MARKOV EQUATION WITH FIBONACCI COMPONENTS

FLORIAN LUCA AND ANITHA SRINIVASAN

ABSTRACT. We find all triples of Fibonacci numbers $(x, y, z) = (F_i, F_j, F_n)$ satisfying the Markov equation $x^2 + y^2 + z^2 = 3xyz$.

1. Introduction

The Markov equation is

$$x^2 + y^2 + z^2 = 3xyz \tag{1.1}$$

in positive integers $x \le y \le z$. A Markov number is any positive integer which is a component of some solution to the Markov equation. Here is the sequence of Markov numbers

$$\mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{13}, 29, \mathbf{34}, \mathbf{89}, 169, 194, \mathbf{233}, \mathbf{610}, 985, 1325, \ldots$$

(sequence A002559 in [2]) appearing as coordinates of the Markov triples

$$(1, 1, 1), (1, 1, 2), (1, 2, 5), (1, 5, 13), (2, 5, 29), (1, 13, 34), (1, 34, 89), (2, 29, 169),$$
$$(5, 13, 194), (1, 89, 233), (5, 29, 433), (1, 233, 610), (2, 169, 985), (13, 34, 1325), \ldots$$

The Fibonacci sequence $\{F_m\}_{m\ge 0}$ starts as $F_0 = 0$, $F_1 = 1$ and satisfies the recurrence $F_{m+2} = F_{m+1} + F_m$ for all $m \ge 0$. Its first few terms are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots$$

(sequence A000045 in [2]). One notices that the Markov numbers seem to contain the odd indexed Fibonacci numbers. The fact that this is so is a consequence of the formula

$$1 + F_{2k-1}^2 + F_{2k+1}^2 = 3F_{2k-1}F_{2k+1} \tag{1.2}$$

valid for all positive integers k. We ask whether there are other solutions $(x, y, z) = (F_i, F_j, F_n)$ to the Markov equation other than the ones arising from (1.2). Here is our main result.

Theorem 1.1. *If $(x, y, z) = (F_i, F_j, F_n)$ is a solution in positive integers to the Markov equation, then it is of the form shown in* (1.2).

Similar problems have been investigated before. For example, it is known that the set of three integers $\{F_{2n}, F_{2n+2}, F_{2n+4}\}$ has the property that the product of any two of them plus one is square since

$$F_{2n}F_{2n+2} + 1 = F_{2n+1}^2, \quad F_{2n+2}F_{2n+4} + 1 = F_{2n+3}^2, \quad F_{2n}F_{2n+4} + 1 = F_{2n+2}^2.$$

In [1], it is shown that if $\{F_{2n}, F_{2n+2}, F_k\}$ has the property that the product of any two plus one is a square, then $k \in \{2n - 2, 2n + 4\}$ except for $n = 2$ when also $k = 1$ is possible.

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2. Preliminary results

Lemma 2.1. *If* $(a,b,c) \neq (1,1,1)$ *satisfies the Markov equation and* $a \leq b \leq c$*, then* $3ab < b+c$*.*

Proof. If $b = c$, then

$$a^2 = 3ab^2 - 2b^2 = b^2(3a-2)$$

and the right–hand side is $> b^2 \geq a^2$ if $a > 1$, which is a contradiction. Thus, $a = 1$ leading to $a = b$, showing that $a = b = c = 1$, which is excluded. Thus, $c > b$, therefore $3abc = a^2 + b^2 + c^2 < 3c^2$, which gives $ab < c$ and hence $a^2 < c$ (as $a \leq b$). Next, from $c(b-1) \geq (b+1)(b-1) = b^2 - 1$, we have $bc - c \geq b^2$. Therefore $a^2 + b^2 < c + b^2 \leq bc$. It follows that $\frac{a^2+b^2}{c} < b$, so that $3ab = \frac{a^2+b^2}{c} + c < b + c$. □

Recall that

$$F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} \qquad \text{where} \qquad (\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} \right) \tag{2.1}$$

for all $k \geq 0$. In particular,

$$\alpha^{k-2} \leq F_k \leq \alpha^{k-1} \qquad \text{holds for all} \qquad k \geq 1. \tag{2.2}$$

3. The proof of Theorem 1.1

Assume $x = F_i,\ y = F_j,\ z = F_n$ with $x \leq y \leq z$. Since $F_1 = F_2 = 1$, we assume that $2 \leq i \leq j \leq n$. Then

$$z - 3xy = -\frac{x^2 + y^2}{z}. \tag{3.1}$$

Inserting the values of x, y, z in the left hand side of (3.1), we get

$$\frac{\alpha^n}{\sqrt{5}} - \frac{3}{5}\alpha^{i+j} = -\frac{F_i^2 + F_j^2}{F_n} + \frac{\beta^n}{\sqrt{5}} - \frac{3}{5}\left(\alpha^i\beta^j + \alpha^j\beta^i - \beta^{i+j}\right).$$

Taking absolute values and using

$$\begin{aligned}
\frac{F_i^2 + F_j^2}{F_n} &\leq \frac{2F_j^2}{F_n} \leq 2\alpha^{2j-n} \leq 2\alpha^j, \\
\left|\frac{\beta^n}{\sqrt{5}}\right| &\leq \frac{\alpha^{-j}}{\sqrt{5}} < \frac{\alpha^j}{5}, \\
\left|\frac{3}{5}\left(\alpha^i\beta^j + \alpha^j\beta^i - \beta^{i+j}\right)\right| &\leq \frac{3}{5}\left(2\alpha^j + 1\right) \leq \frac{9\alpha^j}{5},
\end{aligned}$$

we get that

$$\left|\frac{\alpha^n}{\sqrt{5}} - \frac{3}{5}\alpha^{i+j}\right| \leq \alpha^j \left(2 + \frac{1}{5} + \frac{9}{5}\right) = 4\alpha^j.$$

Dividing across by $\alpha^{i+j}/\sqrt{5}$ we get

$$\left|\alpha^{n-i-j} - \frac{3}{\sqrt{5}}\right| < \frac{4\sqrt{5}}{\alpha^i}. \tag{3.2}$$

Certainly,

$$1 < \frac{3}{\sqrt{5}} < \alpha$$

and

$$\min_{k\in\mathbb{Z}}\left|\alpha^k-\frac{3}{\sqrt{5}}\right|=\left|\alpha-\frac{3}{\sqrt{5}}\right|>0.2763,$$

so (3.2) shows that

$$0.2763<\frac{4\sqrt{5}}{\alpha^i},$$

which gives $\alpha^i<4\sqrt{5}/0.2763$, or $i\le 7$. We record what we have proved.

Lemma 3.1. *If $(x,y,z)=(F_i,F_j,F_n)$ satisfies (1.1) with $i\le j\le n$, then $i\in\{2,3,4,5,6,7\}$. Of these, only $i=2,3,5,7$ lead to $F_i=1,2,5,13$ which are Markov numbers.*

Lemma 3.2. *If $(x,y,z)=(F_i,F_j,F_n)$ satisfies (1.1) with $i\le j\le n$, then n is odd, $j=n-2$ and $F_i=1$.*

Proof. We shall treat the case $F_i=1$ at the end.

Assume that $F_i=2$. Then we have

$$4+F_j^2+F_n^2=6F_jF_n$$

or

$$4+F_j^2=F_n(6F_j-F_n),$$

which gives $F_n<6F_j$. From Lemma 2.1, we have $6F_j<F_j+F_n$ which gives $5F_j<F_n$. Hence, we have

$$5F_j<F_n<6F_j.$$

This is false because

$$F_{j+3}=F_{j+2}+F_{j+1}=2F_{j+1}+F_j=3F_j+2F_{j-1}<5F_j$$

(since $j\ge i\ge 3$), while

$$F_{j+4}=F_{j+3}+F_{j+2}=2F_{j+2}+F_{j+1}=3F_{j+1}+2F_j=5F_j+3F_{j-1}>6F_j,$$

since the last inequality is equivalent to $3F_{j-1}>F_j$, which holds because $F_j=F_{j-1}+F_{j-2}$ and $2F_{j-1}>F_{j-2}$. Thus, for $j\ge 3$, the interval $(5F_j,6F_j)$ does not contain any Fibonacci number. Similarly in the cases $F_i=5$ and $F_i=13$ we get

$$14F_j<F_n<15F_j,$$

and

$$38F_j<F_n<39F_j,$$

respectively, which are false as

$$F_{j+5}=8F_j+5F_{j-1}<14F_j\quad\text{and}\quad F_{j+6}=13F_j+8F_{j-1}>15F_j\quad(j\ge 5)$$

and

$$F_{j+8}=21F_j+13F_{j-1}<38F_j\quad\text{and}\quad F_{j+9}=34F_j+21F_{j-1}>39F_j\quad(j\ge 7),$$

so that the intervals $(14F_j,15F_j)$ and $(38F_j,39F_j)$ do not contain a Fibonacci number for $j\ge 5$ and $j\ge 7$, respectively.

So we have $F_i=1$. In this case, we get

$$F_j<F_n<3F_j$$

which gives $n=j+1$ or $n=j+2$. In the first case, $n\ge 3$ and we have

$$1+F_{n-1}^2+F_n^2=3F_{n-1}F_n$$

or

$$1 + (F_n - F_{n-1})^2 = F_{n-1}F_n,$$

or

$$1 + F_{n-2}^2 = F_{n-1}F_n$$

which is possible only in the case $n = 3$. But this gives the solution $(1, 1, 2) = (F_1, F_1, F_3)$, so it satisfies the conclusion of the theorem. Finally, when $j = n - 2$, we get

$$1 + F_{n-2}^2 + F_n^2 = 3F_{n-2}F_n$$

which implies that n is odd (otherwise one of $n - 2$ or n is a multiple of 4, so one of F_{n-2} or F_n is divisible by 3, in which case the above relation is impossible modulo 3). □

References

[1] B. He, F. Luca and A. Togbé, *Diophantine triples of Fibonacci numbers*, Acta Arithmetica, **175** (2016), 57–70.

[2] N. J. A. Sloane, *On-Line Encyclopedia of Integer Sequences, https://oeis.org/*

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School of Mathematics, University of the Witwatersrand, Private Bag X3, Wits 2050, South Africa;
Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany;
Department of Mathematics, Faculty of Sciences, University of Ostrava, 30 Dubna 22, 701 03 Ostrava 1, Czech Republic

E-mail address: florian.luca@wits.ac.za

Saint Louis University, Madrid Campus, Avenida del Valle 34, 28003 Madrid, Spain

E-mail address: rsrinivasan.anitha@gmail.com