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MATHEMATICAL ENGINEERING AND ARTIFICIAL INTELLIGENCE

Final Degree Project

The generalized Markoff equation

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I hereby declare, under my own responsibility, that the Project presented with the title

The generalized Markoff equation

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Project Summary

Abstract

This project explores the generalized Markoff equation $x^2+y^2+z^2 = 3xyz+m$, focusing on integer solutions whose components belong to the Fibonacci and k-Fibonacci sequences. A symbolic computation engine was implemented to generate and classify these solutions for different values of m, distinguishing between minimal and non-minimal triples. The work reveals new structural properties of these Diophantine solutions and extends existing classifications through symbolic and theoretical analysis.

Keywords: *Markoff triples, generalized Markoff equation, Fibonacci solutions, generalized Fibonacci solutions.*

Introduction

This Final Degree Project addresses the study of the *Generalized Markoff Equation*, a natural extension of the classical Markoff equation, which has a long tradition in number theory. The original equation, introduced by A. A. Markoff [(M1), (M2)], is expressed as

$$x^2 + y^2 + z^2 = 3xyz\,,$$

and its positive integer solutions—the *Markoff triples*—are organized in a tree structure that has been the subject of the famous *Markoff Conjecture*.

The generalized version adds a parameter $m \in \mathbb{Z}$, resulting in the equation

$$x^2 + y^2 + z^2 = 3xyz + m.$$

The solutions to this equation form the so-called *m-triples*, whose structure is organized in forests of trees that vary depending on the value of m. This work focuses on the structural analysis of these solutions and the identification of patterns through a combination of analytic and algebraic number theory techniques, along with symbolic computation tools.

Project Definition

This Final Degree Project consists of an in-depth study of the generalized Markoff equation and its integer solutions when restricted to structured numerical sequences. The work is divided into two main lines of research:

- **Study and classification of m-Markoff triples whose components are Fibonacci numbers**: Based on the concept of *minimality* introduced in [(LS)], the developed symbolic engine allowed exhaustive enumeration and classification of minimal and non-minimal triples for various values of m, identifying structural patterns and relationships between different m-triples, composed entirely of numbers from the Fibonacci sequence, referred to as *Markoff-Fibonacci m-triples*. The outcome of this work led to the writing of the article *A classification of Markoff-Fibonacci m-triples* [(ACMRS1)], accepted for publication in the journal *The Fibonacci Quarterly*.

- **Extension to k-Fibonacci sequences**: The analysis was extended to triples whose components belong to k-Fibonacci sequences, which generalize Fibonacci numbers by adding a factor k. The case $k = 1$ corresponds to the original Fibonacci sequence, and this generalization also includes other known cases such as Pell numbers, which are obtained with $k = 2$. Specific identities were derived, symbolic techniques were used to classify the triples, and detailed case analysis was carried out. This led to the article *Markoff m-triples with k-Fibonacci components* [(ACMRS2)], published in the journal *Mediterranean Journal of Mathematics*.

Additionally, this project included:

- The implementation of symbolic and numerical tools in Python to generate and explore Markoff trees for fixed values of m.
- The design of preliminary datasets and the planning of AI-based strategies to distinguish solution structures, although the implementation and training of the AI models remain pending.

Methodology

The methodology integrates tools from number theory, symbolic computation, and computational exploration:

- Theoretical techniques from analytic and algebraic number theory were applied to bound and simplify the solution space.
- Symbolic engines were implemented in Python to generate and classify both minimal and non-minimal m-Markoff triples.
- Dedicated algorithms explored the structure of the solution trees, allowing systematic identification of all roots and their derived branches for fixed values of m.
- The generation of datasets for AI modeling was planned and executed; however, training and evaluation of learning models were postponed due to time constraints.

Results

The work carried out in this project has produced relevant theoretical and computational results on the generalized Markoff equation. The main results are:

- **Development of a symbolic computation engine**: A symbolic engine was implemented in Python to generate and analyze m-Markoff triples for given values of m. This tool enabled classification of the solutions as minimal or non-minimal, and allowed automated derivation of descendants within a Markoff tree.
- **Classification of m-Markoff triples with Fibonacci components**: Combining theoretical analysis and symbolic computation, the project successfully identified and categorized all m-Markoff triples composed entirely of Fibonacci numbers within practical computational bounds. This included detection of minimal roots and their recursive derivations.
- **Extension to k-Fibonacci components**: The methodology was extended to handle triples constructed from k-Fibonacci numbers, which generalize classical

Fibonacci and Pell sequences. Specific cases were analyzed and classified using symbolic methods and simplifications based on identities.

- **Preparation of datasets for AI models**: The groundwork was laid for applying AI techniques through the construction of labeled datasets from the symbolic engine's output. However, due to time constraints, training and validation of these models were not carried out and are proposed as future work.

All results were obtained through original implementation and experimentation, and the developed symbolic tools serve both as validation mechanisms and exploratory instruments for future research.

Conclusions

This project contributes both concrete classifications and theoretical insights to the study of the generalized Markoff equation. The integration of Fibonacci-type sequences into the framework of m-Markoff solutions enabled the discovery of new algebraic patterns, expanding the understanding of how these solutions behave under recursive constraints.

Although the AI part remains as future work, the symbolic and number theory advances establish a solid foundation for further research. The work opens avenues for applying algebraic invariants, AI-based classification, and new proofs of conjectures in the field of Diophantine equations.

References

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Resumen del proyecto

Resumen

Este proyecto explora la ecuación generalizada de Markoff $x^2 + y^2 + z^2 = 3xyz + m$, centrándose en soluciones enteras cuyos componentes pertenecen a las sucesiones de Fibonacci y k-Fibonacci. Se implementó un motor de computación simbólica para generar y clasificar estas soluciones para distintos valores de m, diferenciando entre triples minimales y no minimales. El trabajo revela nuevas propiedades estructurales de estas soluciones diofánticas y amplía las clasificaciones existentes mediante análisis simbólico y teórico.

Palabras clave: *Triples de Markoff, ecuación generalizada de Markoff, soluciones de Fibonacci, soluciones de Fibonacci generalizadas.*

Introducción

Este Trabajo de Fin de Grado aborda el estudio de la *Ecuación Generalizada de Markoff*, una extensión natural de la ecuación clásica de Markoff, que tiene una larga tradición en la teoría de números. La ecuación original, introducida por A. A. Markoff [(M1), (M2)], se expresa como

$$x^2 + y^2 + z^2 = 3xyz\,,$$

y sus soluciones enteras positivas —los *triples de Markoff*— se organizan en una estructura en árbol que ha sido objeto de la famosa *Conjetura de Markoff*.

La versión generalizada añade un parámetro $m \in \mathbb{Z}$, resultando en la ecuación

$$x^2 + y^2 + z^2 = 3xyz + m.$$

Las soluciones de esta ecuación forman los llamados *m-triples*, cuya estructura se organiza en bosques de árboles que varían dependiendo del valor de m. Este trabajo se centra en el análisis estructural de estas soluciones y en la identificación de patrones mediante una combinación de técnicas de teoría analítica y algebraica de números, junto con herramientas de computación simbólica.

Definición del proyecto

Este Trabajo de Fin de Grado consiste en un estudio en profundidad de la ecuación generalizada de Markoff y sus soluciones enteras cuando se restringen a secuencias numéricas estructuradas. El trabajo se divide en dos líneas principales de investigación:

- **Estudio y clasificación de triples m-Markoff cuyos componentes son números de Fibonacci**: Basado en el concepto de *minimalidad* introducido en [(LS)], el motor simbólico desarrollado permitió la enumeración y clasificación exhaustiva de triples minimales y no minimales para varios valores de m, identificando patrones estructurales y relaciones entre diferentes m-triples, formados completamente por números de la secuencia de Fibonacci, a los cuales se ha denominado como *Markoff-Fibonacci m-triples*. El resultado de este trabajo dio lugar a la redacción del artículo *A classification of Markoff-Fibonacci m-triples* [(ACMRS1)], aceptado para su publicación en la revista *The Fibonacci Quarterly*.

- **Extensión a secuencias k-Fibonacci**: El análisis se amplió a triples cuyos componentes pertenecen a secuencias k-Fibonacci, que generalizan los números de Fibonacci añadiendo un factor k. El caso $k = 1$ se corresponde con la secuencia de Fibonacci original, y esta generalización también engloba otros casos conocidos como los números de Pell, los cuales se consiguen con $k = 2$. Se derivaron identidades específicas, se utilizaron técnicas simbólicas para clasificar los triples y se llevó a cabo un análisis detallado de casos. Esto dio lugar al artículo *Markoff m-triples with k-Fibonacci components* [(ACMRS2)], publicado en la revista *Mediterranean Journal of Mathematics*.

Además, este proyecto incluyó:

- La implementación de herramientas simbólicas y numéricas en Python para generar y explorar árboles de Markoff para valores fijos de m.
- El diseño de conjuntos de datos preliminares y la planificación de estrategias basadas en IA para distinguir estructuras de soluciones, aunque la implementación y el entrenamiento de los modelos de IA quedan pendientes.

Metodología

La metodología integra herramientas de teoría de números, computación simbólica y exploración computacional:

- Se aplicaron técnicas teóricas de teoría analítica y algebraica de números para acotar y simplificar el espacio de soluciones.
- Se implementaron motores simbólicos en Python para generar y clasificar triples m-Markoff tanto minimales como no minimales.
- Algoritmos dedicados exploraron la estructura de los árboles de soluciones, permitiendo la identificación sistemática de todas las raíces y sus ramas derivadas para valores fijos de m.
- SSe planificó y ejecutó la generación de conjuntos de datos para el modelado con IA; sin embargo, el entrenamiento y la evaluación de los modelos de aprendizaje se pospusieron debido a limitaciones de tiempo.

Resultados

El trabajo realizado en este proyecto ha producido resultados teóricos y computacionales relevantes sobre la ecuación generalizada de Markoff. Los principales resultados son:

- **Desarrollo de un motor de computación simbólica**: Se implementó un motor simbólico en Python para generar y analizar triples m-Markoff para valores dados de m. Esta herramienta permitió clasificar las soluciones como minimales o no minimales, y posibilitó la derivación automatizada de descendientes dentro de un árbol de Markoff.
- **Clasificación de triples m-Markoff con componentes de Fibonacci**: Combinando análisis teórico y computación simbólica, el proyecto identificó y categorizó con éxito todos los triples m-Markoff compuestos enteramente por números de Fi-

bonacci dentro de límites computacionales prácticos. Esto incluyó la detección de raíces minimales y sus derivaciones recursivas.

- **Extensión a componentes k-Fibonacci**: La metodología se extendió para manejar triples construidos a partir de números k-Fibonacci, que generalizan las secuencias clásicas de Fibonacci y Pell. Se analizaron y clasificaron casos específicos utilizando métodos simbólicos y simplificaciones basadas en identidades.
- **Preparación de conjuntos de datos para modelos de IA**: Se sentaron las bases para aplicar técnicas de IA mediante la construcción de conjuntos de datos etiquetados a partir de la salida del motor simbólico. No obstante, debido a limitaciones de tiempo, el entrenamiento y la validación de estos modelos no se llevaron a cabo y se proponen como trabajo futuro.

Todos los resultados se obtuvieron mediante implementación y experimentación original, y las herramientas simbólicas desarrolladas sirven tanto como mecanismos de validación como instrumentos exploratorios para futuras investigaciones.

Conclusiones

Este proyecto aporta tanto clasificaciones concretas como ideas teóricas al estudio de la ecuación generalizada de Markoff. La integración de secuencias tipo Fibonacci en el marco de las soluciones m-Markoff permitió descubrir nuevos patrones algebraicos, ampliando la comprensión de cómo se comportan estas soluciones bajo restricciones recursivas.

Aunque la parte de IA queda como trabajo futuro, los avances simbólicos y en teoría de números establecen una base sólida para investigaciones posteriores. El trabajo abre caminos para aplicar invariantes algebraicos, clasificación con IA y nuevas pruebas de conjeturas en el ámbito de las ecuaciones diofánticas.

Referencias

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1 Introduction

The *Markoff equation* $x^2 + y^2 + z^2 = 3xyz$, has attracted substantial interest in number theory due to the interesting structure of its integer solutions, known as *Markoff triples*. These solutions can be organized into a tree-like structure, where each node represents a triple and is related to others via specific algebraic transformations. One of the most prominent open problems in this area is the *Markoff unicity conjecture*, which asserts that no two distinct ordered Markoff triples share the same maximal element.

A natural generalization of this equation introduces a constant term $m \in \mathbb{Z}$, yielding the *generalized Markoff equation*:

$$x^2 + y^2 + z^2 = 3xyz + m.$$

Its integer solutions are known as *Markoff m-triples*, and they inherit a tree structure similar to the classical case. However, the number of trees—and even their existence—depends on the value of m. For some values, there are multiple trees; for others, none. This richer behavior motivates a deeper exploration into the nature and classification of these solutions.

In this work, we focus on a specific class of structured solutions: those composed of elements from the *Fibonacci sequence* and its generalization, the *k-Fibonacci numbers*. These sequences possess a wealth of algebraic identities and recurrence properties that make them especially suitable for rigorous classification within the framework of generalized Markoff equations.

The main objective of this project is to classify all Markoff m-triples composed of Fibonacci or k-Fibonacci numbers. To achieve this, we combine classical number-theoretic techniques with computational tools.

These investigations have led to the development of two research papers: one containing the full classification of Markoff-Fibonacci m-triples, which has been accepted for publication in *The Fibonacci Quarterly*, and another that extends the classification to k-Fibonacci numbers, already published in the *Mediterranean Journal of Mathematics*. Additionally, this project is part of a broader research initiative exploring whether artificial intelligence can help detect hidden algebraic patterns among the solutions of generalized Markoff equations.

The structure of this work is as follows:

- Section 2 introduces the classical Markoff equation, its generalization, and the concept of minimality in m-triples.
- Section 3 develops the theoretical foundation for Fibonacci and k-Fibonacci numbers, including their identities and relevance to Markoff theory.
- Section 4 presents a complete classification of Markoff-Fibonacci m-triples, distinguishing between minimal and non-minimal cases.
- Section 5 extends the results of the previous section to Markoff m-triples formed with k-Fibonacci numbers.
- Section 7 concludes the project and outlines potential directions for future research.

2 Markoff triples and generalized Markoff triples

2.1 Markoff's equation

The Markoff's equation is a diophantine equation of the form

$$x^2 + y^2 + z^2 = 3xyz\,, \tag{2.1}$$

where x, y, and z are positive integers. The equation is named after the mathematician A. A. Markoff, who introduced it in his research [(M1), (M2)]. The equation has been studied in various branches of mathematics, including number theory, algebraic geometry, and combinatorial number theory.

The solutions of the Markoff's equation are known as Markoff triples. A Markoff triple is a solution (x, y, z) of the Markoff's equation, where x, y, and z are positive integers. The solutions are presented in order such that $x \leq y \leq z$. These solutions have the interesting property that they can be arranged in a tree structure, where each node represents a Markoff triple and the edges represent the relationships between them. The tree structure is known as the Markoff tree, and it is a useful tool for visualizing the relationships between different Markoff triples. In the following figure, the first few levels of the Markoff tree can be seen.

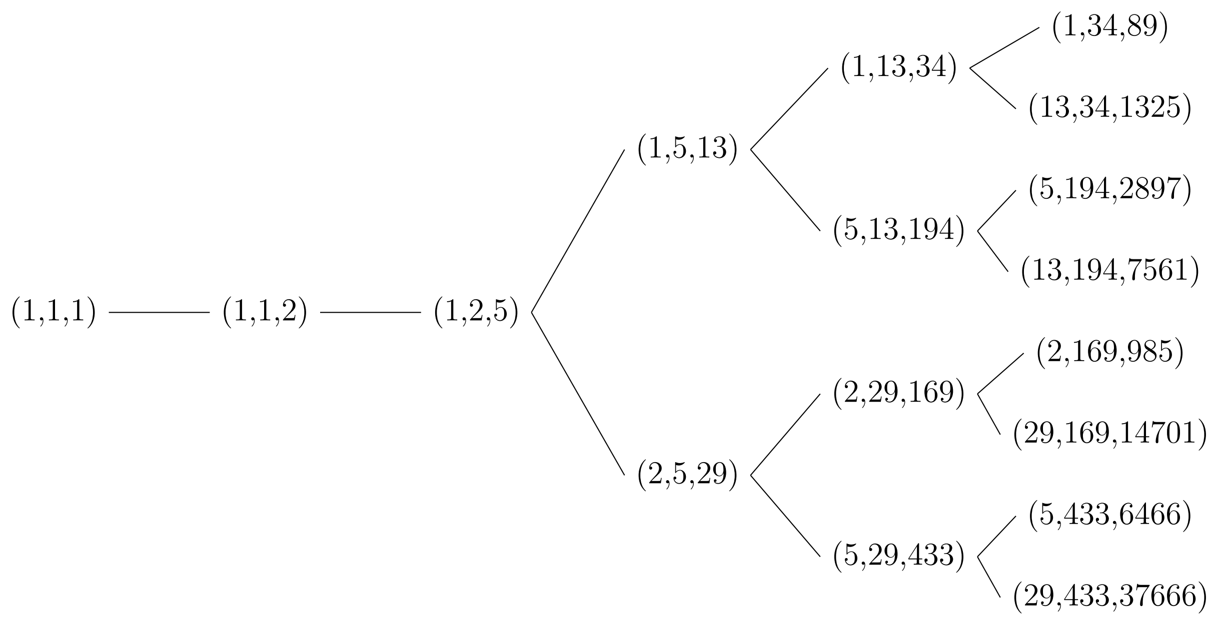

Figure 1: Beginning of the Markoff tree.

This tree structure is generated by the following recurrence relation. Given a Markoff triple (x, y, z), new Markoff triples can be generated by applying the following transformations:

$$\begin{aligned}
(x, y, z) &\to (x, y, 3xy - z)\,,\\
(x, y, z) &\to (x, 3xz - y, z)\,,\\
(x, y, z) &\to (3yz - x, y, z)\,.
\end{aligned}$$

To understand the origin of these transformations, it is essential to examine how they are deduced directly from the Markoff's equation. The Markoff's equation can be rewritten as

$$x^2 + (-3yz)x + (y^2 + z^2) = 0\,.$$

This is a quadratic equation in x. Therefore, it will have two solutions for x, which will be denoted as x_1 and x_2. These solutions relate to each other using one of Vieta's formulas as follows.

$$x_1 + x_2 = 3yz \Longrightarrow x_2 = 3yz - x_1\,.$$

Thus, given the solution (x_1, y, z), the triple $(x_2, y, z) = (3yz - x_1, y, z)$ is also a valid solution. The same reasoning can be applied to the other two variables due to the symmetry of the equation, leading to the other two transformations.

In the Markoff tree, the transformations for x and y result in two children nodes, while the transformation over z results in a single parent node.

2.2 Markoff's unicity conjecture

> *For a given Markov number z, there is exactly one normalized solution having z as its largest element.* [(F)]

This conjecture states that no number can appear more than once in the Markoff tree as the largest element of a Markoff triple. There have been several attempts to prove this conjecture, but it remains unproven. The conjecture is closely related to the properties of the Markoff tree and the relationships between different Markoff triples mentioned above.

2.3 Generalized Markoff's equation

In recent years, many authors have studied generalizations of this equation ([(Mor)], [(GS)]). In [(SC)], Markoff m-triples are introduced as positive integer solutions of the m-Markoff equation

$$x^2 + y^2 + z^2 = 3xyz + m\,, \tag{2.2}$$

where m is a positive integer. As well as with the Markoff's equation, by the symmetry of the equation, any permutation of a solution is also a solution, and hence the Markoff m-triples (x, y, z) are assumed to be ordered with $0 < x \leq y \leq z$. These triples also satisfy the transformations seen in Section 2.1, which leads to these triples being also organized into tree structures. However, in this case, multiple trees or none could exist for a specific value of m.

2.4 Minimality of Markoff triples

The authors in [(SC)] showed that the number of trees for every $m > 0$ is equal to the number of Markoff m-triples (x, y, z) that are minimal, that is to say, those at the root of the tree. These minimal triples must satisfy that their parent nodes do not satisfy the condition of having all positive elements, which translates into satisfying the inequality

$$3xy - z \leq 0\,. \tag{2.3}$$

3 Fibonacci and k-Fibonacci numbers

3.1 Fibonacci numbers

The Fibonacci numbers form a well-known mathematical sequence in which each term is the sum of the two preceding ones. There are several common notations for this sequence, but in this work $F(n)$ will be used to denote the n-th element of the sequence. The recurrence to determine $F(n)$ is defined as follows.

$$\begin{cases} F(0) = 0 \\ F(1) = 1 \\ F(n) = F(n-1) + F(n-2), \quad \forall n \geq 2\,, \end{cases} \tag{3.1}$$

so the start of the sequence is

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Alternatively, the Fibonacci numbers can also be expressed using the Binet's formula [(HW)]. Let

$$\varphi = \frac{1+\sqrt{5}}{2}, \qquad \bar{\varphi} = \frac{1-\sqrt{5}}{2}\,,$$

so that the i-th Fibonacci number can be written as

$$F(i) = \frac{\varphi^i - \bar{\varphi}^i}{\varphi - \bar{\varphi}} = \frac{\varphi^i - \bar{\varphi}^i}{\sqrt{5}}. \tag{3.2}$$

Along this work, both the recurrence and the Binet's formula will be used to proof different identities and properties of the Fibonacci numbers.

3.1.1 Fibonacci identities

The Fibonacci numbers have many properties and identities that can be useful when studying inequalities. Some of these are:

$$\text{(D'Ocagne identity)} \quad (-1)^a F(b-a) = F(b)F(a+1) - F(b+1)F(a)\,, \tag{3.3}$$
$$\text{(Simson identity)} \quad F(n)^2 = F(n+1)F(n-1) - (-1)^n\,, \tag{3.4}$$
$$\text{(Catalan identity)} \quad F(n)^2 = F(n+r)F(n-r) + (-1)^{n-r}F(r)^2\,, \tag{3.5}$$
$$\text{(Vajda identity)} \quad F(n+i)F(n+j) - F(n)F(n+i+j) = (-1)^n F(i)F(j)\,. \tag{3.6}$$

It is worth mentioning that the last identity is a generalization for all the other ones. Also, from this identity, if $n = 1$, $i = a$ and $j = b - 1$, it can be deduced that

$$F(a+b) = F(a+1)F(b) + F(a)F(b-1)\,. \tag{3.7}$$

This expression can also be infered by iteratively applying the Fibonacci recurrence relation (3.1) and taking into account $F(1) = F(2) = 1$.

$$\begin{aligned} F(a+b) &= F(2)F(a+b-1) + F(1)F(a+b-2) \\ &= [F(1)+F(2)]F(a+b-2) + F(2)F(a+b-3) \\ &= F(3)F(a+b-2) + F(2)F(a+b-3) \\ &= [F(2)+F(3)]F(a+b-3) + F(3)F(a+b-4) \\ &= F(4)F(a+b-3) + F(3)F(a+b-4) \\ &= \dots \\ &= F(a+1)F(b) + F(a)F(b-1)\,. \end{aligned}$$

3.1.2 Other Fibonacci properties

Other interesting properties are presented in this section. These properties will be relevant throughout the work to prove several results.

Lemma 3.1. *For each integer $n \geq 0$,*

$$\sum_{k=0}^{n} F(k)^2 = F(n)F(n+1)\,.$$

Proof. Induction will be used to prove the result. For $n = 0$, the identity is true because $F(0) = 0$. Assuming that the result holds for some n, it will be proven for $n+1$, this is,

$$\sum_{k=0}^{n+1} F(k)^2 = F(n+1)F(n+2)\,.$$

From the right hand side it can be deduced that

$$\begin{aligned} F(n+1)F(n+2) &= F(n+1)(F(n+1)+F(n)) \\ &= F(n+1)^2 + F(n)F(n+1) \end{aligned}$$

and, by the induction hypothesis,

$$F(n+1)^2 + F(n)F(n+1) = F(n+1)^2 + \sum_{k=0}^{n} F(k)^2 = \sum_{k=0}^{n+1} F(k)^2\,,$$

which completes the proof. □

Lemma 3.2. *Let a be an integer and let $N > 0$ be an integer. Let*

$$k_{N,a} = \min\left(\frac{F(N)}{F(N+a)}, \frac{F(N+1)}{F(N+1+a)}\right), \qquad K_{N,a} = \max\left(\frac{F(N)}{F(N+a)}, \frac{F(N+1)}{F(N+1+a)}\right).$$

Then, for each $n \geq N$, the following inequalities hold.

$$k_{N,a} \leq \frac{F(n)}{F(n+a)} \leq K_{N,a}.$$

Proof. Let $k, K \in \mathbb{R}$ be any pair of numbers such that for some N

$$k \leq \frac{F(N)}{F(N+a)}, \frac{F(N+1)}{F(N+a+1)} \leq K.$$

It will be proven that

$$kF(n+a) \leq F(n) \leq KF(n+a)$$

for each $n \geq N$. This can be achieved by induction on n. The result clearly holds for $n = N$ and $n = N+1$ by hypothesis. Let $n \geq N+2$ and assume that the statement is true for all n' with $N \leq n' < n$. In particular, it follows that

$$\begin{aligned} kF(n+a-2) \leq F(n-2) \leq KF(n+a-2)\,, \\ kF(n+a-1) \leq F(n-1) \leq KF(n+a-1)\,. \end{aligned}$$

Adding both expressions yields

$$\begin{aligned} kF(n+a) &= kF(n+a-1) + kF(n+a-2) \\ &\leq F(n) = F(n-1) + F(n-2) \\ &\leq KF(n+a-1) + KF(n+a-2) = KF(n+a)\,. \end{aligned}$$

The lemma now follows on taking $k = k_{N,a}$ and $K = K_{N,a}$.

$\square$

Remark 1. The Python notebook *"markoff_fibonacci.ipynb"*, which can be found at `https://github.com/CIAMOD/markoff_fibonacci_m_triples`, was used to provide a table for the values of $k_{N,a}$ and $K_{N,a}$ from Lemma 3.2 for small values of N and a, in which we can find lower and upper bounds for the ratio $\frac{F(n)}{F(n+a)}$ given a certain a and N such as $n \geq N$. These explicit bounds are then used to bound certain expressions in the proofs of some lemmas from Section 4.2.2, especially in Lemma 4.17.

To compute the values of $k_{N,a}$ and $K_{N,a}$, the Binet's formula (3.2) was used to calculate the Fibonacci numbers. Also, in order to optimize the algorithm, a binary exponentiation method was implemented, reducing computational order from $O(n)$ to $O(\log n)$. The pseudo-code for the binary exponentiation is as follows.

```
Function BinaryExponentiation(a, b)
    Input:
        a ← a real or integer number (base)
        b ← an integer (exponent)
    Output:
        result ← a raised to the power b

    If b equals 0 then
        Return 1

    If b is less than 0 then
        a ← 1 / a
        b ← -b

    half_result ← BinaryExponentiation(a, floor(b / 2))
```

```
        result ← half_result x half_result

        If b is odd then
            result ← result x a

        Return result
```

The values of $k_{N,a}$ and $K_{N,a}$ are then computed as the minimum and maximum of the ratios $\frac{F(N)}{F(N+a)}$ and $\frac{F(N+1)}{F(N+1+a)}$.

As we dealt with exact bounds, we implemented two functions that round up and down the numbers to the n-th significant figure, with an auxiliar function that calculates the scale factor to achieve those significant figures. This function calculates, applying a logarithm base 10, where the first significant figure is in the number, and subtracts n to the result in order to get the exponent of the scale factor. Then, the round up and down functions multiply the original number by that scale factor, apply a ceiling and floor methods respectively, and divides again by the scaling factor to get the approximation. A pseudo-code for these functions is shown below.

```
Function CalculateScaleFactor(number, n)
    Input:
        number ← a real or integer number
        n ← target significant figure position (integer)
    Output:
        scale_factor ← multiplier to align nth significant digit

    If number > 0 then
        exponent ← floor(log base 10 of number) - (n - 1)
        scale_factor ← 10 ^ (-exponent)
    Else
        scale_factor ← 0

    Return scale_factor

Function RoundUpToNthSignificant(number, n)
    Input:
        number ← a real or integer number
        n ← number of significant figures to round up to
    Output:
        rounded ← number rounded up to nth significant figure

    scale_factor ← CalculateScaleFactor(number, n)

    If scale_factor not equal to 0 then
        rounded ← ceiling(number x scale_factor) ÷ scale_factor
    Else
        rounded ← number

    Return rounded
```

```
Function RoundDownToNthSignificant(number, n)
    Input:
        number ← a real or integer number
        n ← number of significant figures to round down to
    Output:
        rounded ← number rounded down to nth significant figure

    scale_factor ← CalculateScaleFactor(number, n)

    If scale_factor not equal to 0 then
        rounded ← floor(number x scale_factor) ÷ scale_factor
    Else
        rounded ← number

    Return rounded
```

These functions ensure that the bounds are still true. p has been selected as small as needed by proofs of the lemmas from Section 4.2.2. The tables below show the values of $k_{N,a}$ and $K_{N,a}$ for $N = 2, 3, \ldots, 10$ and $a = 1, 2, \ldots, 9$. The first table shows the lower bounds $k_{N,a}$ and the second table shows the upper bounds $K_{N,a}$. The values are rounded to 4 significant figures.

N \ a	1	2	3	4	5	6	7	8	9
2	0.5000	0.3333	0.2000	0.1250	0.07692	0.04761	0.02941	0.01818	0.01123
3	0.6000	0.3750	0.2307	0.1428	0.08823	0.05454	0.03370	0.02083	0.01287
4	0.6000	0.3750	0.2307	0.1428	0.08823	0.05454	0.03370	0.02083	0.01287
5	0.6153	0.3809	0.2352	0.1454	0.08988	0.05555	0.03433	0.02122	0.01311
6	0.6153	0.3809	0.2352	0.1454	0.08988	0.05555	0.03433	0.02122	0.01311
7	0.6176	0.3818	0.2359	0.1458	0.09012	0.05570	0.03442	0.02127	0.01314
8	0.6176	0.3818	0.2359	0.1458	0.09012	0.05570	0.03442	0.02127	0.01314
9	0.6179	0.3819	0.2360	0.1458	0.09016	0.05572	0.03443	0.02128	0.01315
10	0.6179	0.3819	0.2360	0.1458	0.09016	0.05572	0.03443	0.02128	0.01315

Table 1: Table of lower bounds ($k_{N,a}$)

N \ a	1	2	3	4	5	6	7	8	9
2	0.6667	0.4000	0.2500	0.1539	0.09524	0.05883	0.03637	0.02248	0.01389
3	0.6667	0.4000	0.2500	0.1539	0.09524	0.05883	0.03637	0.02248	0.01389
4	0.6250	0.3847	0.2381	0.1471	0.09091	0.05618	0.03473	0.02146	0.01327
5	0.6250	0.3847	0.2381	0.1471	0.09091	0.05618	0.03473	0.02146	0.01327
6	0.6191	0.3824	0.2364	0.1461	0.09028	0.05580	0.03449	0.02132	0.01318
7	0.6191	0.3824	0.2364	0.1461	0.09028	0.05580	0.03449	0.02132	0.01318
8	0.6182	0.3821	0.2362	0.1460	0.09019	0.05574	0.03445	0.02129	0.01316
9	0.6182	0.3821	0.2362	0.1460	0.09019	0.05574	0.03445	0.02129	0.01316
10	0.6181	0.3820	0.2361	0.1460	0.09018	0.05573	0.03445	0.02129	0.01316

Table 2: Table of upper bounds $(K_{N,a})$

Lemma 3.3. *Let $a, b, c \geq 2$. Then*

$$\begin{aligned} F(c) &\leq 3F(a)F(b) \quad \textit{if and only if} \quad c \leq a+b, &(3.8)\\ F(c) &> 3F(a)F(b) \quad \textit{if and only if} \quad c \geq a+b+1, \textit{ and} &(3.9)\\ F(c) &= 3F(a)F(b) \quad \textit{if and only if} \quad a=b=2,\ c=4\,. &(3.10)\end{aligned}$$

Proof. By equation (3.7), substituting $F(a+1) = F(a) + F(a-1)$ it can be infered

$$F(a+b) = F(a)F(b) + F(a-1)F(b) + F(a)F(b-1) \leq 3F(a)F(b)\,. \tag{3.11}$$

As $F(c)$ is increasing in c, this gives (3.8).

Suppose that $c \geq a+b+1$. Since $F(c) \geq F(a+b+1)$, to prove (3.9) it is enough to show that

$$F(a+b+1) > 3F(a)F(b). \tag{3.12}$$

As before, developing the left-hand side of (3.12) using Vajda's identity (3.7) yields

$$\begin{aligned} F(a+b+1) &= F(a+1)F(b+1) + F(a)F(b)\\ &= 2F(a)F(b) + F(a)F(b-1) + F(a-1)F(b) + F(a-1)F(b-1)\\ &= 2F(a)F(b) + F(a)F(b-1) + 2F(a-1)F(b-1) + F(a-1)F(b-2)\,. \end{aligned}$$

On the other hand, working on the right-hand side of (3.12),

$$\begin{aligned} 3F(a)F(b) &= 2F(a)F(b) + F(a)F(b)\\ &= 2F(a)F(b) + F(a)F(b-1) + F(a)F(b-2)\\ &= 2F(a)F(b) + F(a)F(b-1) + F(a-1)F(b-2) + F(a-2)F(b-2)\,. \end{aligned}$$

Comparing both sides, $2F(a-1)F(b-1) > F(a-2)F(b-2)$, which is clearly true for $a \geq 2$ and $b \geq 2$. Finally, the equality case must be studied. By (3.9), it follows that $c \leq a+b$. As $a+b \geq 4$, then for each $c < a+b$ it holds that $F(c) < F(a+b) \leq 3F(a)F(b)$ consequently $c = a+b$. The equality in (3.11) is only attained if $F(a-1) = F(a)$ and $F(b-1) = F(b)$. That only occurs in the case $a = b = 2$, proving (3.10). □

3.1.3 Relation to the Markoff equation

The Fibonacci numbers are closely related to the Markoff equation. By looking closely at the Markoff tree in Figure 1, it can be seen that one of the branches of the tree is formed by triples with all Fibonacci numbers. This triples are called Markoff-Fibonacci triples.

It was studied in [(LS)] that all Markoff-Fibonacci triples that satisfy the Markoff equation for $m = 0$ are of the form $(1, F(b), F(b+2))$, where b is an odd positive integer. This case is shown in the following figure.

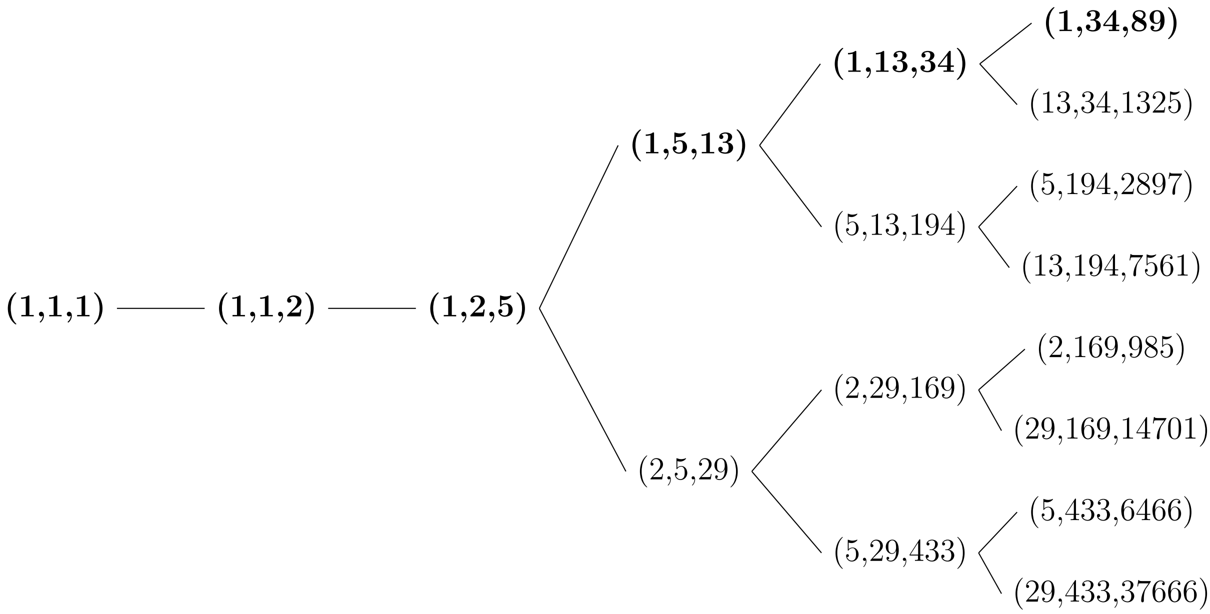

Figure 2: Beginning of the Markoff tree. Marked in bold the triples with all Fibonacci numbers.

3.2 k-Fibonacci numbers

The sequence of the k-Fibonacci numbers is a generalization of the sequence of the Fibonacci numbers, with the following recurrence relation.

$$\begin{cases} F_k(0) = 0 \\ F_k(1) = 1 \\ F_k(n) = k \cdot F_k(n-1) + F_k(n-2) \quad \forall n \geq 2\,, \end{cases} \tag{3.13}$$

where k is a positive integer and $F_k(n)$ denotes the n-th element of the sequence with that k. Therefore, with a given k, the sequence begins as follows:

$$0, 1, k, k^2 + 1, k^3 + 2k, \ldots$$

As for the Fibonacci numbers, there also exists a Binet's formula for the k-Fibonacci numbers. Let

$$\alpha_k = \frac{k + \sqrt{k^2+4}}{2}, \qquad \bar{\alpha}_k = \frac{k - \sqrt{k^2+4}}{2}\,,$$

so that α_k and $\bar{\alpha}_k$ are the roots of the characteristic polynomial $\alpha^2 - k\alpha - 1 = 0$. Then, using the same reasoning as in the Fibonacci case, the n-th k-Fibonacci number can be written as

$$F_k(n) = \frac{\alpha_k^n - \bar{\alpha}_k^n}{D_k}, \tag{3.14}$$

where $k > 0$, $n \geq 0$ and $D_k = \alpha_k - \bar{\alpha}_k = \sqrt{k^2+4}$. In particular, for $k = 1$, $\alpha_1 = \varphi$ and $D_1 = \sqrt{5}$, the classical Binet formula for the Fibonacci numbers is obtained, where φ represents the Golden Ratio. On the other hand, the case $k = 2$ corresponds to the Pell numbers. In fact, in [(KST)] all Markoff-Pell triples are found and classified.

3.2.1 k-Fibonacci identities

The identities presented in Section 3.1.1 correspond to well-known Fibonacci results. However, for the generalized sequence of k-Fibonacci numbers no proofs were found in the literature that showed that these numbers satisfy those properties. Therefore, in this section a demonstration for an analogous version of each property on k-Fibonacci numbers is provided.

Lemma 3.4 (Generalization of Vajda's Identity for k-Fibonacci numbers)**.** *For any positive numbers i, j, k,*

$$F_k(n+i)F_k(n+j) - F_k(n)F_k(n+i+j) = (-1)^n F_k(i)F_k(j).$$

Proof. Multiplying the left hand side by D_k^2 and using Binet's formula (3.14) and the fact that $\alpha_k\bar{\alpha}_k = -1$ yields

$$\begin{aligned}
&D_k^2\left(F_k(n+i)F_k(n+j) - F_k(n)F_k(n+i+j)\right)\\
&\quad = (\alpha_k^{n+i} - \bar{\alpha}_k^{n+i})(\alpha_k^{n+j} - \bar{\alpha}_j^{n+j}) - (\alpha_k^n - \bar{\alpha}_k^n)(\alpha_k^{n+i+j} - \bar{\alpha}_k^{n+i+j})\\
&\quad = -(-1)^n\alpha_k^i\bar{\alpha}_k^j - (-1)^n\bar{\alpha}_k^i\alpha_k^j + (-1)^n\alpha_k^{i+j} + (-1)^n\bar{\alpha}_k^{i+j}\\
&\quad = (-1)^n(\alpha_k^i - \bar{\alpha}_k^i)(\alpha_k^j - \bar{\alpha}_k^j) = D_k^2\left((-1)^nF_k(i)F_k(j)\right).
\end{aligned}$$

□

Corollary 3.5. *The following identities hold for any integers $a, b, n \geq 1$.*

$$F_k(a+b) = F_k(a+1)F_k(b) + F_k(a)F_k(b-1)\,, \tag{3.15}$$

$$F_k(a) \leq \frac{1}{k}F_k(a+1)\,, \tag{3.16}$$

$$F_k(a)F_k(b) \leq F_k(a+b-1)\,, \tag{3.17}$$

$$F_k(a+b-1) \leq F_k(a)F_k(b)\left(1 + \frac{1}{k^2}\right), \tag{3.18}$$

$$(\textit{D'Ocagne identity}) \quad (-1)^aF_k(b-a) = F_k(b)F_k(a+1) - F_k(b+1)F_k(a)\,, \tag{3.19}$$

$$(\textit{Catalan identity}) \quad F_k(n)^2 = F_k(n+r)F_k(n-r) + (-1)^{n-r}F_k(r)^2\,, \tag{3.20}$$

$$(\textit{Simson identity}) \quad F_k(n)^2 = F_k(n+1)F_k(n-1) - (-1)^n\,. \tag{3.21}$$

Moreover, equality holds in the following cases:

(1) *The equality in (3.16) is only attained if $a = 1$.*

(2) *The equality in* (3.17) *is only attained if* $a = 1$ *or* $b = 1$.

(3) *The equality in* (3.18) *is only attained if* $a = b = 2$.

Proof. For (3.15), take $n = 1$, $i = a$ and $j + 1 = b$ in the previous lemma.

For (3.16), it can be deduced from the recurrence relation of the k-Fibonacci numbers (3.13) that

$$F_k(a+1) = kF_k(a) + F_k(a-1) \geq kF_k(a)\,,$$

and equality is only attained if $F_k(a-1) = 0$, i.e., if $a = 1$.

For (3.17), substitute a by $a - 1$ in identity (3.15). Then

$$F_k(a+b-1) = F_k(a)F_k(b) + F_k(a-1)F_k(b-1) \geq F_k(a)F_k(b).$$

Equality is only attained if $F_k(a-1) = 0$ or $F_k(b-1) = 0$, i.e., if $a = 1$ or $b = 1$.

For (3.18), substitute a by $a - 1$ in identity (3.15). Then

$$F_k(a+b-1) = F_k(a)F_k(b) + F_k(a-1)F_k(b-1) \leq F_k(a)F_k(b)\left(1 + \frac{1}{k^2}\right).$$

Equality is only attained if $F_k(a-1) = \frac{1}{k}F_k(a)$ and $F_k(b-1) = \frac{1}{k}F_k(b)$, which only happens if $a = b = 2$.

For the D'Ocagne identity (3.19), take $n = a$, $i = b - a$, $j = 1$ in the previous lemma.

For Catalan's identity (3.20), take $n = n - r$, $i = j = r$ in the previous lemma.

Finally, for the Simson identity (3.21), take $r = 1$ in the Catalan identity (3.20).

□

3.2.2 Other k-Fibonacci properties

As well as with the Fibonacci numbers, the k-Fibonacci numbers also have some interesting properties that will be useful throughout the work.

Lemma 3.6. *For integers* $k \geq 1$ *and* $N \geq 0$,

$$\sum_{n=0}^{N} F_k(n)^2 = \frac{1}{k}F_k(N)F_k(N+1)\,.$$

Proof. Induction will be used to prove the result. For $n = 0$, the identity is true because $F_k(0) = 0$. Assuming that the result holds for some n, it will be proven for $n + 1$, this is

$$\begin{aligned}\frac{1}{k}F_k(n+1)F_k(n+2) &= \frac{1}{k}F_k(n+1)(kF_k(n+1) + F_k(n)) \\ &= F_k(n+1)^2 + \frac{1}{k}F_k(n)F_k(n+1)\,.\end{aligned}$$

And, by the induction hypothesis,

$$F_k(n+1)^2 + \frac{1}{k}F_k(n)F_k(n+1) = F_k(n+1)^2 + \sum_{n=0}^{n} F_k(n)^2 = \sum_{n=0}^{n+1} F_k(n)^2\,,$$

which completes the proof. □

Lemma 3.7. *If $k \geq 4$ and $n \geq 1$, then $4F_k(2n-2) \leq F_k(n)^2$.*

Proof. For $n = 1$, the inequality becomes $0 = 4F_k(0) \leq F_k(1) = 1$, hence the result holds. Assume that $n \geq 2$. Taking $a = b = n-1$ in equation (3.15), and then multiplying by four, it follows that

$$4F_k(2n-2) = 4F_k(n-1)(F_k(n) + F_k(n-2)). \tag{3.22}$$

If $k \geq 5$, then by (3.16), $4F_k(n-1) \leq \frac{4}{5}F_k(n)$ and $F_k(n-2) < \frac{1}{4}F_k(n)$. Combining both inequalities,

$$4F_k(n-1)(F_k(n) + F_k(n-2)) < F_k(n)^2.$$

The above inequality and (3.22) prove the lemma for $k \geq 5$. In the case $k = 4$, using again (3.22),

$$\begin{aligned} 4F_4(2n-2) &= 4F_4(n-1)(F_4(n) + F_4(n-2)) \\ &= (F_4(n) - F_4(n-2))\,(F_4(n) + F_4(n-2)) \\ &= F_4(n)^2 - F_4(n-2)^2 \leq F_4(n)^2, \end{aligned}$$

which proves the result. □

Lemma 3.8. *Let $a, b, c \geq 1$. Then*

$$F_2(c) \geq 3F_2(a)F_2(b) \quad \textit{if and only if} \quad c \geq a+b+1 \ \textit{or}\ (a,b,c) = (2,2,4), \ \textit{and} \tag{3.23}$$

$$F_k(c) \geq 3F_k(a)F_k(b) \quad \textit{if and only if} \quad c \geq a+b, \quad \textit{for all}\ k \geq 3\,. \tag{3.24}$$

Equality is only attained if $k = 2$ and $(a,b,c) = (2,2,4)$, or if $k = 3$ and $(a,b,c) = (1,1,2)$.

Proof. To prove (3.23), by identity (3.15), it can be deduced that

$$\begin{aligned} F_2(a+b+1) &= F_2(a+1)F_2(b+1) + F_2(a)F_2(b) \\ &= (2F_2(a) + F_2(a-1))(2F_2(b) + F_2(b-1)) + F_2(a)F_2(b) \\ &\geq (2^2+1)F_2(a)F_2(b) > 3F_2(a)F_2(b)\,. \end{aligned} \tag{3.25}$$

On the other hand,

$$\frac{F_2(a+b)}{F_2(a)F_2(b)} = \frac{F_2(a+1)F_2(b) + F_2(a)F_2(b-1)}{F_2(a)F_2(b)} = \frac{F_2(a+1)}{F_2(a)} + \frac{F_2(b-1)}{F_2(b)}.$$

It is known that successive quotients of Pell numbers $F_2(n+1)/F_2(n)$ form an oscillating sequence converging to α_2, where the sequence of even terms is decreasing and the sequence of odd terms is increasing. As a consequence, the maximum of $F_2(a+1)/F_2(a)$ is $\frac{5}{2}$ and it is attained only at $a = 2$, and the maximum of $F_2(b-1)/F_2(b)$ is $\frac{1}{2}$ and it is attained only at $b = 2$. Thus,

$$\frac{F_2(a+b)}{F_2(a)F_2(b)} = \frac{F_2(a+1)}{F_2(a)} + \frac{F_2(b-1)}{F_2(b)} \leq \frac{5}{2} + \frac{1}{2} = 3\,, \tag{3.26}$$

and equality is only attained at $(a,b) = (2,2)$. Combining (3.25) and (3.26) and using the fact that the function $F_2(c)$ is strictly increasing in c, inequality (3.23) holds.

Finally, to prove (3.24), by using again (3.15) when $k \geq 3$, it can be infered that

$$\begin{aligned} F_k(a+b) &= F_k(a+1)F_k(b) + F_k(a)F_k(b-1) \\ &= kF_k(a)F_k(b) + F_k(a-1)F_k(b) + F_k(a)F_k(b-1) \\ &\geq 3F_k(a)F_k(b), \end{aligned}$$

with equality if and only if $k = 3$, $F_k(a-1) = 0$ and $F_k(b-1) = 0$, i.e., if $a = b = 1$. Additionally, for all $k \geq 3$ it follows that

$$F_k(a+b-1) = F_k(a)F_k(b) + F_k(a-1)F_k(b-1) \leq 2F_k(a)F_k(b) < 3F_k(a)F_k(b).$$

By the two previous inequalities and since the function $F_k(c)$ is strictly increasing in c, it follows that (3.24) holds.

$\square$

4 A classification of Markoff-Fibonacci m-triples

The first result of this project is the research paper titled *A classification of Markoff-Fibonacci m-triples* [(ACMRS1)], currently accepted for publication in the journal *The Fibonacci Quarterly.* In it, we classified all Markoff-Fibonacci m-triples, using the minimality concept explained in Section 2.4. The main results of the paper are summarized in the following theorem.

Theorem 4.1. *For each $m > 0$, there exists at most one ordered solution to the equation*

$$x^2 + y^2 + z^2 = 3xyz + m\,,$$

composed of Fibonacci numbers, except in the following cases.

- *If $m = 2$, the Fibonacci solutions are $(1, F(b), F(b+2))$ for each even $b \geq 2$.*
- *If $m = 21$, the Fibonacci solutions are the minimal triples $(1, 2, 8)$ and $(2, 2, 13)$.*

Moreover, there exists an infinite number of $m > 0$ admitting exactly one Markoff-Fibonacci m-triple and such triple is always minimal.

Along this section, several proofs will be presented to show the validity of Theorem 4.1.

4.1 Characterization of Markoff-Fibonacci m-triples

It will be denoted

$$m(a, b, c) = F(a)^2 + F(b)^2 + F(c)^2 - 3F(a)F(b)F(c)\,,$$

so that $(F(a), F(b), F(c))$ is a Markoff-Fibonacci m-triple if and only if $m = m(a, b, c) > 0$. Therefore, the conditions on (a, b, c) to ensure $m(a, b, c) > 0$ will be derived in this subsection. Since the purpose is to study ordered triples $(F(a), F(b), F(c))$ composed of positive numbers, without loss of generality, it will be assumed from now on that $2 \leq a \leq b \leq c$.

Lemma 4.2. *If $2 \leq a \leq b \leq c$ with $c \geq a + b + 1$, then $m(a, b, c) > 0$.*

Proof. Since

$$m(a, b, c) = F(a)^2 + F(b)^2 + F(c)\left(F(c) - 3F(a)F(b)\right)\,,$$

and by Lemma 3.3, $F(c) > 3F(a)F(b)$, then all of the terms in the previous factorization are positive, and thus $m(a, b, c) > 0$. □

Lemma 4.3. *If $3 \leq a \leq b \leq c \leq a + b$, then $m(a, b, c) < 0$.*

Proof. As $F(x)$ is an increasing function, $1 < F(a) \leq F(b) \leq F(c) \leq F(a + b)$. Consider the parabola $f(x) = x^2 - 3F(a)F(b)x + F(a)^2 + F(b)^2$. Since $f(x)$ is an upward-opening parabola, its absolute maximum in $F(b) \leq x \leq F(a + b)$ is attained at one of the endpoints. Therefore, to prove that $m(a, b, c) = f(F(c)) < 0$, it suffices to prove that $f(F(b))$ and $f(F(a + b))$ are both negative.

First, it must be proven that $f(F(b)) < 0$. Indeed,

$$f(F(b)) = F(a)^2 + 2F(b)^2 - 3F(a)F(b)^2 \leq F(b)^2(3 - 3F(a)) < 0.$$

Now, it must be proven that $f(F(a+b)) < 0$.

$$f(F(a+b)) = F(a)^2 + F(b)^2 + F(a+b)\left(F(a+b) - 3F(a)F(b)\right).$$

To prove this, two cases will be considered: $a = b$ and $a < b$.
For the first case,

$$f(F(2a)) = m(a,a,2a) = 2F(a)^2 + F(2a)(F(2a) - 3F(a)^2).$$

From basic Fibonacci properties it can be deduced that $F(a) = F(a-1) + F(a-2) \leq 2F(a-1)$. Thus, given equation (3.7),

$$F(2a) = F(a+a) = F(a)^2 + 2F(a)F(a-1) \geq 2F(a)^2. \tag{4.1}$$

On the other hand, for $a \geq 3$ it is true that $F(a-1) < F(a)$, which in this context is equivalent to $F(a-1) \leq F(a) - 1$, so

$$F(2a) = F(a)^2 + 2F(a)F(a-1) \leq F(a)^2 + 2F(a)(F(a) - 1) = 3F(a)^2 - 2F(a).$$

Hence,

$$F(2a) - 3F(a)^2 \leq -2F(a) < -1. \tag{4.2}$$

Therefore, in the case $a = b$, using both equations (4.1) and (4.2), it can be concluded that

$$f(F(2a)) = m(a,a,2a) = 2F(a)^2 + F(2a)(F(2a) - 3F(a)^2) < 2F(a)^2 - F(2a) \leq 0.$$

Thus, it may now be assumed that $a \neq b$. As a reminder, it must be shown that

$$\begin{aligned} f(F(a+b)) &= F(a)^2 + F(b)^2 + F(a+b)\left(F(a+b) - 3F(a)F(b)\right) \\ &= F(a)^2 + F(b)^2 - F(a+b)\left(3F(a)F(b) - F(a+b)\right) < 0. \end{aligned}$$

or equivalently, $F(a)^2 + F(b)^2 < F(a+b)\left(3F(a)F(b) - F(a+b)\right)$. As $a \leq b-1$, it follows that

$$\begin{aligned} F(a)^2 + F(b)^2 &\leq F(b-1)^2 + F(b)^2 \\ &< F(b-1)F(b) + F(b)^2 = F(b+1)F(b) \\ &< F(a+b)F(b). \end{aligned}$$

So, it suffices to show that $F(b) < 3F(a)F(b) - F(a+b)$, or equivalently, $\frac{1}{F(a)} + \frac{F(a+b)}{F(a)F(b)} < 3$. From equation (3.7), the second fraction can be expressed as

$$\frac{F(a+b)}{F(a)F(b)} = \frac{F(a+1)}{F(a)} + \frac{F(b-1)}{F(b)} = 1 + \frac{F(a-1)}{F(a)} + \frac{F(b-1)}{F(b)}.$$

As $a \geq 3$ and $b > a$, using Tables 1 and 2 it follows that

$$\frac{1}{F(a)} + \frac{F(a+b)}{F(a)F(b)} = \frac{1}{F(a)} + 1 + \frac{F(a-1)}{F(a)} + \frac{F(b-1)}{F(b)} \leq 0.5 + 1 + 0.6667 + 0.6667 < 3.$$

Hence

$$F(a)^2 + F(b)^2 < F(a+b)F(b) < F(a+b)(3F(a)F(b) - F(a+b)),$$

giving $f(F(a+b)) = m(a,b,a+b) < 0$. $\square$

Lemma 4.4. *If $b \geq 2$, then $m(2, b, b+1) \leq 0$, and the equality is only attained if $b = 2$.*

Proof. This is equivalent to proving

$$1 + F(b)^2 + F(b+1)^2 \leq 3F(b)F(b+1).$$

By Lemma 3.1 and since $b \geq 2$, then

$$1 + F(b)^2 + F(b+1)^2 = F(1)^2 + F(b)^2 + F(b+1)^2 \leq \sum_{k=1}^{b+1} F(k)^2 = F(b+1)F(b+2)\,.$$

Finally, by equation (3.11) for the case $a = 2$, it can be infered that $F(b+2) \leq 3F(b)$, with equality attained at $b = 2$, so the lemma follows. $\square$

As a consequence of the previous lemmas, a complete classification of Markoff-Fibonacci triples $(F(a), F(b), F(c))$ that are m-triples for a positive m can be established.

Proposition 4.5. *Suppose that $2 \leq a \leq b \leq c$. Then $m(a, b, c) > 0$ if and only if $a + b + 1 \leq c$ or $(a, b, c) = (2, b, b+2)$, for some even $b \geq 2$.*

Proof. If $c \geq a + b + 1$, then $m(a, b, c) > 0$ by Lemma 4.2. If $(a, b, c) = (2, b, b+2)$, using Vajda's identity (see equation (3.6)) for $(i = 1, j = 1, n = b)$ implies

$$\begin{aligned} m(2, b, b+2) &= 1 + F(b)^2 + F(b+2)^2 - 3F(b)F(b+2) && (4.3)\\ &= 1 + (F(b+2) - F(b))^2 - F(b)F(b+2) && (4.4)\\ &= 1 + F(b+1)^2 - F(b)F(b+2) = 1 + (-1)^b\,, && (4.5) \end{aligned}$$

which is positive if and only if b is even. If b is odd, a branch of the Markoff tree ($m = 0$) is obtained. This branch corresponds to the one described in the article [(LS)] and shown in Figure 1. For $b > 2$ even, an analogous branch to the previous one is obtained, in this case, in the 2-Markoff tree with minimal triple $(1, 1, 3)$, which will be shown later in section 4.2.1.

Suppose now that $c \leq a + b$ and that $(a, b, c) \neq (2, b, b+2)$. If $a \geq 3$, then Lemma 4.3 shows that $m(a, b, c) < 0$. If $a = 2$ and $c \neq b + 2$ but $c \leq a + b = b + 2$, then either $(a, b, c) = (2, b, b)$ or $(a, b, c) = (2, b, b+1)$. For $(a, b, c) = (2, b, b)$,

$$m(a, b, c) = m(2, b, b) = 1 + 2F(b)^2 - 3F(b)^2 = 1 - F(b)^2 \leq 0.$$

If $(a, b, c) = (2, b, b+1)$ then Lemma 4.4 shows that $m(a, b, c) = m(2, b, b+1) \leq 0$. $\square$

4.2 Full classification of Markoff-Fibonacci m-triples

Using Proposition 4.5, a specific characterisation of minimal and non-minimal Markoff-Fibonacci m-triples can be obtained, when $m > 0$. Recall that a Markoff m-triple (x, y, z) with $m > 0$ is called minimal (c.f. [(SC), Definition 2.2]) if $z \geq 3xy$.

Proposition 4.6. *Let $a \geq 2$. Then for $a \leq b \leq c$, the m-triple $(F(a), F(b), F(c))$, with $m > 0$, is minimal if and only if either $c \geq a + b + 1$ or $a = b = 2$ and $c = 4$.*

Proof. By Proposition 4.5, the values of (a, b, c) for which $m(a, b, c) > 0$ are those with $c \geq a + b + 1$ or $(a, b, c) = (2, b, b + 2)$, for some even $b \geq 2$. Amongst those triples, the minimal ones are those which also satisfy $F(c) \geq 3F(a)F(b)$. By Lemma 3.3, these are precisely the triples such that $c \geq a + b + 1$ or $(a, b, c) = (2, 2, 4)$. □

Using this proposition, the problem of classifying the Fibonacci solutions to the m-Markoff equation (2.2) can be split into classifying the non-minimal and minimal triples, starting by the classification of non-minimal m-triples.

4.2.1 Non-minimal case

Proposition 4.7. *Every non-minimal Markoff-Fibonacci m-triple with $m > 0$ is a 2-triple of the form $(1, F(b), F(b + 2))$, where $b > 2$ is an even number.*

Proof. By Proposition 4.5, any Markoff-Fibonacci m-triple $(F(a), F(b), F(c))$ with $m > 0$ must either satisfy $c \geq a+b+1$, or be of the form $(1, F(b), F(b+2))$ for some even $b \geq 2$. By Proposition **??**, non-minimal m-triples satisfy $c \leq a + b$ with $(a, b, c) \neq (2, 2, 4)$. Thus, all non-minimal Markoff-Fibonacci m-triples must be of the form $(1, F(b), F(b+2))$ with even $b \geq 3$ and, by equation (5.9), they are all 2-triples. In particular, they form a branch of the 2-Markoff tree spanned by $(1, 3, 8)$. □

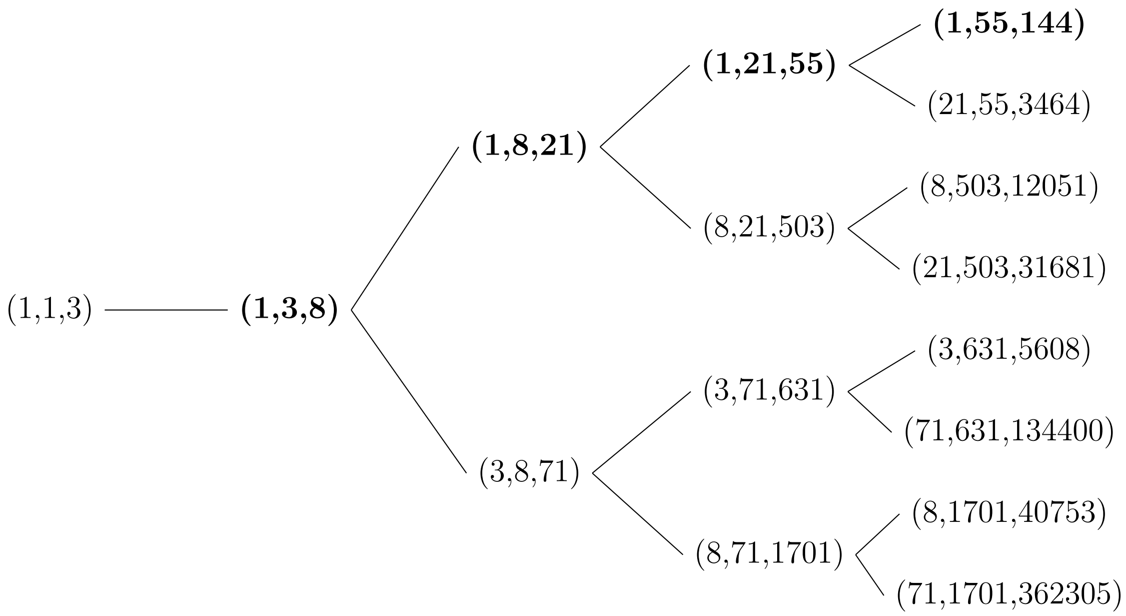

Figure 3: Beginning of the 2-Markoff tree. Marked in bold the non-minimal triples with all Fibonacci numbers.

Remark 2. From [(SC)], it is known that for $m = 2$ there exists exactly one minimal 2-triple, which is the only remaining Fibonacci solution to the m-Markoff equation for $m = 2$, namely, $(1, 1, 3) = (F(2), F(2), F(4))$.

4.2.2 Minimal case

Contrary to the non-minimal Fibonacci triples, which only exist for $m = 2$, the following proposition shows that minimal Markoff-Fibonacci m-triples exist for an infinite number of values of m.

Proposition 4.8. *There exists an infinite number of values of $m > 0$ for which the equation (2.2) admits a solution composed of Fibonacci numbers. Moreover, these solution triples are minimal.*

Proof. By the characterisation of minimal m-triples from Proposition 4.6 and by Proposition 4.5, for each a and b at least 2 and each $c \geq a+b+1$, the triple $(F(a), F(b), F(c))$ is an m-triple for some $m > 0$ and it is minimal. Observe that if by fixing a and b then

$$m(a,b,c) = F(a)^2 + F(b)^2 + F(c)^2 - 3F(a)F(b)F(c)\,,$$

is a quadratic equation in $F(c)$ which is positive and strictly increasing in c for each $c \geq a+b+1$ because $F(a+b+1) > 3F(a)F(b) > \frac{3}{2}F(a)F(b)$ due to Lemma 3.3. This shows that, for each a and b, there is an infinite number of different values of $m > 0$ and c such that $(F(a), F(b), F(c))$ is a minimal m-triple. $\square$

Next, it will be shown that, except for $m = 21$, if $m > 0$ admits a minimal Markoff-Fibonacci m-triple, it is unique up to order. The rest of the paper is devoted to proving such a claim in full generality.

Through the rest of the section, $(F(a), F(b), F(c))$ and $(F(a'), F(b'), F(c'))$ will be assumed to be two different ordered Markoff-Fibonacci triples for the same $m > 0$, i.e., $2 \leq a \leq b \leq c$, $2 \leq a' \leq b' \leq c'$ with $(a,b,c) \neq (a',b',c')$ and

$$m(a,b,c) = m(a',b',c') = m\,,$$

and, without loss of generality, it will be assumed that $c \geq c'$.

By fixing an upper bound for c, there only exists a finite number of pairs of such triples. The following Lemma has been checked computationally and shows that the claim holds if c is small.

Lemma 4.9. *Suppose that $2 \leq a \leq b \leq c < 20$ and $2 \leq a' \leq b' \leq c' \leq c < 20$. If $m = m(a,b,c) = m(a',b',c') > 0$, then either*

- *$a = a'$, $b = b'$, and $c = c'$ or,*
- *$(a,b,c) = (3,3,7)$ and $(a',b',c') = (2,3,6)$, with $m(a,b,c) = m(a',b',c') = 21$ or,*
- *$m = 2$ and (a,b,c) and (a',b',c') are of the form $(2,b,b+2)$, as described by Proposition 4.7.*

Proof. The Python script "*check_minimal_triples.py*", available at `https://github.com/CIAMOD/markoff_fibonacci_m_triples`, was used to check this Lemma, in which we need to prove that $m = 21$ is the only m-value with more than one minimal Fibonacci m-triple. Given the indexes of the Fibonacci elements (a,b,c), this program checks all triples up to $c = 500$. The verification runs in 41 seconds on an Intel(R) Core(TM) i7-1165G7 @ 2.8GHz, and, if run up to $c = 20$ instead, it completes the required verifications for this Lemma in a few milliseconds.

In the code, first, a dictionary with the Fibonacci sequence is generated, where the key is the index of the sequence and the value is the corresponding number. Then, all possible combinations of indexes to generate Fibonacci triples are computed. Given Proposition

4.6, only those triples that satisfy the minimality condition $c \geq a+b+1$ have to be analysed. Finally, all minimal Fibonacci triples are stored in a dictionary, assigned to their respective m-value. With this, it is possible to check if $m = 21$ is truly the only $m > 0$ that has more than one minimal Markoff-Fibonacci m-triple with a small c. $\square$

Given Lemma 4.9, the rest of this work will focus on showing that it is impossible to find two triples $(F(a), F(b), F(c))$ and $(F(a'), F(b'), F(c'))$ with $m(a,b,c) = m(a',b',c')$ if $c \geq 20$. The proof will be divided into two cases. First, pairs of triples with $c > c'$ will be studied and it will be proven that the only possible pair of triples is the known example from Lemma 4.9, namely, $m(3,3,7) = m(2,3,6) = 21$. Then, the case $c = c'$ will be analysed and it will be proven that no pair of distinct Markoff-Fibonacci m-triples can exist with the same maximal element.

Lemma 4.10. *Suppose that $2 \leq a \leq b \leq c$ and $c \geq 5$. Suppose that $2 \leq a \leq a' \leq c$ and $b \leq b' \leq c$. Then*

$$m(a,b,c) \geq m(a',b',c)\,,$$

and the equality holds if and only if $a = a'$ and $b = b'$. In particular, if $(F(a), F(b), F(c))$ is an ordered minimal Markoff-Fibonacci m-triple with $m > 0$, then

$$m(2,2,c) \geq m(a,b,c) \geq m(a, c-a-1, c).$$

Proof. The parabolas

$$\begin{aligned} p(x) &= x^2 - 3F(b)F(c)x + F(b)^2 + F(c)^2 \quad \text{and} \\ q(x) &= x^2 - 3F(a)F(c)x + F(a)^2 + F(c)^2 \end{aligned}$$

have their vertices at $x = \frac{3}{2}F(b)F(c) > F(c)$ and $x = \frac{3}{2}F(a)F(c) > F(c)$ respectively, so they are both strictly decreasing for $0 \leq x \leq F(c)$ for all values of a and b. As the function mapping n to $F(n)$ is increasing, then for any given fixed c the map

$$m_c : (a,b) \mapsto m(a,b,c) = F(a)^2 + F(b)^2 + F(c)^2 - 3F(a)F(b)F(c)$$

is strictly decreasing both in a and b in the region $2 \leq a, b \leq c$. Given a, b, c such that $(F(a), F(b), F(c))$ is a minimal triple, Proposition 4.6 implies that $a + b \leq c - 1$ so $2 \leq a \leq b \leq c - 1 - a < c$ and, therefore

$$m_c(a,b) \geq m_c(a, c-1-a) \quad \text{and} \quad m_c(a,b) \leq m_c(2,b) \leq m_c(2,2)\,.$$

$\square$

Remark 3. An analogue of Lemma 4.10 also holds for non-Fibonacci Markoff m-triples. By defining

$$\tilde{m}(a,b,c) = a^2 + b^2 + c^2 - 3abc\,,$$

so that $m(a,b,c) = \tilde{m}(F(a), F(b), F(c))$, the proof of the previous Lemma can be replicated to show that if $a \leq a' \leq b$ and $b \leq b' \leq c$, then

$$\tilde{m}(a,b,c) \geq \tilde{m}(a',b',c)\,.$$

Now, the maximal element c' will be bounded in a couple of triples with $m(a,b,c) = m(a',b',c')$ and $c' \leq c$.

Lemma 4.11. *Let $m > 0$. Let $A, C \geq 2$ and $t \geq 1$ be three integers. If $A \leq a \leq b \leq c = a + b + t$ and $c \geq C$, then*

$$L_{A,t,C}\frac{1}{5}\varphi^{2c} < m(a,b,c) < U_{A,t,C}\frac{1}{5}\varphi^{2c},$$

where

$$L_{A,t,C} = 1 - \frac{3}{\sqrt{5}}\varphi^{-t} + \left(1 - \frac{3}{\sqrt{5}}\varphi^{t}\right)\left(\varphi^{-2t-2A} + \varphi^{2A-2C}\right) - \left(6 + \frac{3}{\sqrt{5}}\varphi^{t} + \frac{9}{\sqrt{5}}\right)\varphi^{-2C}, \tag{4.6}$$

$$U_{A,t,C} = 1 - \frac{3}{\sqrt{5}}\varphi^{-t} + \left(1 + \frac{3}{\sqrt{5}}\varphi^{t}\right)\left(\varphi^{-2t-2A} + \varphi^{2A-2C}\right) + 9\varphi^{-2C}. \tag{4.7}$$

Proof. Using Binet's formula (3.2) and taking into account that $\varphi\bar{\varphi} = -1$, it can be deduced that

$$F(n)^2 = \frac{1}{5}\varphi^{2n} + \frac{1}{5}\varphi^{-2n} - \frac{2}{5}(-1)^n < \frac{1}{5}\varphi^{2n} + \frac{3}{5}. \tag{4.8}$$

Thus

$$\begin{aligned} m(a,b,c) &= F(c)^2 + F(c-t-a)^2 + F(a)^2 - 3F(c)F(c-t-a)F(a) \\ &< \frac{1}{5}\varphi^{2c} + \frac{1}{5}\varphi^{2c-2t-2a} + \frac{1}{5}\varphi^{2a} + \frac{9}{5} - \frac{3}{5\sqrt{5}}(\varphi^{c} - \bar{\varphi}^{c})(\varphi^{c-t-a} - \bar{\varphi}^{c-t-a})(\varphi^{a} - \bar{\varphi}^{a}). \end{aligned}$$

As $c > t$ and $\varphi\bar{\varphi} = -1$, it follows that

$$\begin{aligned} &(\varphi^{c} - \bar{\varphi}^{c})(\varphi^{c-t-a} - \bar{\varphi}^{c-t-a})(\varphi^{a} - \bar{\varphi}^{a}) \\ &\qquad \geq (\varphi^{c} - \varphi^{-c})(\varphi^{c-t-a} - \varphi^{a-c+t})(\varphi^{a} - \varphi^{-a}) \\ &\qquad = \varphi^{2c-t} - \varphi^{2c-t-2a} - \varphi^{2a+t} - \varphi^{-2c+t} + \varphi^{t} - \varphi^{-t} + \varphi^{-2a-t} + \varphi^{2a-2c+t} \\ &\qquad > \varphi^{2c-t} - \varphi^{2c-t-2a} - \varphi^{2a+t} - \varphi^{-2c+t} + 1 \\ &\qquad > \varphi^{2c-t} - \varphi^{2c-t-2a} - \varphi^{2a+t}. \end{aligned}$$

Therefore,

$$\begin{aligned} m(a,b,c) &< \frac{1}{5}\varphi^{2c} + \frac{1}{5}\varphi^{2c-2t-2a} + \frac{1}{5}\varphi^{2a} + \frac{9}{5} - \frac{3}{5\sqrt{5}}(\varphi^{2c-t} - \varphi^{2c-t-2a} - \varphi^{2a+t}) \\ &= \frac{1}{5}\left(1 - \frac{3}{\sqrt{5}}\varphi^{-t}\right)\varphi^{2c} + \frac{1}{5}\left(1 + \frac{3}{\sqrt{5}}\varphi^{t}\right)(\varphi^{2c-2t-2a} + \varphi^{2a}) + \frac{9}{5}. \end{aligned}$$

As φ^x is a convex function and $a \geq A$, by applying Karamata's inequality [(K)], it is true to say that

$$\varphi^{2c-2t-2a} + \varphi^{2a} \leq \varphi^{2c-2t-2A} + \varphi^{2A}. \tag{4.9}$$

Since the factor $1 + \frac{3}{\sqrt{5}}\varphi^{t}$ is positive and $c \geq C$, then

$$\begin{aligned} m(a,b,c) &< \frac{1}{5}\left(1 - \frac{3}{\sqrt{5}}\varphi^{-t}\right)\varphi^{2c} + \frac{1}{5}\left(1 + \frac{3}{\sqrt{5}}\varphi^{t}\right)(\varphi^{2c-2t-2A} + \varphi^{2A}) + \frac{9}{5} \\ &= \left(1 - \frac{3}{\sqrt{5}}\varphi^{-t} + \left(1 + \frac{3}{\sqrt{5}}\varphi^{t}\right)(\varphi^{-2t-2A} + \varphi^{2A-2c}) + 9\varphi^{-2c}\right)\frac{1}{5}\varphi^{2c} \leq U_{A,t,C}\frac{1}{5}\varphi^{2c}. \end{aligned}$$

Analogously,

$$F(n)^2 = \frac{1}{5}\varphi^{2n} + \frac{1}{5}\varphi^{-2n} - \frac{2}{5}(-1)^n > \frac{1}{5}\varphi^{2n} - \frac{2}{5},$$

and, since $c > t$

$$\begin{aligned}(\varphi^c - \bar{\varphi}^c)&(\varphi^{c-t-a} - \bar{\varphi}^{c-t-a})(\varphi^a - \bar{\varphi}^a)\\ &\leq (\varphi^c + \varphi^{-c})(\varphi^{c-t-a} + \varphi^{a-c+t})(\varphi^a + \varphi^{-a})\\ &= \varphi^{2c-t} + \varphi^{2c-t-2a} + \varphi^{2a+t} + \varphi^t + \varphi^{-t} + \varphi^{-2a-t} + \varphi^{2a-2c+t} + \varphi^{-2c+t}\\ &< \varphi^{2c-t} + \varphi^{2c-t-2a} + \varphi^{2a+t} + \varphi^t + 3,\end{aligned}$$

so

$$\begin{aligned}m(a,b,c) &> \frac{1}{5}\varphi^{2c} + \frac{1}{5}\varphi^{2c-2t-2a} + \frac{1}{5}\varphi^{2a} - \frac{6}{5} - \frac{3}{5\sqrt{5}}(\varphi^{2c-t} + \varphi^{2c-t-2a} + \varphi^{2a+t} + \varphi^t + 3)\\ &= \frac{1}{5}\left(1 - \frac{3}{\sqrt{5}}\varphi^{-t}\right)\varphi^{2c} + \frac{1}{5}\left(1 - \frac{3}{\sqrt{5}}\varphi^t\right)(\varphi^{2c-2t-2a} + \varphi^{2a}) - \frac{6}{5} - \frac{3}{5\sqrt{5}}\varphi^t - \frac{9}{5\sqrt{5}}.\end{aligned}$$

Using again Karamata's inequality (5.10), and taking into account that the factor $1 - \frac{3}{\sqrt{5}}\varphi^t$ is negative and that $c \geq C$, the following result is obtained

$$\begin{aligned}&m(a,b,c)\\ &> \frac{1}{5}\left(1 - \frac{3}{\sqrt{5}}\varphi^{-t}\right)\varphi^{2c} + \frac{1}{5}\left(1 - \frac{3}{\sqrt{5}}\varphi^t\right)(\varphi^{2c-2t-2A} + \varphi^{2A}) - \frac{6}{5} - \frac{3}{5\sqrt{5}}\varphi^t - \frac{9}{5\sqrt{5}}\\ &= \left(1 - \frac{3}{\sqrt{5}}\varphi^{-t} + \left(1 - \frac{3}{\sqrt{5}}\varphi^t\right)(\varphi^{-2t-2A} + \varphi^{2A-2c}) - \left(6 + \frac{3}{\sqrt{5}}\varphi^t + \frac{9}{\sqrt{5}}\right)\varphi^{-2c}\right)\frac{1}{5}\varphi^{2c}\\ &\geq L_{A,t,C}\frac{1}{5}\varphi^{2c}.\end{aligned}$$

□

Lemma 4.12. *Let $m > 0$. Let $(F(a), F(b), F(c))$ and $(F(a'), F(b'), F(c'))$ be two ordered minimal Markoff-Fibonacci m-triples with $a \leq b \leq c$, $a' \leq b' \leq c'$ and $c \geq c'$. If $a \geq 4$ and $c \geq 9$ then either $c' = c$ or $c' = c - 1$.*

Proof. Assuming $m(a,b,c) = m = m(a',b',c')$, from Lemma (4.10) and Lemma 4.11, if $L_{A,t,C}$ is the lower bound given by (4.6), then

$$m = m(a,b,c) \geq m(a, c-1-a, c) \geq L_{4,1,9}\frac{1}{5}\varphi^{2c}.$$

Running a quick calculation in MATLAB, it can be verified that the following holds

$$L_{4,1,9} = 1 - \frac{3}{\sqrt{5}}\varphi^{-1} + \left(1 - \frac{3}{\sqrt{5}}\varphi\right)2\varphi^{-10} - 6\varphi^{-18} - \frac{3}{\sqrt{5}}\varphi^{-17} - \frac{9}{\sqrt{5}}\varphi^{-18} > \varphi^{-4},$$

so

$$m(a,b,c) > L_{4,1,9}\frac{1}{5}\varphi^{2c} > \frac{1}{5}\varphi^{2c-4} = \frac{1}{5}\varphi^{2(c-2)}. \tag{4.10}$$

On the other hand, by Lemma (4.10) and equation (4.8), it follows

$$m = m(a',b',c') \leq m(2,2,c') = F(c')^2 - 3F(c') + 2 < \frac{1}{5}\varphi^{2c'} + \frac{3}{5} - 1 < \frac{1}{5}\varphi^{2c'}. \tag{4.11}$$

Using equations (4.10) and (4.11) together yields $\varphi^{2(c-2)} < 5m < \varphi^{2c'}$. Thus, $c' > c - 2$. As we assumed $c' \leq c$, then either $c' = c$ or $c' = c - 1$. □

Lemma 4.13. *Let $(F(a), F(b), F(c))$ and $(F(a'), F(b'), F(c'))$ be two minimal Markoff-Fibonacci m-triples with: $a' \leq b' \leq c' \leq c$, $a \leq b \leq c$ and $a = 2$ or 3. Then, $c' \geq c - 1$ except for the case $a = 3$ and $c = b + 4$, where $c' \geq c - 2$.*

Proof. First of all, we can deduce that $c' \leq 4$ is discarded in the following way: if $c' \leq 4$, then $F(c') \leq F(4) = 3$. Since $(F(a'), F(b'), F(c'))$ is minimal, the only possible solution is $F(a') = F(b') = 1$ and $F(c') = 3$. But, this case corresponds to $m = 2$ and, for this value of m, there are not two minimal Markoff-Fibonacci m-triples. As a result, we will assume for the rest of the proof that $c' \geq 5$.

The Markoff-Fibonacci m-triples satisfy

$$m(a, b, c) = m(a', b', c'). \tag{4.12}$$

Dividing (4.12) by $F(c)^2$, and arranging we obtain

$$\begin{aligned}
&\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2} \\
&= 1 + \frac{F(a)^2}{F(c)^2} + \frac{F(b)^2}{F(c)^2} - 3\frac{F(a)F(b)}{F(c)} - \left(\frac{F(c')^2}{F(c)^2} + \frac{F(a')^2}{F(c)^2} + \frac{F(b')^2}{F(c)^2} - 3\frac{F(a')F(b')F(c')}{F(c)^2}\right) \\
&= 0.
\end{aligned} \tag{4.13}$$

We also have the inequality $F(n+1) > \sqrt{2}F(n)$, for $n \geq 2$ (see, for instance, Table 2). We will divide now the proof in to cases: a = 2 and a = 3.

First, we assume that $a = 2$ and we know that $c \geq c' \geq 5$. Then, we obtain the upper bounds $a' \leq b' \leq c' - 3$ and $b \leq c - 3$ by the minimality of the triples (Proposition 4.6).

If $c' \leq c - 2$, then $a' \leq b' \leq c - 5$, which by the inequality mentioned before implies that $F(c) > 2F(c')$ and $F(c) > 4\sqrt{2}F(b') \geq 4\sqrt{2}F(a')$. As a result, $F(c) > 4\sqrt{2}F(b)$ if we first assume $c \geq b + 5$, obtaining the following lower bound for the left side of (4.13):

$$\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2} \tag{4.14}$$

$$> 1 - 3\frac{F(b)}{F(c)} - \left(\frac{F(c')^2}{F(c)^2} + \frac{F(a')^2}{F(c)^2} + \frac{F(b')^2}{F(c)^2}\right) \tag{4.15}$$

$$> 1 - \frac{3}{4\sqrt{2}} - \left(\frac{1}{4} + \frac{1}{32} + \frac{1}{32}\right) > 0, \tag{4.16}$$

contradicting equation (4.13). Therefore, $c = b + 3$ or $c = b + 4$. If $c = b + 3$, we obtain the following lower bound for the left side of (4.13):

$$\begin{aligned}
&\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2} \\
&\qquad > 1 + \frac{1}{F(c)^2} + \frac{F(c-3)^2}{F(c)^2} - 3\frac{F(c-3)}{F(c)} - \left(\frac{1}{4} + \frac{1}{32} + \frac{1}{32}\right) \\
&\qquad > 1 + 0.2307^2 - 3(0.2381) - 1/4 - 1/16 > 0,
\end{aligned}$$

for $c \geq 7$ (using Tables 1 and 2). As we considered $c \geq c' + 2 \geq 7$, the case $c = b + 3$ is discarded for contradicting (4.13).

This method will be used along the following cases to help prove the lemma.

If $c = b + 4$, then the lower bound of the left side of (4.13) is

$$\begin{aligned}\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2}& \\ &> 1 + \frac{1}{F(c)^2} + \frac{F(c-4)^2}{F(c)^2} - 3\frac{F(c-4)}{F(c)} - \frac{5}{16} \\ &> 1 + 0.1428^2 - 3(0.1539) - 5/16 > 0.2461 > 0\end{aligned}$$

for $c \geq 7$, contradicting again (4.13), so the case $c = b+4$ is not possible. Therefore, only the case $c' \geq c - 1$ is possible for $a = 2$.

Now, moving on to $a = 3$, consider the case $c' \leq c - 2$. Without loss of generality, we can assume that $c \geq 7$ and $b \geq a = 3$, as in the previous case. On the other hand, since $(2, F(b), F(c))$ is minimal, then $b \leq c - 4$ by Proposition 4.6.

If $c \geq b + 5 \geq 8$, we obtain the following lower bound for the left side of (4.13)

$$\begin{aligned}\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2} &> 1 + \frac{4}{F(c)^2} + \frac{F(c-5)^2}{F(c)^2} - 6\frac{F(c-5)}{F(c)} - \frac{5}{16} \\ &> 1 - 6\frac{F(c-5)}{F(c)} - \frac{5}{16} \\ &\geq 1 - 6(0.09524) - 5/16 > 0.1160 > 0\end{aligned}$$

for $c \geq 8$, which again contradicts (4.13). This implies that $c' \geq c - 1$ for $c \geq b + 5$.

On the other hand, if $c = b + 4 \geq 7$ and $c' \leq c - 3$, then $a' \leq b' \leq c - 6$. As a result, we obtain a lower bound for the left side of (4.13)

$$\begin{aligned}\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2}& \\ &> 1 + \frac{4}{F(c)^2} + \frac{F(c-4)^2}{F(c)^2} - 6\frac{F(c-4)}{F(c)} - \left(\frac{F(c-3)^2}{F(c)^2} + 2\frac{F(c-6)^2}{F(c)^2}\right) \\ &> 1 + 0.1428^2 - 6(0.1539) - (0.2381^2 + 2 \cdot 0.07693^2) > 0.02846 > 0\end{aligned}$$

for $c \geq 7$, obtaining a contradiction. Thus $c' \geq c - 2$ in this case, concluding the lemma. □

Lemma 4.14. *Let $(F(a), F(b), F(c))$ and $(F(a'), F(b'), F(c'))$ be two ordered minimal Markoff m-triples with $m > 0$ such that $c \geq c'$. If $a = 2$ or $a = 3$, then $c' = c$ or $c' = c - 1$.*

Proof. According to Lemma 4.13, we only need to prove that it is impossible to have two such triples with $a = 3$, $c = b+4$ and $c' = c-2$. The two m-triples are $(F(a), F(b), F(b+4))$ and $(F(a'), F(b'), F(b+2))$. The equation relating both is

$$\begin{aligned}F(a)^2 + F(b)^2 + F(b+4)^2 + 3F(a')F(b')F(b+2)& \\ = F(a')^2 + F(b')^2 + F(b+2)^2 + 3F(a)F(b)F(b+4).& \quad (4.17)\end{aligned}$$

Using $F(b+4) = 5F(b)+3F(b-1)$ and $F(b+2) = 2F(b)+F(b-1)$ in (4.17), simplifying we get

$$\begin{aligned} F(a)^2 + 8F(b-1)^2 + 3F(a')F(b')F(b+2) \\ = F(a')^2 + F(b')^2 + F(b)^2(15F(a)-22) + F(b)F(b-1)(9F(a)-26). \end{aligned} \tag{4.18}$$

As $F(a) = 2$, equation (4.18) reduces to

$$4 + 8F(b-1)^2 + 3F(a')F(b')F(b+2) = F(a')^2 + F(b')^2 + 8F(b)^2 - 8F(b)F(b-1)$$

giving

$$\begin{aligned} 4 + 8F(b)F(b-1) + 3F(a')F(b')F(b+2) &= F(a')^2 + F(b')^2 + 8(F(b)^2 - F(b-1)^2) \\ &= F(a')^2 + F(b')^2 + 8F(b+1)F(b-2), \end{aligned}$$

consequently,

$$\begin{aligned} 4 + 3F(a')F(b')F(b+2) &= F(a')^2 + F(b')^2 + 8(F(b+1)F(b-2) - F(b)F(b-1)) \\ &= F(a')^2 + F(b')^2 \pm 8, \end{aligned} \tag{4.19}$$

where we applied Vajda's identity with $n = b-2$, $i = 2$ and $j = 1$ (see Remark 3.6).

Note that for any ordered minimal m-triple (x, y, z), it follows from [(SC), Lema 2.2] that $x^2 + y^2 \leq m < z^2$ and hence

$$F(a')^2 + F(b')^2 - 3F(a')F(b')F(b+2) = m - F(b+2)^2 < 0. \tag{4.20}$$

Considering that, by equation (4.20), $F(a')^2 + F(b')^2 < 3F(a')F(b')F(b+2)$, the equation (4.19) reduces to

$$3F(a')F(b')F(b+2) = F(a')^2 + F(b')^2 + 4,$$

which is not possible as

$$3F(b')F(b+2) \leq 3F(a')F(b')F(b+2) = F(a')^2 + F(b')^2 + 4 \leq 2F(b')^2 + 4.$$

Therefore

$$3F(b+2) \leq 2F(b') + \frac{4}{F(b')} \leq 2F(b') + 4 < 2F(c') + 4 = 2F(b+2) + 4\,,$$

giving $F(b+2) < 4$, a contradiction as $b = c-4 \geq 3$.

□

Lemma 4.15. *Let $m > 0$. Let $(F(a), F(b), F(c))$ and $(F(a'), F(b'), F(c'))$ be two ordered minimal Markoff-Fibonacci m-triples with $a \leq b \leq c$, $a' \leq b' \leq c'$ and $c' = c-1$. If $c \geq 7$ then $a + b = c - 1$.*

Proof. By Proposition 4.6, $a + b \leq c - 1$. Suppose that $a + b \leq c - 2$. Let us prove that in this case $m(a, b, c) > m(a', b', c')$. This will lead to a contradiction, proving that $a + b = c - 1$. We will work analogously to Lemma 4.12. The same proof of equation (4.11) from that lemma shows that

$$m(a', b', c') < \frac{1}{5}\varphi^{2c'} = \frac{1}{5}\varphi^{2c-2}.$$

On the other hand, by Lemma 4.10 and Lemma 4.11, since $b \leq c-2-a < c$, $a \geq 2$ and $c \geq 7$, then

$$m(a,b,c) \geq m(a,c-2-a,c) > L_{2,2,7}\frac{1}{5}\varphi^{2c},$$

where $L_{A,t,C}$ is the constant defined at equation (4.6), which admits the following lower bound.

$$L_{2,2,7} = 1-\frac{3}{\sqrt{5}}\varphi^{-2}+\left(1-\frac{3}{\sqrt{5}}\varphi^{2}\right)(\varphi^{-8}+\varphi^{-10})-\left(6+\frac{3}{\sqrt{5}}\varphi^{2}+\frac{9}{\sqrt{5}}\right)\varphi^{-14} > 1.04\varphi^{-2} > \varphi^{-2}.$$

Consequently,

$$m(a,b,c) > L_{2,2,7}\frac{1}{5}\varphi^{2c} > \frac{1}{5}\varphi^{2c-2} > m(a',b',c')\,.$$

□

Lemma 4.16. *Let $m > 0$. Let $(F(a),F(b),F(c))$ and $(F(a'),F(b'),F(c'))$ be two ordered minimal Markoff m-triples such that $c' = c-1$ and $(a,a') \neq (2,2)$. Then, if $c \geq 19$ the following hold.*

- *If $a \neq 2,4$, then $a'+b' = c'-1$.*
- *If $a = 4$, then either $a'+b' = c'-1$ or $a'+b' = c'-2$.*
- *If $a = 2$, then $a'+b'+1 \leq c' \leq a'+b'+5$.*

Proof. First, assume that $a' \geq 3$. For two such triples, we have that $c' = c-1$, $c = a+b+1$ (Lemma 4.15) and $c' \geq a'+b'+1$ (Proposition 4.6). Since $b \geq a$, if we first assume that $a \geq 6$, then $c \geq b+7$ and $c \geq 2a+1$, so $a \leq \lfloor\frac{c-1}{2}\rfloor$. Since we are considering $c \geq 19$, we can affirm that $a \leq \lfloor\frac{c-1}{2}\rfloor \leq c-10$. Hence, we obtain the following upper bound for the left side of (4.13):

$$\begin{aligned}&\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2}\\ &\leq 1+\frac{F(c-7)^2}{F(c)^2}+\frac{F(c-10)^2}{F(c)^2}-3\frac{F(a)F(b)}{F(c)}-\left(\frac{F(c-1)^2}{F(c)^2}-\frac{3F(a')F(b')}{F(c)}\frac{F(c-1)}{F(c)}\right).\end{aligned} \tag{4.21}$$

We derive the following bounds for $F(n)$ based on Binet's formula (3.2)

$$\begin{aligned}F(n) &\leq \frac{1}{\sqrt{5}}\left(\varphi^n+\frac{1}{\varphi^n}\right), \quad F(n) \leq \frac{1}{\sqrt{5}}(\varphi^n+1)\,,\\ F(n) &\geq \frac{1}{\sqrt{5}}\left(\varphi^n-\frac{1}{\varphi^n}\right), \quad F(n) \geq \frac{1}{\sqrt{5}}(\varphi^n-1)\,.\end{aligned}$$

These bounds allow us to establish constraints for the crossed terms of (4.21) assuming that $c' \geq a'+b'+2$ (and then $c' \geq b'+5$):

$$\begin{aligned}3\frac{F(a)F(b)}{F(c)} &\geq \frac{3}{\sqrt{5}}\frac{(\varphi^a-\frac{1}{\varphi^a})(\varphi^b-\frac{1}{\varphi^b})}{\varphi^c+1}\\ &= \frac{3}{\sqrt{5}}\frac{\varphi^{a+b}-\frac{\varphi^b}{\varphi^a}-\frac{\varphi^a}{\varphi^b}+\frac{1}{\varphi^{a+b}}}{\varphi^c+1} \geq \frac{3}{\sqrt{5}}\frac{\varphi^{c-1}-\frac{\varphi^{c-7}}{\varphi^6}-1+\frac{1}{\varphi^{c-1}}}{\varphi^c+1}\,,\end{aligned}$$

$$\frac{3F(a')F(b')}{F(c)} \leq \frac{3}{\sqrt{5}} \frac{\left(\varphi^{a'}+\frac{1}{\varphi^{a'}}\right)\left(\varphi^{b'}+\frac{1}{\varphi^{b'}}\right)}{\varphi^{c}-1} = \frac{3}{\sqrt{5}} \frac{\varphi^{a'+b'}+\frac{\varphi^{b'}}{\varphi^{a'}}+\frac{\varphi^{a'}}{\varphi^{b'}}+\frac{1}{\varphi^{a'+b'}}}{\varphi^{c}-1}$$
$$\leq \frac{3}{\sqrt{5}} \frac{\varphi^{c'-2}+\frac{\varphi^{c'-5}}{\varphi^{3}}+2}{\varphi^{c}-1} = \frac{3}{\sqrt{5}} \frac{\varphi^{c-3}+\frac{\varphi^{c-6}}{\varphi^{3}}+2}{\varphi^{c}-1}.$$

Consequently, we obtain the following upper bound for the right side of (4.21), where we also have proceeded as in the proof of Lemma 4.13 to find the value of c. Using the Tables mentioned in Remark 1, we have

$$\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2} \leq 1 + \frac{F(c-7)^2}{F(c)^2} + \frac{F(c-10)^2}{F(c)^2} - \frac{3}{\sqrt{5}} \frac{\varphi^{c-1}-\varphi^{c-13}-1+\frac{1}{\varphi^{c-1}}}{\varphi^{c}+1}$$
$$- \left(\frac{F(c-1)^2}{F(c)^2} - \frac{3}{\sqrt{5}} \frac{\varphi^{c-3}+\varphi^{c-9}+2}{\varphi^{c}-1} \frac{F(c-1)}{F(c)}\right) \leq -0.0001. \quad (4.22)$$

for $c \geq 19$. This contradicts (4.13) for these values of c.

For $a = 5$, it follows that $b = c-6$ and we obtain the upper bound

$$\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2} \leq 1 + \frac{F(c-6)^2}{F(c)^2} + \frac{25}{F(c)^2} - \frac{15F(c-6)}{F(c)}$$
$$- \left(\frac{F(c-1)^2}{F(c)^2} - \frac{3}{\sqrt{5}} \frac{\varphi^{c-3}+\varphi^{c-9}+2}{\varphi^{c}-1} \frac{F(c-1)}{F(c)}\right) \leq -0.003 \quad (4.23)$$

for $c \geq 13$, which contradicts (4.13), for $c \geq 19$.

For $a = 3$, it follows that $b = c-4$ and then

$$\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2} \leq 1 + \frac{F(c-4)^2}{F(c)^2} + \frac{4}{F(c)^2} - \frac{6F(c-4)}{F(c)}$$
$$- \left(\frac{F(c-1)^2}{F(c)^2} - \frac{3}{\sqrt{5}} \frac{\varphi^{c-3}+\varphi^{c-9}+2}{\varphi^{c}-1} \frac{F(c-1)}{F(c)}\right) \leq -0.009 \quad (4.24)$$

for $c \geq 10$, consequently, we find again a contradiction with (4.13) for $c \geq 19$. This implies that $c' = a'+b'+1$ for $a \geq 5$ or $a = 3$ if $a' \geq 3$ and the other conditions are fulfilled, as desired.

For $a = 4$, then $b = c-5$, obtaining

$$\frac{m(a,b,c)}{F(c)^2} = 1 + \frac{F(c-5)^2}{F(c)^2} + \frac{9}{F(c)^2} - \frac{9F(c-5)}{F(c)}.$$

On the other hand, assuming that $c' \geq a'+b'+3$ we obtain that $c = c'+1 \geq a'+b'+4 \geq b'+7$, $c = c'+1 \geq a'+b'+4 \geq 2a'+4$, therefore

$$\frac{3F(a')F(b')}{F(c)} \leq \frac{3}{\sqrt{5}} \frac{\varphi^{a'+b'}+\frac{\varphi^{b'}}{\varphi^{a'}}+\frac{\varphi^{a'}}{\varphi^{b'}}+\frac{1}{\varphi^{a'+b'}}}{\varphi^{c}-1} \leq \frac{3}{\sqrt{5}} \frac{\varphi^{c-4}+\varphi^{c-10}+2}{\varphi^{c}-1}.$$

The upper bound for (4.13) is

$$\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2} \leq 1 + \frac{F(c-5)^2}{F(c)^2} + \frac{9}{F(c)^2} - \frac{9F(c-5)}{F(c)}$$
$$- \left(\frac{F(c-1)^2}{F(c)^2} - \frac{3}{\sqrt{5}} \frac{\varphi^{c-4}+\varphi^{c-10}+2}{\varphi^{c}-1} \frac{F(c-1)}{F(c)}\right) \leq -0.007 \quad (4.25)$$

for $c \geq 9$, so we get a contradiction. As a result, $a' + b' + 1 \leq c' \leq a' + b' + 2$. Since the parameters are integers, either $c' = a' + b' + 1$ or $c' = a' + b' + 2$, as desired in this case.

For $a = 2$, then $b = c - 3$ and

$$\frac{m(a,b,c)}{F(c)^2} = 1 + \frac{F(c-3)^2}{F(c)^2} + \frac{1}{F(c)^2} - \frac{3F(c-3)}{F(c)}.$$

On the other hand, assuming that $c' \geq a'+b'+6$ we find that $c = c'+1 \geq a'+b'+7 \geq b'+10$, in consequence

$$\frac{3F(a')F(b')}{F(c)} \leq \frac{3}{\sqrt{5}} \frac{\varphi^{a'+b'} + \frac{\varphi^{b'}}{\varphi^{a'}} + \frac{\varphi^{a'}}{\varphi^{b'}} + \frac{1}{\varphi^{a'+b'}}}{\varphi^c - 1} \leq \frac{3}{\sqrt{5}} \frac{\varphi^{c-7} + \varphi^{c-13} + 2}{\varphi^c - 1}.$$

This gives the following upper bound for (4.13)

$$\begin{aligned}\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2} &\leq 1 + \frac{F(c-3)^2}{F(c)^2} + \frac{1}{F(c)^2} - \frac{3F(c-3)}{F(c)} \\ &\quad - \left(\frac{F(c-1)^2}{F(c)^2} - \frac{3}{\sqrt{5}} \frac{\varphi^{c-7} + \varphi^{c-13} + 2}{\varphi^c - 1} \frac{F(c-1)}{F(c)}\right) \leq -0.0007\end{aligned}$$

for $c \geq 13$, a contradiction with (4.13). This implies that $a' + b' + 1 \leq c' \leq a' + b' + 5$, as desired in this case.

Assume that $a' = 2$, $a \geq 3$. Now, if $c' \geq a' + b' + 2$, then $c' \geq b' + 4$. This implies that $b' \leq c - 5$, so

$$\begin{aligned}&\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2} \\ &\quad \leq 1 + \frac{F(c-5)^2}{F(c)^2} + \frac{9}{F(c)^2} - \frac{9F(c-5)}{F(c)} - \left(\frac{F(c-1)^2}{F(c)^2} - \frac{3F(c-5)}{F(c)}\frac{F(c-1)}{F(c)}\right) \\ &\quad \leq -0.014\end{aligned}$$

for $c \geq 11$, a contradiction with (4.13). Hence $c' = a' + b' + 1$ also in this case and we are done. □

Lemma 4.17. *Let $(F(a), F(b), F(c))$ and $(F(a'), F(b'), F(c'))$ be two ordered minimal Markoff-Fibonacci m-triples such that $c' = c - 1$. If $c \geq 11$, then $(a, a') \neq (2, 2)$.*

Proof. Assume that $m(2, b, c) = m(2, b', c - 1)$ for some b, b' and c, with $c \geq 11$. By Lemma 4.15, $b = c - 3$. Thus, we obtain

$$\begin{aligned}1 + F(c-3)^2 + F(c)^2 - 3F(c-3)F(c) &= m(2, c-3, c) \\ &= m(2, b', c-1) \\ &= 1 + F(b')^2 + F(c-1)^2 - 3F(b')F(c-1).\end{aligned}$$

We can simplify this equality to get

$$\begin{aligned}F(b')^2 - 3F(b')F(c-1) &= F(c-3)^2 + F(c)^2 - F(c-1)^2 - 3F(c-3)F(c) \\ &= F(c-3)^2 + F(c+1)F(c-2) - 3F(c-3)F(c) \\ &= k.\end{aligned}$$

By Vajda's identity (3.6) we have $F(c+1)F(c-2) = F(c+2)F(c-3) + 3(-1)^{c-3}$, so

$$\begin{aligned} k &= F(c-3)\left(F(c-3) + F(c+2) - 3F(c)\right) + 3(-1)^{c-3} \\ &= -F(c-3)F(c-4) + 3(-1)^{c-3} . \end{aligned}$$

The equation $F(b')^2 - 3F(b')F(c-1) = k$ is a quadratic equation in $F(b')$. Let $p(x) = x^2 - 3F(c-1)x$. Since $k < 0$ for all $c \geq 7$, the parabola $p(x) = k$ has two positive roots. One of them is greater than its vertex, $\frac{3}{2}F(c-1)$, which is greater than $F(c-1)$, so it cannot be $F(b')$, since $b' \leq c-1$. We will prove that the other root, belonging to the interval $(0, \frac{3}{2}F(c-1))$ is not a Fibonacci number by showing that there exists a Fibonacci number $F(b^*) < F(c-1)$ such that $p(F(b^*-1)) > k > p(F(b^*))$. Since the function $p(x)$ is strictly decreasing in the interval $(0, F(c-1))$, this will prove that the root belongs to the open interval $(F(b^*-1), F(b^*))$ and, therefore, that it cannot be a Fibonacci number. Concretely, we will show that for all $c \geq 11$

$$p(F(c-9)) > k > p(F(c-8)) .$$

Dividing both sides of the equation by $F(c-1)^2$, this inequality is equivalent to

$$\frac{F(c-9)^2}{F(c-1)^2} - 3\frac{F(c-9)}{F(c-1)} > -\frac{F(c-3)}{F(c-1)}\frac{F(c-4)}{F(c-1)} + \frac{3(-1)^{c-3}}{F(c-1)^2} > \frac{F(c-8)^2}{F(c-1)^2} - 3\frac{F(c-8)}{F(c-1)} . \tag{4.26}$$

Using the Tables 1 and 2, we can bound above and below the left-hand side, right-hand side and middle part of the previous equation as follows. For each $c \geq 11$, the following holds

$$\begin{aligned} LHS &= \frac{F(c-9)^2}{F(c-1)^2} - 3\frac{F(c-9)}{F(c-1)} \geq -0.06711 , \\ RHS &= \frac{F(c-8)^2}{F(c-1)^2} - 3\frac{F(c-8)}{F(c-1)} \leq -0.09978 , \end{aligned}$$

$$\begin{aligned} -0.08909 &\geq -\frac{F(c-4)}{F(c-1)}\frac{F(c-3)}{F(c-1)} + \frac{3}{F(c-1)^2} \\ &\geq -\frac{F(c-4)}{F(c-1)}\frac{F(c-3)}{F(c-1)} + \frac{3(-1)^{c-3}}{F(c-1)^2} \\ &\geq -\frac{F(c-4)}{F(c-1)}\frac{F(c-3)}{F(c-1)} - \frac{3}{F(c-1)^2} \geq -0.09132 . \end{aligned}$$

As

$$LHS \geq -0.06711 > -0.08909 \geq \frac{k}{F(c-1)^2} \geq -0.09132 > -0.09978 \geq RHS ,$$

the lemma follows. □

We can now combine all the previous results to show that no pair of minimal Markoff-Fibonacci m-triples can exist when $c' < c$, if c is big enough.

Lemma 4.18. *Let $(F(a), F(b), F(c))$ and $(F(a'), F(b'), F(c'))$ be two ordered minimal Markoff-Fibonacci m-triples. If $c \geq 20$ then $c = c'$.*

Proof. As a consequence of Lemma 4.12 and Lemma 4.14, we know that either $c' = c$ or $c' = c - 1$. It is therefore enough to prove that the case $c' = c - 1$ is impossible. Assume that $c' = c - 1$. By Lemma 4.15, we know that $a + b = c - 1$. Due to Lemma 4.17, we know that $(a, a') \neq (2, 2)$, so we can apply Lemma 4.16 and obtain that $a' + b' = c' - 1$ if $a \neq 4$ and that $a' + b' = c' - 1$ or $a' + b' = c' - 2$ if $a = 4$ or that $c - 6 = c' - 5 \leq a' + b' \leq c' - 1 = c - 2$ if $a = 2$.

Let us first analyse the case where $a \geq 3$ and $a' + b' = c' - 1 = c - 2$. Suppose that

$$m(a, c - 1 - a, c) = m(a', c - 2 - a', c - 1) .$$

Applying Lemma 4.11, we have that, as $a \geq 3$ and $c \geq 9$, then

$$m(a, c - 1 - a, c) > L_{3,1,9} \frac{1}{5} \varphi^{2c} ,$$
$$m(a', c - 2 - a', c - 1) < U_{2,1,8} \frac{1}{5} \varphi^{2(c-1)} = \varphi^{-2} U_{2,1,8} \frac{1}{5} \varphi^{2c} .$$

A direct computation shows that $L_{3,1,9} > 0.14 > 0.139 > \varphi^{-2} U_{2,1,8}$, so $m(a, c - 1 - a - c) > m(a', c - 2 - a', c - 1)$ and, therefore, the case $a' + b' = c' - 1$ is impossible for $a \geq 3$. By Lemma 4.16, the only two remaining cases to prove the result are the following: either $a = 2$ and $c - 6 = c' - 5 \leq a' + b' \leq c' - 1 = c - 2$ or $a = 4$ and $a' + b' = c' - 2 = c - 3$.

Let us begin analysing the $a = 2$ case, where for each $t = 1, \ldots, 5$, we will consider that

$$m(a, c - 1 - a, c) = m(a', c - 1 - t - a', c - 1) . \tag{4.27}$$

Here, we find

$$\begin{aligned} m(a, c - 1 - a, c) &= m(2, c - 3, c) = 1 + F(c - 3)^2 + F(c)^2 - 3F(c - 3)F(c) \\ &> 1 + \frac{1}{5}\varphi^{2c} + \frac{1}{5}\varphi^{2c-6} - \frac{4}{5} - \frac{3}{5}(\varphi^{c-3} + \varphi^{-c+3})(\varphi^c + \varphi^{-c}) \\ &> \frac{1}{5}\varphi^{2c} + \frac{1}{5}\varphi^{2c-6} - \frac{3}{5}\varphi^{2c-3} - \frac{3}{5}\varphi^{-3} - \frac{3}{5}\varphi^{3} - \frac{3}{5}\varphi^{-2c+3} . \end{aligned}$$

Thus, for $c \geq 11$

$$\frac{5m(a, c - 1 - a, c)}{\varphi^{2c}} > 1 + \varphi^{-6} - 3\varphi^{-3} - 3\varphi^{-25} - 3\varphi^{-19} - 3\varphi^{-41} > 0.347 .$$

On the other hand, since $(a, a') \neq (2, 2)$, we can deduce that $a' \geq 3$. Using Lemma 4.11 for each $t = 1, \ldots, 5$ it follows that

$$\frac{5m(a', c - 1 - t - a', c - 1)}{\varphi^{2c}} < \varphi^{-2} - \frac{3}{\sqrt{5}}\varphi^{-2-t} + \left(1 + \frac{3}{\sqrt{5}}\varphi^{t}\right)(\varphi^{-2t-6} + \varphi^{6-2c})\varphi^{-2} + 9\varphi^{-2c-2} .$$

The maximum of the right hand side for $1 \leq t \leq 5$ and $c' = c - 1 \geq C = 10$ is attained at $t = 5$ and $c = 11$, and yields

$$\frac{5\, m(a', c - 1 - t - a', c - 1)}{\varphi^{2c}} < \varphi^{-2} - \frac{3}{\sqrt{5}}\varphi^{-7} + \left(1 + \frac{3}{\sqrt{5}}\varphi^{5}\right)(\varphi^{-16} + \varphi^{-18}) + 9\varphi^{-22} < 0.346 .$$

Therefore,

$$\frac{5\, m(a, c - 1 - a, c)}{\varphi^{2c}} > 0.347 > 0.346 > \frac{5\, m(a', c - 1 - t - a', c - 1)}{\varphi^{2c}} ,$$

so the case $a = 2$ is impossible as contradiction with (4.27).

Now, let us examine the case $a = 4$ and $c' = a' + b' + 2$. In this instance, the value of $\frac{m(a,b,c)}{F(c)^2}$ can be derived as a consequence of Lemma 4.15

$$\frac{m(a,b,c)}{F(c)^2} = 1 + \frac{F(c-5)^2}{F(c)^2} + \frac{9}{F(c)^2} - \frac{9F(c-5)}{F(c)}. \tag{4.28}$$

On the other hand, if $a' \geq 7$, then $c - 1 = c' = a' + b' + 2 \geq b' + 9$, implying $b' \leq c - 10$. Also, $c - 1 = c' = a' + b' + 2 \geq 2a' + 2$, hence $a' \leq \lfloor \frac{c-3}{2} \rfloor$. Since we are considering $c \geq 20$, it is true to say that $a' \leq \lfloor \frac{c-3}{2} \rfloor \leq c - 12$. Consequently, the left-hand side of equation (4.13) has the following lower bound:

$$\begin{aligned}\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2} \geq{}& 1 + \frac{F(c-5)^2}{F(c)^2} + \frac{9}{F(c)^2} - \frac{9F(c-5)}{F(c)} \\ & - \left(\frac{F(c-1)^2}{F(c)^2} + \frac{F(c-12)^2}{F(c)^2} + \frac{F(c-10)^2}{F(c)^2} - 3\frac{F(a')F(b')}{F(c)}\frac{F(c-1)}{F(c)}\right).\end{aligned}$$

The last term of this expression is bounded from below by

$$\begin{aligned}3\frac{F(a')F(b')}{F(c)}\frac{F(c-1)}{F(c)} &\geq \frac{3}{\sqrt{5}}\frac{(\varphi^{a'}-1)(\varphi^{b'}-1)}{\varphi^c+1}\frac{F(c-1)}{F(c)} \\ &= \frac{3}{\sqrt{5}}\frac{(\varphi^{a'+b'}-\varphi^{a'}-\varphi^{b'}+1)}{\varphi^c+1}\frac{F(c-1)}{F(c)} \\ &\geq \frac{3}{\sqrt{5}}\frac{(\varphi^{c-2}-\varphi^{c-12}-\varphi^{c-10}+1)}{\varphi^c+1}\frac{F(c-1)}{F(c)}.\end{aligned}$$

This gives the following lower bound for $\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2}$.

$$\begin{aligned}&\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2} \geq 1 + \frac{F(c-5)^2}{F(c)^2} + \frac{9}{F(c)^2} - \frac{9F(c-5)}{F(c)} \\ &- \left(\frac{F(c-1)^2}{F(c)^2} + \frac{F(c-12)^2}{F(c)^2} + \frac{F(c-10)^2}{F(c)^2} - \frac{3}{\sqrt{5}}\frac{(\varphi^{c-3}-\varphi^{c-12}-\varphi^{c-10}+1)}{\varphi^c+1}\frac{F(c-1)}{F(c)}\right) \\ &\geq 0.0009\end{aligned}$$

for $c \geq 20$. This contradicts (4.13).

For the case $a' = 6$, then $b' = c' - 8 = c - 9$ and it can be seen that the value of $\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2}$ has the following lower bound

$$\begin{aligned}\frac{m(a,b,c)}{F(c)^2} - \frac{m(a',b',c')}{F(c)^2} ={}& 1 + \frac{F(c-5)^2}{F(c)^2} + \frac{9}{F(c)^2} - \frac{9F(c-5)}{F(c)} \\ & - \left(\frac{F(c-1)^2}{F(c)^2} + \frac{64}{F(c)^2} + \frac{F(c-9)^2}{F(c)^2} - 24\frac{F(c-9)F(c-1)}{F(c)^2}\right) \geq 0.001161 > 0\end{aligned}$$

for $c \geq 12$, contradicting again (4.13).

For the case $a'=5$, then $b'=c'-7=c-8$ and we can see that the value of $\frac{m(a,b,c)}{F(c)^2}-\frac{m(a',b',c')}{F(c)^2}$ has the following lower bound

$$\begin{aligned}\frac{m(a,b,c)}{F(c)^2}-\frac{m(a',b',c')}{F(c)^2} &= 1+\frac{F(c-5)^2}{F(c)^2}+\frac{9}{F(c)^2}-\frac{9F(c-5)}{F(c)}\\ &-\left(\frac{F(c-1)^2}{F(c)^2}+\frac{25}{F(c)^2}+\frac{F(c-8)^2}{F(c)^2}-15\frac{F(c-8)F(c-1)}{F(c)^2}\right)\geq 0.002911>0\end{aligned}$$

for $c\geq 11$, contradicting again (4.13).

For the case $a'=4$, then $b'=c'-6=c-7$ and we can see that the value of $\frac{m(a,b,c)}{F(c)^2}-\frac{m(a',b',c')}{F(c)^2}$ has the following lower bound

$$\begin{aligned}\frac{m(a,b,c)}{F(c)^2}-\frac{m(a',b',c')}{F(c)^2} &= 1+\frac{F(c-5)^2}{F(c)^2}+\frac{9}{F(c)^2}-\frac{9F(c-5)}{F(c)}\\ &-\left(\frac{F(c-1)^2}{F(c)^2}+\frac{9}{F(c)^2}+\frac{F(c-7)^2}{F(c)^2}-9\frac{F(c-7)F(c-1)}{F(c)^2}\right)\geq 0.002857>0\end{aligned}$$

for $c\geq 12$, contradicting again (4.13).

For the case $a'=3$, then $b'=c'-5=c-6$ and we can see that the value of $\frac{m(a,b,c)}{F(c)^2}-\frac{m(a',b',c')}{F(c)^2}$ has the following lower bound

$$\begin{aligned}\frac{m(a,b,c)}{F(c)^2}-\frac{m(a',b',c')}{F(c)^2} &= 1+\frac{F(c-5)^2}{F(c)^2}+\frac{9}{F(c)^2}-\frac{9F(c-5)}{F(c)}\\ &-\left(\frac{F(c-1)^2}{F(c)^2}+\frac{4}{F(c)^2}+\frac{F(c-6)^2}{F(c)^2}-6\frac{F(c-6)F(c-1)}{F(c)^2}\right)\geq 0.002604>0\end{aligned}$$

for $c\geq 9$, contradicting again (4.13).

Finally, if $a'=2$, then $b'=c'-4=c-5$ and the value of $\frac{m(a,b,c)}{F(c)^2}-\frac{m(a',b',c')}{F(c)^2}$ now has the following upper bound

$$\begin{aligned}\frac{m(a,b,c)}{F(c)^2}-\frac{m(a',b',c')}{F(c)^2} &= 1+\frac{F(c-5)^2}{F(c)^2}+\frac{9}{F(c)^2}-\frac{9F(c-5)}{F(c)}\\ &-\left(\frac{F(c-1)^2}{F(c)^2}+\frac{1}{F(c)^2}+\frac{F(c-5)^2}{F(c)^2}-3\frac{F(c-5)F(c-1)}{F(c)^2}\right)\leq -0.007283<0\end{aligned}$$

for $c\geq 9$, a contradiction. □

To complete our result, we need to prove that there do not exist two minimal Markoff-Fibonacci triples with the same highest element and the same m. In other words, let us prove now that if

$$m(a,b,c)=m(a',b',c)\,,$$

then $a=a'$ and $b=b'$.

Let us suppose that $m(a,b,c)=m(a',b',c)$, for some $(a,b)\neq(a',b')$. Assume without loss of generality that $a\leq a'$. By Lemma 4.10, if $b\leq b'$ and $(a,b)\neq(a',b')$ then $m(a,b,c)>m(a',b',c)$. Thus, we must have $a\leq a'\leq b'<b$. On the other hand, if $a=a'$ and $b'<b$, then again by Lemma 4.10 it follows that $m(a,b',c)>m(a,b,c)$. Consequently, we can infer without loss of generality that

$$a<a'\leq b'<b\leq c\,.$$

Lemma 4.19. *Let $(F(a), F(b), F(c))$ and $(F(a'), F(b'), F(c))$ be two ordered minimal Markoff-Fibonacci m-triples with $2 \leq a < a' \leq b' < b \leq c$, then $a + b = a' + b'$.*

Proof. Rearranging the equation

$$F(a)^2 + F(b)^2 + F(c)^2 - 3F(a)F(b)F(c) = F(a')^2 + F(b')^2 + F(c)^2 - 3F(a')F(b')F(c)$$

yields

$$F(a)^2 + F(b)^2 - F(a')^2 - F(b')^2 = 3F(c)\left(F(a)F(b) - F(a')F(b')\right) . \tag{4.29}$$

The left-hand side is always positive because, as $b \geq 4$ and $a' \leq b' < b$, by Table 2 it follows that

$$F(b)^2 > 2F(b-1)^2 \geq F(b')^2 + F(a')^2.$$

Let us see that this is impossible if $a' + b' > a + b$. Assume that $a' + b' = a + b + t$ with $t > 0$. As a result

$$\begin{aligned}
\frac{F(a')F(b')}{F(a)F(b)} &= \frac{(\varphi^{a'} - \bar{\varphi}^{a'})(\varphi^{b'} - \bar{\varphi}^{b'})}{(\varphi^{a} - \bar{\varphi}^{a})(\varphi^{b} - \bar{\varphi}^{b})} \\
&\geq \frac{(\varphi^{a'} - \varphi^{-a'})(\varphi^{b'} - \varphi^{-b'})}{(\varphi^{a} + \varphi^{-a})(\varphi^{b} + \varphi^{-b})} \\
&= \frac{\varphi^{a'+b'} - \varphi^{b'-a'} - \varphi^{a'-b'} + \varphi^{-a'-b'}}{\varphi^{a+b} + \varphi^{b-a} + \varphi^{a-b} + \varphi^{-a-b}} .
\end{aligned}$$

Let $s = a + b$, then $a' + b' = s + t$. Dividing the numerator and denominator by φ^s yields

$$\begin{aligned}
\frac{\varphi^{a'+b'} - \varphi^{b'-a'} - \varphi^{a'-b'} + \varphi^{-a'-b'}}{\varphi^{a+b} + \varphi^{b-a} + \varphi^{a-b} + \varphi^{-a-b}} &= \frac{\varphi^{t} - \varphi^{t-2a'} - \varphi^{t-2b'} + \varphi^{-2s-t}}{1 + \varphi^{-2a} + \varphi^{-2b} + \varphi^{-2s}} \\
&= \varphi^{t}\frac{1 - \varphi^{-2a'} - \varphi^{-2b'} + \varphi^{-2s-2t}}{1 + \varphi^{-2a} + \varphi^{-2b} + \varphi^{-2s}} .
\end{aligned}$$

As $2 \leq a < a' \leq b' < b$, then $a \geq 2$, $a' \geq 3$, $b' \geq 3$, $b \geq 4$ and $s = a + b \geq 6$. Thus

$$\varphi^{t}\frac{1 - \varphi^{-2a'} - \varphi^{-2b'} + \varphi^{-2s-2t}}{1 + \varphi^{-2a} + \varphi^{-2b} + \varphi^{-2s}} \geq \varphi\frac{1 - 2\varphi^{-6}}{1 + \varphi^{-4} + \varphi^{-8} + \varphi^{-12}} \geq 1.22 > 1 .$$

Therefore, $F(a')F(b') > F(a)F(b)$, which contradicts the positivity of both sides of equation (5.13).

Thus, we must have $a + b \geq a' + b'$. Assume that $a' + b' = a + b - t$ with $t > 0$ and let $s = a + b$ as before. Analogously to the previous case,

$$\begin{aligned}
\frac{F(a')F(b')}{F(a)F(b)} &\leq \frac{(\varphi^{a'} + \varphi^{-a'})(\varphi^{b'} + \varphi^{-b'})}{(\varphi^{a} - \varphi^{-a})(\varphi^{b} - \varphi^{-b})} \\
&= \varphi^{-t}\frac{1 + \varphi^{-2a'} + \varphi^{-2b'} + \varphi^{-2s-2t}}{1 - \varphi^{-2a} - \varphi^{-2b} + \varphi^{-2s}} \\
&\leq \varphi^{-1}\frac{1 + 2\varphi^{-6} + \varphi^{-14}}{1 - \varphi^{-4} - \varphi^{-8}} < 0.83 < \frac{8}{9} .
\end{aligned}$$

As a consequence, it follows that

$$1-\frac{F(a')F(b')}{F(a)F(b)}>1-\frac{8}{9}=\frac{1}{9}\geq\frac{1}{9F(a)^2}\,.$$

Multiplying both sides by $3F(a)F(b)F(c)$, yields

$$3F(c)\left(F(a)F(b)-F(a')F(b')\right)>\frac{F(c)F(b)}{3F(a)}\,.$$

As we assumed that $(F(a),F(b),F(c))$ is minimal, then $F(c)\geq 3F(a)F(b)$ therefore

$$3F(c)\left(F(a)F(b)-F(a')F(b')\right)>\frac{F(c)F(b)}{3F(a)}\geq F(b)^2>F(b)^2-F(b')^2+F(a)^2-F(a')^2\,.$$

This contradicts equation (5.13), so $a+b$ cannot be greater than $a'+b'$. Thus $a+b=a'+b'$. □

4.2.3 Proof of the main theorem

Finally, we combine all the previous results to establish the main theorem of the paper (Theorem 4.1), proving first an intermediary proposition.

Proposition 4.20. *For each $m>0$, except $m=21$, there exists at most one minimal Markoff-Fibonacci m-triple. For $m=21$, there exist exactly two minimal Fibonacci triples: $(F(3),F(3),F(7))$ and $(F(2),F(3),F(6))$.*

Proof. Let $(F(a),F(b),F(c))$ and $(F(a'),F(b'),F(c'))$ be a pair of ordered minimal Markoff-Fibonacci m-triples with $2\leq a\leq b\leq c$, $2\leq a'\leq b'\leq c'$ contradicting the proposition. Assume without loss of generality that $c\geq c'$. From the computational verification stated in Lemma 4.9, we know that any counterexample to this theorem must have $c\geq 20$. By Lemma 4.18 it follows that $c=c'$. Moreover, by Lemma 5.11, we must have $a+b=a'+b'$. Taking $n=a$, $i=b'-a$ and $j=b-b'=a'-a$ in Vajda's identity (3.6), we can transform equation (5.13) into

$$\begin{aligned}F(a)^2+F(b)^2-F(a')^2-F(b')^2&=3F(c)\left(F(a)F(b)-F(a')F(b')\right)\\&=(-1)^{a+1}3F(c)F(b'-a)F(b-b')\,,\end{aligned}$$

From the proof of Lemma 5.11, we know that the left-hand side of this equality is positive, therefore a is odd, and then

$$F(a)^2+F(b)^2-F(a')^2-F(b')^2=3F(c)F(b'-a)F(b-b')\,.\tag{4.30}$$

However, using Lemma 3.3, we know that

$$F(b)\leq 3F(b')F(b-b')\leq 9F(a)F(b'-a)F(b-b')\,.$$

Multiplying by $F(b)$ and using minimality, $3F(a)F(b)\leq F(c)$, yields

$$F(b)^2\leq 3F(c)F(b'-a)F(b-b')$$

and, therefore,

$$F(b)^2-F(b')^2+F(a)^2-F(a')^2<F(b)^2\leq 3F(c)F(b'-a)F(b-b')\,,$$

which contradicts equation (5.16). Consequently, there is no possible counterexample to the theorem with $c\geq 20$ and the result follows. □

Proof of Theorem 4.1: Suppose that $m = 2$. By Proposition 4.7, the non-minimal Markoff-Fibonacci 2-triples are $(1, F(b), F(b+2))$ for even $b > 2$. By [(SC)], there only exists a minimal 2-triple, which is $(1, 1, 3) = (1, F(2), F(4))$. Thus all 2-triples are given by $(1, F(b), F(b+2))$, for even b.

For $m = 21$, [(SC)] showed that there exist exactly two minimal 21-triples, which are $(1, 2, 8) = (F(2), F(3), F(6))$ and $(2, 2, 13) = (F(3), F(3), F(7))$, so both of them are Markoff-Fibonacci triples. From Proposition 4.7 we know that all the non-minimal Markoff-Fibonacci triples have $m = 2$, so there are no more Markoff-Fibonacci m-triples for $m = 21$.

Let us assume now that $m > 0$, $m \neq 2$ and $m \neq 21$. Again, from Proposition 4.7, we know that m cannot admit a non-minimal Markoff-Fibonacci triple. Thus, any Markoff-Fibonacci m-triple must be minimal and, by Proposition 4.20, one such triple must exist at most.

Finally, by Proposition 4.8, we know that there is an infinite number of values of m for which the m-Markoff equation admits a Markoff-Fibonacci m-triple and only two values (2 and 21) admit more than one triple. The rest admit exactly one solution which, according to the previous argument, must be a minimal triple.

5 Markoff m-triples with k-Fibonacci components

The second result of this project is the research paper titled *Markoff m-triples with k-Fibonacci components* [(ACMRS2)], currently published in the journal *Mediterranean Journal of Mathematics.*

As a reminder, when $k = 1$, the sequence corresponds to the classic Fibonacci numbers, and for $k = 2$, it yields Pell numbers. Some particular cases of Markoff m-triples with k-Fibonacci components have already been studied: ($k = 1$, $m = 0$), was studied in [(LS)]; ($k = 2$, $m = 0$), was examined in [(KST)]; ($k > 1$, $m = 0$), was treated in [(Gom)]; the case $m = 0$, with Lucas sequences in [(AL)], [(RSP)] and, finally, the case ($k = 1$, $m > 0$) was dealt with in [(ACMRS1)], which corresponds to the previous section. Because of this, henceforth, we will assume that $m > 0$ and $k \geq 2$.

In it, we classified all Markoff m-triples with k-Fibonacci components, using the minimality concept explained in Section 2.4. The main results of the paper are summarized in the following theorem.

Theorem 5.1. *Every non-minimal Markoff m-triple with k-Fibonacci components and $m > 0$ is a Markoff 8-triple of the form $(F_2(2), F_2(2n), F_2(2n+2))$, for $n \geq 2$.*

In particular, the non-minimal Markoff m-triples with k-Fibonacci components are situated on the upper branch of the 8-tree with minimal triple $(2, 2, 12)$. The triples in this branch are composed of Pell numbers, as shown in Figure 4.

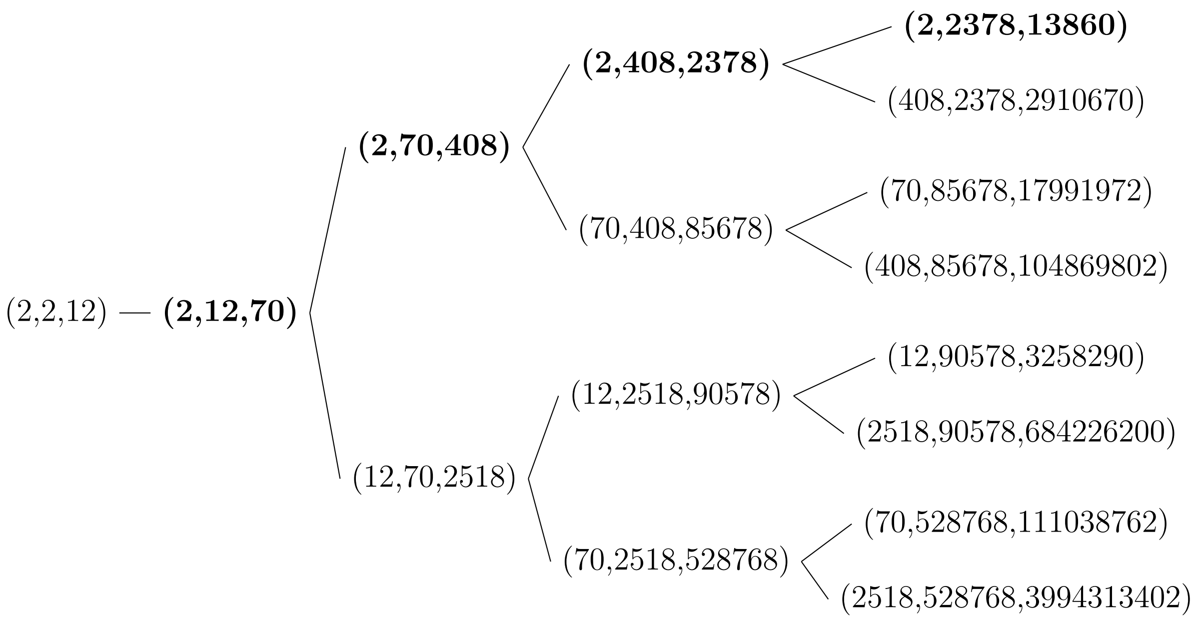

Figure 4: Beginning of the Markoff 8-tree with minimal triple $(2, 2, 12)$. The sequence of non-minimal 8-Markoff triples with 2-Fibonacci components (Pell components) is represented in bold.

Theorem 5.2. *If $m > 0$ admits a minimal Markoff m-triple with k-Fibonacci components, then it is unique, except for $k = 3$ and all pairs of triples $(F_3(a), F_3(b), F_3(a+b))$, $(F_3(a+$*

$1), F_3(b-1), F_3(a+b))$, *for a odd and b even with $b \geq a+3$.*

Along this section, several proofs will be presented to show the validity of both Theorem 5.1 and Theorem 5.2.

5.1 Non-minimal case

Analogously to previous Chapter, for positive integers a, b, c, we shall denote

$$m_k(a,b,c) = F_k(a)^2 + F_k(b)^2 + F_k(c)^2 - 3F_k(a)F_k(b)F_k(c),$$

so that $(F_k(a), F_k(b), F_k(c))$ is a Markoff m-triple with k-Fibonacci components if and only if $m_k(a,b,c) > 0$. In this section, after deriving conditions on (a,b,c) for which $m_k(a,b,c) \leq 0$, as a straightforward consequence, Theorem 5.1 will be proven, showing that there exists only one branch of non-minimal Markoff m-triples with k-Fibonacci components. Note that $k \geq 2$ will be considered, since the case $k=1$ was previously treated in Chapter 4.

Lemma 5.3.

(1) *For $a \geq 3$, if $c \leq a+b$, then $m_2(a,b,c) \leq 0$.*

(2) *For $a \geq 1$, if $c < a+b$, then $m_k(a,b,c) \leq 0$, for all $k \geq 3$.*

Proof. We start with (2). We have

$$2F_k(a+1) = 2(kF_k(a) + F_k(a-1)) \leq 2(k+1)F_k(a) \leq 3kF_k(a), \tag{5.1}$$

for $k \geq 2$. Next, from equation (3.15) and (5.1) above, we obtain

$$F_k(a+b) \leq 2F_k(a+1)F_k(b) \leq 3kF_k(a)F_k(b). \tag{5.2}$$

Also, since $c \leq a+b-1$, from (5.2) above,

$$F_k(c+1)F_k(c) \leq F_k(a+b)F_k(c) \leq 3kF_k(a)F_k(b)F_k(c). \tag{5.3}$$

Now, by Lemma 3.6, assuming a, b, c distinct or $a = b < c-1$, we have

$$F_k(a)^2 + F_k(b)^2 + F_k(c)^2 \leq \frac{F_k(c+1)F_k(c)}{k}. \tag{5.4}$$

Then, (5.3) and (5.4) yield

$$F_k(a)^2 + F_k(b)^2 + F_k(c)^2 \leq 3F_k(a)F_k(b)F_k(c),$$

which is equivalent to $m_k(a,b,c) \leq 0$.

Observe that in the case $a \leq b = c$, we trivially have $m_k(a,b,c) \leq 0$, as shown below

$$\begin{aligned} m_k(a,b,c) &= F_k(a)^2 + 2F_k(b)^2 - 3F_k(a)F_k(b)^2 \\ &\leq F_k(b)^2 + 2F_k(b)^2 - 3F_k(a)F_k(b)^2 \\ &= 3F_k(b)^2 - 3F_k(a)F_k(b)^2 \leq 0\,, \end{aligned}$$

since $F_k(a) \geq 1$.

Next, we prove the remaining case $a = b = c-1$. As $F_k(c) \leq (k+1)F_k(c-1)$, we have

$$2F_k(c-1)^2 + F_k(c)^2 \leq 2F_k(c-1)^2 + (k+1)^2 F_k(c-1)^2 = F_k(c-1)^2 \left(2 + (k+1)^2\right). \quad (5.5)$$

Since $c \leq a+b-1 = 2(c-1)-1$, we can suppose that $c \geq 3$, which leads to

$$2 + (k+1)^2 = k^2 + 2k + 3 < k^2 + 2k^2 + 3 = 3(k^2+1) = 3F_k(3) \leq 3F_k(c).$$

As a result,

$$F_k(c-1)^2 \left(2 + (k+1)^2\right) < F_k(c-1)^2\, 3\, F_k(c). \quad (5.6)$$

Combining equations (5.5) and (5.6), we obtain

$$2F_k(c-1)^2 + F_k(c)^2 < 3\, F_k(c-1)^2\, F_k(c),$$

which can also be expressed as $m_k(c-1, c-1, c) < 0$.

For $a = 1$, since $c \leq a+b-1 = b$, then $c = b$, therefore $m_k(1,b,b) = 1 + 2F_k(b)^2 - 3F_k(b)^2 = 1 - F_k(b)^2 \leq 0$ because $F_k(b)^2 \geq 1$, with equality in the case $b = 1$.

Finally, we prove (1). The only case to be checked is $c = a+b$ because the proof above is valid if $c \geq a+b+1$. We aim to prove

$$F_2(a)^2 + F_2(b)^2 + F_2(a+b)^2 \leq 3F_2(a)F_2(b)F_2(a+b).$$

Adding $2F_2(a)F_2(b)$ on both sides,

$$\left(F_2(a) + F_2(b)\right)^2 + F_2(a+b)^2 \leq F_2(a)F_2(b)\left(3F_2(a+b) + 2\right).$$

Since $\left(F_2(a) + F_2(b)\right)^2 \leq 4F_2(b)^2$, it suffices to prove

$$4F_2(b)^2 + F_2(a+b)^2 \leq 3F_2(a)F_2(b)F_2(a+b).$$

Rearranging terms,

$$4F_2(b)^2 \leq F_2(a+b)\left(3F_2(a)F_2(b) - F_2(a+b)\right).$$

Developing $F_2(a+b)$ on the right-hand side, using (3.15),

$$4F_2(b)^2 \leq F_2(a+b)\left(3F_2(a)F_2(b) - F_2(a+1)F_2(b) - F_2(a)F_2(b-1)\right).$$

Using $3F_2(a) - F_2(a+1) = F_2(a-1) + F_2(a-2)$, we obtain

$$4F_2(b)^2 \leq F_2(a+b)\left(F_2(b)(F_2(a-1) + F_2(a-2)) - F_2(a)F_2(b-1)\right),$$

and thus, reordering terms on the right-hand side we have

$$4F_2(b)^2 \leq F_2(a+b)\left(F_2(b)F_2(a-2) + F_2(b)F_2(a-1) - F_2(a)F_2(b-1)\right).$$

Now, applying D'Ocagne identity (3.19) to $a-1$ and $b-1$,

$$4F_2(b)^2 \leq F_2(a+b)\left(F_2(b)F_2(a-2) + (-1)^a F_2(b-a)\right). \quad (5.7)$$

To prove the inequality above, we distinguish two cases: a being even and odd. If a is even, since $a \geq 4$, then $4F_2(b) \leq F_2(a+b) < 2F_2(a+b) \leq F_2(a-2)F(a+b)$. Consequently,

$$4F_2(b)^2 < F_2(a+b)F_2(a-2)F_2(b) \leq F_2(a+b)\left(F_2(b)F_2(a-2) + F_2(b-a)\right)$$

and (5.7) holds. If a is odd, since $a \geq 3$, we have $12F_2(b) \leq F_2(a+b)$, and for proving (5.7) it is enough to prove

$$F_2(b) \leq 3F_2(b)F_2(a-2) - 3F_2(b-a).$$

in other words,

$$F_2(b) + 3F_2(b-a) \leq 3F_2(b)F_2(a-2)$$

and this holds because $3F_2(b-a) \leq 3F_2(b-3) \leq \frac{F_2(b)}{4}$ and $F_2(a-2) \geq 1$.

□

Lemma 5.4. *The following hold.*

(1) *$m_2(1, b, b+1) \leq 0$, for any b, and equality holds only for $b = 1, 2$.*

(2) *$m_2(2, b, b+1) < 0$, for any $b \geq 2$.*

Proof. For (1), it suffices to prove

$$1 + F_2(b)^2 + F_2(b+1)^2 \leq 3F_2(b)F_2(b+1).$$

If $b = 1$, the equation above holds as an equality. If $b > 1$, by applying Lemma 3.6 to the left-hand side, the above is equivalent to

$$1 + F_2(b)^2 + F_2(b+1)^2 \leq \frac{1}{2}F_2(b+1)F_2(b+2) \leq 3F_2(b)F_2(b+1). \tag{5.8}$$

Equivalently, looking at the right inequality of (5.8), we have

$$F_2(b+1)(2F_2(b+1) + F_2(b)) \leq 6F_2(b)F_2(b+1).$$

Dividing by $F_2(b+1) \neq 0$, we obtain $2F_2(b+1) \leq 5F_2(b)$, but this inequality holds because $2F_2(b+1) = 4F_2(b) + 2F_2(b-1)$ and $F_2(b) \geq 2F_2(b-1)$. In this case, equality is only achieved when $b = 2$.

Next, (2) is equivalent to

$$4 + F_2(b)^2 + F_2(b+1)^2 < 6F_2(b)F_2(b+1).$$

If $b = 2$, we can verify the above inequality numerically $(4 + 4 + 25 < 60)$. For $b > 2$, by Lemma 3.6, and equation (5.8), we see that

$$4 + F_2(b)^2 + F_2(b+1)^2 \leq \frac{1}{2}F_2(b+1)F_2(b+2) \leq 3F_2(b)F_2(b+1) < 6F_2(b)F_2(b+1).$$

□

Theorem 5.5 (Theorem 5.1)**.** *Every non-minimal Markoff m-triple with k-Fibonacci components is an Markoff* 8*-triple of the form* $(F_2(2), F_2(2n), F_2(2n+2))$, *for* $n \geq 2$.

Proof. First, we start with the case $k \geq 3$. If a Markoff m-triple with k-Fibonacci components $(F_k(a), F_k(b), F_k(c))$ is not minimal then $c < a + b$, by Lemma 3.8. However, by Lemma 5.3 (2), for $k \geq 3$ this restriction implies that $m_k(a, b, c) \leq 0$. Therefore, non-minimal Markoff m-triples with k-Fibonacci components do not exist for $k \geq 3$.

In the case $k = 2$, if a Markoff m-triple with 2-Fibonacci components $(F_2(a), F_2(b), F_2(c))$ is not minimal, then $c \leq a + b$, by Lemma 3.8. This restriction forces $F_2(a)$ to be equal to 1 or 2, because of Lemma 5.3 (1).

If $F_2(a) = 1$, then $a = 1$ and $c \leq b + 1$. In the case $b = c$, we have that

$$m_2(1, b, b) = 1 + 2F_2(b)^2 - 3F_2(b)^2 = 1 - F_2(b)^2 \leq 0.$$

as $F_2(b)^2 \geq 1$, with equality in the case $b = 1$. In the case $c = b + 1$, it follows that $m_2(1, b, b + 1) \leq 0$ by Lemma 5.4 (1).

Finally, if $F_2(a) = 2$, then $a = 2$, and $c \leq b + 2$. In the case $b = c$, we have that

$$m_2(2, b, b) = 4 + 2F_2(b)^2 - 6F_2(b)^2 = 4 - 4F_2(b)^2 < 0.$$

as $F_2(b) \geq 2$ for $b \geq 2$. In the case $c = b + 1$, we have that $m_2(2, b, b + 1) < 0$ by Lemma 5.4 (2). Lastly, if $c = b + 2$, the triple is of the form $(2, b, b + 2)$. Now, we prove that b is an even number. Indeed,

$$\begin{aligned} m_2(2, b, b + 2) &= 4 + F_2(b)^2 + F_2(b + 2)^2 - 6F_2(b)F_2(b + 2) \\ &= 4 + (F_2(b + 2) - F_2(b))^2 - 4F_2(b)F_2(b + 2) \\ &= 4 + 4F_2(b + 1)^2 - 4F_2(b)F_2(b + 2) \\ &= 4(1 - (-1)^{b+1}) \end{aligned} \tag{5.9}$$

is positive if and only if b is even, where the last equality is a consequence of the Simson identity (3.21) for $n = b + 1$.

As a result, all the triples must be of the form $(F_2(2), F_2(2n), F_2(2n + 2))$ for $n \geq 1$. Taking into account $\alpha_2 = 1 + \sqrt{2}$ and $\bar{\alpha}_2^n = (-1)^n \alpha_2^{-n}$, using Binet's formula (3.14) we have that

$$\begin{aligned} m_2(2, 2n, 2n + 2) &= 4 + F_2(2n)^2 + F_2(2n + 2)^2 - 6F_2(2n)F_2(2n + 2) \\ &= 4 + 4F_2(2n + 1)^2 - 4F_2(2n)F_2(2n + 2) \\ &= 4 + \frac{\alpha_2^{4n+2}}{2} + \frac{\alpha_2^{-4n-2}}{2} + 1 - \frac{(\alpha_2^{2n} - \alpha_2^{-2n})(\alpha_2^{2n+2} - \alpha_2^{-2n-2})}{2} \\ &= 4 + 1 + \frac{\alpha_2^{-2}}{2} + \frac{\alpha_2^{2}}{2} = 8, \end{aligned}$$

which means that all non-minimal Markoff m-triples with k-Fibonacci components for $k > 1$ are 8-triples and it is straightforward to check that they all lie in a branch of the Markoff 8-tree with minimal triple $(2, 2, 12)$ (See Fig. 4). For $m = 8$, this tree is unique because there are no more minimal triples than $(2, 2, 12)$ as shown in Table 1 of [(SC)]. □

5.2 Minimal case

We recall that if (x, y, z) is a minimal Markoff m-triple, i.e. a solution of the Markoff m-equation (2.2), with $z \geq 3xy$, then

$$m = z(z - 3xy) + x^2 + y^2 > 0.$$

Let a, b be any pair of positive integers with $a \leq b$ and let $c = a + b + t$. By Lemma 3.8, if $t \geq 1$ for $k = 2$, or $t \geq 0$ for $k \geq 3$, then $(F_k(a), F_k(b), F_k(c))$ is minimal, therefore $m_k(a, b, c) > 0$. Consequently, there exists an infinite number of minimal Markoff triples with k-Fibonacci components. Clearly they cannot all correspond to a finite number of values of m, as the number of minimal triples is finite for each m [(SC)]. Hence there are infinitely many values of m that admit minimal Markoff m-triples with k-Fibonacci components. In the rest of the section, we will prove that any $m > 0$ admits at most one minimal Markoff m-triple with k-Fibonacci components, except when $k = 3$, $c = a + b$, a is odd, b is even and $b \geq a + 3$, where $m_3(a, b, a + b)$ admits two such triples.

Lemma 5.6. *Let $1 \leq a \leq b$. Suppose that $k = 2$ and $c = a+b+1$, or $k \geq 3$ and $c = a+b$. Then*

$$m_k(a, b, c) > L_k \frac{\alpha_k^{2c}}{D_k^2},$$

where $D_k = \alpha_k - \bar{\alpha}_k = \sqrt{k^2 + 4}$ and

$$\begin{aligned}
L_2 &= \left(1 - \frac{3}{D_2}\alpha_2^{-1}\right) + 2\left(1 - \frac{3}{D_2}\alpha_2\right)\alpha_2^{-4} - \left(6 + \frac{3}{D_2}\alpha_2 + \frac{9}{D_2}\right)\alpha_2^{-6},\\
L_3 &= \left(1 - \frac{3}{D_3}\right)(1 + 2\alpha_3^{-2}) - \left(6 + \frac{12}{D_3}\right)\alpha_2^{-4},\\
L_k &= 1 - \frac{3}{D_k}, \qquad \forall k \geq 4.
\end{aligned}$$

Proof. Using Binet's formula (3.14) and taking into account that $\alpha_k \bar{\alpha}_k = -1$, it follows that for any $k \geq 1$

$$F_k(n)^2 = \frac{1}{D_k^2}\left(\alpha_k^{2n} + \alpha_k^{-2n} - 2 \cdot (-1)^n\right) > \frac{1}{D_k^2}\left(\alpha_k^{2n} - 2\right).$$

If $k = 2$ and $b = c - 1 - a$, we have

$$\begin{aligned}
m_2(a, b, c) &= F_2(c)^2 + F_2(c - 1 - a)^2 + F_2(a)^2 - 3F_2(c)F_2(c - 1 - a)F_2(a)\\
&> \frac{1}{D_2^2}\left(\alpha_2^{2c} + \alpha_2^{2c-2-2a} + \alpha_2^{2a} - 6\right) - \frac{3}{D_2^3}(\alpha_2^c - \bar{\alpha}_2^c)(\alpha_2^{c-1-a} - \bar{\alpha}_2^{c-1-a})(\alpha_2^a - \bar{\alpha}_2^a).
\end{aligned}$$

As $c = a + b + 1 > 1$ and $\alpha_2\bar{\alpha}_2 = -1$, we conclude that

$$\begin{aligned}
&(\alpha_2^c - \bar{\alpha}_2^c)(\alpha_2^{c-1-a} - \bar{\alpha}_2^{c-1-a})(\alpha_2^a - \bar{\alpha}_2^a)\\
&\qquad \leq (\alpha_2^c + \alpha_2^{-c})(\alpha_2^{c-1-a} + \alpha_2^{a-c+1})(\alpha_2^a + \alpha_2^{-a})\\
&\qquad = \alpha_2^{2c-1} + \alpha_2^{2c-1-2a} + \alpha_2^{2a+1} + \alpha_2 + \alpha_2^{-1} + \alpha_2^{-2a-1} + \alpha_2^{2a-2c+1} + \alpha_2^{-2c+1}\\
&\qquad < \alpha_2^{2c-1} + \alpha_2^{2c-1-2a} + \alpha_2^{2a+1} + \alpha_2 + 3.
\end{aligned}$$

Hence

$$\begin{aligned}
m_2(a, b, c) &> \frac{1}{D_2^2}\left(\alpha_2^{2c} + \alpha_2^{2c-2-2a} + \alpha_2^{2a} - 6\right) - \frac{3}{D_2^3}(\alpha_2^{2c-1} + \alpha_2^{2c-1-2a} + \alpha_2^{2a+1} + \alpha_2 + 3)\\
&= \frac{1}{D_2^2}\alpha_2^{2c}\left[\left(1 - \frac{3}{D_2}\alpha_2^{-1}\right) + \left(1 - \frac{3}{D_2}\alpha_2\right)\left(\alpha_2^{-2-2a} + \alpha_2^{2a-2c}\right) - \left(6 + \frac{3}{D_2}\alpha_2 + \frac{9}{D_2}\right)\alpha_2^{-2c}\right].
\end{aligned}$$

As $f(x)=\alpha_2^x$ is a convex function, $c>1$ and $a\geq 1$, by applying Karamata's inequality [(K)], we obtain

$$\alpha_2^{-2-2a}+\alpha_2^{2a-2c}\leq \alpha_2^{-2-2}+\alpha_2^{2-2c}=\alpha_2^{-4}+\alpha_2^{2-2c}. \tag{5.10}$$

Since

$$1-\frac{3}{D_2}\alpha_2=1-\frac{6+3\sqrt{8}}{2\sqrt{8}}<1-\frac{3}{2}<0$$

and $c\geq a+b+1\geq 3$, we have

$$\begin{aligned}
&m_2(a,b,c)\\
&>\frac{1}{D_2^2}\alpha_2^{2c}\left[\left(1-\frac{3}{D_2}\alpha_2^{-1}\right)+\left(1-\frac{3}{D_2}\alpha_2\right)\left(\alpha_2^{-2-2a}+\alpha_2^{2a-2c}\right)-\left(6+\frac{3}{D_2}\alpha_2+\frac{9}{D_2}\right)\alpha_2^{-2c}\right]\\
&\geq \frac{1}{D_2^2}\alpha_2^{2c}\left[\left(1-\frac{3}{D_2}\alpha_2^{-1}\right)+\left(1-\frac{3}{D_2}\alpha_2\right)\left(\alpha_2^{-4}+\alpha_2^{2-2c}\right)-\left(6+\frac{3}{D_2}\alpha_2+\frac{9}{D_2}\right)\alpha_2^{-2c}\right]\\
&\geq L_2\frac{1}{D_2^2}\alpha_2^{2c},
\end{aligned}$$

as the coefficient of α_2^{-2c} is clearly negative in the previous expression, and therefore its minimum for $c\geq 3$ is attained at $c=3$.

Analogously, if we assume that $k\geq 3$ and $c=a+b$, we have

$$\begin{aligned}
(\alpha_k^c-\bar{\alpha}_k^c)&(\alpha_k^{c-a}-\bar{\alpha}_k^{c-a})(\alpha_k^a-\bar{\alpha}_k^a)\\
&\leq (\alpha_k^c+\alpha_k^{-c})(\alpha_k^{c-a}+\alpha_k^{a-c})(\alpha_k^a+\alpha_k^{-a})\\
&=\alpha_k^{2c}+\alpha_k^{2c-2a}+\alpha_k^{2a}+2+\alpha_k^{-2a}+\alpha_k^{2a-2c}+\alpha_k^{-2c}\\
&<\alpha_k^{2c}+\alpha_k^{2c-2a}+\alpha_k^{2a}+4.
\end{aligned}$$

Hence

$$\begin{aligned}
m_k(a,b,c)&>\frac{1}{D_k^2}\left(\alpha_k^{2c}+\alpha_k^{2c-2a}+\alpha_k^{2a}-6\right)-\frac{3}{D_k^3}(\alpha_k^{2c}+\alpha_k^{2c-2a}+\alpha_k^{2a}+4)\\
&=\frac{1}{D_k^2}\alpha_k^{2c}\left[\left(1-\frac{3}{D_k}\right)\left(1+\alpha_k^{-2a}+\alpha_k^{2a-2c}\right)-\left(6+\frac{12}{D_k}\right)\alpha_k^{-2c}\right].
\end{aligned}$$

Now, the factor $1-\frac{3}{D_k}=1-\frac{3}{\sqrt{k^2+4}}$ becomes positive for $k\geq 3$, so this time we need to apply the opposite Karamata bound [(K)] (which becomes simply Jensen's inequality in this case)

$$\alpha_k^{-2a}+\alpha_k^{2a-2c}\geq 2\alpha_k^{\frac{-2a+2a-2c}{2}}=2\alpha_k^{-c},$$

yielding

$$m_k(a,b,c)>\frac{1}{D_k^2}\alpha_k^{2c}\left[\left(1-\frac{3}{D_k}\right)\left(1+2\alpha_k^{-c}\right)-\left(6+\frac{12}{D_k}\right)\alpha_k^{-2c}\right].$$

Let us consider the polynomial

$$p_k(x)=2\left(1-\frac{3}{D_k}\right)x-\left(6+\frac{12}{D_k}\right)x^2.$$

Then, our bound can be written as

$$m_k(a,b,c)>\frac{1}{D_k^2}\alpha_k^{2c}\left[1-\frac{3}{D_k}+p_k(\alpha_k^{-c})\right].$$

We know that $c = a + b \geq 2$, so $\alpha_k^{-c} \in (0, \alpha_k^{-2}]$, as $\alpha_k > 1$, and therefore, $\lim_{c\to\infty} \alpha_k^{-c} = 0$. The polynomial $p_k(x)$ is a parabola with a negative leading coefficient, so its minimum in the interval $[0, \alpha_k^{-2}]$ is attained at one of the ends of the interval.

For $k = 3$, a direct computation shows that $p_3(\alpha_3^{-2}) < 0 = p_3(0)$, and hence

$$m_3(a,b,c) > \frac{1}{D_3^2}\alpha_3^{2c}\left[1 - \frac{3}{D_3} + p_3(\alpha_3^{-2})\right] = L_3\frac{1}{D_3^2}\alpha_3^{2c}.$$

On the other hand, for $k \geq 4$, we can prove that $p_k(\alpha_k^{-2}) > 0 = p_k(0)$ as follows. The expression

$$\alpha_k^4 p_k(\alpha_k^{-2}) = 2\alpha_k^2\left(1 - \frac{3}{D_k}\right) - \left(6 + \frac{12}{D_k}\right)$$

is clearly increasing in k, because α_k and D_k are both increasing functions of k. A direct computation shows that for $k = 4$ we have $\alpha_4^4 p_4(\alpha_4^{-2}) > 0$, so $p_k(\alpha_k^{-2})$ must be positive for all $k \geq 4$. As a consequence,

$$\begin{aligned} m_k(a,b,c) &> \frac{1}{D_k^2}\alpha_k^{2c}\left[1 - \frac{3}{D_k} + p_k(\alpha_k^{-c})\right] > \frac{1}{D_k^2}\alpha_k^{2c}\left[1 - \frac{3}{D_k} + p_k(0)\right] \\ &= \frac{1}{D_k^2}\alpha_k^{2c}\left(1 - \frac{3}{D_k}\right) = L_k\frac{1}{D_k^2}\alpha_k^{2c}. \end{aligned}$$

□

We have the following lower bound for the constant L_k in the lemma above.

Lemma 5.7. *For each $k \geq 2$, the constant L_k satisfies*

$$L_k > \alpha_k^{-2}.$$

Proof. For $k = 2, 3$, a direct computation in MATLAB shows that $\alpha_2^2 L_2 > 1$ and $\alpha_3^2 L_3 > 1$, so $L_k > \alpha_k^{-2}$ for $k = 2, 3$. For $k \geq 4$ we wish to prove that

$$L_k = 1 - \frac{3}{D_k} > \alpha_k^{-2}.$$

Rearranging the equation, this is equivalent to proving that for all $k \geq 4$

$$1 > \frac{3}{D_k} + \alpha_k^{-2} = \frac{3}{\sqrt{k^2+4}} + \frac{4}{(k+\sqrt{k^2+4})^2}.$$

The right-hand side of this expression is decreasing in k and for $k = 4$ a direct computation in MATLAB shows that

$$\frac{3}{D_4} + \alpha_4^{-2} < 1,$$

and hence the inequality holds for all $k \geq 4$.

□

Lemma 5.8. *Let $1 \leq a \leq b \leq c$ and $c \geq 3$. Suppose that $a \leq a' \leq c$ and $b \leq b' \leq c$. Then*

$$m_k(a,b,c) \geq m_k(a',b',c)$$

and equality holds if and only if $a = a'$ and $b = b'$. In particular, if $(F_k(a), F_k(b), F_k(c))$ is an ordered minimal Markoff-Fibonacci m-triple, then

$$m_k(1, 1, c) \geq m_k(a, b, c) \geq m_k(a, c - a - s, c),$$

where $s = 1$, for $k = 2$ and $s = 0$, for $k \geq 3$.

Proof. The lemma and its proof are entirely analogous to Lemma 4.10 in Chapter 4, which addresses the case $k = 1$. In this lemma, the starting point is $a = 2$ because $F_1(2) = F_1(1) = 1$. In our situation, with $k \geq 2$, the case $a = 1$ is also valid since $F_k(2) > F_k(1) = 1$. □

Lemma 5.9. *If $(F_k(a), F_k(b), F_k(c))$ and $(F_k(a'), F_k(b'), F_k(c'))$ are two ordered minimal Markoff-Fibonacci m-triples with $c \geq c'$, then $c = c'$.*

Proof. Assume that $m_k(a, b, c) = m = m_k(a', b', c')$. By applying Lemma 5.8 and Lemma 5.6, it follows that

$$m = m_2(a, b, c) \geq m_2(a, c - a - 1, c) > L_2 \frac{1}{D_2^2} \alpha_2^{2c}$$

if $k = 2$ and

$$m = m_k(a, b, c) \geq m_k(a, c - a, c) > L_k \frac{1}{D_k^2} \alpha_k^{2c},$$

for any other $k \geq 3$. From Lemma 5.7 we know that $L_k > \alpha_k^{-2}$ for all $k \geq 2$, so

$$m_k(a, b, c) > L_k \frac{1}{D_k^2} \alpha_k^{2c} > \frac{1}{D_k^2} \alpha_k^{2c-2}. \tag{5.11}$$

On the other hand, from Lemma 5.8 we deduce that

$$\begin{aligned} m = m_k(a', b', c') \leq m_k(1, 1, c') &= F_k(c')^2 - 3F_k(c') + 2 \\ &< \frac{1}{D_k^2} \alpha_k^{2c'} + \frac{1}{D_k^2} \bar{\alpha}_k^{2c'} + \frac{2}{D_k^2} (-1)^{c'} - 1 < \frac{1}{D_k^2} \alpha_k^{2c'}. \end{aligned} \tag{5.12}$$

Using equations (5.11) and (5.12) together, we obtain $\alpha_k^{2(c-1)} < D_k^2 m < \alpha_k^{2c'}$. Thus, $c' > c - 1$. As we assumed $c' \leq c$, we conclude that $c' = c$. □

Lemma 5.10. *Let $(F_k(a), F_k(b), F_k(c))$ and $(F_k(a'), F_k(b'), F_k(c))$ be two distinct ordered minimal Markoff-Fibonacci m-triples with the same third element. If $a \leq a'$, then $a < a' \leq b' < b$.*

Proof. Suppose first that $a = a'$. Then, by Lemma 5.8, the equality $m_k(a, b, c) = m_k(a', b', c') = m_k(a, b', c)$ is only possible if $b = b'$, in which case $(a, b, c) = (a', b', c')$, contradicting the assumption that the two m-triples are distinct. Thus $a < a'$. If $b \leq b'$, then Lemma 5.8 implies $m(a, b, c) < m(a', b', c)$, which is not possible as both are m-triples for the same m. Therefore, it follows that $a < a' \leq b' < b$. □

Lemma 5.11. *Let $(F_k(a), F_k(b), F_k(c))$ and $(F_k(a'), F_k(b'), F_k(c))$ be two ordered minimal Markoff-Fibonacci m-triples. Then $a + b = a' + b'$.*

Proof. By Lemma 5.10 we can assume without loss of generality that $1 \le a < a' \le b' < b \le c$. In particular, $b \ge 3$. Rearranging the equation $m_k(a,b,c) = m_k(a',b',c)$, yields

$$F_k(a)^2 + F_k(b)^2 - F_k(a')^2 - F_k(b')^2 = 3F_k(c)\left(F_k(a)F_k(b) - F_k(a')F_k(b')\right). \qquad (5.13)$$

Since $b \ge 3$ and $a' \le b' < b$ we have

$$F_k(b)^2 \ge k^2 F_k(b-1)^2 > 2F_k(b-1)^2 \ge F_k(b')^2 + F_k(a')^2,$$

so the left-hand side of equation (5.13) is always positive and, thus, so is the right-hand side. Let us see that this is impossible if $a' + b' > a + b$. Indeed,

$$\begin{aligned} \frac{F_k(a')F_k(b')}{F_k(a)F_k(b)} &= \frac{(\alpha_k^{a'} - \bar{\alpha}_k^{a'})(\alpha_k^{b'} - \bar{\alpha}_k^{b'})}{(\alpha_k^{a} - \bar{\alpha}_k^{a})(\alpha_k^{b} - \bar{\alpha}_k^{b})} \\ &\ge \frac{(\alpha_k^{a'} - \alpha_k^{-a'})(\alpha_k^{b'} - \alpha_k^{-b'})}{(\alpha_k^{a} + \alpha_k^{-a})(\alpha_k^{b} + \alpha_k^{-b})} \\ &= \frac{\alpha_k^{a'+b'} - \alpha_k^{b'-a'} - \alpha_k^{a'-b'} + \alpha_k^{-a'-b'}}{\alpha_k^{a+b} + \alpha_k^{b-a} + \alpha_k^{a-b} + \alpha_k^{-a-b}}. \end{aligned}$$

Assume that $a' + b' = a + b + r$ with $r > 0$ and let $s = a + b$. Then $a' + b' = s + r$. Dividing the numerator and denominator by α_k^s yields

$$\begin{aligned} \frac{\alpha_k^{a'+b'} - \alpha_k^{b'-a'} - \alpha_k^{a'-b'} + \alpha_k^{-a'-b'}}{\alpha_k^{a+b} + \alpha_k^{b-a} + \alpha_k^{a-b} + \alpha_k^{-a-b}} &= \frac{\alpha_k^{r} - \alpha_k^{r-2a'} - \alpha_k^{r-2b'} + \alpha_k^{-2s-r}}{1 + \alpha_k^{-2a} + \alpha_k^{-2b} + \alpha_k^{-2s}} \\ &= \alpha_k^r \frac{1 - \alpha_k^{-2a'} - \alpha_k^{-2b'} + \alpha_k^{-2s-2r}}{1 + \alpha_k^{-2a} + \alpha_k^{-2b} + \alpha_k^{-2s}}. \end{aligned}$$

As $1 \le a < a' \le b' < b$, we have $a \ge 1$, $a' \ge 2$, $b' \ge 2$, $b \ge 3$ and $s = a + b \ge 4$. Thus

$$\alpha_k^r \frac{1 - \alpha_k^{-2a'} - \alpha_k^{-2b'} + \alpha_k^{-2s-2r}}{1 + \alpha_k^{-2a} + \alpha_k^{-2b} + \alpha_k^{-2s}} \ge \alpha_k \frac{1 - 2\alpha_k^{-4}}{1 + \alpha_k^{-2} + \alpha_k^{-6} + \alpha_k^{-8}} \ge 1.92 > 1.$$

Therefore, $F_k(a')F_k(b') > F_k(a)F_k(b)$, which contradicts the positivity of both sides of equation (5.13).

Therefore, we must have $a + b \ge a' + b'$. Suppose that $a' + b' = a + b - r$ with $r > 0$ and let $s = a + b$ as before. Following the same logic as in the previous case,

$$\begin{aligned} \frac{F_k(a')F_k(b')}{F_k(a)F_k(b)} &= \frac{(\alpha_k^{a'} - \bar{\alpha}_k^{a'})(\alpha_k^{b'} - \bar{\alpha}_k^{b'})}{(\alpha_k^{a} - \bar{\alpha}_k^{a})(\alpha_k^{b} - \bar{\alpha}_k^{b})} \le \frac{(\alpha_k^{a'} + \alpha_k^{-a'})(\alpha_k^{b'} + \alpha_k^{-b'})}{(\alpha_k^{a} - \alpha_k^{-a})(\alpha_k^{b} - \alpha_k^{-b})} \\ &= \frac{\alpha_k^{a'+b'} + \alpha_k^{b'-a'} + \alpha_k^{a'-b'} + \alpha_k^{-a'-b'}}{\alpha_k^{a+b} - \alpha_k^{b-a} - \alpha_k^{a-b} + \alpha_k^{-a-b}} = \alpha_k^{-r} \frac{1 + \alpha_k^{-2a'} + \alpha_k^{-2b'} + \alpha_k^{-2s-2r}}{1 - \alpha_k^{-2a} - \alpha_k^{-2b} + \alpha_k^{-2s}} \\ &\le \alpha_k^{-1} \frac{1 + 2\alpha_k^{-4} + \alpha_k^{-10}}{1 - \alpha_k^{-2} - \alpha_k^{-6}} < 0.53 < \frac{8}{9}. \end{aligned}$$

As a result,

$$1 - \frac{F_k(a')F_k(b')}{F_k(a)F_k(b)} > 1 - \frac{8}{9} = \frac{1}{9} \ge \frac{1}{9F_k(a)^2}.$$

Multiplying both sides by $3F_k(a)F_k(b)F_k(c)$, results in

$$3F_k(c)\left(F_k(a)F_k(b) - F_k(a')F_k(b')\right) > \frac{F_k(c)F_k(b)}{3F_k(a)}\,.$$

Since $(F_k(a), F_k(b), F_k(c))$ is minimal, we have $F_k(c) \geq 3F_k(a)F_k(b)$. Consequently,

$$\begin{aligned}3F_k(c)\left(F_k(a)F_k(b) - F_k(a')F_k(b')\right) &> \frac{F_k(c)F_k(b)}{3F_k(a)} \geq F_k(b)^2 \\ &> F_k(b)^2 - F_k(b')^2 + F_k(a)^2 - F_k(a')^2\,.\end{aligned}$$

This contradicts equation (5.13), and thus $a'+b' \geq a+b$ and therefore $a+b = a'+b'$. □

Lemma 5.12. *If a is odd, b is even, $b \geq a+3$ then*

$$m_3(a,b,a+b) = m_3(a+1,b-1,a+b).$$

Proof. Using Simson identity (3.21) for a odd,

$$\begin{aligned}F_3(a)^2 - F_3(a+1)^2 &= F_3(a)^2 - F_3(a)F_3(a+2) + (-1)^{a+1} \\ &= F_3(a)(F_3(a) - F_3(a+2)) + (-1)^{a+1} \\ &= -3F_3(a)F_3(a+1) + 1\,.\end{aligned}$$

Using a similar argument for b even, we have

$$F_3(b)^2 - F_3(b-1)^2 = 3F_3(b)F_3(b-1) - 1.$$

Adding both expressions yields

$$F_3(a)^2 + F_3(b)^2 - F_3(a+1)^2 - F_3(b-1)^2 = 3\left(F_3(b)F_3(b-1) - F_3(a)F_3(a+1)\right). \tag{5.14}$$

Following with the assumption that a is odd and b is even, applying Vajda's identity (see Lemma 3.4) with $n = b-a-1$, $i = a$ and $j = a+1$:

$$F_3(b)F_3(b-1) - F_3(a+b)F_3(b-a-1) = (-1)^{b-a-1}F_3(a)F_3(a+1) = F_3(a)F_3(a+1)$$

and with $n = a$, $i = 1$ and $j = b-a-1$:

$$F_3(a+1)F_3(b-1) - F_3(a)F_3(b) = (-1)^a F_3(1)F_3(b-a-1) = -F_3(b-a-1)\,.$$

Thus,

$$\begin{aligned}F_3(b)F_3(b-1) - F_3(a)F_3(a+1) &= F_3(a+b)F_3(b-1-a) \\ &= F_3(a+b)(F_3(a)F_3(b) - F_3(a+1)F_3(b-1)).\end{aligned}$$

Substituting back in (5.14) yields

$$F_3(a)^2 + F_3(b)^2 - F_3(a+1)^2 - F_3(b-1)^2 = 3F_3(a+b)(F_3(a)F_3(b) - F_3(a+1)F_3(b-1)).$$

Rearranging this equation we have

$$\begin{aligned}m_3(a,b,a+b) &= F_3(a)^2 + F_3(b)^2 - 3F_3(a)F_3(b)F_3(a+b) \\ &= F_3(a+1)^2 + F_3(b-1)^2 - 3F_3(a+1)F_3(b-1)F_3(a+b) \\ &= m_3(a+1,b-1,a+b),\end{aligned}$$

obtaining the desired result. □

Theorem 5.13 (Theorem 5.2). *If m admits a minimal Markoff m-triple with k-Fibonacci components then it is unique except for $k = 3$ and all pairs of triples $(F_3(a), F_3(b), F_3(a+b))$, $(F_3(a+1), F_3(b-1), F_3(a+b))$, for a odd, b even and $b \geq a+3$.*

Proof. Let $(F_k(a), F_k(b), F_k(c))$ and $(F_k(a'), F_k(b'), F_k(c'))$ be a pair of ordered minimal m-triples contradicting the theorem. By Lemma 5.9, it follows that $c = c'$. Moreover, by Lemma 5.10 we can assume without loss of generality that $1 \leq a < a' \leq b' < b \leq c$ and by Lemma 5.11 we must have $a + b = a' + b'$. Taking $n = a$, $i = b' - a$ and $j = b - b' = a' - a$ in Vajda's identity (Lemma 3.4), we transform equation (5.13) into

$$\begin{aligned} F_k(a)^2 + F_k(b)^2 - F_k(a')^2 - F_k(b')^2 &= 3F_k(c)\left(F_k(a)F_k(b) - F_k(a')F_k(b')\right) \\ &= (-1)^{a+1} 3F_k(c)F_k(b'-a)F_k(b-b'). \end{aligned} \tag{5.15}$$

From the proof of Lemma 5.11, the left-hand side of this equality is positive, therefore a is odd, and hence

$$F_k(a)^2 + F_k(b)^2 - F_k(a')^2 - F_k(b')^2 = 3F_k(c)F_k(b'-a)F_k(b-b'). \tag{5.16}$$

In the case $k = 2$, using (3.23) from Lemma 3.8 twice, we obtain that

$$F_2(b) \leq 3F_2(b')F_2(b-b') \leq 9F_2(a)F_2(b'-a)F_2(b-b').$$

Multiplying by $F_2(b)$ and by minimality, $3F_2(a)F_2(b) \leq F_2(c)$, it follows that

$$F_2(b)^2 \leq 9F_2(a)F_2(b)F_2(b'-a)F_2(b-b') \leq 3F_2(c)F_2(b'-a)F_2(b-b')$$

and as a consequence

$$F_2(b)^2 - F_2(b')^2 + F_2(a)^2 - F_2(a')^2 < F_2(b)^2 \leq 3F_2(c)F_2(b'-a)F_2(b-b'),$$

which contradicts equation (5.16).

In the case $k \geq 4$, suppose that $c = a + b$. We want to prove

$$F_k(b)^2 - F_k(b')^2 + F_k(a)^2 - F_k(a')^2 > 3F_k(c)F_k(b'-a)F_k(b-b'), \tag{5.17}$$

contradicting (5.16).

First, since $F_k(b) \geq kF_k(b-1) \geq 4F_k(b')$ by equation (3.16), we have

$$F_k(a')^2 + F_k(b')^2 \leq 2F_k(b')^2 \leq \frac{1}{8}F_k(b)^2 < \frac{F_k(b)^2}{4}. \tag{5.18}$$

Now, using equation (3.17) twice, it follows that

$$3F_k(a+b)F_k(b-b')F_k(b'-a) \leq 3F_k(a+b)F_k(b-a-1) \leq 3F_k(2b-2).$$

The inequality above and (5.18) give

$$F_k(a')^2 + F_k(b')^2 + 3F_k(a+b)F_k(b-b')F_k(b'-a) < \frac{F_k(b)^2}{4} + 3F_k(2b-2),$$

and by Lemma 3.7

$$\frac{F_k(b)^2}{4} + 3F_k(2b-2) \leq \frac{F_k(b)^2}{4} + \frac{3}{4}F_k(b)^2 = F_k(b)^2.$$

Due to the two inequalities above, (5.17) holds.

In the case $k = 3$, suppose that $c = a + b$ and $b' \leq b - 2$. We want to prove

$$F_3(b)^2 > F_3(a')^2 + F_3(b')^2 + 3F_3(a+b)F_3(b'-a)F_3(b-b'), \tag{5.19}$$

which contradicts equation (5.16). Repeating the argument above,

$$3F_3(a+b)F_3(b'-a)F_3(b-b') \leq 3F_3(2b-2) \leq \frac{3}{4}F_3(b)^2.$$

On the other hand, if $a' \leq b' \leq b - 2$, since $F_3(b) \geq 9F_3(b-2)$, we have

$$F_3(a')^2 + F_3(b')^2 \leq 2F_3(b')^2 \leq 2F_3(b-2)^2 \leq \frac{2}{9}F_3(b)^2 < \frac{1}{4}F_3(b)^2.$$

Adding the two inequalities above, (5.19) holds.

In the case $k \geq 3$, we first consider $c \geq a + b + 1$. We will show that

$$F_k(b)^2 - F_k(b')^2 + F_k(a)^2 - F_k(a')^2 < 3F_k(c)F_k(b'-a)F_k(b-b'), \tag{5.20}$$

which contradicts equation (5.16). Then, since $F_k(b') > F_k(a)$ it is enough to show that

$$F_k(b)^2 < 3F_k(a+b+1)F_k(b'-a)F_k(b-b'). \tag{5.21}$$

By using equation (3.18) twice, we obtain

$$\begin{aligned} 3F_k(a+b+1)F_k(b'-a)F_k(b-b') &\geq 3F_k(a+b+1)\frac{1}{(1+\frac{1}{9})}F_k(b-a-1) \\ &\geq \frac{3}{\left(1+\frac{1}{9}\right)^2}F_k(2b-1) > F_k(2b-1). \end{aligned}$$

On the other hand, applying formula (3.15) to $b-1$ and b, it follows that

$$F_k(2b-1) = F_k(b)^2 + F_k(b-1)^2 > F_k(b)^2.$$

The two inequalities above show that

$$\begin{aligned} F_k(b)^2 - F_k(b')^2 + F_k(a)^2 - F_k(a')^2 &< F_k(b) < F_k(2b-1) \\ &\leq 3F_k(a+b+1)F_k(b'-a)F_k(b-b') \\ &\leq 3F_k(c)F_k(b'-a)F_k(b-b'), \end{aligned}$$

which shows that (5.21) holds.

Finally, we study the last case; $k = 3$, $c = a+b$, $b' = b-1$ and a odd (see equation (5.15)). This is precisely addressed in Lemma 5.12, which identifies the minimal pairs of Markoff m-triples with k-Fibonacci components satisfying $m = m_3(a, b, a+b) = m_3(a+1, b-1, a+b)$, where b is even. Note that the condition $b \geq a + 3$ in that lemma implies that the triple $(F_3(a+1), F_3(b-1), F_3(a+b))$ is ordered, so $(F_3(a+1), F_3(b-1), F_3(a+b))$ and $(F_3(a), F_3(b), F_3(a+b))$ are distinct. This, however, does not hold if $b = a + 1$, which would also break the condition $a < a' \leq b' < b$ obtained in Lemma 5.10. If b were odd, following the same procedure as in Lemma 5.12, we would obtain

$$F_3(a)^2 + F_3(b)^2 - F_3(a+1)^2 - F_3(b-1)^2 = 3F_3(a+b)(F_3(a)F_3(b) - F_3(a+1)F_3(b-1)) + 2.$$

Therefore, if b were odd, $m_3(a, b, a+b) > m_3(a+1, b-1, a+b)$. $\square$

6 Algebraic invariants detection through AI

Additionally, the possibility of using AI techniques to detect algebraic invariants in the solutions of the generalized Markoff equation has been explored. The idea is to train a model that can learn from the patterns in the solutions generated by the symbolic computation engine, allowing it to identify new algebraic relationships and invariants that may not be immediately apparent through traditional analytical methods.

In this case, the goal is to find an algebraic invariant that distinguishes triples depending on whether or not they belong to the same tree. To do this, a dataset of triples generated by the symbolic computation engine has been created, and is being used to train a siamese neural network, which will determine if two triples come from the same root solution or not.

6.1 Dataset generation

The first step in this process is to generate a dataset of triples that can be used to train the AI model. The symbolic computation engine is used to generate a large number of m-Markoff triples for various values of m. Each triple is represented as a tuple of integers (x, y, z), and the dataset will include both minimal and non-minimal triples. The dataset will include two columns:

- *triple_pair*: Each entry will contain the two Markoff m-triples that are being compared in that instance. It will have the following structure: $((x_i, y_i, z_i), (x'_i, y'_i, z'_i))$.
- *same_root*: Binary label indicating whether the two triples come from the same root solution or not — 1 if they do, and 0 if they do not.

Since the elements of the Markoff m-triples increase rapidly, the components of the triples are generated up to a certain limit. This way, we reduce the possibility of exploding gradients during training and we can control the size of the dataset. Also, the dataset is balanced, meaning that it contains an equal number of pairs of triples that come from the same root and pairs that do not.

6.2 Model architecture

The model architecture chosen for this task is a siamese neural network, which is particularly suitable for tasks involving similarity learning. The siamese network consists of two identical subnetworks that share the same weights and parameters. Each subnetwork processes one of the input triples, extracting features that are then compared to determine whether the two triples belong to the same root solution. The architecture of each subnetwork is as follows:

- **Input layer**: Takes a triple as input, represented as a tuple of integers (x, y, z).
- **Embedding layer**: As the goal is to generate a polynomial that distinguishes between triples, this layer maps the input triple (x, y, z) to a higher-dimensional Van der Monde tensor that represents all monomials up to degree d in x, y, and z. The process is the following:

(1) For each component of the triple, we apply the following transformation:

$$\begin{aligned} V : \mathbb{R} &\mapsto \mathbb{R}^{d+1} \\ t &\mapsto (1, t, t^2, \ldots, t^d)\,. \end{aligned}$$

(2) Then, we construct the Van der Monde tensor by taking the outer product of the individual embeddings:

$$V_{\bar{x}} = V_{(x,y,z)} = V(x) \otimes V(y) \otimes V(z) \in \left(\mathbb{R}^{d+1}\right)^{\otimes 3}\,.$$

where $\bar{x}$ is the notation used to denote the triple (x, y, z).

Observe that, by construction, the coordinate (i, j, k) of the tensor $V_{\bar{x}}$ corresponds to the term $x^i y^j z^k$. Thus any polynomial $P(x, y, z) = \sum_{i,j,k} a_{i,j,k}\, x^i y^j z^k$ with maximum degree d in x, y and z can be obtained as a linear combination of the coefficients of $V_{\bar{x}}$.

This allows the network to capture polynomial relationships between the components of the triple.

- **Linear layer**: It reduces the dimensionality of the embedding to a single number, applying a linear transformation to the embedded vector. This is equivalent to evaluating the polynomial at the point $\bar{x} = (x, y, z)$, where the coefficients of the polynomial are learned during training. This layer can be represented as:

$$P(x, y, z) = \sum_{i,j,k} a_{i,j,k}\, x^i y^j z^k = A \cdot V_{\bar{x}}\,,$$

where $A \in \left(\mathbb{R}^{d+1}\right)^{\otimes 3}$ is the tensor of coefficients for the polynomial and $\cdot$ is the standard dot product. However, in this case, a mask was applied to the tensor A to ensure that only the coefficients corresponding to monomials of degree d or less are considered, as the goal is to learn a polynomial of degree d. Therefore, it was established that

$$A \in U \subset \left(\mathbb{R}^{d+1}\right)^{\otimes 3}\,,$$

where U is defined as

$$U = \{\, a_{i,j,k} \;\mid\; a_{i,j,k} = 0 \quad \forall\, i + j + k > d \,\}\,.$$

The output of each subnetwork is a single number, which represents the polynomial evaluated at the input triple. The outputs of the two subnetworks are then compared using a distance metric, such as Euclidean distance or cosine similarity, to determine whether the two triples belong to the same root solution.

6.3 Experiments

Since mathematically the existence of the invariant polynomial is not yet proven, in order to test the model a different approach has been taken. Instead of looking for a polynomial that distinguishes between triples in different trees, as an initial proof of concept, the model was trained to distinguish between triples according to their m value.

This means that the polynomial learned by the model should be the Markoff equation, as $m = x^2 + y^2 + z^2 - 3xyz$ is the expression that defines the value of m for a given triple.

For this experiment, a similar dataset was created, but instead of using the label *same_root*, it was replaced with *same_m*. This new label indicates whether the two triples have the same value of m or not. The dataset is balanced, meaning that it contains an equal number of pairs of triples that have the same value of m and pairs that do not. It is worth remarking that the m labels are not in the dataset, so a classical polynomial regression approach to this problem is not feasible. This is important in order to explore the extension of this approach to the study of triples belonging to different trees, as no polynomial invariant is known for these classes and, therefore, it would be impossible to provide such labels.

However, results have not been satisfactory as several obstacles were found due to lack of stability and exploding gradients for larger degree monomials. Several ideas are already in mind to try alternative approaches in order to overcome those setbacks and further investigate the detection of algebraic invariants using AI techniques. Nevertheless, due to time constraints, this line of research has been established as a topic for future exploration.

6.4 Alternative approach

Nonetheless, another approach was tried without AI, which consisted in calculating the null space of the matrix formed by the subtracted embeddings of the triples.

Again, as before, an experiment was done to check if the method was feasible. Similarly to the previous experiment, the objective is to calculate a polynomial that distinguishes between triples that have the same value of m from those that do not. The polynomial should compute the same value given two triples with the same value of m, for all n entries of the batch. This can be expressed as follows:

$$P(x_i, y_i, z_i) = P(x_i', y_i', z_i') \iff m_i = m_i' \quad \forall i \in \{1, 2, \ldots, n\},$$

where P is the polynomial we want to find, and (x_i, y_i, z_i) and (x_i', y_i', z_i') are two triples with values m_i and m_i' for m, respectively. As explained before, we have that

$$P(x, y, z) = A \cdot V_{\bar{x}},$$

where $A \in \left(\mathbb{R}^{d+1}\right)^{\otimes 3}$ is the vector of coefficients that we want to find, $V_{\bar{x}} = V(x) \otimes V(y) \otimes V(z) \in \left(\mathbb{R}^{d+1}\right)^{\otimes 3}$ and $\cdot$ is the standard dot product. The goal is to find a vector A such that the polynomial P evaluates to the same value for all triples with the same value of m. Therefore, we can express the condition as follows:

$$\left(V_{\bar{x}_i} - V_{\bar{x}_i'}\right) \cdot A = 0 \iff m_i = m_i' \quad \forall i \in \{1, 2, \ldots, n\},$$

which translates into calculating the nullifier of $\left(V_{\bar{x}_i} - V_{\bar{x}_i'}\right)$ $\forall i \in \{1, 2, \ldots, n\}$, which is equivalent to determining the null space of the $n \times (d+1)^3$ matrix whose rows are the coordinates of $V_{\bar{x}_i} - V_{\bar{x}_i'}$ for each $i \in \{1, 2, \ldots, n\}$.

To do this, the dataset created for the previous experiment was used, but only the pairs of triples that have the same value of m were selected. Then, all pairs were treated as a batch and the previous process was applied. The result was satisfactory, as the null space included the vector of coefficients of the Markoff equation and the vector of constants — which always works.

Since the approach was successful, the method was then tried with the original dataset, with the goal of distinguishing between triples in different trees. As the degree was increased, the null space began to include new vectors, but these disappeared as soon as more data was included in the dataset. This indicated that there did not exist an algebraic invariant for low degree, and the method started to become computationally unattainable. This suggested that this approach was not suitable for the original goal, which led to the decision of focusing back on the AI approach, which as mentioned, has been established as a topic for future exploration.

7 Conclusion and future work

In this work, we have studied the generalized Markoff equation

$$x^2 + y^2 + z^2 = 3xyz + m,$$

focusing on those solutions—called Markoff m-triples—that are composed entirely of Fibonacci and k-Fibonacci numbers.

We began by studying Markoff m-triples composed of Fibonacci numbers, aiming to identify and classify all such solutions. The main result, formalized in Theorem 4.1, establishes that for every $m > 0$, there exists at most one ordered Markoff-Fibonacci m-triple, with two exceptions: for $m = 2$, there is an infinite family of non-minimal triples of the form $(1, F(b), F(b+2))$ for even $b \geq 2$; and for $m = 21$, there exist exactly two minimal triples: $(F(2), F(3), F(6)) = (1, 2, 8)$ and $(F(3), F(3), F(7)) = (2, 2, 13)$. For all other values of m, if a Markoff-Fibonacci triple exists, it must be minimal and unique up to permutation.

Next, we extended the analysis to k-Fibonacci numbers. In this generalized setting, we proved in Theorem 5.1 that the only non-minimal Markoff m-triples composed of k-Fibonacci numbers for $m > 0$ are 8-triples of the form $(F_2(2), F_2(2n), F_2(2n+2))$ for $n \geq 2$. These correspond to a single infinite branch of the 8-Markoff tree rooted at the minimal triple $(2, 2, 12)$, and are formed by Pell numbers.

Finally, in Theorem 5.2, we demonstrated that, with the exception of a specific family when $k = 3$, every minimal Markoff m-triple with k-Fibonacci components is unique for that specific value of m. The exceptional case corresponds to pairs of the form $(F_3(a), F_3(b), F_3(a+b))$ and $(F_3(a+1), F_3(b-1), F_3(a+b))$, for a odd, b even, and $b \geq a + 3$.

These results provide a complete classification of Markoff-Fibonacci and k-Fibonacci m-triples for $m > 0$, highlighting their scarcity and structural rigidity. In all cases, our approach combined theoretical bounds with symbolic computation to reduce the search to a finite, tractable set.

As an avenue for future exploration, as explained in Section 6, we are currently investigating the application of artificial intelligence to the study of generalized Markoff equations. Specifically, an experimental Siamese neural network is being developed to detect algebraic invariants that can help distinguish between Markoff triples that belong to different trees, even if they share the same value of m. Although preliminary experiments are ongoing and results are not yet conclusive, this line of research remains promising and could offer powerful tools for addressing classification problems where traditional mathematical techniques reach their limits.

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