$$

and let $H^1_*(\Omega)$ be the clousure in $H^1(\Omega)$ of the set of functions $C^\infty_{\Gamma_0\cup\Gamma_1^{\mathcal{D}}}(\Omega)$, that is

$$H^1_*(\Omega)=\overline{C^\infty_{\Gamma_0\cup\Gamma_1^{\mathcal{D}}}(\Omega)}^{H^1(\Omega)}.$$

By construction if $u\in H^1_*(\Omega)$, then $u=0$ on $\Gamma_0\cup\Gamma_1^{\mathcal{D}}$. In the same way, taking into account that $\partial\Omega_0=\Gamma_0^0\cup\Gamma_1=\Gamma_0^0\cup\Gamma_1^{\mathcal{D}}\cup\Gamma_1^{\mathcal{N}}$, we denote

$$C^\infty_{\Gamma_0^0\cup\Gamma_1^{\mathcal{D}}}(\Omega_0):=\left\{\phi:\bar{\Omega}_0\to\mathbb{R}\,:\,\phi\in C^\infty(\Omega_0)\cap C(\bar{\Omega}_0)\ \wedge\ supp\,\phi\subset\bar{\Omega}_0\setminus(\Gamma_0^0\cup\Gamma_1^{\mathcal{D}})\right\}$$

and

$$H^1_*(\Omega_0)=\overline{C^\infty_{\Gamma_0^0\cup\Gamma_1^{\mathcal{D}}}(\Omega_0)}^{H^1(\Omega_0)}.$$

Also we denote

$$\tilde{H}^1_*(\Omega_0):=\left\{\varphi:\bar{\Omega}\to\mathbb{R}\,:\,\varphi\in H^1_*(\Omega_0)\ \wedge\ \varphi=0\text{ in }\Omega^+\cup\Gamma_0\right\},$$

that is, any function belonging to $\tilde{H}^1_*(\Omega_0)$ is the extension by 0 to $\bar{\Omega}$ of a previous function belonging to $H^1_*(\Omega_0)$. By definition, if $u\in\tilde{H}^1_*(\Omega_0)$ then $u=0$ in $\Gamma_1^{\mathcal{D}}\cup\Gamma_0^0\cup\Omega^+\cup\Gamma_0$. Also, by construction it is clear that

$$\tilde{H}^1_*(\Omega_0)\subsetneq H^1_*(\Omega)\,. \tag{1.4}$$

By a *positive weak solution* of (1.1) we mean any function $\varphi\in H^1_*(\Omega)$ satisfying

$$\varphi>0\,,\qquad\int_{\Omega^+}a(x)\varphi^{q+1}<\infty\,,$$

and such that for each $\xi\in C^\infty_{\Gamma_0\cup\Gamma_1^{\mathcal{D}}}(\Omega)$, or $\xi\in H^1_*(\Omega)$, the following holds

$$\int_\Omega\nabla\varphi\,\nabla\xi+\int_\Omega a(x)\varphi^q\xi=\lambda\int_\Omega\varphi\xi\,. \tag{1.5}$$

In particular, taking $\xi=\varphi\in H^1_*(\Omega)$ we have that

$$\int_\Omega|\nabla\varphi|^2+\int_\Omega a(x)\varphi^{q+1}=\lambda\int_\Omega\varphi^2\,. \tag{1.6}$$

Thus, since any positive weak solution of (1.1) can not be constant, it follows from (1.6) that if (1.1) possesses a positive weak solution φ for the value λ of the parameter, then

$$\lambda=\frac{\int_\Omega|\nabla\varphi|^2+\int_\Omega a(x)\varphi^{q+1}}{\int_\Omega\varphi^2}\geq\frac{\int_\Omega|\nabla\varphi|^2}{\int_\Omega\varphi^2}>0\,, \tag{1.7}$$

and therefore, $\lambda>0$ is a necessary condition for the existence of positive weak solutions of (1.1).

Hereafter we denote $\mathfrak{B}^{\mathcal{N}}$, $\mathfrak{B}^*(\Gamma_1^{\mathcal{N}})$ and $\mathfrak{B}^*_0(\Gamma_1^{\mathcal{N}})$ the boundary operators defined by

$$\mathfrak{B}^{\mathcal{N}}u:=\begin{cases}u & \text{on }\Gamma_0\\ \partial u & \text{on }\Gamma_1\end{cases},\qquad\mathfrak{B}^*(\Gamma_1^{\mathcal{N}})u:=\begin{cases}u & \text{on }\Gamma_0\\ \partial u & \text{on }\Gamma_1^{\mathcal{N}}\\ u & \text{on }\Gamma_1^{\mathcal{D}}\end{cases},$$

and

$$\mathfrak{B}_0^*(\Gamma_1^{\mathcal{N}})u := \begin{cases} u & \text{on} \quad \Gamma_0^0 \\ \partial u & \text{on} \quad \Gamma_1^{\mathcal{N}} \\ u & \text{on} \quad \Gamma_1^{\mathcal{D}} \end{cases},$$

and by $\mathfrak{D}$ the Dirichlet boundary operator on $\partial\Omega$. Clearly, $\mathfrak{B}^{\mathcal{N}} = \mathfrak{B}^*(\Gamma_1)$. Also we denote

$$W^2(\Omega) := \bigcap_{p>1} W_p^2(\Omega)\,.$$

In the sequel we will say that a function $u \in W_p^2(\Omega)$, $p > N$ is *strongly positive* in Ω, and we will denote it by $u \gg 0$, if $u(x) > 0$ for each $x \in \Omega \cup \Gamma_1$ and $\partial u(x) < 0$ for each $x \in \Gamma_0$ such that $u(x) = 0$.

Let us consider the eigenvalue problem

$$\begin{cases} -\Delta\varphi = \sigma\varphi & \text{in} \quad \Omega, \\ \mathfrak{B}^{\mathcal{N}}\varphi = \bar{0} & \text{on} \quad \partial\Omega. \end{cases} \tag{1.8}$$

By a principal eigenvalue of (1.8) we mean any eigenvalue of it which possesses a one-signed eigenfunction and in particular a positive eigenfunction. Owing to the results in [1, Theorem 12.1] it is known that (1.8) possesses a unique principal eigenvalue, denoted in the sequel by $\sigma_1^\Omega[-\Delta, \mathfrak{B}^{\mathcal{N}}]$, which is the least eigenvalue of (1.8) and it is simple. Moreover, the positive eigenfunction $\varphi_1^{\mathcal{N}}$ associated to it, unique up multiplicative constant, satisfies

$$\varphi_1^{\mathcal{N}} \gg 0 \quad \text{in} \quad \Omega, \tag{1.9}$$

and in addition

$$\varphi_1^{\mathcal{N}} \in W^2(\Omega) \subset C^{1+\alpha}(\bar{\Omega}) \qquad \text{for all} \quad \alpha \in (0,1)\,. \tag{1.10}$$

A function $\varphi \in W_p^2(\Omega)$ for $p > N$ is said to be a positive strict supersolution of the problem $(-\Delta, \Omega, \mathfrak{B}^{\mathcal{N}})$, if $\varphi > 0$ in Ω and the following hold

$$\begin{cases} -\Delta\varphi \geq 0 & \text{in} \quad \Omega, \\ \mathfrak{B}^{\mathcal{N}}\varphi \geq 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with some of the inequalities strict. Since any positive constant $\mu > 0$ is a positive strict supersolution of the problem $(-\Delta, \Omega, \mathfrak{B}^{\mathcal{N}})$, it follows from the characterization of the strong maximum principle given in [2, Theorem 2.4] that

$$\sigma_1^\Omega[-\Delta, \mathfrak{B}^{\mathcal{N}}] > 0\,. \tag{1.11}$$

Now, for any $K \in L_\infty(\Omega)$, let us denote $\mathcal{L}_K := -\Delta + K$ and let us consider the eigenvalue problem with mixed boundary conditions and glued Dirichlet-Neumann boundary conditions on Γ_1 given by

$$\begin{cases} \mathcal{L}_K\varphi = \mu\varphi & \text{in} \quad \Omega\,, \\ \mathfrak{B}^*(\Gamma_1^{\mathcal{N}})\varphi = \bar{0} & \text{on} \quad \partial\Omega\,. \end{cases} \tag{1.12}$$

A function φ is said to be a weak solution of (1.12) if $\varphi \in H_*^1(\Omega)$ and for each $\xi \in H_*^1(\Omega)$ the following holds

$$\int_\Omega \nabla\varphi\,\nabla\xi + \int_\Omega K\varphi\,\xi = \mu\int_\Omega \varphi\,\xi\,.$$

The value μ is an eigenvalue of (1.12), if there exists a weak solution $\varphi \neq 0$ of (1.12) associated to μ. In that case, it is said that φ is a weak eigenfunction of (1.12) associated to the eigenvalue μ. By a principal eigenvalue of (1.12) we mean any eigenvalue of it which possesses a one-signed eigenfunction and in particular a positive eigenfunction.

Owing to the results in [5, Theorem 1.1] it is known that (1.12) possesses a unique principal eigenvalue, denoted in the sequel by $\sigma_1^\Omega[\mathcal{L}_K, \mathfrak{B}^*(\Gamma_1^{\mathcal{N}})]$, which is simple and the smallest eigenvalue of all others eigenvalues of (1.12). Moreover, the positive eigenfunction φ^* associated to it, unique up multiplicative constant, satisfies that $\varphi^* \in H_*^1(\Omega)$ and

$$\varphi^*(x) > 0 \qquad \text{a.e. in} \quad \Omega\,. \tag{1.13}$$

Furthermore, $\sigma_1^\Omega[\mathcal{L}_K, \mathfrak{B}^*(\Gamma_1^{\mathcal{N}})]$ comes characterized by

$$\sigma_1^\Omega[\mathcal{L}_K, \mathfrak{B}^*(\Gamma_1^{\mathcal{N}})] = \inf_{\varphi\in H_*^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla\varphi|^2 + \int_\Omega K\varphi^2}{\int_\Omega \varphi^2} = \frac{\int_\Omega |\nabla\varphi^*|^2 + \int_\Omega K(\varphi^*)^2}{\int_\Omega (\varphi^*)^2} \text{(cf. [5, (2.27)])}. \tag{1.14}$$

Also, owing to [5, Corollary 3.5] the following hold

$$\sigma_1^\Omega[\mathcal{L}_K, \mathfrak{B}^{\mathcal{N}}] < \sigma_1^\Omega[\mathcal{L}_K, \mathfrak{B}^*(\Gamma_1^{\mathcal{N}})] < \sigma_1^\Omega[\mathcal{L}_K, \mathfrak{D}]\,. \tag{1.15}$$

In the same way, substituting in (1.12) Ω by Ω_0 and $\mathfrak{B}^*(\Gamma_1^{\mathcal{N}})$ by $\mathfrak{B}_0^*(\Gamma_1^{\mathcal{N}})$, owing to [5, Theorem 1.1] we obtain the following variationl characterization for $\sigma_1^{\Omega_0}[\mathcal{L}_K, \mathfrak{B}_0^*(\Gamma_1^{\mathcal{N}})]$

$$\sigma_1^{\Omega_0}[\mathcal{L}_K, \mathfrak{B}_0^*(\Gamma_1^{\mathcal{N}})] := \inf_{\varphi\in H_*^1(\Omega_0)\setminus\{0\}} \frac{\int_{\Omega_0} |\nabla\varphi|^2 + \int_{\Omega_0} K\varphi^2}{\int_{\Omega_0} \varphi^2}\,. \tag{1.16}$$

In the particular case when $K = 0$, that is, when $\mathcal{L}_0 := -\Delta$, set

$$\sigma_1^\Omega[\mathcal{D}] := \sigma_1^\Omega[-\Delta, \mathcal{D}]\,, \qquad \sigma_1^\Omega[\mathfrak{B}^{\mathcal{N}}] := \sigma_1^\Omega[-\Delta, \mathfrak{B}^{\mathcal{N}}],$$

and

$$\sigma_1^\Omega[\mathfrak{B}^*(\Gamma_1^{\mathcal{N}})] := \sigma_1^\Omega[-\Delta, \mathfrak{B}^*(\Gamma_1^{\mathcal{N}})]\,, \qquad \sigma_1^{\Omega_0}[\mathfrak{B}_0^*(\Gamma_1^{\mathcal{N}})] := \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}_0^*(\Gamma_1^{\mathcal{N}})].$$

Owing to (1.11) and (1.14)–(1.16) the following hold

$$\sigma_1^\Omega[\mathfrak{B}^*(\Gamma_1^{\mathcal{N}})] := \inf_{\varphi\in H_*^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega \varphi^2} = \frac{\int_\Omega |\nabla\varphi^*|^2}{\int_\Omega (\varphi^*)^2}, \tag{1.17}$$

$$\sigma_1^{\Omega_0}[\mathfrak{B}_0^*(\Gamma_1^{\mathcal{N}})] := \inf_{\varphi\in H_*^1(\Omega_0)\setminus\{0\}} \frac{\int_{\Omega_0} |\nabla\varphi|^2}{\int_{\Omega_0} \varphi^2} = \frac{\int_{\Omega_0} |\nabla\varphi_0^*|^2}{\int_{\Omega_0} (\varphi_0^*)^2}, \tag{1.18}$$

and

$$0 < \sigma_1^\Omega[\mathfrak{B}^{\mathcal{N}}] < \sigma_1^\Omega[\mathfrak{B}^*(\Gamma_1^{\mathcal{N}})] < \sigma_1^\Omega[\mathfrak{D}], \tag{1.19}$$

where φ^* and φ_0^* stand for the positive principal eigenfunctions associated to $\sigma_1^\Omega[\mathfrak{B}^*(\Gamma_1^{\mathcal{N}})]$ and $\sigma_1^{\Omega_0}[\mathfrak{B}_0^*(\Gamma_1^{\mathcal{N}})]$, respectively, unique up multiplicative constant. Taking into account (1.4) and the variational characterizations (1.17) and (1.18), it is clear that

$$\sigma_1^\Omega[\mathfrak{B}^*(\Gamma_1^{\mathcal{N}})] \leq \sigma_1^{\Omega_0}[\mathfrak{B}_0^*(\Gamma_1^{\mathcal{N}})]. \tag{1.20}$$

The statements and proofs of the main findings of this work appear in Proposition 1 and Theorem 1. The main technical tools used to carry out the mathematical analysis of this work are variational and monotonicity techniques.

The distribution of the rest of this paper is the following. In Section 2 is given a necessary condition for the existence of positive weak solutions of (1.1), sharper than (1.7), and some results about the pointing profile of such solutions. In Section 3 is given a sufficient condition for the exsitence of positive weak solutions of (1.1) depending on the λ-parameter, the spatial dimension $N \geq 2$ and the exponent $q > 1$ of the reaction term.

2. Necessary condition for the existence of positive weak solutions of (1.1)

In this section is given a necessary condition for the existence of positive weak solutions of (1.1) sharper than (1.7), and some partial results about the pointing profile and regularity of the weak positive solutions of (1.1). The main result of this section establishes the following

Proposition 1. *Let u be a positive weak solution of* (1.1) *for the value λ of the parameter. Then,*

$$0 < \sigma_1^{\Omega}[\mathfrak{B}^*(\Gamma_1^N)] < \lambda \tag{2.1}$$

and

$$u > 0 \qquad in \qquad \Omega^+. \tag{2.2}$$

Moreover:

a) If $u \in L_\infty(\Omega^+)$, then

$$\lambda \leq \sigma_1^{\Omega_0}[\mathfrak{B}_0^*(\Gamma_1^N)] \tag{2.3}$$

and

$$u(x) > 0 \qquad a.e.\ in \qquad \Omega. \tag{2.4}$$

b) If

$$N \geq 3 \qquad and \qquad 1 < q < \frac{N}{N-2}, \tag{2.5}$$

then $u \in H^2(\Omega')$ for any subdomain $\Omega' \subset\subset \Omega$.

c) If

$$N = 3 \qquad and \qquad 1 < q < 3, \tag{2.6}$$

then $u \in C(K)$ in any compact subset $K \subset \Omega$.

Proof. Let us denote

$$\sigma_1^* := \sigma_1^{\Omega}[\mathfrak{B}^*(\Gamma_1^N)], \qquad \sigma_0^* := \sigma_1^{\Omega_0}[\mathfrak{B}_0^*(\Gamma_1^N)].$$

Owing to (1.19) and (1.20) we know that

$$0 < \sigma_1^* \leq \sigma_0^*.$$

To prove (2.1), let $u \in H_*^1(\Omega)$ be a positive weak solution of (1.1) for the value λ of the parameter. Then

$$\lambda = \frac{\int_\Omega |\nabla u|^2 + \int_\Omega a(x) u^{q+1}}{\int_\Omega u^2} \text{ (cf. (1.7)).} \tag{2.7}$$

Now, since $u(x) \geq 0$ a.e. in Ω and $a(x) > 0$ for all $x \in \Omega^+$, we have that

$$\int_\Omega a(x)u^{q+1} = \int_{\Omega^+} a(x)u^{q+1} \geq 0\,, \tag{2.8}$$

and hence, since $u \in H^1_*(\Omega) \setminus \{0\}$, it follows from (2.7), (2.8) and (1.17) that

$$\lambda = \frac{\int_\Omega |\nabla u|^2 + \int_\Omega a(x)u^{q+1}}{\int_\Omega u^2} \geq \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} \geq \inf_{\varphi \in H^1_*(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla \varphi|^2}{\int_\Omega \varphi^2} = \sigma_1^* \tag{2.9}$$

and therefore,

$$\lambda \geq \sigma_1^*. \tag{2.10}$$

We now prove that (1.1) does not possess a positive weak solution for $\lambda = \sigma_1^*$. To prove it we will argue by contradiction. Let us assume that $v \in H^1_*(\Omega)$ is a positive weak solution of (1.1) for $\lambda = \sigma_1^*$ and let φ^* be the positive principal eigenfunction associated to σ_1^*, normalized so that $\int_\Omega (\varphi^*)^2 = 1$. Owing to (1.13) and (1.17) we know that

$$\varphi^*(x) > 0 \qquad \text{a.e. in } \Omega \tag{2.11}$$

and

$$\sigma_1^* = \frac{\int_\Omega |\nabla \varphi^*|^2}{\int_\Omega (\varphi^*)^2}.$$

Since $v \in H^1_*(\Omega)$ is a positive weak solution of (1.1) for $\lambda = \sigma_1^* > 0$, we have that

$$\int_\Omega a(x)\, v^{q+1} < \infty,$$

and for any $\xi \in H^1_*(\Omega)$ the following holds

$$\int_\Omega \nabla v\, \nabla \xi + \int_\Omega a(x)\, v^q\, \xi = \sigma_1^* \int_\Omega v\, \xi \text{ (cf. (1.5))}. \tag{2.12}$$

Also, it follows from (1.6) that

$$\int_\Omega |\nabla v|^2 + \int_\Omega a(x)v^{q+1} = \sigma_1^* \int_\Omega v^2. \tag{2.13}$$

Moreover, necessarily

$$v > 0 \qquad \text{in} \quad \Omega^+, \tag{2.14}$$

because on the contrary, if

$$v = 0 \qquad \text{in} \quad \Omega^+, \tag{2.15}$$

then for all $\xi \in H^1_*(\Omega)$ we have that

$$\int_\Omega a(x)\, v^q\, \xi = \int_{\Omega^+} a(x)\, v^q\, \xi = 0$$

and (2.12) becomes

$$\int_\Omega \nabla v\, \nabla \xi = \sigma_1^* \int_\Omega v \xi \qquad \forall\, \xi \in H^1_*(\Omega). \tag{2.16}$$

Then, $v \in H^1_*(\Omega)$ is a weak positive eigenfunction associated to σ_1^* and therefore, owing to the simplicity of σ_1^* guaranted by [5, Theorem 1.1], there exists $\alpha > 0$ such that

$$v = \alpha\varphi^* \qquad \text{in} \quad \Omega. \tag{2.17}$$

Now, it follows from (2.11) and (2.17) that $v(x) > 0$ a.e. in Ω^+ which contradicts (2.15). This completes the proof of (2.14). Then, since (2.14) holds, we have that

$$\int_\Omega a(x)v^{q+1} = \int_{\Omega^+} a(x)v^{q+1} > 0, \tag{2.18}$$

and hence, (2.13) and (2.18) imply that

$$\sigma_1^* = \frac{\int_\Omega |\nabla v|^2 + \int_\Omega a(x)v^{q+1}}{\int_\Omega v^2} > \frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2},$$

which contradicts the variational characterization of σ_1^* given by (1.17), and completes the proof of the fact that (1.1) does not possess a positive weak solution for $\lambda = \sigma_1^*$. This fact, together with (2.10) and (1.19), complete the proof of (2.1).

We now prove (2.2). To prove it we will argue by contradiction. Indeed, let $v \in H^1_*(\Omega)$ be a positive weak solution of (1.1) for the value λ of the parameter and let assume that $v = 0$ in Ω^+. Then, (2.1) holds,

$$\int_\Omega \nabla v\, \nabla\xi + \int_\Omega a(x)v^q\xi = \lambda \int_\Omega v\xi \qquad \forall\, \xi \in H^1_*(\Omega) \text{ (cf. (1.5))} \tag{2.19}$$

and since

$$\int_\Omega a(x)v^q\xi = \int_{\Omega^+} a(x)v^q\xi = 0 \qquad \forall\xi \in H^1_*(\Omega),$$

(2.19) becomes

$$\int_\Omega \nabla v\, \nabla\xi = \lambda \int_\Omega v\,\xi \qquad \forall\ \xi \in H^1_*(\Omega). \tag{2.20}$$

Now, since $v > 0$ in Ω, it follows from (2.20) that (λ, v) is a principal eigenpair of the problem

$$\begin{cases} -\Delta\varphi = \mu\varphi & \text{in} \quad \Omega, \\ \mathfrak{B}^*(\Gamma_1^{\mathcal{N}})\varphi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.21}$$

and owing to the uniqueness of the principal eigenvalue of (2.21) guaranteed by [5, Theorem 1.1], we have that $\lambda = \sigma_1^*$, which contradicts (2.1) and completes the proof of (2.2).

We now prove (2.3) and (2.4). If $u \in H^1_*(\Omega) \cap L_\infty(\Omega^+)$ is a positive weak solution of (1.1) for the value λ of the parameter, then u is a positive weak solution of the eigenvalue problem

$$\begin{cases} \left(-\Delta + a(x)u^{q-1}\right)u = \lambda u & \text{in} \quad \Omega, \\ \mathfrak{B}^*\left(\Gamma_1^{\mathcal{N}}\right)u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.22}$$

where the potential

$$K = a(x)u^{q-1} \in L_\infty(\Omega).$$

Now, since (2.22) fits into the abstract framework of (1.12), it follows from the uniqueness of the principal eigenvalue of (2.22) and the structure of its positive eigenfunction, unique up multiplicative constant (cf. [5, Theorem 1.1]) that

$$\lambda = \sigma_1^{\Omega}[-\Delta + a(x)u^{q-1}, \mathfrak{B}^*(\Gamma_1^{\mathcal{N}})] \tag{2.23}$$

and

$$u(x) > 0 \qquad \text{a.e. in} \quad \Omega.$$

This completes the proof of (2.4). Now, taking into account the variational characterization of the principal eigenvalue $\sigma_1^{\Omega}[-\Delta + a(x)u^{q-1}, \mathfrak{B}^*(\Gamma_1^{\mathcal{N}})]$ given by

$$\sigma_1^{\Omega}[-\Delta + a(x)u^{q-1}, \mathfrak{B}^*(\Gamma_1^{\mathcal{N}})] = \inf_{\varphi\in H^1_*(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla\varphi|^2 + \int_\Omega a(x)u^{q-1}\varphi^2}{\int_\Omega \varphi^2} \text{ (cf. (1.14))},$$

(2.23), the definition of $\tilde{H}^1_*(\Omega_0)$, (1.4) and (1.18), the following hold

$$\begin{aligned}
\lambda &= \inf_{\varphi\in H^1_*(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla\varphi|^2 + \int_\Omega a(x)u^{q-1}\varphi^2}{\int_\Omega \varphi^2} \\
&\le \inf_{\varphi\in \tilde{H}^1_*(\Omega_0)\setminus\{0\}} \frac{\int_\Omega |\nabla\varphi|^2 + \int_\Omega a(x)u^{q-1}\varphi^2}{\int_\Omega \varphi^2} \\
&= \inf_{\varphi\in H^1_*(\Omega_0)\setminus\{0\}} \frac{\int_{\Omega_0} |\nabla\varphi|^2 + \int_{\Omega_0} a(x)u^{q-1}\varphi^2}{\int_{\Omega_0} \varphi^2} \\
&= \inf_{\varphi\in H^1_*(\Omega_0)\setminus\{0\}} \frac{\int_{\Omega_0} |\nabla\varphi|^2}{\int_{\Omega_0} \varphi^2} \\
&= \sigma_0^*,
\end{aligned}$$

which completes the proof of (2.3).

We now prove b). Let $u \in H^1_*(\Omega)$ be a positive weak solution of (1.1). Owing to the Rellich-Kondrachov theorem we have that under condition (2.5) the following holds

$$H^1(\Omega) \subset L_{2q}(\Omega). \tag{2.24}$$

Then, since $u \in H^1_*(\Omega)$, it follows from (2.24) that $u^q \in L_2(\Omega)$ and since $a \in \mathcal{C}(\bar{\Omega})$, we have that the function

$$f = -au^q \in L_2(\Omega). \tag{2.25}$$

Now, since $u \in H^1(\Omega)$ satisfies

$$-\Delta u - \lambda u = -au^q \qquad \text{in} \quad \Omega$$

in the weak sense, owing to (2.25) it follows from [6, Theorem 8.8] that $u \in H^2(\Omega')$ for any subdomain $\Omega' \subset\subset \Omega$, which completes the proof of b).

We now prove c). Let u be a positive weak solution of (1.1) and let K be a compact subset of Ω. Let pick up Ω' a subdomain of Ω satisfying

$$K \subset \Omega' \subset\subset \Omega. \tag{2.26}$$

Owing to (2.6) it follows from b) that

$$u \in H^2(\Omega'). \tag{2.27}$$

Now, since for $N = 3$ under the general assumptions we have that

$$H^2(\Omega') \subset C(\Omega') \text{ (cf. [6, Eq (7.30)])}, \tag{2.28}$$

the result follows from (2.26)–(2.28). This completes the proof of c).

This completes the proof. □

Remark 1. *It should be pointed out that owing to* (1.19)*,* (2.1) *provides us with a necessary condition for the existence of positive weak solution of* (1.1) *sharper than* (1.7)*. In fact, as it will be shown in the following section, the lower bound about the λ-parameter for the existence of positive weak solution of* (1.1) *given by* (2.1) *is optimal.*

3. Sufficient condition for the existence of positive weak solutions of (1.1)

In this section is given a sufficient condition for the existence of positive weak solutions of (1.1) depending on the λ parameter, on the exponent $q > 1$ of the reaction term and on the spatial dimension $N \geq 2$. To prove it are used some of the arguments given in [7, Theorem 2]. The main result of this section establishes the following

Theorem 1. *Assume that*

$$\sigma_1^{\Omega}[\mathfrak{B}^*(\Gamma_1^{\mathcal{N}})] < \lambda < \sigma_1^{\Omega_0}[\mathfrak{B}_0^*(\Gamma_1^{\mathcal{N}})] \tag{3.1}$$

and either:

i) $N = 2$ *(and* $q > 1$*), or*
ii) $N \geq 3$ *and* $1 < q < \frac{N+2}{N-2}$.

Then, (1.1) *posseses a positive weak solution. Moreover, if v stands for such a positive weak solution of* (1.1)*, then $v > 0$ in Ω^+.*

Proof. At the beginning we remark that (1.19) and (3.1) imply that

$$\lambda > \sigma_1^{\Omega}[\mathfrak{B}^*(\Gamma_1^{\mathcal{N}})] > \sigma_1^{\Omega}[\mathfrak{B}^{\mathcal{N}}] > 0. \tag{3.2}$$

To prove the existence of a weak positive solution of (1.1) for each λ satisfying (3.1), we will consider the functional

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{q+1}\int_\Omega a(x)|u|^{q+1} - \frac{\lambda}{2}\int_\Omega u^2,$$

and we will show that it reaches its minimum in a positive function of $H_*^1(\Omega)$. Before proving the existence of such a global minimum $\varphi_m \in H_*^1(\Omega)$ of Φ in $H_*^1(\Omega)$, we will prove that if it exists, then it is nontrivial, that is, $\varphi_m \neq 0$, and it may be considered positive. Indeed, since (3.1) holds, taking into account the variational characterization of the principal eigenvalue $\sigma_1^{\Omega}[\mathfrak{B}^*(\Gamma_1^{\mathcal{N}})]$ we have that

$$\sigma_1^{\Omega}[\mathfrak{B}^*(\Gamma_1^{\mathcal{N}})] = \inf_{u \in H_*^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} < \lambda.$$

Hence, there exists $\tilde{\varphi} \in H^1_*(\Omega) \setminus \{0\}$ such that

$$\sigma_1^\Omega[\mathfrak{B}^*(\Gamma_1^{\mathcal{N}})] < \frac{\int_\Omega |\nabla\tilde{\varphi}|^2}{\int_\Omega \tilde{\varphi}^2} < \lambda,$$

and therefore,

$$\int_\Omega |\nabla\tilde{\varphi}|^2 - \lambda \int_\Omega \tilde{\varphi}^2 < 0\,. \tag{3.3}$$

We can assume that $\tilde{\varphi} > 0$ because on the contrary we can replace $\tilde{\varphi}$ by $|\tilde{\varphi}|$.

Since $\tilde{\varphi} \in H^1_*(\Omega)$, it follows from the Rellich-Kondrachov Theorem that under condition *i*), that is, $N = 2$ and $q > 1$, or under condition *ii*), that is $N \geq 3$ and $q \in \left(1, \frac{N+2}{N-2}\right)$, we have that $\tilde{\varphi} \in L_{q+1}(\Omega)$ and hence

$$\left|\int_\Omega a(x)\tilde{\varphi}^{q+1}\right| \leq \|a\|_{L_\infty(\Omega)}\|\tilde{\varphi}\|^{q+1}_{L_{q+1}(\Omega)} < \infty. \tag{3.4}$$

Now, for each $\varepsilon > 0$, let us consider the positive function

$$\tilde{\varphi}_\varepsilon := \varepsilon\tilde{\varphi} \in H^1_*(\Omega).$$

We have that

$$\Phi(\tilde{\varphi}_\varepsilon) = \varepsilon^2\left(\frac{1}{2}\int_\Omega |\nabla\tilde{\varphi}|^2 - \frac{\lambda}{2}\int_\Omega \tilde{\varphi}^2 + \frac{\varepsilon^{q-1}}{q+1}\int_\Omega a(x)\tilde{\varphi}^{q+1}\right)$$

and hence, owing to (3.3) and (3.4), we infer that $\Phi(\tilde{\varphi}_\varepsilon) < 0$ for $\varepsilon > 0$ small enough. Then, a possible minimum $\varphi_m \in H^1_*(\Omega)$ of Φ must be nontrivial. Moreover, we can assume that $\varphi_m > 0$ because on the contrary, since $\Phi(\varphi_m) = \Phi(|\varphi_m|)$, we can replace $\varphi_m \neq 0$ by $|\varphi_m| > 0$.

Now, in order to prove the existence of the global minimum of Φ in $H^1_*(\Omega)$, we will prove that Φ is coercive and weakly lower semicontinuous.

To prove that Φ is coercive we will argue by contradiction, assuming the existence of a sequence $u_n \in H^1_*(\Omega)$, $n \geq 1$ satisfying

$$\lim_{n\to\infty} \|u_n\|_{H^1_*(\Omega)} = \infty, \tag{3.5}$$

and

$$\Phi(u_n) = \frac{1}{2}\int_\Omega |\nabla u_n|^2 + \frac{1}{q+1}\int_\Omega a(x)|u_n|^{q+1} - \frac{\lambda}{2}\int_\Omega u_n^2 \leq C \tag{3.6}$$

for some $C > 0$. Then, it follows from (3.5) and (3.6) that

$$\lim_{n\to\infty}\int_\Omega u_n^2 = \infty, \tag{3.7}$$

because on the contrary, if there exists a subsequence on u_n, again labeled by n such that

$$\int_\Omega u_n^2 \leq D\,, \qquad n \geq 1, \tag{3.8}$$

for some positive constant $D > 0$, then, since $\lambda > 0$ (cf. (3.2)), it follows from (3.6) and (3.8) that

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 \leq \frac{1}{2}\int_\Omega |\nabla u_n|^2 + \frac{1}{q+1}\int_\Omega a(x)|u_n|^{q+1} \leq C + \frac{\lambda}{2}\int_\Omega u_n^2 \leq C + \frac{\lambda}{2}D,$$

and hence
$$\int_\Omega |\nabla u_n|^2 \le 2C + \lambda D,$$
which implies, together with (3.8), that u_n is bounded in $H_*^1(\Omega)$, which contradicts (3.5). This proves that under conditions (3.5) and (3.6), (3.7) holds.

Now, set
$$v_n = \frac{u_n}{\|u_n\|_{L_2(\Omega)}}, \qquad n \ge 1.$$
By construction,
$$\|v_n\|_{L_2(\Omega)} = 1, \qquad n \ge 1. \tag{3.9}$$
Taking into account the definition of v_n, it follows from (3.6) and (3.9) that
$$\frac{1}{2}\int_\Omega |\nabla v_n|^2 + \frac{1}{q+1}\int_\Omega a(x)|v_n|^{q+1}\|u_n\|_{L_2(\Omega)}^{q-1} \le \frac{C}{\|u_n\|_{L_2(\Omega)}^2} + \frac{\lambda}{2} \tag{3.10}$$
and hence,
$$\int_\Omega |\nabla v_n|^2 \le \frac{2C}{\|u_n\|_{L_2(\Omega)}^2} + \lambda. \tag{3.11}$$
Now, (3.7), (3.9) and (3.11) imply that v_n is a bounded sequence in $H_*^1(\Omega)$ and therefore, along some subsequence of v_n, again labeled by v_n. we have that v_n converges strongly in $L_2(\Omega)$, that is
$$\lim_{n\to\infty} \|v_n - v\|_{L_2(\Omega)} = 0, \qquad v \in L_2(\Omega), \tag{3.12}$$
and v_n converges weakly in $H^1(\Omega)$,
$$v_n \rightharpoonup v \qquad \text{in} \quad H^1(\Omega). \tag{3.13}$$
It follows from (3.9) and (3.12) that
$$\|v\|_{L_2(\Omega)} = 1. \tag{3.14}$$
Also, owing to (3.10) the following holds
$$\frac{1}{q+1}\int_\Omega a(x)|v_n|^{q+1}\|u_n\|_{L_2(\Omega)}^{q-1} \le \frac{C}{\|u_n\|_{L_2(\Omega)}^2} + \frac{\lambda}{2}.$$
Hence,
$$\int_{\Omega^+} a(x)|v_n|^{q+1} = \int_\Omega a(x)|v_n|^{q+1} \le \frac{(q+1)C}{\|u_n\|_{L_2(\Omega)}^{q+1}} + \frac{\lambda(q+1)}{2\|u_n\|_{L_2(\Omega)}^{q-1}} \tag{3.15}$$
and therefore, owing to (3.7) it follows from (3.15) that
$$\lim_{n\to\infty}\int_{\Omega^+} a(x)|v_n|^{q+1} = 0. \tag{3.16}$$
Now, owing to the Fatou Lemma, it follows from (3.12) and (3.16) that
$$\int_{\Omega^+} a(x)|v|^{q+1} = 0,$$

and since $a(x) > 0$ for all $x \in \Omega^+$ we have that

$$v = 0 \qquad \text{a.e. in} \quad \Omega^+. \tag{3.17}$$

Thus, owing to (3.17) and (3.7), letting $n \to \infty$ in (3.11) we obtain that

$$\int_{\Omega_0} |\nabla v|^2 = \int_{\Omega} |\nabla v|^2 \leq \lambda. \tag{3.18}$$

On the other hand, it follows from (3.14) and (3.17) that

$$\|v\|_{L_2(\Omega_0)} = 1. \tag{3.19}$$

Also, since by construction $v_n = 0$ on $\Gamma_0 \cup \Gamma_1^{\mathcal{D}}$, $n \geq 1$, taking into account (3.12) we have that

$$v = 0 \qquad \text{on} \quad \Gamma_0 \cup \Gamma_1^{\mathcal{D}} \tag{3.20}$$

in the sense of traces. We now show that

$$v = 0 \qquad \text{on} \quad \Gamma_0^0. \tag{3.21}$$

Since $\Gamma_0^0 = \partial\Omega_0 \cap \Omega = \partial\Omega^+ \cap \Omega$, let us consider the trace operator on Γ_0^0, $\tilde{\gamma} \in \mathcal{L}(H^1(\Omega^+), L_2(\Gamma_0^0))$. Owing to the continuity of $\tilde{\gamma}$, it follows from (3.17) the existence of $\tilde{K} > 0$ such that

$$\|v|_{\Gamma_0^0}\|_{L_2(\Gamma_0^0)} \leq \tilde{K}\|v\|_{H^1(\Omega^+)} = 0,$$

and therefore $v = 0$ on Γ_0^0, which proves (3.21). Then, (3.18)–(3.21) imply that $v \in H_*^1(\Omega_0)$ and since $\|v\|_{L_2(\Omega_0)} = 1$, it follows from (3.18) and the variational characterization for $\sigma_1^{\Omega_0}[\mathfrak{B}_0^*(\Gamma_1^{\mathcal{N}})]$ that

$$\sigma_1^{\Omega_0}[\mathfrak{B}_0^*(\Gamma_1^{\mathcal{N}})] \leq \int_{\Omega_0} |\nabla v|^2 \leq \lambda,$$

which contradicts (3.1) and proves that Φ is coercive.

We now prove that Φ is weakly lower semicontinuous in $H_*^1(\Omega)$. To prove it, let u_n be a sequence such that $u_n \rightharpoonup u$. Then, u_n is bounded in $H^1(\Omega)$ and

$$\|u\|_{H^1(\Omega)} \leq \liminf_{n\to\infty} \|u_n\|_{H^1(\Omega)} \text{ (cf. [3, Proposition III.5]).} \tag{3.22}$$

By compactness we have that $u_n \to u$ in $L_2(\Omega)$ and hence,

$$\lim_{n\to\infty} \|u_n\|_{L_2(\Omega)} = \|u\|_{L_2(\Omega)}, \tag{3.23}$$

and

$$u_n(x) \to u(x) \qquad \text{a.e. in } \Omega. \tag{3.24}$$

Then, it follows from (3.22) and (3.23) that

$$\int_{\Omega} |\nabla u|^2 \leq \liminf_{n\to\infty} \int_{\Omega} |\nabla u_n|^2. \tag{3.25}$$

Now, let us consider the sequence $f_n = a|u_n|^{q+1} \geq 0$ and $f = a|u|^{q+1}$. Owing to (3.24) $f_n(x) \to f(x)$ a.e. in Ω and hence, the Fatou's Lemma implies that

$$\int_\Omega f \leq \liminf_{n\to\infty} \int_\Omega f_n,$$

that is,

$$\int_\Omega a|u|^{q+1} \leq \liminf_{n\to\infty} \int_\Omega a|u_n|^{q+1}. \tag{3.26}$$

Now, (3.23), (3.25) and (3.26) imply that

$$\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n),$$

and therefore Φ is weakly lower semicontinuous.

Then, since Φ is coercive and weakly lower semicontinuos, it follows from [8], [4] that Φ reaches a global minimun φ_m in $H^1_*(\Omega)$ and, as it was remarked at the beginning of the proof, it may be considered positive, that is, $\varphi_m > 0$. Now, differentiating Φ at φ_m in any direction $\xi \in H^1_*(\Omega)$ we obtain that

$$\frac{d}{dt}\Phi(\varphi_m + t\xi)|_{t=0} = \int_\Omega \nabla\varphi_m \nabla\xi + \int_\Omega a(x)\varphi_m^q \xi - \lambda \int_\Omega \varphi_m \xi\,, \tag{3.27}$$

and since Φ reaches its global minimum at φ_m, it follows from (3.27) that

$$\int_\Omega \nabla\varphi_m \nabla\xi + \int_\Omega a(x)\varphi_m^q \xi - \lambda \int_\Omega \varphi_m \xi = 0,$$

which proves, under condition *i*) or *ii*), the existence of a weak positive solution φ_m of (1.1) for any λ satisfying (3.1). The fact that $v = \varphi_m > 0$ in Ω^+ follows from Proposition 1.

This completes the proof. □

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Conflict of interest

The author declares no conflict of interest.

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