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ORIGINAL PAPER

Jacobi–Piñeiro Markov chains

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Abstract
Given a non-negative recursion matrix describing higher order recurrence relations for multiple orthogonal polynomials of type II and corresponding linear forms of type I, a general strategy for constructing a pair of stochastic matrices, dual to each other, is provided. The Karlin–McGregor representation formula is extended to both dual Markov chains and applied to the discussion of the corresponding generating functions and first-passage distributions. Recurrent or transient character of the Markov chain is discussed. The Jacobi–Piñeiro multiple orthogonal polynomials are taken as a case study of the described results. The region of parameters where the recursion matrix is non-negative is given. Moreover, two stochastic matrices, describing two dual Markov chains are given in terms of the recursion matrix and the values of the multiple orthogonal polynomials of type II and corresponding linear forms of type I at the point $x = 1$. The region of parameters where the Markov chains are recurrent or transient is given, and the connection between both dual Markov chains is discussed at the light of the Poincaré's theorem.

Keywords Multiple orthogonal polynomials · Non-negative bounded recursion matrices · Christoffel–Darboux formula · Markov chains · Stochastic matrices · Karlin–McGregor representation formula · Recurrent states · First-passage times · Asymptotic ratio Poincaré's theorem for linear recurrences · Jacobi–Piñeiro multiple orthogonal polynomials

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1 Introduction

In this paper we will show that multiple orthogonality, of both types I and II, are related to Markov chains beyond birth and death chains. This paper is a revised and shortened version of our previous prepublication [8].

The interplay of orthogonal polynomials and stochastic processes is quite old. We can refer the role played by Hermite polynomials in the theory of stochastic processes and the integration with respect to the Wiener process [22, 40].

The 1950s witnessed important advances in the understanding of the links between orthogonal polynomials and stochastic processes. Indeed, highly influential papers on the role of spectral representation of birth and death processes probabilities appeared in those years. Let us mention the celebrated papers by Kendal, Ledermman and Reuter [27–29] and also the seminal works by Samuel Karlin and James McGregor. In particular, the works [23, 24] were devoted to birth and death Markov processes touching differential and classification aspects. In these works the authors presenting an integral representation of the transition probability matrix reveals the intimate relation between the theory of birth and death processes and the theory of the Stieltjes moment problem. Moreover, in [25] the authors studied random walks, that is uncountable Markov chains, and presented one of theirs key findings: the Karlin–McGregor representation formula. These formula allow an interpretation of relevant probabilistic objects in terms of orthogonal polynomials and their recurrence. For an account on this see [35] as well as [12, 19].

Nowadays, a *birth-death polynomial sequence* is defined as a standard orthogonal polynomial sequence which is orthogonal with respect to a measure on $[-1, 1]$ and which is such that the three term recurrence relation coefficient are nonnegative, and the corresponding Jacobi matrix is stochastic. Random walk polynomials have become a classical subject in the literature on orthogonal polynomials, for example, see [21, Chapter 4] for a recent account of some of its relevant aspects.

The contents of the paper are as follows. In Sect. 2 we define type I and II multiple orthogonal polynomials, their biorthogonality, and for the step line case we present the homogeneous linear recurrence relations satisfied by both types of polynomials as well as the reproducing kernel and the corresponding Christoffel–Darboux formula within the sequence of multiple orthogonal polynomials.

The next two sections are the core of the paper. In Sect. 3 we give a general strategy for constructing stochastic matrices once a non negative recursion matrix is known. For this aim we assume the zeros of the orthogonal polynomials of type II and linear forms of type I belong to a bounded set, which is satisfied as an example for AT-systems. In Theorem 2 we provide a candidate for an invariant distribution constructed in terms of the orthogonal polynomials of type II and linear forms of type I. Then, in Theorem 3, we extend to multiple orthogonal polynomials of type I and II the representation formula of Karlin and McGregor, and in Theorem 4 the generating functions for transition probabilities and first passage are given. Theorem 5 gives the integral formula characterizing recurrent and transient Markov chains.

Finally, in Sect. 4 we use the Jacobi–Piñeiro multiple orthogonal polynomials as a case study. We recall the explicit expressions for Jacobi–Piñeiro multiple orthogonal polynomials of type II, and in Theorem 6 we give the explicit expressions for the two families of Jacobi–Piñeiro multiple orthogonal polynomials of type I. To our best knowledge this is the first time such an expression is found. In Theorem 8 the region of parameters in which the Jacobi–Piñeiro's recursion matrix is nonnegative is determined. In Theorem 9 and Corollary 3 the stochastic matrix of type II is given and the coefficients are explicitly determined as rational functions in the Jacobi–Piñeiro parameters $\alpha_1, \alpha_2, \gamma$ and n. Then, in Theorem 10 the Jacobi–Piñeiro type I stochastic matrix is given. In Proposition 3 it is proven, using Poincaré's theorem and the Christoffel–Darboux formula that, in the large n-limit, the dual stochastic matrices are transposed to each other. In Proposition 4 the region of parameters classifying recurrent and transient Markov chains is determined. Finally, some particular examples, of types I and II, recurrent and transient, are discussed in more detail.

2 Multiple orthogonal polynomials

Let $\mathcal{M}(\Delta)$ denote the finite Borel measures which have support, with infinitely many points in the interval $\Delta \subset \mathbb{R}$, where they do not change sign. A weight on Δ is a real integrable function defined on Δ which does not change its sign on Δ. For a finite Borel measure $\mu \in \mathcal{M}(\Delta)$ we consider a system of weights $\vec{w} = (w_1, \dots, w_p)$ on Δ, with $p \in \mathbb{N}$, and a multi-index $\vec{\nu} = (\nu_1, \dots, \nu_p) \in \mathbb{N}_0^p$, and denote $|\vec{\nu}| = \nu_1 + \cdots + \nu_p$. Then, there exist polynomials, $A_{\vec{\nu},1}, \dots, A_{\vec{\nu},p}$, not all identically equal to zero, which satisfy the following orthogonality relations

$$\int_\Delta x^j \sum_{a=1}^{p} A_{\vec{\nu},a}(x) w_a(x) \mathrm{d}\,\mu(x) = 0, \quad \deg A_{\vec{\nu},a} \leq \nu_a - 1, \quad j \in \{0, \dots, |\vec{\nu}| - 2\}. \tag{1}$$

Analogously, there exists a polynomial $B_{\vec{\nu}}$ not identically equal to zero, such that

$$\int_\Delta B_{\vec{\nu}}(x) w_a(x) x^j \mathrm{d}\,\mu(x) = 0, \quad \deg B_{\vec{\nu}} \leq |\vec{\nu}|, \quad j = 0, \dots, \nu_a - 1, \quad a = 1, \dots, p. \tag{2}$$

These families of polynomials are, respectively called, type I and type II multiple orthogonal polynomials, with respect to the combination $(\mu, \vec{w}, \vec{\nu})$ of the measure μ, the system of weights $\vec{w}$ and the multi-index $\vec{\nu}$. We also refer to $Q_{\vec{\nu}(n+1)}(x) = \sum_{a=1}^{p} A_{\vec{\nu}(n+1),a}(x) w_a(x)$ as type I linear forms associated with $(\mu, \vec{w}, \vec{\nu})$.

When $p = 1$ both definitions coincide with standard orthogonal polynomials on the real line. The existence of a system of polynomials $(A_{\vec{\nu},1}, \dots, A_{\vec{\nu},p})$ and a polynomial $B_{\vec{\nu}}$ defined from (1) and (2) respectively, is ensured by solving a system of $|\vec{\nu}|$ linear homogeneous equations with $|\vec{\nu}| + 1$ unknown coefficients.

From the theory of orthogonal polynomials it is well known that when $p = 1$ each polynomial $A_1 \equiv B$ has exactly degree $|\vec{\nu}| = \nu_1$; for $p > 1$ that is not true in general. For instance, if $\vec{w} = (w_1, w_1, \dots, w_1)$ the solution linear space has dimension bigger than one, and we can find two solutions which are linearly independent. Hence, there is at least an $a \in \{1, \dots, p\}$ such that $\deg A_{\vec{\nu},a} < \nu_a - 1$ and $\deg B < |\vec{\nu}|$. Given a measure $\mu \in \mathcal{M}(\Delta)$ and a system of weights $\vec{w}$ on Δ a multi-index $\vec{\nu}$ is called type I or type II normal if $\deg A_{\vec{\nu},a}$ is equal to $\nu_a - 1$, $a = 1, \dots, p$ or $\deg B$ must be equal to $|\vec{\nu}| - 1$, respectively. These two conditions are equivalent (cf. [38]) When for a pair $(\mu, \vec{w})$ all the multi-indices are type I normal (or equivalently, type II normal), then the pair is called perfect.

Multiple orthogonal polynomials satisfy a finite order recurrence relation but, since we are working with multi-indices, there are several ways to decrease or increase the degree of the multiple orthogonal polynomials. In this work we stay on the step line case, i.e.

$$\begin{aligned}\vec{\nu}(j) &\in \{(0, \dots, 0), (1, 0, \dots, 0), \dots, (1, \dots, 1, 0), (1, \dots, 1), (2, 1, \dots, 1), \dots\},\\ |\vec{\nu}(j)| &= j,\end{aligned}$$

for $j \in \{0, 1, \dots\}$. In this case we denote

$$B^{(n)}(x) = B_{\vec{\nu}(n)}(x) \quad \text{and} \quad Q^{(n)}(x) = Q_{\vec{\nu}(n+1)}(x).$$

The following multiple biorthogonality relations

$$\int_0^1 B^{(l)}(x) Q^{(k)}(x) \mathrm{d}\,\mu(x) = \delta_{l,k}, \quad l, k \in \mathbb{N}_0, \tag{3}$$

hold.

Vector of type II multiple orthogonal polynomials and associated type I linear forms are defined by

$$B := \begin{bmatrix} B^{(0)} \\ B^{(1)} \\ \vdots \end{bmatrix}, \quad Q := \begin{bmatrix} Q^{(0)} \\ Q^{(1)} \\ \vdots \end{bmatrix}.$$

For further information on multiple orthogonal polynomials, we recommend consulting [1, 9, 21, 32].

It is well known (cf. [38]) that the type II multiple orthogonal polynomials and the type I linear forms verify recurrence relations for the step line case. Defining

$$T = \begin{bmatrix} T_{0,0} & 1 & 0 & \cdots & \\ \vdots & & \ddots & \ddots & \\ T_{N,0} & & & & \ddots \\ 0 & T_{N+1,1} & & & \\ \vdots & \ddots & \ddots & & \end{bmatrix}$$

the recursion relations can be written as the following eigenvalue properties that the vector type II multiple orthogonal polynomials and the corresponding vector type I multiple orthogonal polynomials verify

$$TB = xB, \quad T^{\top} Q = xQ.$$

That is, for $n \in \mathbb{N}_0$, the following order $N+1$ homogeneous linear recurrence relations are satisfied

$$\begin{aligned} T_{n,n-N} B^{(n)} + \cdots + T_{n,n} B^{(n)} + B^{(n+1)} &= x B^{(n)}, \\ Q^{(n-1)} + T_{n,n} Q^{(n)} + \cdots + T_{N+n,n} Q^{(N+n)} &= x Q^{(n)} \quad \text{where } Q^{(-1)} = 0. \end{aligned}$$

Notice that in order to have multiple orthogonality, it is necessary that $T_{N+i,i} \neq 0$ for $i \in N_0$. This condition implies the irreducibility of the matrix. Consequently, there cannot exist a permutation matrix that, upon conjugation, transforms the matrix into the direct sum of two blocks. In other words, there are no proper invariant subspaces in such a scenario.

When dealing with the case where matrix T is nonnegative, this condition is tantamount to the existence of an integer q for which T^q becomes a positive matrix.

Daems and Kuijlaars found a CD formula for near neighbours (cf. [16, 17]). Notice that this formula involves polynomials that are not in the step-line. In our analysis we required a CD formula adapted to the step-line. In this context Sorokin and Van Iseghem [37] derived such a CD formula in a general context, that can be particularized to multiple orthogonal polynomials scenario (see also [3, 13, 15]). The Christoffel–Darboux (CD) kernel is given by

$$K^{(n)}(x,y) := \sum_{m=0}^{n-1} B^{(m)}(y) Q^{(m)}(x),$$

fulfills a reproducing property and Christoffel–Darboux formulas, i.e.

$$(y-x)K^{(n)}(x,y) = Q^{(n-1)}(x)B^{(n)}(y) - \begin{bmatrix} Q^{(n)}(x) & \cdots\cdots & Q^{(n+N-1)}(x) \end{bmatrix} \\ \times \begin{bmatrix} T_{n,n-N} & \cdots & \cdots & T_{n,n-1} \\ 0 & T_{n+1,n-N+1} & \cdots & T_{n+1,n-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots\ 0 & & T_{n+N-1,n-1} \end{bmatrix} \begin{bmatrix} B^{(n-N)}(y) \\ \vdots \\ B^{(n-1)}(y) \end{bmatrix}.$$

For example, for $N = 2$, the CD formula reads as follows

$$\begin{aligned}(y-x)K^{(n)}(x,y) = Q^{(n-1)}(x)B^{(n)}(y) \\ &- Q^{(n)}(x)\big(T_{n,n-2}B^{(n-2)}(y) + T_{n,n-1}B^{(n-1)}(y)\big) \\ &- Q^{(n+1)}(x)T_{n+1,n-1}B^{(n-1)}(y).\end{aligned}$$

Taking $y = x$ in the previous CD formulas leads to

$$\begin{aligned}Q^{(n-1)}(x)B^{(n)}(x) &= Q^{(n)}(x)\big(T_{n,n-2}B^{(n-2)}(x) + T_{n,n-1}B^{(n-1)}(x)\big) \\ &+ Q^{(n+1)}(x)T_{n+1,n-1}B^{(n-1)}(x).\end{aligned} \quad (4)$$

3 Stochastic matrices, Markov chains for AT systems

A set of functions $\{\varphi\}_{i=1}^{s}$is said a Chebyshev system in $[a,b]$ if the set is linearly independent and any linear combination $a_1\varphi_1 + \cdots + a_s\varphi_s$ has at most $s-1$ zeros in $[a,b]$. We say that $\{\mu, \vec{w}\}$ is an algebraic Chebyshev system (AT system, T is from the French transliteration Tchebycheff) if for any index $\vec{\nu} = (\nu_1, \ldots, \nu_p)$ it holds that $\{w_1, \ldots, x^{\nu_1-1}w_1, \ldots, w_p, \ldots, x^{\nu_p-1}w_p\}$ is a Chebyshev system in the support of μ, $[a,b]$ (cf. [32, 38]). In this case the zeros of type II polynomials and the ones of the type I linear forms are confined to (a,b), and the index $(\nu_1, \ldots, \nu_p)$ is normal, so the system $\{\mu, \vec{w}\}$ is perfect.

We give two definitions of bounded banded Hessenberg semi-infinite matrices:

(i) A semi-infinite stochastic matrix P_{II} is said to be a multiple stochastic matrix of type II if it has the form

$$P_{II} = \begin{bmatrix} P_{0,0} & P_{0,1} & 0 & \cdots & \\ P_{1,0} & P_{1,1} & P_{1,2} & \ddots & \\ \vdots & & & \ddots & \ddots \\ P_{N,0} & & & & \ddots \\ 0 & P_{N+1,1} & & & \\ \vdots & \ddots & \ddots & & \end{bmatrix}.$$

(ii) A semi-infinite stochastic matrix P_I is said to be a multiple stochastic matrix of type I if it has the form

$$P_I = \begin{bmatrix} P_{0,0} & P_{0,1} & \cdots\cdots & P_{0,N} & 0 & \cdots\cdots\cdots \\ P_{1,0} & P_{1,1} & & & \ddots & \\ 0 & P_{2,1} & \ddots & & & \ddots \\ \vdots & \ddots & & \ddots & & \\ \vdots & & & & & \end{bmatrix}.$$

The same definition holds for the semi-stochastic case.

Now, we describe a method for deriving two stochastic matrices from a set of multiple biorthogonal polynomials associated with an AT system.

Given the corresponding recursion matrix T we will get a stochastic matrix P using the linear recurrence relation in its eigenvalue form $TB(x) = xB(x)$, and consequently using properties of the sequence of multiple orthogonal polynomials of type II $\{B^{(l)}(x)\}_{l=0}^{\infty}$. These ideas extend to the transposed recursion matrix $T^\top$, and we will use the eigenvalue property $T^\top Q(x) = xQ(x)$ for the type I linear form $Q(x) = \sum_{a=1}^{p} A_a(x)w_a(x)$.

Theorem 1 (Multiple stochastic matrix) *Let us assume that* $\{B^{(n)}\}$ *and* $\{Q^{(n)}\}$ *are the type II monic multiple orthogonal polynomial and type I linear form sequences, with respect to an AT system* $\{\mu, \vec{w}\}$ *on the interval* $[0, 1]$. *If the recursion matrix* T *is nonnegative and* $Q^{(n)}(1) > 0$, *then*

$$\sigma_{II} = \operatorname{diag}\begin{bmatrix}\sigma_{II,0} & \sigma_{II,1} \cdots\cdots\end{bmatrix}, \qquad \sigma_{II,n} := \frac{1}{B^{(n)}(1)},$$
$$\sigma_{I} = \operatorname{diag}\begin{bmatrix}\sigma_{I,0} & \sigma_{I,1} \cdots\cdots\end{bmatrix}, \qquad \sigma_{I,n} := \frac{1}{Q^{(n)}(1)}$$

is such that

$$P_{II} := \sigma_{II} T \sigma_{II}^{-1}, \quad P_{II,n,m} = \frac{B^{(m)}(1)}{B^{(n)}(1)} T_{n,m},$$
$$P_{I} := \sigma_{I} T^\top \sigma_{I}^{-1}, \quad P_{I,n,m} = \frac{Q^{(m)}(1)}{Q^{(n)}(1)} T_{m,n}$$

are, respectively, a multiple stochastic matrix of type II, and multiple stochastic matrix of type I.

Proof Notice that each of the type II multiple orthogonal polynomials $B^{(n)}$ is a monic polynomial whose zeros are in the interval $(0, 1)$, cf. [32, Corollary of Theorem 4.3], so $B^{(n)}(1) > 0$ and we can assert that $B(1)$ is a positive vector. By definition of σ_{II}, we have $\mathbf{1} = \sigma_{II} B(1)$. As for $n \in \mathbb{N}_0$ we have $\sigma_{II,n} > 0$, and by hypothesis $\sigma_{I,n} > 0$, the matrices P_{II} and P_I are nonnegative and

$$P_{II}\,\mathbf{1} = \sigma_{II} T \sigma_{II}^{-1} \sigma_{II} B(1) = \sigma_{II} T\, B(1) = \sigma_{II} B(1) = \mathbf{1},$$
$$P_{I}\,\mathbf{1} = \sigma_{I} T^\top \sigma_{I}^{-1} \sigma_{I} Q(1) = \sigma_{I} T\, Q(1) = \sigma_{I} Q(1) = \mathbf{1},$$

as desired. □

As we have seen, we have two stochastic matrices P_I and P_{II}, we say that they are dual stochastic matrices. The corresponding Markov chains are said to be dual.

Corollary 1 *Both dual stochastic matrices, P_I and P_{II} are connected by*

$$\sigma_I^{-1} P_I \sigma_I = \left(\sigma_{II}^{-1} P_{II} \sigma_{II}\right)^\top.$$

That is, the stochastic matrices coefficients fulfill

$$P_{I,n,n-k} = \frac{B^{(n-k)}(1) Q^{(n-k)}(1)}{B^{(n)}(1) Q^{(n)}(1)} P_{II,n-k,n}, \tag{5}$$

with $m = n - k$, $k \in \{-1, 0, 1, \dots, N\}$, where N is the number of nonzero subdiagonals of the recursion matrix T.

We now present a candidate for an invariant distribution.

Theorem 2 (Invariant measure) *Let us assume conditions in Theorem 1. Then,*

(i) *The row vector $\boldsymbol{\kappa} = (B^{(0)}(1) Q^{(0)}(1), B^{(1)}(1) Q^{(1)}(1), \dots)$ is a nonnegative vector, which is a left eigenvector of both dual multiple stochastic matrices P_{II} and P_I with unit eigenvalue $\boldsymbol{\kappa} P_{II} = \boldsymbol{\kappa}$ and $\boldsymbol{\kappa} P_I = \boldsymbol{\kappa}$.*

(ii) *If*

$$\lim_{n\to\infty} K^{(n)}(1,1) = \sum_{k=0}^{\infty} B^{(k)}(1) Q^{(k)}(1) < \infty,$$

then $\boldsymbol{\pi} = \frac{\boldsymbol{\kappa}}{\|\boldsymbol{\kappa}\|_1}$, is an invariant distribution for both dual Markov chains. In this situation the Markov chain is positive recurrent, and whenever the chain is aperiodic is also ergodic.

(iii) *If*

$$\lim_{n\to\infty} \frac{B^{(n+1)}(1) Q^{(n+1)}(1)}{B^{(n)}(1) Q^{(n)}(1)} < 1$$

we have $\boldsymbol{\pi} = \frac{\boldsymbol{\kappa}}{\|\boldsymbol{\kappa}\|_1}$, is an invariant distribution.

Proof (i) We have the eigenvalue property of the recursion matrix $(Q(1))^\top = (Q(1))^\top T$, so that

$$(Q(1))^\top \sigma_{II}^{-1} = (Q(1))^\top \sigma_{II}^{-1} \sigma_{II} T \sigma_{II}^{-1}.$$

Thus, as $(Q(1))^\top \sigma_{II}^{-1} = \boldsymbol{\kappa}$, we get one the assertions. For the other we recall $(B(1))^\top = (B(1))^\top T^\top$ so that

$$(B(1))^\top \sigma_I^{-1} = (B(1))^\top \sigma_I^{-1} \sigma_I T^\top \sigma_I^{-1}.$$

and now, we notice $(B(1))^\top \sigma_I^{-1} = \boldsymbol{\kappa}$, and we get the other case.

(ii) If $\boldsymbol{\kappa} \in \ell_1$, as $\|\boldsymbol{\kappa}\|_1 = \boldsymbol{\kappa}\,\mathbf{1}$, the row vector $\boldsymbol{\pi} = \frac{\boldsymbol{\kappa}}{\|\boldsymbol{\kappa}\|_1}$ is a probability vector, $\boldsymbol{\pi}\,\mathbf{1} = 1$. Therefore, we have found an invariant distribution.

Finally, applying d'Alembert's ratio test we see that (iii) holds true. □

Remark 1 Now we comment on ergodic states as discussed in [25]. The π_n introduced in [25] correspond to $B^{(n)}(1) Q^{(n)}(1)$ in the standard tridiagonal scenario, observe also that the corresponding vector $(\pi_0, \pi_1, \dots)$ in [25] is not a probability vector yet as it needs to be normalized. Recall also that for a birth and death Markov chain; i.e., a tridiagonal case, we have $1/\rho = \sum_{n=0}^{\infty} \pi_n = \lim_{n\to\infty} K^{(n)}(1,1)$, being $\rho = C(1)$, the Christoffel function evaluated at 1. Therefore, following [36] for the standard situation described in [25] the process is ergodic if and only if 1 is a mass point of the measure $\mathrm{d}\,\psi$, in the notation used in [25]. This is what in [25] is referred as a jump of ψ.

We now bring the results of Karlin and McGregor concerning tridiagonal stochastic matrices [25] to the multi-diagonal situation of multiple orthogonal polynomials.

We have seen in Theorem 1 that certain sets of multiple orthogonal polynomials give two families of stochastic matrices P_{II} and P_I. Here P_{II} models a Markov chain with allowed jumps backward farther than near neighbors, and P_I models a random walk with allowed jumps forward farther than near neighbors.

Theorem 3 (KMcG representation formula) *Let us assume the conditions requested in Theorem 1. Then, for a Markov chain with transition matrix P_{II}, the probability, after r steps from state n to state m is given by*

$$P^r_{II,nm} = \frac{B_{\vec{\nu}(m)}(1)}{B_{\vec{\nu}(n)}(1)} \int_0^1 x^r B_{\vec{\nu}(n)}(x) Q_{\vec{\nu}(m+1)}(x) \mathrm{d}\,\mu(x). \tag{6}$$

Moreover, for a Markov chain with transition matrix P_I, the probability, after r steps from state n to state m is given by

$$P^r_{I,nm} = \frac{Q_{\vec{\nu}(m+1)}(1)}{Q_{\vec{\nu}(n+1)}(1)} \int_0^1 x^r B_{\vec{\nu}(m)}(x) Q_{\vec{\nu}(n+1)}(x) \mathrm{d}\,\mu(x). \tag{7}$$

Proof In terms of the recursion matrix T we have

$$P^r_{II,nm} = (T^r)_{n,m} \frac{B^{(m)}(1)}{B^{(n)}(1)}.$$

But, $T^r B(x) = x^r B(x)$ so that $\sum_{m=0}^{\infty} (T^r)_{n,m} B^{(m)}(x) = x^r B^{(n)}(x)$ and using biorthogonality (3) we get

$$(T^r)_{n,m} = \int_0^1 x^r B^{(n)}(x) Q^{(m)}(x) \mathrm{d}\,\mu(x),$$

and (6) follows.

In terms of the transposed recursion matrix $T^\top$ we have

$$P^r_{I,nm} = ((T^\top)^r)_{n,m} \frac{Q^{(m)}(1)}{Q^{(n)}(1)} = (T^r)_{m,n} \frac{Q^{(m)}(1)}{Q^{(n)}(1)}$$

and we obtain (7). □

For this discussion see for instance [18, 25]. The generating functions of the probability P^n_{ij} and first-passage-time probability f^n_{ij} given by $P_{ij}(s) = \sum_{n=0}^{\infty} P^n_{ij} s^n$, $F_{ij}(s) = \sum_{n=1}^{\infty} f^n_{ij} s^n$, are connected by $P_{ij}(s) = F_{ij}(s) P_{ij}(s)$, for $i \neq j$, and $P_{jj}(s) = 1 + F_{jj}(s) P_{jj}(s)$. That allows us to express the generating functions of the first time passage distributions after n transitions in terms of the generating functions for the transition probability after n transitions.

Theorem 4 *In the conditions of Theorem 1, then for $|s| < 1$, the transition probability generating function reads as*

$$P_{II,nm}(s) = \frac{B_{\vec{\nu}(m)}(1)}{B_{\vec{\nu}(n)}(1)} \int_0^1 \frac{B_{\vec{\nu}(n)}(x) Q_{\vec{\nu}(m+1)}(x)}{1 - sx} \mathrm{d}\,\mu(x),$$

while for the first passage generating function we have

$$F_{II,nm}(s) = \frac{B_{\vec{\nu}(m)}(1)}{B_{\vec{\nu}(n)}(1)} \frac{\int_0^1 \frac{B_{\vec{\nu}(n)}(x)Q_{\vec{\nu}(m+1)}(x)}{1-sx}\mathrm{d}\,\mu(x)}{\int_0^1 \frac{B_{\vec{\nu}(m)}(x)Q_{\vec{\nu}(m+1)}(x)}{1-sx}\mathrm{d}\,\mu(x)}, \quad n \neq m,$$

$$F_{nn}(s) = 1 - \frac{1}{\int_0^1 \frac{B_{\vec{\nu}(n)}(x)Q_{\vec{\nu}(n+1)}(x)}{1-sx}\mathrm{d}\,\mu(x)}.$$

Moreover, for $|s| < 1$, *the transition probability generating function reads as*

$$P_{I,nm}(s) = \frac{Q_{\vec{\nu}(m+1)}(1)}{Q_{\vec{\nu}(n+1)}(1)} \int_0^1 \frac{B_{\vec{\nu}(m)}(x)Q_{\vec{\nu}(n+1)}(x)}{1-sx}\mathrm{d}\,\mu(x),$$

while for the first passage generating function we have

$$F_{I,nm}(s) = \frac{Q_{\vec{\nu}(m+1)}(1)}{Q_{\vec{\nu}(n+1)}(1)} \frac{\int_0^1 \frac{B_{\vec{\nu}(m)}(x)Q_{\vec{\nu}(n+1)}(x)}{1-sx}\mathrm{d}\,\mu(x)}{\int_\Delta \frac{B_{\vec{\nu}(m)}(x)Q_{\vec{\nu}(m+1)}(x)}{1-sx}\mathrm{d}\,\mu(x)}, \quad n \neq m,$$

$$F_{nn}(s) = 1 - \frac{1}{\int_0^1 \frac{B_{\vec{\nu}(n)}(x)Q_{\vec{\nu}(n+1)}(x)}{1-sx}\mathrm{d}\,\mu(x)}.$$

Proof Let us denote by $P_{II}(s) = \big(P_{II,nm}(s)\big)_{n,m\in\mathbb{N}_0}$ the semi-infinite matrix whose coefficients are the transition probability generating functions. Then, as $|s| < 1$ and $\|P_{II}\|_\infty = 1$, we know that for the first Neumann type expansion $P_{II}(s) = \sum_{k=0}^{\infty} P_{II}^k s^k = \big(I - sP_{II}\big)^{-1}$, uniformly in norm. Now, as

$$P_{II}\sigma_{II}B(x) = \sigma_{II}T\sigma_{II}^{-1}\sigma_{II}B(x) = x\sigma_{II}B(x),$$

we conclude that

$$\big(I - sP_{II}\big)\sigma_{II}B(x) = \Big(1 - sx\Big)\sigma_{II}B(x),$$

and, consequently,

$$\big(I - sP_{II}\big)^{-1}\sigma_{II}B(x) = \frac{1}{1-sx}\sigma_{II}B(x).$$

Therefore,

$$\sum_{m=0}^{\infty} P_{II,nm}(s)\sigma_{II,m}B^{(m)}(x) = \frac{1}{1-sx}\sigma_{II,n}B^{(n)}(x).$$

Now, using biorthogonality (3) we get

$$P_{II,nm}(s) = \frac{\sigma_{II,n}}{\sigma_{II,m}} \int_0^1 \frac{1}{1-sx}B^{(n)}(x)Q^{(m)}(x)\mathrm{d}\,\mu(x).$$

For the type I we proceed analogously. For $|s| < 1$, the following first Neumann expansion converges uniformly $P_I(s) = \sum_{k=0}^{\infty} P_I^k s^k = \big(I - sP_I\big)^{-1}$. Now, analogously to the previous discussion, we have

$$P_I\sigma_I Q(x) = \sigma_I T^\top \sigma_I^{-1}\sigma_I Q(x) = x\sigma_I Q(x),$$

so that

$$\big(I - sP_I\big)\sigma_I Q(x) = \Big(1 - sx\Big)\sigma_I Q(x)$$

and, consequently,

$$\big(I - sP_I\big)^{-1}\sigma_I Q(x) = \frac{1}{1-sx}\sigma_I Q(x).$$

Hence,

$$\sum_{m=0}^{\infty} P_{I,nm}(s)\sigma_{I,m} Q^{(m)}(x) = \frac{1}{1-sx}\sigma_{I,n} Q^{(n)}(x)$$

and biorthogonality (3) leads to

$$P_{I,nm}(s) = \frac{\sigma_{I,n}}{\sigma_{I,m}} \int_0^1 \frac{1}{1-sx} B^{(m)}(x) Q^{(n)}(x) \mathrm{d}\,\mu(x),$$

as desired. □

Lemma 1 *For both Markov chains, the n-th state is recurrent or transient whenever*

$$\int_0^1 \frac{B_{\vec{\nu}(n)}(x) Q_{\vec{\nu}(n+1)}(x)}{1-x} \mathrm{d}\,\mu(x) \tag{8}$$

diverges or converges, respectively.

Proof The limit $F^{\infty}_{nn} = \lim_{s\to 1^-} F_{nn}(s)$ describes the probability that the n-th state is visited again, that is that the state is recurrent.

According to the previous results,

$$\lim_{s\to 1^-} F_{nn}(s) = 1 - \frac{1}{\int_0^1 \frac{B_{\vec{\nu}(n)}(x) Q_{\vec{\nu}(n+1)}(x)}{1-x} \mathrm{d}\,\mu(x)}.$$

Thus, the n-th state is visited again whenever the integral in (8) diverges to $+\infty$, so that $\lim_{s\to 1^-} F_{nn}(s) = 1$, as we wanted to show. □

Lemma 2 *The Markov chains corresponding to the stochastic matrices P_{II} and P_I are irreducible; that is, they have only one class.*

Proof Since the recurrence matrix T is irreducible, both P_{II} and P_I, being conjugations of T by positive diagonal matrices, are also irreducible. Consequently, these Markov chains have only one class. □

Theorem 5 *The Markov chains corresponding to the stochastic matrices P_{II} and P_I are recurrent if and only if the integral*

$$\int_0^1 \frac{w_1(x)}{1-x} \mathrm{d}\,\mu(x)$$

diverges. Both dual Markov chains are transient whenever the integral converges.

Proof As there is only one class of states, one needs only to check if the state 0 is recurrent or transient. Using the previous Lemma 1 the result follows. □

4 The Jacobi–Piñeiro Markov chains

In this section we analyze an example of multiple orthogonal polynomials that fulfill the requirements of Theorem 1 in the previous section. Therefore, we will have type I and II multiple stochastic matrices and corresponding Markov chains.

The general context is as follows. In our notation we take $p = 2$ and two recursion type weights

$$w_1(x) = x^{\alpha_1}, \quad w_2(x) = x^{\alpha_2}, \quad \mathrm{d}\,\mu(x) = (1-x)^{\gamma}\,\mathrm{d}\,x$$

supported in $\Delta = [0, 1]$. It is known [32] that $\{(1-x)^{\gamma}, (x^{\alpha_1}, x^{\alpha_2}\}$ is an AT system whenever $\alpha_1, \alpha_2, \gamma > -1$ and $\alpha_1 - \alpha_2 \notin \mathbb{Z}$. These polynomials for $\gamma = 0$ where considered for the first time by Luis Piñeiro in [33] and in [32] the general situation was studied.

4.1 Jacobi–Piñeiro multiple orthogonal polynomials of type II

Here we follow Van Assche and Coussement [39] and Aptekarev et al. [2]. In [2], using the Rodrigues formula, the polynomials of type II where computed (cf. [2, 32, 33, 39]).

Proposition 1 *The monic Jacobi–Piñeiro multiple orthogonal polynomials of type II are*

$$B^{(2n)}(x) = B_{(n,n)}(x), \quad B^{(2n+1)}(x) = B_{(n+1,n)}(x),$$

where

$$\begin{aligned} B_{(n_1,n_2)}(x) &= N_{n_1,n_2} \sum_{k=0}^{n_1} \sum_{j=0}^{n_2} B^{k,j}_{n_1,n_2} (x-1)^{j+k} x^{-j-k+n_2+n_1}, \\ &= N_{n_1,n_2} \sum_{k=0}^{n_1} \sum_{j=0}^{n_2} \tilde{B}^{k,j}_{n_1,n_2} \sum_{i=0}^{n_1+n_2-j-k} \binom{n_1+n_2-k-j}{i} (-1)^{j+k} x^{k+j+i}, \end{aligned} \tag{9}$$

with

$$B^{k,j}_{n_1,n_2} := \frac{(\gamma+j+k+1)_{n_2-j}(\gamma+k+n_2+1)_{n_1-k}(\alpha_1-k+n_1+1)_k(\alpha_2-j-k+n_2+n_1+1)_j}{j!k!(n_2-j)!(n_1-k)!},$$

$$\tilde{B}^{k,k}_{n_1,n_2} := \frac{(\gamma+n_1+n_2-j-k+1)_j(\gamma+n_1+n_2-k+1)_k(\alpha_1+k+1)_{n_1-k}(\alpha_2+j+k+1)_{n_2-j}}{j!k!(n_2-j)!(n_1-k)!},$$

$$N_{n_1,n_2} := \frac{1}{\sum_{k=0}^{n_1}\sum_{j=0}^{n_2} B^{k,j}_{n_1,n_2}} = \frac{n_1!n_2!}{(n_1+n_2+\alpha_1+\gamma+1)_{n_1}(n_1+n_2+\alpha_2+\gamma+1)_{n_2}},$$

where we have used the Pochhammer symbol $(z)_k = z(z+1)\cdots(z+k-1)$, $z \in \mathbb{C}$ *and* $k \in \mathbb{Z}$.

Proof These results have been proved in the references quoted. However, the value of N_{n_1,n_2} requires a discussion. Recalling the definition of the binomial function

$$\binom{z}{k} := \frac{(z-k+1)_k}{k!}$$

we immediately see that

$$B^{k,j}_{n_1,n_2} = \binom{\gamma+k+n_2}{n_2-j}\binom{\gamma+n_1+n_2}{n_1-k}\binom{\alpha_1+n_1}{k}\binom{\alpha_2+n_1+n_2-k}{j}$$

in agreement with §3.3. in [2]. Thus, the normalization factor is

$$
\begin{aligned}
N^{-1}_{n_1,n_2} &= \sum_{k=0}^{n_1}\sum_{j=0}^{n_2} B^{k,j}_{n_1,n_2} = \sum_{k=0}^{n_1}\binom{\alpha_1+n_1}{k}\binom{\gamma+n_1+n_2}{n_1-k}\\
&\quad\times\sum_{j=0}^{n_2}\binom{\alpha_2+n_1+n_2-k}{j}\binom{\gamma+k+n_2}{n_2-j}\\
&= \sum_{k=0}^{n_1}\binom{\alpha_1+n_1}{k}\binom{\gamma+n_1+n_2}{n_1-k}\binom{\alpha_2+\gamma+n_1+2n_2}{n_2}\\
&= \binom{\alpha_1+\gamma+2n_1+n_2}{n_1}\binom{\alpha_2+\gamma+n_1+2n_2}{n_2}\\
&= \frac{(\alpha_1+\gamma+n_1+n_2+1)_{n_1}(\alpha_2+\gamma+n_1+n_2+1)_{n_2}}{n_1!n_2!},
\end{aligned}
$$

where we have used the Chu–Vandermonde identity twice. A similar argument was used in [39]. □

Corollary 2 *The values of the multiple orthogonal polynomials at* 1 *is*

$$
B_{(n_1,n_2)}(1) = \frac{(\gamma+1)_{n_2+n_1}}{(\alpha_1+\gamma+n_2+n_1+1)_{n_1}(\alpha_2+\gamma+n_2+n_1+1)_{n_2}}. \tag{10}
$$

Moreover, at the origin we have

$$
B_{(n_1,n_2)}(0) = (-1)^{n_2+n_1}\frac{(\alpha_1+1)_{n_1}(\alpha_2+1)_{n_2}}{(\alpha_1+\gamma+n_2+n_1+1)_{n_1}(\alpha_2+\gamma+n_2+n_1+1)_{n_2}}.
$$

Proof It follows from (9). □

Using the generalized hypergeometric function ${}_3F_2$ we have [38]

$$
\begin{aligned}
(1-x)^{\gamma}B_{(n,m)}(x) &= \frac{(-1)^{n+m}(\alpha+1)_n(\beta+1)_m}{(n+m+\alpha+\gamma+1)_n(n+m+\beta+\gamma+1)_m}\\
&\quad\times {}_3F_2\left[\begin{matrix}-n-m-\gamma,\ \alpha+n+1,\ \beta+n+1\\ \alpha+1,\ \beta+1\end{matrix};x\right].
\end{aligned}
$$

4.2 Jacobi–Piñeiro multiple orthogonal polynomials of type I

Theorem 6 *The Jacobi–Piñeiro multiple orthogonal polynomials of type I are*

$$
\begin{aligned}
A_{(n_1,n_2),i}(x) &= (-1)^{n_1+n_2-1}\\
&\quad\times\frac{(\alpha_1+\gamma+n_1+n_2)_{n_1}(\alpha_2+\gamma+n_1+n_2)_{n_2}}{(n_i-1)!(\hat{\alpha}_i-\alpha_i)_{\hat{n}_i}}\frac{\Gamma(\alpha_i+\gamma+n_1+n_2)}{\Gamma(\gamma+n_1+n_2)\Gamma(\alpha_i+1)}\\
&\quad\times {}_3F_2\left[\begin{matrix}-n_i+1,\alpha_i+\gamma+n_1+n_2,\alpha_i-\hat{\alpha}_i-\hat{n}_i+1\\ \alpha_i+1,\alpha_i-\hat{\alpha}_i+1\end{matrix};x\right] = \sum_{l=0}^{n_i-1}C^{l}_{(n_1,n_2),i}x^l
\end{aligned} \tag{11}
$$

where we have defined $\hat{\alpha}_i \equiv \delta_{i,2}\alpha_1+\delta_{i,1}\alpha_2$, $\hat{n}_i \equiv \delta_{i,2}n_1+\delta_{i,1}n_2$ *and*

$$
\begin{aligned}
C^l_{(n_1,n_2),i} &\equiv (-1)^{n_1+n_2-1}\\
&\times \frac{(\alpha_1+\gamma+n_1+n_2)_{n_1}(\alpha_2+\gamma+n_1+n_2)_{n_2}}{(n_i-1)!(\hat{\alpha}_i-\alpha_i)_{\hat{n}_i}}\frac{\Gamma(\alpha_i+\gamma+n_1+n_2)}{\Gamma(\gamma+n_1+n_2)\Gamma(\alpha_i+1)}\\
&\times \frac{(-n_i+1)_l}{l!}\frac{(\alpha_i+\gamma+n_1+n_2)_l(\alpha_i-\hat{\alpha}_i-\hat{n}_i+1)_l}{(\alpha_i+1)_l(\alpha_i-\hat{\alpha}_i+1)_l}
\end{aligned}
$$

In order to prove the orthogonality relations we need the following theorem due to Karp and Prilepkina [26, Theorem 2.2]

Theorem 7 *Let be* $r \in \mathbb{N}_0$, $a, b, f_1, \dots, f_r \in \mathbb{C}$, $p \in \mathbb{N}$, $m_1, \dots, m_r \in \mathbb{N}_0$. *If* $\operatorname{Re}(p-a-m_1-\cdots-m_r) > 0$ *then*

$$
\begin{aligned}
&{}_{r+2}F_{r+1}\left[\begin{matrix} a, b, f_1+m_1, \dots, f_r+m_r \\ b+p, f_1, \dots, f_r \end{matrix}; 1\right]\\
&= \frac{\Gamma(1-a)\Gamma(b+p)}{(p-1)!\Gamma(b-a+1)}\frac{(f_1-b)_{m_1}\cdots(f_r-b)_{m_r}}{(f_1)_{m_1}\cdots(f_r)_{m_r}}\\
&\times {}_{r+2}F_{r+1}\left[\begin{matrix} -p+1, b, -f_1+b+1, \dots, -f_r+b+1 \\ b-a+1, -f_1+b+1-m_1, \dots, -f_r+b+1-m_r \end{matrix}; 1\right].
\end{aligned}
$$

and the following lemma

Lemma 3 *Given a polynomial* $q(x)$ *with* $\deg q < n$ *then*

$$
\sum_{l=0}^{n}(-1)^l\binom{n}{l}q(x+l) = 0.
$$

Proof Is a well known fact in the theory of finite differences that any polynomial $p(x)$ with $\deg p \leq n-2$ is such that

$$
\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}p(j) = 0.
$$

This can be shown by taken consecutive derivatives of the relation

$$
(x+1)^{n-1} = \sum_{j=0}^{n-1}\binom{n-1}{j}x^j,
$$

up to the $(n-2)$-th derivative, evaluating at $x=1$ and taking linear combinations to get

$$
\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}j^k = 0, \quad k \in \{0, 1, \dots, n-2\}.
$$

Obviously $p(x) = q(\alpha+x)$ is a polynomial with $\deg q \leq n-2$. □

Now, we can prove the orthogonality relations

Proposition 2 *The polynomials described at* (11) *satisfy the orthogonality relations*

$$
\int_0^1 \Big(A_{(n_1,n_2),1}(x)w_1(x) + A_{(n_1,n_2),2}(x)w_2(x)\Big)x^j \,\mathrm{d}\mu(x) = 0 \text{ if } j \in \{0, \dots, n_1+n_2-2\}
$$

$$\int_0^1 \left(A_{(n_1,n_2),1}(x)w_1(x) + A_{(n_1,n_2),2}(x)w_2(x)\right)x^{n_1+n_2-1}\mathrm{d}\,\mu(x) = 1$$

respect to the weight functions $w_i(x) = x^{\alpha_i}$, $i = 1, 2$ *and measure* $\mathrm{d}\,\mu(x) = (1-x)^\gamma \mathrm{d}\,x$.

Proof Let's start finding a convenient expression for

$$I^j_{(n_1,n_2),i} \equiv \int_0^1 A_{(n_1,n_2),i}(x)w_i(x)x^j \mathrm{d}\,\mu(x)$$

Replacing the polynomials and the weight functions, we have

$$\begin{aligned} I^j_{(n_1,n_2),i} &= \sum_{l=0}^{n_i-1} C^l_{(n_1,n_2),i} \int_0^1 x^{\alpha_i+l+j}(1-x)^\gamma \mathrm{d}\,x \\ &= \sum_{l=0}^{n_i-1} C^l_{(n_1,n_2),i} \frac{\Gamma(\alpha_i+l+j+1)\Gamma(\gamma+1)}{\Gamma(\alpha_i+\gamma+l+j+2)} \end{aligned}$$

Replacing the coefficients $C^l_{(n_1,n_2),i}$ and simplifying, we find

$$\begin{aligned} I^j_{(n_1,n_2),i} &= (-1)^{n_1+n_2-1}\frac{(\alpha_i+\gamma+n_1+n_2)_{n_i}(\hat{\alpha}_i+\gamma+n_1+n_2)_{\hat{n}_i}}{(\gamma+1)_{n_1+n_2-1}} \\ &\quad\times \frac{(\alpha_i+1)_j(\alpha_i+\gamma+j+2)_{n_1+n_2-2-j}}{(n_i-1)!(\hat{\alpha}_i-\alpha_i)_{\hat{n}_i}} \\ &\quad\times {}_4F_3\left[\begin{matrix} -n_i+1, \alpha_i-\hat{\alpha}_i-\hat{n}_i+1, \alpha_i+1+j, \alpha_i+\gamma+n_1+n_2 \\ \alpha_i-\hat{\alpha}_i+1, \alpha_i+1, \alpha_i+\gamma+j+2 \end{matrix}; 1\right]. \end{aligned}$$

Now if $j < n_1 + n_2 - 1$ we can apply Theorem 7 over the previous ${}_4F_3$ function to get

$$\begin{aligned} &{}_4F_3\left[\begin{matrix} -n_i+1, \alpha_i-\hat{\alpha}_i-\hat{n}_i+1, \alpha_i+1+j, \alpha_i+\gamma+n_1+n_2 \\ \alpha_i-\hat{\alpha}_i+1, \alpha_i+1, \alpha_i+\gamma+j+2 \end{matrix}; 1\right] \\ &= \frac{(n_i-1)!}{(\hat{n}_i-1)!}\frac{\Gamma(\alpha_i-\hat{\alpha}_i+1)}{\Gamma(\alpha_i-\hat{\alpha}_i-\hat{n}_i+1+n_i)}\frac{(\hat{\alpha}_i+\hat{n}_i)_j(\hat{\alpha}_i+\gamma+\hat{n}_i+j+1)_{n_1+n_2-2-j}}{(\alpha_i+1)_j(\alpha_i+\gamma+j+2)_{n_1+n_2-2-j}} \\ &\quad\times {}_4F_3\left[\begin{matrix} -\hat{n}_i+1, \alpha_i-\hat{\alpha}_i-\hat{n}_i+1, -\hat{\alpha}_i-\hat{n}_i+1, -\hat{\alpha}_i-\gamma-\hat{n}_i-j \\ \alpha_i-\hat{\alpha}_i-\hat{n}_i+1+n_i, -\hat{\alpha}_i-\hat{n}_i+1-j, -\hat{\alpha}_i-\gamma-\hat{n}_i-n_1-n_2+2 \end{matrix}; 1\right]. \end{aligned}$$

Finally we can apply the following formula

$$\begin{aligned} &{}_{p+1}F_q\left[\begin{matrix} -n, a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; 1\right] \\ &\quad= (-1)^n\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\,{}_{p+1}F_q\left[\begin{matrix} -n, -b_1-n+1, \ldots, -b_q-n+1 \\ -a_1-n+1, \ldots, -a_p-n+1 \end{matrix}; 1\right] \end{aligned}$$

to get that

$$\begin{aligned} &{}_4F_3\left[\begin{matrix} -n_i+1, \alpha_i-\hat{\alpha}_i-\hat{n}_i+1, \alpha_i+1+j, \alpha_i+\gamma+n_1+n_2 \\ \alpha_i-\hat{\alpha}_i+1, \alpha_i+1, \alpha_i+\gamma+j+2 \end{matrix}; 1\right] \\ &= \frac{(n_i-1)!}{(\hat{n}_i-1)!}\frac{\Gamma(\alpha_i-\hat{\alpha}_i+1)}{\Gamma(\alpha_i-\hat{\alpha}_i-\hat{n}_i+1+n_i)}\frac{(\hat{\alpha}_i+\hat{n}_i)_j(\hat{\alpha}_i+\gamma+\hat{n}_i+j+1)_{n_1+n_2-2-j}}{(\alpha_i+1)_j(\alpha_i+\gamma+j+2)_{n_1+n_2-2-j}} \\ &\quad\times (-1)^{\hat{n}_i-1}\frac{(\alpha_i-\hat{\alpha}_i-\hat{n}_i+1)_{\hat{n}_i-1}}{(\alpha_i-\hat{\alpha}_i-\hat{n}_i+1+n_i)_{\hat{n}_i-1}}\frac{(-\hat{\alpha}_i-\hat{n}_i+1)_{\hat{n}_i-1}}{(-\hat{\alpha}_i-\hat{n}_i+1-j)_{\hat{n}_i-1}} \end{aligned}$$

$$\times \frac{(-\hat{\alpha}_i - \gamma - \hat{n}_i - j)_{\hat{n}_i - 1}}{(-\hat{\alpha}_i - \gamma - \hat{n}_i - n_1 - n_2 + 2)_{\hat{n}_i - 1}}$$
$$\times\, {}_4F_3\left[\begin{matrix} -\hat{n}_i + 1, \hat{\alpha}_i - \alpha_i - n_i + 1, \hat{\alpha}_i + j + 1, \hat{\alpha}_i + \gamma + n_1 + n_2 \\ \hat{\alpha}_i - \alpha_i + 1, \hat{\alpha}_i + 1, \hat{\alpha}_i + \gamma + j + 2 \end{matrix}; 1\right].$$

So, taking $i = 1$, we can replace the previous expression in (4.2) and simplifying we get for $j = 0, \ldots, n_1 + n_2 - 2$

$$I^j_{(n_1,n_2),1} = -I^j_{(n_1,n_2),2}.$$

If $j = n_1 + n_2 - 1$ we can rewrite the discrete integral as

$$\begin{aligned} I^{n_1+n_2-1}_{(n_1,n_2),i} &= (-1)^{n_1+n_2-1} \frac{(\alpha_1 + \gamma + n_1 + n_2)_{n_1} (\alpha_2 + \gamma + n_1 + n_2)_{n_2}}{(\gamma + 1)_{n_1+n_2-1}} \\ &\quad \times \frac{1}{(n_i - 1)!} \sum_{l=0}^{n_i - 1} (-1)^{\hat{n}_i + l} \binom{n_i - 1}{l} \\ &\quad \times \frac{(\alpha_i + 1 + l)_{n_1+n_2-1}}{(\alpha_i + \gamma + n_1 + n_2 + l)(\alpha_i - \hat{\alpha}_i - \hat{n}_i + l + 1)_{\hat{n}_i}}. \end{aligned} \tag{12}$$

Let's remind now that if $f(z)$ is a polynomial of degree m then we can decompose

$$\begin{aligned} \frac{f(z)}{(z - b)(z - a)_n} &= q(z) + \frac{f(b)}{(z - b)(b - a)_n} \\ &\quad + \frac{1}{(n - 1)!} \sum_{p=0}^{n-1} (-1)^p \binom{n - 1}{p} \frac{f(a - p)}{(a - p - b)(z - a + p)} \end{aligned}$$

with $q(z)$ a polynomial of degree $m - n - 1$ if $m \geq n + 1$ and $q(z) = 0$ if $m < n + 1$.

We can apply this to the fraction within the sum of (12) to get that

$$\begin{aligned} &\frac{(\alpha_i + 1 + l)_{n_1+n_2-1}}{(\alpha_i + \gamma + n_1 + n_2 + l)(\alpha_i - \hat{\alpha}_i - \hat{n}_i + l + 1)_{\hat{n}_i}} \\ &\quad = q(\alpha_i + l) + \frac{(-\gamma - n_1 - n_2 + 1)_{n_1+n_2-1}}{(-\gamma - n_1 - n_2 - \hat{\alpha}_i - \hat{n}_i + 1)_{\hat{n}_i}} \frac{1}{(\alpha_i + l + \gamma + n_1 + n_2)} \\ &\quad + \frac{1}{(\hat{n}_i - 1)!} \sum_{p=0}^{\hat{n}_i - 1} (-1)^p \binom{\hat{n}_i - 1}{p} \frac{1}{(\alpha_i + l - \hat{\alpha}_i - \hat{n}_i + 1 + p)} \\ &\quad \times \frac{(\hat{\alpha}_i + \hat{n}_i - p)_{n_1+n_2-1}}{(\hat{\alpha}_i + \gamma + n_1 + n_2 + \hat{n}_i - 1 - p)} \end{aligned} \tag{13}$$

where q is a polynomial with $\deg q = n_i - 2$. Replacing (13) in (12) for $i = 1$ we find that

$$\begin{aligned} I^{n_1+n_2-1}_{(n_1,n_2),1} &= (-1)^{n_1+n_2-1} \frac{(\alpha_1 + \gamma + n_1 + n_2)_{n_1} (\alpha_2 + \gamma + n_1 + n_2)_{n_2}}{(\gamma + 1)_{n_1+n_2-1}} \\ &\quad \times \frac{1}{(n_1 - 1)!} \sum_{l=0}^{n_1 - 1} (-1)^{n_2 + l} \binom{n_1 - 1}{l} \\ &\quad \times \left(q(\alpha_1 + l) + \frac{(-\gamma - n_1 - n_2 + 1)_{n_1+n_2-1}}{(-\gamma - n_1 - n_2 - \alpha_2 - n_2 + 1)_{n_2}} \frac{1}{(\alpha_1 + l + \gamma + n_1 + n_2)} \right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(n_2-1)!}\sum_{p=0}^{n_2-1}(-1)^p\binom{n_2-1}{p}\frac{1}{(\alpha_1+l-\alpha_2-n_2+1+p)}\\
&\times\frac{(\alpha_2+n_2-p)_{n_1+n_2-1}}{(\alpha_2+\gamma+n_1+n_2+n_2-1-p)}\Bigg)\\
=&(-1)^{n_1+n_2-1}\frac{(\alpha_1+\gamma+n_1+n_2)_{n_1}(\alpha_2+\gamma+n_1+n_2)_{n_2}}{(\gamma+1)_{n_1+n_2-1}}\frac{(-1)^{n_2}}{(n_1-1)!}\\
&\times\overbrace{\sum_{l=0}^{n_1-1}(-1)^l\binom{n_1-1}{l}q(\alpha_1+l)}^{0\text{ by lemma }3}\\
&+\overbrace{\frac{(-1)^{n_1+n_2-1}(-\gamma-n_1-n_2+1)_{n_1+n_2-1}}{(\gamma+1)_{n_1+n_2-1}}}^{1}\overbrace{\frac{(-1)^{n_2}(\alpha_2+\gamma+n_1+n_2)_{n_2}}{(-\gamma-n_1-n_2-\alpha_2-n_2+1)_{n_2}}}^{1}\\
&\times(\alpha_1+\gamma+n_1+n_2)_{n_1}\underbrace{\frac{1}{(n_1-1)!}\sum_{l=0}^{n_1-1}(-1)^l\binom{n_1-1}{l}\frac{1}{(\alpha_1+l+\gamma+n_1+n_2)}}_{\frac{1}{(\alpha_1+\gamma+n_1+n_2)_{n_1}}}\\
&-(-1)^{n_1+n_2-1}\frac{(\alpha_1+\gamma+n_1+n_2)_{n_1}(\alpha_2+\gamma+n_1+n_2)_{n_2}}{(\gamma+1)_{n_1+n_2-1}}\\
&\times\frac{1}{(n_2-1)!}\sum_{p=0}^{n_2-1}(-1)^{n_2-1-p}\binom{n_2-1}{p}\frac{(\alpha_2+n_2-1-p+1)_{n_1+n_2-1}}{(\alpha_2+\gamma+n_1+n_2+n_2-1-p)}\\
&\times\underbrace{\frac{1}{(n_1-1)!}\sum_{l=0}^{n_1-1}(-1)^l\binom{n_1-1}{l}\frac{1}{(\alpha_1+l-\alpha_2-n_2+1+p)}}_{\frac{1}{(\alpha_1-\alpha_2-n_2+1+p)_{n_1}}=\frac{(-1)^{n_1}}{(\alpha_2-\alpha_1-n_1+1+n_2-1-p)_{n_1}}}.
\end{aligned}
$$

Finally, doing the index change $p\to n_2-1-p$ we find that

$$
\begin{aligned}
I^{n_1+n_2-1}_{(n_1,n_2),1}=&1-(-1)^{n_1+n_2-1}\frac{(\alpha_1+\gamma+n_1+n_2)_{n_1}(\alpha_2+\gamma+n_1+n_2)_{n_2}}{(\gamma+1)_{n_1+n_2-1}}\\
&\times\frac{1}{(n_2-1)!}\sum_{p=0}^{n_2-1}(-1)^{n_1+p}\binom{n_2-1}{p}\\
&\times\frac{(\alpha_2+p+1)_{n_1+n_2-1}}{(\alpha_2+\gamma+n_1+n_2+p)}\frac{1}{(\alpha_2-\alpha_1-n_1+1+p)_{n_1}}=1-I^{n_1+n_2-1}_{(n_1,n_2),2}
\end{aligned}
$$

which ends the proof. □

Remark 2 To the best of our knowledge, the initial instance of the explicit presentation of the Jacobi–Piñeiro multiple orthogonal polynomials of type I can be traced back to the prepublication [8]. Subsequently, in [4], we provided an alternative simplified derivation for the Hahn multiple orthogonal polynomials of type I. Through utilization of the multiple Askey scheme, we derived the corresponding expressions for its subsequent forms, encompassing the noteworthy Jacobi–Piñeiro case.

The proof presented here is an adaptation of the aforementioned proof, tailored to the context of the Jacobi–Piñeiro multiple orthogonal polynomials of type I. For the original proof, we direct the reader to [8].

4.3 The recursion matrix

The coefficients of the recursion matrix

$$T = \begin{bmatrix} b_{0,0}^{\alpha_1,\alpha_2,\gamma} & 1 & 0 & 0 & 0 & \cdots \\ c_{1,0}^{\alpha_1,\alpha_2,\gamma} & b_{1,0}^{\alpha_1,\alpha_2,\gamma} & 1 & 0 & 0 & \ddots \\ d_{1,1}^{\alpha_1,\alpha_2,\gamma} & c_{1,1}^{\alpha_1,\alpha_2,\gamma} & b_{1,1}^{\alpha_1,\alpha_2,\gamma} & 1 & 0 & \ddots \\ 0 & d_{2,1}^{\alpha_1,\alpha_2,\gamma} & c_{2,1}^{\alpha_1,\alpha_2,\gamma} & b_{2,1}^{\alpha_1,\alpha_2,\gamma} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{14}$$

were determined in [39]. Inspired by [2] we obtain, for $n = 0, 1, \ldots,$

$$\begin{aligned} b_{n,n}^{\alpha_1-1,\alpha_2-1,\gamma-1} &= \frac{B_{n,n}}{\tilde{B}_{n,n}}, & b_{n+1,n}^{\alpha_1-1,\alpha_2-1,\gamma-1} &= \frac{B_{n+1,n}}{\tilde{B}_{n+1,n}}, \\ c_{n+1,n+1}^{\alpha_1-1,\alpha_2-1,\gamma-1} &= \frac{C_{n+1,n+1}}{\tilde{C}_{n+1,n+1}}, & c_{n+1,n}^{\alpha_1-1,\alpha_2-1,\gamma-1} &= \frac{C_{n+1,n}}{\tilde{C}_{n+1,n}}, \\ d_{n+1,n+1}^{\alpha_1-1,\alpha_2-1,\gamma-1} &= \frac{D_{n+1,n+1}}{\tilde{D}_{n+1,n+1}}, & d_{n+2,n+1}^{\alpha_1-1,\alpha_2-1,\gamma-1} &= \frac{D_{n+2,n+1}}{\tilde{D}_{n+2,n+1}}, \end{aligned}$$

with

$$\begin{aligned} B_{n,n} &= \frac{(\alpha_1+n)(\alpha_2+\gamma+2n-1)(\alpha_1+\gamma+2n-1)}{(\alpha_1+\gamma+3n)(\alpha_2+\gamma+3n-1)} \\ &\quad + \frac{n(\gamma+2n-1)(\alpha_1+\gamma+2n-1)}{(\alpha_2+\gamma+3n-2)(\alpha_2+\gamma+3n-1)} \\ &\quad + \frac{n(\gamma+2n-1)(\alpha_2+\gamma+2n-2)}{(\alpha_1+\gamma+3n-2)(\alpha_2+\gamma+3n-2)}, \\ \tilde{B}_{n,n} &= \alpha_1+\gamma+3n-1, \\ B_{n+1,n} &= (\alpha_2^2+(\gamma+3n-1)\alpha_2+2n(\gamma+2n))\alpha_1^2 \\ &\quad + (2\gamma+5n)(\alpha_2^2+(\gamma+3n-1)\alpha_2+2n(\gamma+2n))\alpha_1 \\ &\quad + (\gamma+2n)(18n^3+(14\alpha_2+15\gamma+5)n^2+(2\alpha_2+\gamma+2)(2\alpha_2+3\gamma-1)n \\ &\quad + (\alpha_2+1)(\gamma+1)(\alpha_2+\gamma-1)), \\ \tilde{B}_{n+1,n} &= (\alpha_1+\gamma+3n)(\alpha_1+\gamma+3n+1)(\alpha_2+\gamma+3n-1)(\alpha_2+\gamma+3n+1), \\ C_{n+1,n+1} &= n(\gamma+2n+1)(\alpha_1+\gamma+2n)(\alpha_2+\gamma+2n)\Big((\alpha_2+n)(\alpha_2+\gamma+2n) \\ &\quad + \frac{(\alpha_1-\alpha_2+n+1)(\gamma+2n)(\alpha_2+\gamma+3n+1)}{\alpha_1+\gamma+3n} \\ &\quad + \frac{(\alpha_2+n)(\alpha_1+\gamma+2n+1)(\alpha_1+\gamma+3n+1)}{\alpha_2+\gamma+3n+2}\Big), \\ \tilde{C}_{n+1,n+1} &= (\alpha_1+\gamma+3n+1)^2(\alpha_1+\gamma+3n+2)(\alpha_2+\gamma+3n)(\alpha_2+\gamma+3n+1)^2, \\ C_{n+1,n} &= (\gamma+2n)(-1+\alpha_1+\gamma+2n)(-1+\alpha_2+\gamma+2n)\Big(\alpha_1^3 n(-1+\alpha_2+n) \end{aligned}$$

$$
\begin{aligned}
&+\alpha_1^2 n(-1+\alpha_2+n)(-1+3\gamma+8n)\\
&+\alpha_1\Big(\alpha_2^3(1+n)+\gamma^3(1+n)+\alpha_2^2(1+n)(-3+3\gamma+8n)\\
&+3\gamma^2(-1+n+4n^2)+\gamma(2+n(-13-9n+42n^2))\\
&+\alpha_2\big(2+\gamma^2(3+6n)+\gamma(-6+9n+33n^2)+n(-15+n+44n^2)\big)\\
&+n\big(6+n(-13+n(-26+45n))\big)\Big)\\
&+n\Big(\alpha_2^3(1+n)+\alpha_2^2(1+n)(-3+3\gamma+8n)\\
&+(-1+\gamma+3n)(\gamma+3n)(-2+\gamma+3\gamma n+6n^2)\\
&+\alpha_2\big(2+\gamma^3+3\gamma^2(1+4n)+\gamma(-7+9n+42n^2)+n(-17+n(2+45n))\big)\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\tilde{C}_{n+1,n} &= (-1+\alpha_1+\gamma+3n)(\alpha_1+\gamma+3n)^2(1+\alpha_1+\gamma+3n)(-2+\alpha_2+\gamma+3n)\\
&\quad\times(-1+\alpha_2+\gamma+3n)^2(\alpha_2+\gamma+3n),\\
D_{n+1,n+1} &= (n+1)(\alpha_1+n)(\alpha_1-\alpha_2+n+1)(\gamma+2n)(\gamma+2n+1)(\alpha_1+\gamma+2n-1)\\
&\quad\times(\alpha_1+\gamma+2n)(\alpha_2+\gamma+2n-1)(\alpha_2+\gamma+2n)\\
\tilde{D}_{n+1,n+1} &= (\alpha_1+\gamma+3n-1)(\alpha_1+\gamma+3n)^2(\alpha_1+\gamma+3n+1)^2(\alpha_1+\gamma+3n+2)\\
&\quad\times(\alpha_2+\gamma+3n-1)(\alpha_2+\gamma+3n)(\alpha_2+\gamma+3n+1),\\
D_{n+2,n+1} &= n(\alpha_2+n)(-\alpha_1+\alpha_2+n+1)(\gamma+2n+1)(\gamma+2n+2)(\alpha_1+\gamma+2n)\\
&\quad\times(\alpha_1+\gamma+2n+1)(\alpha_2+\gamma+2n)(\alpha_2+\gamma+2n+1),\\
\tilde{D}_{n+2,n+1} &= (\alpha_1+\gamma+3n+1)(\alpha_1+\gamma+3n+2)(\alpha_1+\gamma+3n+3)\\
&\quad\times(\alpha_2+\gamma+3n)(\alpha_2+\gamma+3n+1)^2\\
&\quad\times(\alpha_2+\gamma+3n+2)^2(\alpha_2+\gamma+3n+3).
\end{aligned}
$$

Lemma 4 *Let us consider the coefficients of the recursion matrix T given in* (14) *for the Jacobi–Piñeiro multiple orthogonal polynomials given in* (9)*. Then*:

- (i) *The coefficient $b_{n,n}^{\alpha_1-1,\alpha_2-1,\gamma-1}$ is positive for $n\in\mathbb{N}_0$ and $\alpha_1,\alpha_2,\gamma>0$*
- (ii) *The coefficient $c_{n,n}^{\alpha_1-1,\alpha_2-1,\gamma-1}$ is positive for $n\in\mathbb{N}$, $\alpha_1,\alpha_2,\gamma>0$ and $\alpha_1-\alpha_2+1>0$.*
- (iii) *The coefficient $c_{n+1,n}^{\alpha_1-1,\alpha_2-1,\gamma-1}$ is positive for $n\in\mathbb{N}_0$ and $\alpha_1,\alpha_2,\gamma>0$.*
- (iv) *The coefficient $d_{n,n}^{\alpha_1-1,\alpha_2-1,\gamma-1}$ is positive for $n\in\mathbb{N}$, $\alpha_1,\alpha_2,\gamma>0$ and $\alpha_1-\alpha_2+1>0$.*
- (v) *The coefficient $d_{n+1,n}^{\alpha_1-1,\alpha_2-1,\gamma-1}$ is positive for $n\in\mathbb{N}$, $\alpha_1,\alpha_2,\gamma>0$ and $-\alpha_1+\alpha_2+1>0$.*

Proof (i) For $n\in\mathbb{N}$ we immediately see that the denominator $\tilde{B}_{n,n}$ is positive. The numerator $B_{n,n}$ for $n\in\mathbb{N}$ is the sum of three positive rational expressions in $\alpha_1,\alpha_2,\gamma>0$. For $n=0$ we have $b_{0,0}^{\alpha_1-1,\alpha_2-1,\gamma-1}=\frac{\alpha_1}{\alpha_1+\gamma}$.

(ii) For $n\in\mathbb{N}$ the denominator $\tilde{B}_{n+1,n}$ is positive. The numerator $B_{n+1,n}$ requires more analysis. We need to check the positivity of

$$
\begin{aligned}
T &= 18n^3+(14\alpha_2+15\gamma+5)n^2+(2\alpha_2+\gamma+2)(2\alpha_2+3\gamma-1)n\\
&\quad+(\alpha_2+1)(\gamma+1)(\alpha_2+\gamma-1),
\end{aligned}
$$

that ensures the positivity of $B_{n,n+1}$. To understand that this positivity is not obvious we write T as follows

$$
T=18n^3+(14\alpha_2+15\gamma+5)n^2+(2\alpha_2+\gamma+2)(2\alpha_2+3\gamma)n
$$

$$+ (\alpha_2+1)(\gamma+1)(\alpha_2+\gamma) - (2\alpha_2+\gamma+2)n - (\alpha_2+1)(\gamma+1).$$

But further manipulation do show that is a positive term. Indeed,

$$\begin{aligned}
T &= 18n^3 + (12\alpha_2+14\gamma+3)n^2 + (2\alpha_2+\gamma+2)(2\alpha_2+3\gamma)n \\
&\quad + (\alpha_2+1)(\gamma+1)(\alpha_2+\gamma) + (2\alpha_2+\gamma+2)n(n-1) - (\alpha_2+1)(\gamma+1) \\
&= 18n^3 + (12\alpha_2+14\gamma+3)n^2 + \big((\alpha_2+\gamma+1)(2\alpha_2+3\gamma) + 2(\alpha_2+1)(\alpha_2+\gamma)\big)n \\
&\quad + (\alpha_2+1)(\gamma+1)(\alpha_2+\gamma) + (2\alpha_2+\gamma+2)n(n-1) + (\alpha_2+1)\gamma(n-1) - (\alpha_2+1) \\
&= 18n^3 + (11\alpha_2+14\gamma+2)n^2 + \big((\alpha_2+\gamma+1)(2\alpha_2+3\gamma) + 2(\alpha_2+1)(\alpha_2+\gamma)\big)n \\
&\quad + (\alpha_2+1)(\gamma+1)(\alpha_2+\gamma) \\
&\quad + (2\alpha_2+\gamma+2)n(n-1) + (\alpha_2+1)\gamma(n-1) + (\alpha_2+1)(n^2-1),
\end{aligned}$$

and we see that T is a positive number whenever $n \in \mathbb{N}$ and $\alpha_1, \alpha_2, \gamma > 0$, and so is $b_{n+1,n}^{\alpha_1-1,\alpha_2-1,\gamma-1}$. Now, for $n=0$ we find

$$b_{1,0}^{\alpha_1-1,\alpha_2-1,\gamma-1} = \frac{\alpha_2\alpha_1^2 + 2\alpha_2\gamma\alpha_1 + (\alpha_2+1)\gamma(\gamma+1)}{(\alpha_1+\gamma)(\alpha_1+\gamma+1)(\alpha_2+\gamma+1)},$$

which is again positive.

(iii) For $n \in \mathbb{N}$, the positivity of the denominator $\tilde{C}_{n,n}$ for $\alpha_1, \alpha_2, \gamma > 0$ is obvious by inspection. For the numerator $C_{n,n}$, the positivity is ensured whenever $\alpha_1 - \alpha_2 + 1 > 0$.

(iv) For $n \in \mathbb{N}$, the positivity of the denominator $\tilde{C}_{n,n}$ for $\alpha_1, \alpha_2, \gamma > 0$ is obvious. The numerator $C_{n+1,n}$ after inspection of its long expression, is also seen to be positive after checking the positivity of all its summands for $n = 1, 2, \ldots$. Moreover, for $n=0$ we have

$$c_{1,0}^{\alpha_1-1,\alpha_2-1,\gamma-1} = \frac{\alpha_1\gamma}{(\alpha_1+\gamma)^2(\alpha_1+\gamma+1)}.$$

Therefore, the positivity holds in this case, as well.

(v) For $n \in \{2, 3, \ldots\}$, the positivity of the denominator $\tilde{D}_{n,n}$ for $\alpha_1, \alpha_2, \gamma > 0$ is immediate. For the numerator $D_{n,n}$, the positivity is ensured whenever $\alpha_1 - \alpha_2 + 1 > 0$. For $n = 1$ we have

$$d_{1,1}^{\alpha_1-1,\alpha_2-1,\gamma-1} = \frac{\alpha_1(\alpha_1-\alpha_2+1)\gamma(\gamma+1)}{(\alpha_1+\gamma)(\alpha_1+\gamma+1)^2(\alpha_1+\gamma+2)(\alpha_2+\gamma+1)}$$

and the result is proven.

(vi) For $n \in \mathbb{N}$, the denominator $\tilde{D}_{n+1,n}$ for $\alpha_1, \alpha_2, \gamma > 0$ is positive. The numerator $D_{n+1,n}$ is positive whenever $-\alpha_1 + \alpha_2 + 1 > 0$.

Which completes the proof. □

Theorem 8 *For the Jacobi–Piñeiro multiple orthogonal polynomials as in* (9) *the corresponding recursion matrix given in* (14) *is a nonnegative matrix whenever* $\alpha_1, \alpha_2, \gamma > -1$ *and* $|\alpha_1 - \alpha_2| < 1$.

Proof From Lemma 4 (recall the shift of the parameters $\alpha_1 \to \alpha_1 - 1$, $\alpha_2 \to \alpha_2 - 1$, $\gamma \to \gamma - 1$) we see that when $\alpha_1, \alpha_2, \gamma > -1$:

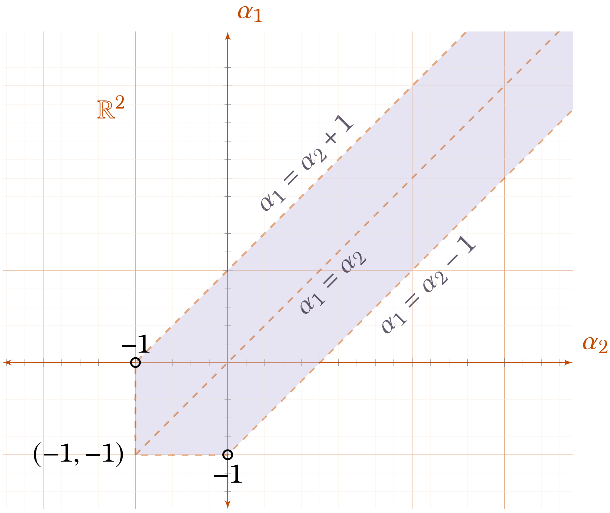
(i) The coefficients $b_{n,n}^{\alpha_1,\alpha_2,\gamma}$ and $c_{n+1,n}^{\alpha_1,\alpha_2,\gamma}$ are positive.

(ii) The coefficients $c_{n,n}^{\alpha_1,\alpha_2,\gamma}$ and $d_{n,n}^{\alpha_1,\alpha_2,\gamma}$ are positive for $\alpha_1 - \alpha_2 + 1 > 0$.

(iii) The coefficient $d_{n+1,n}^{\alpha_1,\alpha_2,\gamma}$ is positive if $-\alpha_1 + \alpha_2 + 1 > 0$.

Hence, for $x = \alpha_1 - \alpha_2$, we need to fulfill the couple of inequalities $x+1 > 0$ and $-x+1 > 0$, that is $x \in (-1, 1)$. □

Next we present a picture illustrating the possible values of the couple of parameters (α_1, α_2), the filled region that represents the possible values is an infinite band.



Parameter region for a non negative Jacobi–Piñeiro's recursion matrix

The dashed lines are excluded of the allowed region for the parameters α_1 and α_2, as those lines correspond to resonances, i.e., the difference $\alpha_1 - \alpha_2 = \pm 1, 0$, over those semi-lines.

It is easy to see that, cf. for example [14]:

$$\begin{aligned} &\lim_{n\to\infty} b_{n,n} = \lim_{n\to\infty} b_{n+1,n} = 3\kappa, \quad \lim_{n\to\infty} c_{n,n} = \lim_{n\to\infty} c_{n+1,n} = 3\kappa^2, \\ &\lim_{n\to\infty} d_{n,n} = \lim_{n\to\infty} d_{n+1,n} = \kappa^3, \end{aligned} \tag{15}$$

with $\kappa = \frac{4}{27}$.

4.4 Type II Jacobi–Piñeiro's

Now, we explain how to turn stochastic the recursion matrix for the type II given in (14). The zeros of the Jacobi–Piñeiro polynomials, being an AT-system, are in $(0, 1)$. Moreover, their density distribution of zeros fills that open interval and accumulate at the boundaries [31].

Theorem 9 *Let us assume for the Jacobi–Piñeiro system that* $\alpha_1, \alpha_2, \gamma > -1$, $\alpha_1 \neq \alpha_2$ *and* $|\alpha_1 - \alpha_2| < 1$. *Then, the semi-infinite matrix*

$$P_{II} = \begin{bmatrix} P_{II,0,0} & P_{II,0,1} & 0 & 0 & 0 & \cdots \\ P_{II,1,0} & P_{II,1,1} & P_{II,1,2} & 0 & 0 & \ddots \\ P_{II,2,0} & P_{II,2,1} & P_{II,2,2} & P_{II,2,3} & 0 & \ddots \\ 0 & P_{II,3,1} & P_{II,3,2} & P_{II,3,3} & P_{II,3,4} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

with coefficients given in terms of the coefficients of the recursion matrix (14) *and the multiple orthogonal polynomials of type II evaluated at* $x = 1$, $B_{\vec{\nu}}(1)$, $\vec{\nu} = (n+1, n)$ *or* (n, n), *for* $n = 0, 1, \dots,$ *as follows*

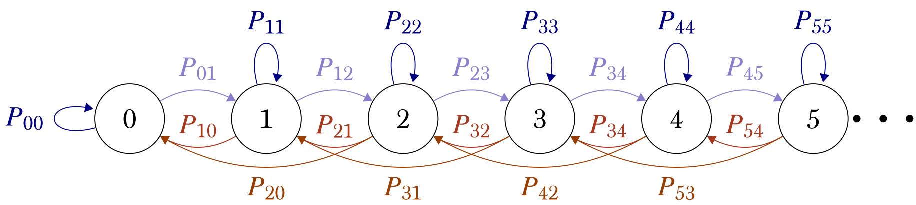
$$
\begin{aligned}
P_{II,2n,2n+1} &= \frac{B_{(n+1,n)}(1)}{B_{(n,n)}(1)}, \quad P_{II,2n+1,2n+2} = \frac{B_{(n+1,n+1)}(1)}{B_{(n+1,n)}(1)},\\
P_{II,2n,2n} &= b_{n,n}, \quad P_{II,2n+1,2n+1} = b_{n+1,n},\\
P_{II,2n+2,2n+1} &= \frac{B_{(n+1,n)}(1)}{B_{(n+1,n+1)}(1)} c_{n+1,n+1}, \quad P_{II,2n+1,2n} = \frac{B_{(n,n)}(1)}{B_{(n+1,n)}(1)} c_{n+1,n},\\
P_{II,2n+2,2n} &= \frac{B_{(n,n)}(1)}{B_{(n+1,n+1)}(1)} d_{n+1,n+1}, \quad P_{II,2n+3,2n+1} = \frac{B_{(n+1,n)}(1)}{B_{(n+2,n+1)}(1)} d_{n+2,n+1}.
\end{aligned}
$$

is a multiple stochastic matrix of type II. Here the b, c *and* d *coefficients are those in the recursion matrix* (14). *The corresponding Markov chain is irreducible.*

Proof From the explicit expression (10) we know that $B^{(n)}(1)$ is a strictly positive number, from the AT property for the system $\{x^{\alpha_1}, x^{\alpha_2}\}$. Hence, using Theorem 1 we can normalize at $x = 1$ to get the stochastic recursion matrix using the factors $\sigma_{II,n} = \frac{1}{B^{(n)}(1)}$.

The irreducibility follows from the fact that all the elements in the band of the Markov matrix are positive. □

The diagram for this Markov chain is



Type II Jacobi–Piñeiro's Markov chain diagram

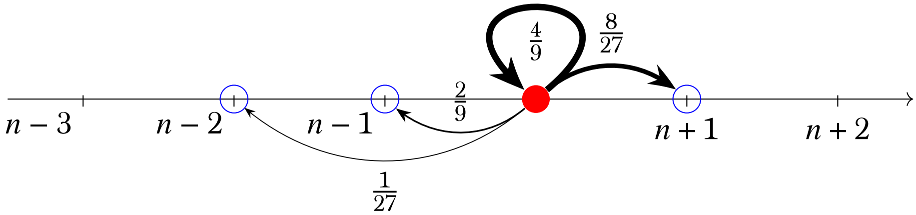
Corollary 3 *The explicit expressions for the type II multiple stochastic matrix coefficients are*

$$
\begin{aligned}
P_{II,2n,2n+1} &= \frac{(\gamma+2n+1)(\alpha_1+\gamma+2n+1)(\alpha_2+\gamma+2n+1)}{(\alpha_1+\gamma+3n+1)(\alpha_1+\gamma+3n+2)(\alpha_2+\gamma+3n+1)},\\
P_{II,2n+1,2n+2} &= \frac{(\gamma+2n+2)(\alpha_1+\gamma+2n+2)(\alpha_2+\gamma+2n+2)}{(\alpha_1+\gamma+3n+3)(\alpha_2+\gamma+3n+2)(\alpha_2+\gamma+3n+3)},\\
P_{II,2n,2n} &= \frac{(\alpha_1+n+1)(\alpha_1+1+\gamma+2n)(\alpha_2+\gamma+2n+1)}{(\alpha_1+\gamma+3n+1)(\alpha_1+\gamma+3n+2)(\alpha_2+\gamma+3n+1)}\\
&\quad + \frac{n(\gamma+2n)(\alpha_1+\gamma+2n+1)}{(\alpha_1+\gamma+3n+1)(\alpha_2+\gamma+3n)(\alpha_2+\gamma+3n+1)}\\
&\quad + \frac{n(\gamma+2n)(\alpha_2+\gamma+2n)}{(\alpha_1+\gamma+3n)(\alpha_1+\gamma+3n+1)(\alpha_2+\gamma+3n)},\\
P_{II,2n+1,2n+1} &= -\frac{n(\alpha_2+n)(-\alpha_1+\alpha_2+n)}{(\alpha_2-\alpha_1)(\alpha_2+\gamma+3n)}\\
&\quad + \frac{n(\alpha_2+n)(-\alpha_1+\alpha_2+n)(\alpha_1+\gamma+3n+1)}{(\alpha_2-\alpha_1)(\alpha_1-\alpha_2+1)(\alpha_2+\gamma+3n)(\alpha_2+\gamma+3n+1)}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{(n+1)(\alpha_1+n+1)(\alpha_1-\alpha_2+n+1)(\alpha_2+\gamma+3n+2)}{(\alpha_2-\alpha_1)(\alpha_1-\alpha_2+1)(\alpha_1+\gamma+3n+2)(\alpha_1+\gamma+3n+3)} \\
&+ \frac{(n+1)(\alpha_2+n+1)(-\alpha_1+\alpha_2+n+1)}{(\alpha_2-\alpha_1)(\alpha_2+\gamma+3n+3)},
\end{aligned}
$$

$$
\begin{aligned}
P_{II,2n,2n-1} &= \frac{n(\alpha_1-\alpha_2+n)(\gamma+2n-1)}{(\alpha_1+\gamma+3n-1)(\alpha_1+\gamma+3n)(\alpha_1+\gamma+3n+1)} \\
&+ \frac{n(\alpha_2+n)(\alpha_2+\gamma+2n)}{(\alpha_1+\gamma+3n)(\alpha_1+\gamma+3n+1)(\alpha_2+\gamma+3n)} \\
&+ \frac{n(\alpha_2+n)(\alpha_1+\gamma+2n+1)}{(\alpha_1+\gamma+3n+1)(\alpha_2+\gamma+3n)(\alpha_2+\gamma+3n+1)}, \\
P_{II,2n+1,2n} &= \frac{n(\alpha_1-\alpha_2-n)(\alpha_2+n)(\alpha_1+\gamma+3n+1)}{(\alpha_1-\alpha_2+1)(\alpha_2+\gamma+3n)(\alpha_2+\gamma+3n+1)(\alpha_2+\gamma+3n+2)} \\
&+ \frac{(n+1)(\alpha_1-\alpha_2+n+1)(\alpha_1+n+1)}{(\alpha_1-\alpha_2+1)(\alpha_1+\gamma+3n+2)(\alpha_1+\gamma+3n+3)} \\
P_{II,2n,2n-2} &= \frac{n(\alpha_1-\alpha_2+n)(\alpha_1+n)}{(\alpha_1+\gamma+3n-1)(\alpha_1+\gamma+3n)(\alpha_1+\gamma+3n+1)}, \\
P_{II,2n+1,2n-1} &= \frac{n(-\alpha_1+\alpha_2+n)(\alpha_2+n)}{(\alpha_2+\gamma+3n)(\alpha_2+\gamma+3n+1)(\alpha_2+\gamma+3n+2)}.
\end{aligned}
$$

The corresponding transition diagram for large n is



Asymptotic type II Jacobi–Piñeiro's Markov chain diagram

4.5 Type I Jacobi–Piñeiro's

In the next theorem we show how to turn stochastic the recursion matrix for the type I given in (14).

Theorem 10 *Let us assume for the Jacobi–Piñeiro system that* $\alpha_1, \alpha_2, \gamma > -1$, $\alpha_1 \neq \alpha_2$ *and* $|\alpha_1 - \alpha_2| < 1$. *Then, the semi-infinite matrix*

$$
P_I = \begin{bmatrix}
P_{I,0,0} & P_{I,0,1} & P_{I,0,2} & 0 & 0 & \cdots \\
P_{I,1,0} & P_{I,1,1} & P_{I,1,2} & P_{I,1,3} & 0 & \ddots \\
0 & P_{I,2,1} & P_{I,2,2} & P_{I,2,3} & P_{I,2,4} & \ddots \\
0 & 0 & P_{I,3,2} & P_{I,3,3} & P_{I,3,4} & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
$$

with coefficients expressed in terms of the coefficients of the recursion matrix (14) *and the linear forms of type I evaluated at* $x = 1$, $Q_{\vec{\nu}}(1)$ *with* $\vec{\nu} = (n+1, n)$ *or* (n, n), *for* $n = 0, 1, \ldots,$ *as follows*

$$
\begin{aligned}
P_{I,2n,2n+2} &= \frac{Q_{(n+2,n+1)}(1)}{Q_{(n+1,n)}(1)} d_{n+1,n+1}, \quad P_{2n+1,2n+3} = \frac{Q_{(n+2,n+2)}(1)}{Q_{(n+1,n+1)}(1)} d_{n+2,n+1},\\
P_{I,2n,2n+1} &= \frac{Q_{(n+1,n+1)}(1)}{Q_{(n+1,n)}(1)} c_{n+1,n}, \quad P_{I,2n+1,2n+2} = \frac{Q_{(n+2,n+1)}(1)}{Q_{(n+1,n+1)}(1)} c_{n+1,n+1},\\
P_{I,2n,2n} &= b_{n,n}, \quad P_{I,2n+1,2n+1} = b_{n+1,n},\\
P_{I,2n+2,2n+1} &= \frac{Q_{(n+1,n+1)}(1)}{Q_{(n+2,n+1)}(1)} \quad P_{I,2n+1,2n} = \frac{Q_{(n+1,n)}(1)}{Q_{(n+1,n+1)}(1)},
\end{aligned}
$$

is a multiple stochastic matrix of type I. The corresponding Markov chain is irreducible.

Proof According to [32] the system $\{x^{\alpha_1}, \dots, x^{\nu_1-1+\alpha_1}, x^{\alpha_2}, \dots, x^{\nu_2-1+\alpha_2}\}$ is a Chebyshev system in any closed interval of the positive semiaxis $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. Consequently, the linear form $Q_{\vec{\nu}}(x) = A_{\vec{\nu},1}(x)x^{\alpha_1} + A_{\vec{\nu},2}(x)x^{\alpha_2}$ has at most $|\vec{\nu}| - 1$ zeros in any closed interval $[a, b] \subset \mathbb{R}_+$. Thus, the maximum number of zeros in $\mathbb{R}_+$ will be $|\vec{\nu}|-1$. As $\{x^{\alpha_1}, x^{\alpha_2}\}$ conforms an AT-system on [0, 1] it has $|\vec{\nu}| - 1$ zeros in its interior, the open interval (0, 1), see [32, 38, 39]. Therefore, $Q_{\vec{\nu}}(x)$ has no zeros for $x \geq 1$.

Now, we analyze the behavior of the linear form $Q_{\vec{\nu}}(x)$ for $x \to +\infty$. For $\vec{\nu} = (n+1, n)$, $n \in \mathbb{N}$ we have for $x \to +\infty$

$$
\begin{aligned}
A_{(n+1,n),1}(x)x^{\alpha_1} &= C^n_{(n+1,n),1} x^{n+\alpha_1} + O(x^{n+\alpha_1-1}),\\
A_{(n+1,n),2}(x)x^{\alpha_2} &= C^n_{(n+1,n),2} x^{n+\alpha_2-1} + O(x^{n+\alpha_2-2}),
\end{aligned}
$$

where $C^n_{(n+1,n),1}$ and $C^n_{(n+1,n),2}$ are the conductor coefficients, accompanying the leading terms, of the polynomials $A_{(n+1,n),1}(x)$ and $A_{(n+1,n),2}(x)$, respectively. Observing that $n + \alpha_1 - (n + \alpha_2 - 1) = \alpha_1 - \alpha_2 + 1 > 0$, we see that for $x \to +\infty$ we have $Q_{(n+1,n)}(x) = C^n_{(n+1,n),1} x^{n+\alpha_1} + O(x^{n+\alpha_1-1})$. According to Theorem 6 we have

$$
C^n_{(n+1,n),1} \equiv \frac{\Gamma(\alpha_1 + \gamma + 3n + 2)}{\Gamma(\gamma + 2n + 1)\Gamma(\alpha_1 + n + 1)} \frac{(\alpha_2 + \gamma + 2n + 1)_n(\alpha_1 + \gamma + 2n + 1)_n}{n!(\alpha_1 - \alpha_2 + 1)_n} > 0
$$

and, consequently, for $n \in \mathbb{N}$, we find $\lim_{x\to+\infty} Q_{(n+1,n)}(x) = +\infty$. For $n = 0$, we have $Q_{(1,0)} = \frac{\Gamma(\alpha_1+\gamma+2)}{\Gamma(\gamma+1)\Gamma(\alpha_1+1)}$, that is always positive. Hence $Q_{(n+1,n)}(x) > 0$ for $x \geq 1$.

For $\vec{\nu} = (n, n)$ we have for $x \to +\infty$

$$
\begin{aligned}
A_{(n,n),1}(x)x^{\alpha_1} &= C^{n-1}_{(n,n),1} x^{n-1+\alpha_1} + O(x^{n-2+\alpha_1}),\\
A_{(n,n),2}(x)x^{\alpha_2} &= C^{n-1}_{(n,n),1} x^{n-1+\alpha_2} + O(x^{n-2+\alpha_2}).
\end{aligned}
$$

Hence, the dominant behavior at $+\infty$ of the linear form $Q_{(n,n)}(x)$ depends on whether $\alpha_1 \lessgtr \alpha_2$. Let us assume, in the first place, that $\alpha_1 > \alpha_2$. Then, for $x \to +\infty$, we find $Q_{(n,n)}(x) = C^{n-1}_{(n,n),1} x^{n+\alpha_1-1} + o(x^{n+\alpha_1-1})$. According to Theorem 6 we have

$$
C^{n-1}_{(n,n),1} = \frac{\Gamma(3n - 1 + \alpha_1 + \gamma)}{\Gamma(n + \alpha_1)\Gamma(2n + \gamma)} \frac{(2n + \alpha_1 + \gamma)_n(2n + \alpha_2 + \gamma)_n}{(n - 1)!(\alpha_1 - \alpha_2)_n},
$$

that is positive for $\alpha_1 > \alpha_2$. Thus, for $n \in \{1, 2, 3, \dots\}$ we have $\lim_{x\to+\infty} Q_{(n,n)}(x) = +\infty$. Finally, when $\alpha_2 > \alpha_1$, for $x \to +\infty$ we have

$$
Q_{(n,n)}(x) = C^{n-1}_{(n,n),2} x^{n+\alpha_2-1} + o(x^{n+\alpha_2-1}).
$$

Recalling that $A^{\alpha_1,\alpha_2}_{(n,n),2} = A^{\alpha_2,\alpha_1}_{(n,n),1}$ we use the previous result interchanging α_1 and α_2 to get for $n \in \{1, 2, 3, \dots\}$ that $\lim_{x\to+\infty} Q_{(n,n)}(x) = +\infty$, for $\alpha_2 > \alpha_1$. Therefore, $Q_{(n,n)}(x) > 0$ for $x \geq 1$.

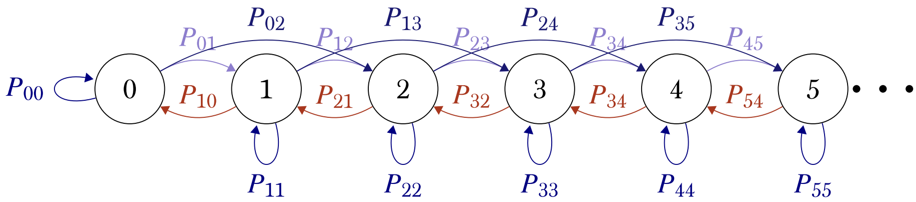
We conclude that

$$\sigma_{I,l} = \frac{1}{Q^{(l)}(1)} > 0,$$

so that $P_I = \sigma_I T^\top \sigma_I^{-1}$ is a multiple stochastic matrix of type I, i.e., $P_I \mathbf{1} = \mathbf{1}$.

Finally, the irreducibility follows from the fact that all the elements in the band of the Markov matrix are positive. □

The diagram for this Markov chain is



Type I Jacobi–Piñeiro's Markov chain diagram

Here we require of the Poincaré theory for the ratio asymptotics of homogeneous linear recurrences [34].

Proposition 3 (Large n limit for the dual Jacobi–Piñeiro Markov chains) *The large n limit of the Jacobi–Piñeiro stochastic matrices of type I and II are the same after transposition, i.e.,*

$$\lim_{n\to\infty} P_{I,n,n+k} = \lim_{n\to\infty} P_{II,n+k,n}, \quad k \in \{-2, -1, 0, 1\}.$$

Proof According to (5) we need to show that

$$\frac{B^{(n-k)}(1)Q^{(n-k)}(1)}{B^{(n)}(1)Q^{(n)}(1)} \xrightarrow[n\to\infty]{} 1, \quad k = 2, 1, -1. \tag{16}$$

From (10) we directly deduce that

$$\frac{B^{(2n+1)}(1)}{B^{(2n)}(1)} = \frac{(\gamma+1+2n)(\alpha_1+\gamma+2n+1)(\alpha_2+\gamma+2n+1)}{(\alpha_1+\gamma+3n+1)(\alpha_2+\gamma+3n+1)(\alpha_2+\gamma+3n+2)} \xrightarrow[n\to\infty]{} \frac{8}{27},$$
$$\frac{B^{(2n+2)}(1)}{B^{(2n+1)}(1)} = \frac{(\gamma+2+2n)(\alpha_2+\gamma+2n+2)(\alpha_1+\gamma+2n+2)}{(\alpha_2+\gamma+3n+3)(\alpha_1+\gamma+3n+2)(\alpha_1+\gamma+3n+3)} \xrightarrow[n\to\infty]{} \frac{8}{27}.$$

Now, from (4) we get

$$\begin{aligned} Q^{(n-1)}(1)B^{(n)}(1) &= Q^{(n)}(1)(T_{n,n-2}B^{(n-2)}(1) + T_{n,n-1}B^{(n-1)}(1)) \\ &\quad + Q^{(n+1)}(1)T_{n+1,n-1}B^{(n-1)}(1), \end{aligned}$$

so that we have for the linear forms of type I the following lower degree homogeneous linear recurrence

$$-Q^{(n-1)}(1) + a_n Q^{(n)}(1) + b_n Q^{(n+1)}(1) = 0,$$

with

$$a_n = T_{n,n-2}\frac{B^{(n-2)}(1)}{B^{(n)}(1)} + T_{n,n-1}\frac{B^{(n-1)}(1)}{B^{(n)}(1)} \xrightarrow[n\to\infty]{} \frac{4^3}{27^3}\frac{27^2}{8^2} + 3\frac{4^2}{27^2}\frac{27}{8} = \frac{7}{27},$$

$$b_n = T_{n+1,n-1}\frac{B^{(n-1)}(1)}{B^{(n)}(1)} \xrightarrow[n\to\infty]{} \frac{4^3}{27^3}\frac{27}{8} = \frac{8}{729},$$

where we have used the previous result, $\lim_{n\to\infty}\frac{B^{(n+1)}(1)}{B^{(n)}(1)} = \frac{8}{27}$, and (15). The characteristic polynomial is

$$-1 + \frac{7}{27}r + \frac{8}{729}r^2 = \frac{8}{729}(r+27)\Big(r - \frac{27}{8}\Big).$$

Therefore, from Poincaré's theorem, having its characteristic roots $\{-27, \frac{27}{8}\}$ distinct absolute value, as the linear forms of type I are positive at 1, we get $\lim_{n\to\infty}\frac{Q^{(n+1)}(1)}{Q^{(n)}(1)} = \frac{27}{8}$ and (16) is satisfied. □

Proposition 4 (Recurrent and transient Jacobi–Piñeiro) *Both dual Jacobi–Piñeiro Markov chains are recurrent whenever* $-1 < \gamma < 0$ *and transient for* $\gamma \geq 0$.

Proof Is a direct consequence of the irreducibility of the Jacobi–Piñeiro Markov chains and Theorem 5 as the divergence of the integral coincides with the divergence of $\int_a^1 (1-x)^{\gamma-1}\mathrm{d}\,x$, for $0 < a < 1$, that happens for $\gamma < 0$. □

4.6 Two type II examples: recurrent and transient Markov chains

For $\alpha_1 = -\frac{1}{4}, \alpha_2 = \gamma = -\frac{1}{2}$, that gives a recurrent Markov chain, we get the following transition matrix coefficients

$$\begin{aligned}
P_{II,2n,2n+1} &= \frac{4(4n+1)(8n+1)}{3(144n^2+72n+5)}, \quad P_{II,2n+1,2n+2} = \frac{(2n+1)(8n+5)}{6(9n^2+9n+2)}, \qquad n \geq 0,\\
P_{II,2n,2n} &= \frac{96n^2+16n-17}{6\left(36n^2+3n-5\right)}, \quad P_{II,2n+1,2n+1} = \frac{32n^2+32n+7}{72n^2+78n+20}, \qquad n \geq 1,\\
P_{II,2n,2n-1} &= \frac{576n^3-552n^2+94n+7}{6(432n^3-360n^2+51n+7)},\\
P_{II,2n+1,2n} &= \frac{192n^3+104n^2-22n-11}{8(108n^3+45n^2-12n-5)}, \qquad n \geq 1,\\
P_{II,2n,2n-2} &= \frac{4n(4n+1)}{3(12n-7)(12n+1)}, \quad P_{II,2n+1,2n-1} = \frac{8n^2-6n+1}{24(9n^2-1)}, \qquad n \geq 1,
\end{aligned}$$

with $P_{II,0,0} = \frac{3}{5}$ and $P_{II,1,0} = \frac{2}{5}$. Hence, the corresponding transition matrix looks as follows

$$P_{II} = \begin{bmatrix}
\frac{3}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & 0 & \cdots\\
\frac{4}{15} & \frac{19}{60} & \frac{5}{12} & 0 & 0 & 0 & 0 & \ddots\\
\frac{4}{39} & \frac{25}{156} & \frac{95}{204} & \frac{60}{221} & 0 & 0 & 0 & \ddots\\
0 & \frac{1}{64} & \frac{263}{1088} & \frac{71}{170} & \frac{13}{40} & 0 & 0 & \ddots\\
0 & 0 & \frac{24}{425} & \frac{173}{850} & \frac{133}{290} & \frac{204}{725} & 0 & \ddots\\
0 & 0 & 0 & \frac{1}{40} & \frac{271}{1160} & \frac{199}{464} & \frac{5}{16} & \ddots\\
\vdots & \ddots & \ddots & \ddots & & & & \ddots
\end{bmatrix}$$

For $\alpha_1 = -\frac{1}{4}, \alpha_2 = -\frac{1}{2}, \gamma = \frac{1}{2}$, that gives a transient Markov chain, we get the following transition matrix coefficients

$$
\begin{aligned}
P_{II,2n,2n+1} &= \frac{2(2n+1)(8n+5)}{3(36n^2+27n+5)}, \quad P_{II,2n+1,2n+2} = \frac{(4n+5)(8n+9)}{3(36n^2+63n+26)}, \qquad n \geq 0,\\
P_{II,2n,2n} &= \frac{32n^2+16n+1}{72n^2+30n+2}, \quad P_{II,2n+1,2n+1} = \frac{96n^2+128n+25}{6(36n^2+51n+13)}, \qquad n \geq 1,\\
P_{II,2n,2n-1} &= \frac{576n^3+120n^2-74n-5}{6(432n^3+360n^2+87n+5)},\\
P_{II,2n+1,2n} &= \frac{192n^3+328n^2+146n+17}{8(3n+1)(3n+2)(12n+13)}, \qquad n \geq 1,\\
P_{II,2n,2n-2} &= \frac{4n(4n+1)}{3\left(144n^2+72n+5\right)}, \quad P_{II,2n+1,2n-1} = \frac{8n^2-6n+1}{24(9n^2+9n+2)}, \qquad n \geq 1,
\end{aligned}
$$

and $P_{II,0,0} = \frac{1}{3}$ and $P_{II,1,0} = \frac{2}{3}$. The corresponding transition matrix is

$$
P_{II} = \begin{bmatrix}
\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & \cdots\\
\frac{4}{39} & \frac{25}{78} & \frac{15}{26} & 0 & 0 & 0 & 0 & \ddots\\
\frac{20}{663} & \frac{617}{5304} & \frac{49}{104} & \frac{13}{34} & 0 & 0 & 0 & \ddots\\
0 & \frac{1}{160} & \frac{683}{4000} & \frac{83}{200} & \frac{51}{125} & 0 & 0 & \ddots\\
0 & 0 & \frac{24}{725} & \frac{47}{290} & \frac{23}{50} & \frac{10}{29} & 0 & \ddots\\
0 & 0 & 0 & \frac{1}{64} & \frac{451}{2368} & \frac{95}{222} & \frac{325}{888} & \ddots\\
\vdots & \ddots & \ddots & \ddots & & & \ddots & \ddots
\end{bmatrix}.
$$

Inspection of both transition matrices, we see that the probabilities to go to the left are bigger in the recurrent situation, with $\gamma = -\frac{1}{2}$, than in the transient case example with $\gamma = \frac{1}{2}$.

4.7 Two type I examples: recurrent and transient Markov chains

For $\alpha_1 = -\frac{1}{4}, \alpha_2 = \gamma = -\frac{1}{2}$, that gives a recurrent Markov chain, we get the following approximate transition matrix in decimal form with a precision of four significant digits (now is not possible to find closed rational expressions and several sums involving the Euler's Gamma function are required)

$$
P_I \approx \begin{bmatrix}
0.6000 & 0.2531 & 0.1469 & 0 & 0 & 0 & 0 & 0 & \cdots\\
0.4215 & 0.3167 & 0.2419 & 0.0199 & 0 & 0 & 0 & 0 & \ddots\\
0 & 0.2760 & 0.4657 & 0.2036 & 0.0547 & 0 & 0 & 0 & \ddots\\
0 & 0 & 0.3223 & 0.4176 & 0.2341 & 0.0260 & 0 & 0 & \ddots\\
0 & 0 & 0 & 0.2826 & 0.4586 & 0.2110 & 0.0478 & 0 & \ddots\\
0 & 0 & 0 & 0 & 0.3115 & 0.4289 & 0.4289 & 0.0291 & \ddots\\
\vdots & \ddots & \ddots & \ddots & \ddots & & & \ddots & \ddots
\end{bmatrix}.
$$

For $\alpha_1 = -\frac{1}{4}, \alpha_2 = -\frac{1}{2}, \gamma = \frac{1}{2}$, that gives a transient Markov chain, we get the following approximate expression for the transition matrix

$$P_I \approx \begin{bmatrix} 0.3333 & 0.3198 & 0.3469 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0.2138 & 0.3205 & 0.4289 & 0.0368 & 0 & 0 & 0 & 0 & \ddots \\ 0 & 0.1565 & 0.4711 & 0.2726 & 0.0998 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0.2395 & 0.4150 & 0.3061 & 0.0394 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0.2160 & 0,4600 & 0.2542 & 0.0697 & 0 & \ddots \\ 0 & 0 & 0 & 0 & 0.2583 & 0.4279 & 0.2746 & 0.0391 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & & \ddots \end{bmatrix}.$$

Conclusions and outlook

Considering higher-order recurrence relationships for type II multiple orthogonal polynomials, we focus on a non-negative recursion matrix, along with its counterpart: linear forms of type I. In this context, we reveal a clear strategy to create a pair of dual stochastic matrices.

Our exploration goes beyond the usual as we expand the Karlin–McGregor representation formula to include both of these connected dual Markov chains. This expanded approach helps us to understand the details of generating functions and first-passage distributions. As we go deeper into the complexities of the Markov chains, we get a better picture of their nature.

To make our findings more solid, we focus on the Jacobi–Piñeiro multiple orthogonal polynomials as a good example. This in-depth look at a practical case allows us to better understand the ideas we've introduced. We can determine the exact ranges of parameters where the recursion matrix stays positive, setting the limits of its usefulness.

It's worth noting that the ideas we've talked about have been used to create finite Markov chains in [6]. In fact, that work really explored many examples in the finite setting, finding out lots of conclusions about randomness. It also been looked at breaking down the transition matrices into smaller parts that are easy to understand, as it is done in [7]. This could be important when we want to study these Markov chains in certain situations, like in [20] for the Jacobi–Piñeiro case. Also, in [5], it is done similar developments with hypergeometric multiple orthogonal polynomials [30].

Following ideas from [25], in [10], we also expanded our view to include Hessenberg recurrence matrices with a positive bidiagonal factorization. We understood the structure of these Markov chains, even when they're not as simple as birth and death. We took this even further in [11], looking at banded bounded matrices, and showing how they can be broken down too. But, we still don't know how to do a similar construction with more complex situations involving mixed multiple orthogonal polynomials.

Lastly, by looking at real examples of mixed multiple orthogonal polynomials, using generalized hypergeometric series and theirs extensions, we can make specific types of Markov chains that break free from the Hessenberg restriction. This is part of a bigger effort to create these kinds of math expressions and use them in new ways. Stepping into the domain of higher-order recurrence relations for multiple orthogonal polynomials of type II, the spotlight falls on a non-negative recursion matrix, accompanied by its counterpart—-the linear forms of type I. Within this framework, we lay bare a comprehensive strategy that shapes the creation of a distinct dual pair of stochastic matrices.

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