Research Article

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A singular perturbation result for a class of periodic-parabolic BVPs

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Abstract: In this article, we obtain a very sharp version of some singular perturbation results going back to Dancer and Hess [*Behaviour of a semilinear periodic-parabolic problem when a parameter is small,* Lecture Notes in Mathematics, Vol. 1450, Springer-Verlag, Berlin, 1990, pp. 12–19] and Daners and López-Gómez [*The singular perturbation problem for the periodic-parabolic logistic equation with indefinite weight functions*, J. Dynam. Differential Equations **6** (1994), 659–670] valid for a general class of semilinear periodic-parabolic problems of logistic type under general boundary conditions of mixed type. The results of Dancer and Hess [*Behaviour of a semilinear periodic-parabolic problem when a parameter is small,* Lecture Notes in Mathematics, Vol. 1450, Springer-Verlag, Berlin, 1990, pp. 12–19] and [*The singular perturbation problem for the periodic-parabolic logistic equation with indefinite weight functions*, J. Dynam. Differential Equations **6** (1994), 659–670] were found, respectively, for Neumann and Dirichlet boundary conditions with $\mathfrak{L} = -\Delta$. In this article, $\mathfrak{L}$ stands for a general second-order elliptic operator.

Keywords: positive solutions, periodic-parabolic problems, singular perturbations

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1 Introduction

In this article, we study the periodic-parabolic problem

$$\begin{cases} \partial_t u + \mathrm{d}\kappa(t)\mathscr{L}u = m(x,t)u - a(x,t)u^p, & \text{in } \Omega \times \mathbb{R}, \\ \mathscr{B}u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,0) = u(x,T), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $p > 1$ and $d > 0$ are the constants, under the following conditions:

(i) Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, of class $C^{2+\theta}$ for some $\theta \in (0,1)$, with boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are two disjoint open and closed subsets of $\partial\Omega$. As they are disjoint, Γ_0 and Γ_1 are of class $C^{2+\theta}$.

(ii) For a given $T > 0$, $\mathscr{L}$ stands for the autonomous linear second-order differential operator

$$\mathscr{L} = \mathscr{L}(x) := -\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{N} b_j(x)\frac{\partial}{\partial x_j},$$

with $a_{ij} = a_{ji}$, $b_j \in C^\theta(\bar{\Omega}; \mathbb{R})$ for all $1 \leq i, j \leq N$. Moreover, $\mathscr{L}$ is uniformly elliptic in $\bar{\Omega}$, i.e., there exists $\mu > 0$ such that

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$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2, \quad \text{for all } (x,\xi) \in \bar{\Omega} \times \mathbb{R}^N,$$

where $|\cdot|$ stands for the Euclidean norm of $\mathbb{R}^N$.

(iii) $\kappa(t)$ is a T-periodic Hölder continuous function in $\mathbb{R}$ such that $\kappa(t) > 0$ for all $t \in \mathbb{R}$. Moreover, setting

$$F := \left\{u \in C^{\theta,\frac{\theta}{2}}(\bar{\Omega} \times \mathbb{R}; \mathbb{R}) : u(\cdot, T+t) = u(\cdot, t), \quad \text{for all } t \in \mathbb{R}\right\},$$

$a \in F$ satisfies $a(x,t) > 0$ for all $(x,t) \in \bar{\Omega} \times \mathbb{R}$, and $m \in F$ may change of sign in $\bar{\Omega} \times \mathbb{R}$.

(iv) $\mathscr{B} : C(\Gamma_0) \oplus C^1(\Omega \cup \Gamma_1) \to C(\partial\Omega)$ stands for the boundary operator

$$\mathscr{B}\xi := \begin{cases} \xi, & \text{on } \Gamma_0, \\ \dfrac{\partial \xi}{\partial \nu} + \beta(x)\xi, & \text{on } \Gamma_1, \end{cases}$$

for every $\xi \in C(\Gamma_0) \oplus C^1(\Omega \cup \Gamma_1)$, where $\beta \in C^\theta(\Gamma_1)$ and

$$\nu = (\nu_1, \dots, \nu_N) \in C^{1+\theta}(\partial\Omega; \mathbb{R}^N)$$

is an outward pointing nowhere tangent vector field.

As it will become apparent in Section 3, though in this article, $\beta(x)$ can change of sign on Γ_1, one can assume, without loss of generality, that

$$\beta(x) > 0, \quad \text{for all } x \in \Gamma_1. \tag{1.2}$$

Note that since Γ_1 is smooth, it must consist of finitely many components, say $\Gamma_{1,j}$ with $j \in \{1, \dots, q\}$ for some integer $q \geq 1$.

Throughout this article, for every continuous T-periodic function $V : [0,T] \to \mathbb{R}$, we will denote by

$$\overline{V} := \frac{1}{T}\int_0^T V(s)\mathrm{d}s$$

the mean of V in $[0,T]$.

Our main goal in this article is to obtain the following singular perturbation result, where $\theta_{[m,a,d]}$ stands for the unique positive solution of the semilinear periodic-parabolic problem (1.1). According to Theorem 5.2, $\theta_{[m,a,d]}$ exists for sufficiently small $d > 0$ if $\overline{m}(x_0) > 0$ for some $x_0 \in \Omega$.

Theorem 1.1. *Assume that there exists $x_0 \in \Omega$ such that $\overline{m}(x_0) > 0$, and let $K \subset \Omega \cup \Gamma_1$ be a compact subset. Then, the following conditions are satisfied*:

(*i*) *If $\overline{m}(x) \leq 0$ for all $x \in K$, then*

$$\lim_{d\downarrow 0} \theta_{[m,a,d]} = 0, \quad \textit{uniformly in } K \times [0,T].$$

(*ii*) *If $\overline{m}(x) > 0$ for all $x \in K$ and $K \subset \Omega$, then*

$$\lim_{d\downarrow 0} \theta_{[m,a,d]} = \alpha_{[m,a]}, \quad \textit{uniformly in } K \times [0,T], \tag{1.3}$$

where $\alpha_{[m,a]}$ stands for the unique positive periodic solution of the associated kinetic model

$$\begin{cases} \partial_t u = m(x,t)u - a(x,t)u^p, & t \in \mathbb{R}, \\ u(x,0) = u(x,T). \end{cases} \tag{1.4}$$

(*iii*) *If $\overline{m}(x) > 0$ for all $x \in K$ and there exists a nonempty subset $\mathscr{I} \subset \{1, \dots, q\}$ such that*

$$\partial K \cap \Gamma_1 = \bigcup_{i \in \mathscr{I}} \Gamma_{1,i}, \quad \operatorname{dist}(\partial K \cap \Omega, \Gamma_1) > 0,$$

and $(m,a) = (m(t), a(t))$ on a neighborhood of $\partial K \cap \Gamma_1$, then (1.3) *holds.*

This result is a substantial extension of some not well-known findings of Dancer and Hess [5], and Theorem 1.3 of Daners and López-Gómez [6], which are very simple counterparts of Theorem 1.1 for $\mathfrak{L} = -\Delta$ and either $\partial\Omega = \Gamma_1$ with $\beta = 0$, or $\partial\Omega = \Gamma_0$, respectively. Some very recent elliptic counterparts of Theorem 1.1 valid for general elliptic operators $(\mathfrak{L}, \mathfrak{B}, \Omega)$ have been given by Fernández-Rincón and López-Gómez [7]. In this article, it remains an open problem to ascertain whether, or not, the condition that $(m, a) = (m(t), a(t))$ on a neighborhood of $\partial K \cap \Gamma_1$ in Part (iii) is of a technical nature.

Theorem 1.1 is of a huge interest in population dynamics, where the behavior of the species for small diffusion coefficients provides us with an idealized behavior of many real systems. A simple glance to the pioneering article of Hutson et al. [9] will convince the reader of it very easily. Actually, [9] generated a huge industry in the field under the auspices of Y. Lou, W. M. Ni, and their coworkers. The reader should compare the results of Section 2 of Hutson et al. [9] with Theorem 1.1 of Lou [13].

The condition $a(x, t) > 0$ for all $(x, t) \in \bar{\Omega} \times \mathbb{R}$ is imperative for the existence of a positive solution of (1.1) for small $d > 0$ even for the simplest elliptic counterpart of (1.1)

$$\begin{cases} -d\Delta u = mu - a(x)u^p, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $m > 0$ is a constant. Indeed, if $a(x)$ vanishes on some nice smooth subdomain of Ω, say Ω_0, with $\bar{\Omega}_0 \subset \Omega$, then, according to [12, Ch. 4], it is well-known that (1.5) possesses a positive solution if, and only if,

$$\mathrm{d}\sigma_1[-\Delta, \mathscr{D}, \Omega] < m < \mathrm{d}\sigma_1[-\Delta, \mathscr{D}, \Omega_0], \tag{1.6}$$

where we are denoting $\mathscr{D} \equiv \mathscr{B}$ if $\Gamma_1 = \emptyset$. Since (1.6) can be equivalently expressed as

$$\frac{m}{\sigma_1[-\Delta, \mathscr{D}, \Omega_0]} < d < \frac{m}{\sigma_1[-\Delta, \mathscr{D}, \Omega]},$$

it is apparent that (1.5) cannot admit a positive solution for sufficiently small $d > 0$. Therefore, Theorem 1.1 cannot be applied in the degenerate case when $a(x, t)$ vanishes somewhere in $\Omega \times \mathbb{R}$, because (1.1) might not admit any positive solution for sufficiently small $d > 0$.

Throughout this article, for any given open bounded subset, $D \subset \mathbb{R}^N$, $N \geq 1$, and any continuous function $f : \bar{D} \to \mathbb{R}$, we denote

$$f_L \coloneqq \min_{\bar{D}} f \quad \text{and} \quad f_M \coloneqq \max_{\bar{D}} f,$$

and, for any compact subset $K \subset D$, we set

$$f_{L,K} \coloneqq \min_{K} f \quad \text{and} \quad f_{M,K} \coloneqq \max_{K} f.$$

Naturally, we are denoting $f_L = f_{L,\bar{D}}$ and $f_M = f_{M,\bar{D}}$. Also, for every $d > 0$, we will consider the periodic-parabolic operator

$$\mathscr{P}_d \coloneqq \partial_t + \mathrm{d}\kappa(t)\mathscr{L},$$

and for any subdomain $D \subset \Omega$, we denote by D_T the parabolic cylinder $D_T \coloneqq D \times [0, T]$. In particular, $\Omega_T = \Omega \times [0, T]$.

This study is organized as follows. In Section 2, we analyze the associated kinetic problem (1.4). In Section 3, we show that, without loss of generality, one can assume that (1.2) holds in (1.1). In Section 4, we study some pivotal properties of the underlying principal eigenvalues associated with the periodic-parabolic problem (1.1). In Section 5, we study the existence and the uniqueness of the positive solution of (1.1) for small $d > 0$. In Section 6 we construct some supersolutions for problem (1.1). In Section 7, we construct some subsolutions of (1.1) in the special case when $m(x, t) = m(t)$ and $a(x, t) = a(t)$. The construction of φ in the proof of Theorem 7.1 is a technical device borrowed from López-Gómez [10]. In Section 8, we deliver an auxiliary result to prove Theorem 1.1(iii). Finally, in Section 9, the proof of Theorem 1.1 is completed.

2 Associated kinetic problem

This section analyzes the existence of (T-periodic) positive solutions of

$$\begin{cases} \partial_t u = m(x,t)u - a(x,t)u^p, & t \in \mathbb{R}, \\ u(x,0) = u(x,T), \end{cases} \tag{2.1}$$

where $x \in \bar{\Omega}$ is regarded as a parameter. Its main result can be stated as follows.

Proposition 2.1. *For every $x \in \bar{\Omega}$,* (2.1) *possesses a T-periodic positive solution if, and only if,*

$$\overline{m}(x) = \frac{1}{T}\int_0^T m(x,t)\mathrm{d}t > 0. \tag{2.2}$$

In such case, it is unique and given through

$$\alpha_{[m,a;\,x]}(t) = e^{\int_0^t m(x,s)\mathrm{d}s}\left(A(x) + (p-1)\int_0^t a(x,s)e^{(p-1)\int_0^s m(x,\tau)\mathrm{d}\tau}\mathrm{d}s\right)^{\frac{-1}{p-1}}, \tag{2.3}$$

where

$$A(x) := \frac{(p-1)\int_0^T a(x,s)e^{(p-1)\int_0^s m(x,\tau)\mathrm{d}\tau}\mathrm{d}s}{e^{(p-1)\int_0^T m(x,s)\mathrm{d}s} - 1}.$$

If (2.2) *fails, i.e., $\overline{m}(x) \le 0$, then, $\alpha_{[m,a;\,x]}(t) \equiv 0$ is the unique non-negative T-periodic solution of* (2.1).

Proof. Since $p > 1$, for every $x \in \bar{\Omega}$, any solution of (2.1) satisfies

$$\partial_t u = [m(x,t) - a(x,t)u^{p-1}]u,$$

and hence,

$$u(t) = e^{\int_0^t [m(x,s) - a(x,s)u^{p-1}(x,s)]\mathrm{d}s}u(0),$$

for all $t \in \mathbb{R}$. Thus, $u(t) > 0$ for all $t \in [0,T]$ if $u(0) > 0$, $u(t) = 0$ for all $t \in [0,T]$ if $u(0) = 0$, and $u(t) < 0$ for all $t \in [0,T]$ if $u(0) < 0$.

Suppose that $u(t)$ is a positive solution of (2.1) for some $x \in \bar{\Omega}$. Then, the change of variable $v = u^{1-p}$ transforms (2.1) into the linear problem

$$\begin{cases} v' + (p-1)m(x,t)v = (p-1)a(x,t), & t \in \mathbb{R}, \\ v(0) = v(T). \end{cases} \tag{2.4}$$

Solving the linear differential equation of (2.4), we have that

$$v(t) = e^{(1-p)\int_0^t m(x,s)\mathrm{d}s}\left[v(0) + \int_0^t (p-1)a(x,s)e^{(p-1)\int_0^s m(x,\tau)\mathrm{d}\tau}\mathrm{d}s\right]. \tag{2.5}$$

Thus, imposing $v(0) = v(T)$, the following identity must be satisfied:

$$v(0) = v(T) = e^{(1-p)\int_0^T m(x,s)\mathrm{d}s}\left[v(0) + (p-1)\int_0^T a(x,s)e^{(p-1)\int_0^s m(x,\tau)\mathrm{d}\tau}\mathrm{d}s\right],$$

and hence,

$$v(0) = \frac{(p-1)\int_0^T a(x,s)e^{(p-1)\int_0^s m(x,\tau)\mathrm{d}\tau}\mathrm{d}s}{e^{(p-1)\int_0^T m(x,s)\mathrm{d}s} - 1} = A(x). \tag{2.6}$$

Consequently, substituting (2.6) into (2.5) and taking into account that $u = v^{\frac{1}{1-p}}$, (2.3) is satisfied.

Now, we will show that $\alpha_{[m,a;\,x]}$ is positive if and only if (2.2) holds. Suppose that $\overline{m}(x) > 0$. Then, since $p > 1$ and $a(x,t) > 0$ for all $(x,t) \in \bar{\Omega} \times \mathbb{R}$, it follows from (2.6) that $v(0) > 0$. Consequently, by (2.5), we find that $v(t) > 0$ for all $t \in \mathbb{R}$. Therefore, $u(t) > 0$ for all $t \in \mathbb{R}$. Conversely, suppose that (2.1) has a T-periodic positive solution u. Then, $u(t) > 0$ for all $t \in [0,T]$, and $v = u^{1-p}$ satisfies $v(t) > 0$ for all $t \in [0,T]$, as well as (2.4) and (2.6). As, in particular, $v(0) > 0$, necessarily, $\overline{m}(x) > 0$.

Finally, the uniqueness of the T-periodic positive solution comes from the fact that it must be given by (2.3). This ends the proof. □

Subsequently, we denote by $\alpha_{[m,a]}$ the function

$$\begin{aligned} \alpha_{[m,a]} : \bar{\Omega} \times [0,T] &\longrightarrow [0,\infty) \\ (x,t) &\mapsto \alpha_{[m,a;\,x]}(t), \end{aligned} \tag{2.7}$$

where $\alpha_{[m,a;\,x]}(t) > 0$ for all $t \in [0,T]$ if $\overline{m}(x) > 0$ and $\alpha_{[m,a;\,x]} \equiv 0$ if $\overline{m}(x) \leq 0$. The next result collects some of its properties.

Proposition 2.2. *The function $\alpha_{[m,a]}$ defined in* (2.7) *is continuous in $(x,t) \in \bar{\Omega} \times [0,T]$. Thus, it satisfies the following properties*:

(*i*) *If $\overline{m}(x_0) > 0$ for some $x_0 \in \bar{\Omega}$, then there exists a neighborhood, $\mathcal{U}$, of x_0 in $\bar{\Omega}$ such that $\alpha_{[m,a]}(x,t) > 0$ for all $(x,t) \in \mathcal{U}_T \equiv \mathcal{U} \times [0,T]$.*

(*ii*) *If $\overline{m}(x_0) < 0$ for some $x_0 \in \bar{\Omega}$, then there exists a neighborhood, $\mathcal{U}$, of x_0 in $\bar{\Omega}$ such that $\alpha_{[m,a]}(x,t) = 0$ for all $(x,t) \in \mathcal{U}_T$.*

(*iii*) *Let $K \subset \bar{\Omega}$ be a compact subset such that $\overline{m}(x) > 0$ for all $x \in K$. Then,*

$$(\alpha_{[m,a]})_{L,K_T} \coloneqq \min_{(x,t)\in K_T} \alpha_{[m,a]}(x,t) > 0.$$

Proof. The continuity of the map (2.7) is a direct consequence of the continuity of $\alpha_{[m,a;\,x]}(t)$ with respect to $x \in \bar{\Omega}$ and $t \in \mathbb{R}$. This follows easily from (2.3) by the continuity of $a(x,t)$ and $m(x,t)$ if $\overline{m}(x) > 0$. Similarly, it follows from our definition of $\alpha_{[m,a]}$ if $\overline{m}(x) < 0$. However, the case when $\overline{m}(x) = 0$ is more delicate. The continuity in this case relies on the fact that

$$\lim_{\substack{y\to x \\ \overline{m}(y)>0}} \alpha_{[m,a;\,y]}(t) = 0, \quad \text{for all } t \in [0,T]. \tag{2.8}$$

Since

$$\lim_{\substack{y\to x \\ \overline{m}(y)>0}} A(y) = \infty,$$

Property (2.8) can be also derived from (2.3). The remaining assertions are direct consequences from this continuity. □

The following result establishes the monotonicity of $\alpha_{[m,a]}$ with respect to m and a.

Proposition 2.3. *Let $m_i, a_i \in F$, $i = 1,2$, be such that $m_1 \leq m_2$ and $a_1 \geq a_2$ in D_T, for some open subset $D \subset \Omega$ with $\overline{m}_1(x) > 0$ for all $x \in D$. Then,*

$$\alpha_{[m_1,a_1]} \leq \alpha_{[m_2,a_2]}, \quad \text{in } D_T. \tag{2.9}$$

Proof. By the assumptions,

$$0 < \overline{m}_1(x) \leq \overline{m}_2(x), \quad \text{for all } x \in D.$$

Thus, thanks to Proposition 2.1,

$$\alpha_{[m_i,a_i;\,x]}(t) = \alpha_{[m_i,a_i]}(x,t) > 0, \quad \text{for all } (x,t) \in D_T,\ i = 1,2.$$

Unfortunately, Estimate (2.9) cannot be obtained directly from (2.3), because the character of the integral

$$\int_0^t a(x,s)e^{(p-1)\int_0^s m(x,\tau)\mathrm{d}\tau}\mathrm{d}s$$

is unclear when a decreases and m increases. Thus, to prove (2.9), in this case, we use the following argument. Setting

$$u_1 := \alpha_{[m_1,a_1]}, \quad u_2 := \alpha_{[m_2,a_2]}, \quad \text{and} \quad \omega := u_2 - u_1,$$

we have that

$$\begin{aligned}\omega' &= m_2u_2 - a_2u_2^p - (m_1u_1 - a_1u_1^p) \geq m_2u_2 - a_2u_2^p - (m_2u_1 - a_2u_1^p)\\ &= m_2(u_2-u_1) - a_2(u_2^p - u_1^p) = m_2(u_2-u_1) - a_2\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}(su_2 + (1-s)u_1)^p\mathrm{d}s\\ &= \left(m_2 - a_2p\int_0^1 [su_2 + (1-s)u_1]^{p-1}\mathrm{d}s\right)\omega.\end{aligned}$$

In other words, setting

$$\gamma(t) := m_2(x,t) - a_2(x,t)p\int_0^1 [su_2(t) + (1-s)u_1(t)]^{p-1}\mathrm{d}s, \quad t \in \mathbb{R},$$

we find that

$$\omega'(t) \geq \gamma(t)\omega(t), \quad \text{for all } t \in \mathbb{R}. \tag{2.10}$$

Hence, performing the change of variable

$$\omega(t) = e^{\int_0^t \gamma(s)\mathrm{d}s}z(t), \quad t \in \mathbb{R},$$

in (2.10), it readily follows that $z'(t) \geq 0$ for all $t \in \mathbb{R}$.

On the other hand, since $u_2(t) > 0$ for all $t \in \mathbb{R}$, integrating in $[0,T]$ the identity

$$\frac{u_2'(t)}{u_2(t)} = m_2(x,t) - a_2(x,t)u_2^{p-1}(x,t),$$

it follows from the fact that u_2 is T-periodic that

$$\int_0^T m_2(x,t)\mathrm{d}t = \int_0^T a_2(x,t)u_2^{p-1}(x,t)\mathrm{d}t. \tag{2.11}$$

Consequently, since $u_1(t) > 0$ for all $t \in \mathbb{R}$, (2.11) implies that

$$\begin{aligned}\int_0^T \gamma(t)\mathrm{d}t &= \int_0^T m_2(x,t)\mathrm{d}t - p\int_0^T a_2(x,t)\int_0^1 [su_2(t) + (1-s)u_1(t)]^{p-1}\mathrm{d}s\mathrm{d}t\\ &< \int_0^T m_2(x,t)\mathrm{d}t - p\int_0^T a_2(x,t)\int_0^1 [su_2(t)]^{p-1}\mathrm{d}s\mathrm{d}t\\ &= \int_0^T m_2(x,t)\mathrm{d}t - \int_0^T a_2(x,t)u_2^{p-1}(t)\mathrm{d}t = 0.\end{aligned}$$

Therefore, $\int_0^T \gamma(t)\mathrm{d}t < 0$.

Next, going back to the change of variable and taking into account that $\omega(t)$ and $\gamma(t)$ are T-periodic, it becomes apparent that, for every integer $n \in \mathbb{Z}$,

$$\omega(0) = \omega(nT) = e^{\int_0^{nT} \gamma(s)\mathrm{d}s} z(nT) = e^{n\int_0^{T} \gamma(s)\mathrm{d}s} z(nT).$$

Thus,

$$z(nT) = \omega(0) e^{-n\int_0^T \gamma(s)\mathrm{d}s}.$$

As $\int_0^T \gamma(s)\mathrm{d}s < 0$, $e^{-n\int_0^T \gamma(s)\mathrm{d}s}$ is increasing with respect to n. Moreover, $z(nT)$ is non-decreasing with respect to n, because $z' \geq 0$. Hence,

$$\omega(0) = u_2(0) - u_1(0) \geq 0,$$

and consequently, $z(0) = \omega(0) \geq 0$. So, since $z' \geq 0$, we find that $z(t) \geq 0$ for all $t \geq 0$. Therefore,

$$\omega(t) = e^{\int_0^t \gamma(s)\mathrm{d}s} z(t) \geq 0, \quad \text{for all } \; t \geq 0.$$

As $\omega(t)$ is T-periodic, this entails that

$$\omega(t) = u_2(t) - u_1(t) \geq 0,$$

for all $t \in \mathbb{R}$, and this ends the proof. □

3 Pivotal change of variable

When (1.2) fails, one can proceed as follows. Since $\Omega \in C^{2+\theta}$, it follows from [11, Le. 2.1] and [7, Th. 1.3] that there exists $\psi \in C^{2+\theta}(\mathbb{R}^N)$ such that $\psi(x) < 0$ for all $x \in \Omega$, $\psi(x) > 0$ for all $x \in \mathbb{R}^N \setminus \bar{\Omega}$, $\psi^{-1}(0) = \partial\Omega$, and

$$(\partial_\nu \psi)_L := \min_{\partial\Omega} \frac{\partial \psi}{\partial \nu} \equiv (\partial_\nu \psi)_{L,\partial\Omega} > 0.$$

Then, setting

$$h(x) = e^{\mu\psi(x)}, \quad x \in \Omega,$$

for some constant $\mu > 0$ to be determined later, we have that $h \in C^{2+\theta}(\mathbb{R}^N)$ satisfies $h(x) > 0$ for all $x \in \mathbb{R}^N$. Thus, making the change of variable

$$w(x,t) = \frac{u(x,t)}{h(x)}, \quad (x,t) \in \bar{\Omega} \times \mathbb{R}, \tag{3.1}$$

where

$$u \in E := \left\{ u \in C^{2+\theta, 1+\frac{\theta}{2}}(\bar{\Omega} \times \mathbb{R}; \mathbb{R}) \;:\; u(\cdot, T+t) = u(\cdot, t), \;\; \text{for all } \; t \in \mathbb{R} \right\},$$

it is apparent that $w \in E$ and

$$\mathscr{L}u = \mathscr{L}(hw) = h\mathscr{L}_h w, \tag{3.2}$$

where $\mathscr{L}_h$ stands for the differential operator

$$\mathscr{L}_h := -\sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^N b_{j,h}(x) \frac{\partial}{\partial x_j} + c_h(x), \tag{3.3}$$

with

$$b_{i,h} := b_i - \frac{2}{h} \sum_{j=1}^N a_{ij} \frac{\partial h}{\partial x_j}, \quad c_h := \frac{\mathscr{L}h}{h}, \quad i \in \{1, \ldots, N\}.$$

The reader is sent to Section 1.7 of [11] for any further details on the change (3.1), going back to the generalized maximum principle of Protter and Weinberger [14]. Note that (3.3) satisfies similar properties as $\mathscr{L}$. In particular, its coefficients also belong to F.

Since $h(x)$ does not depend on t, by (3.2), the change of variable (3.1) transforms the periodic-parabolic equation of (1.1) into

$$\partial_t w + \mathrm{d}\kappa(t)\mathscr{L}_h w = m(x,t)w - a_h(x,t)w^p, \quad \text{where } a_h(x,t) \coloneqq h^{p-1}(x)a(x,t).$$

Since $a \in F$ and $h \in C^{2+\theta}(\bar{\Omega})$, with $h(x) > 0$ for all $x \in \bar{\Omega}$, the function a_h lies in F.

Similarly, one has that

$$\mathscr{B}(hw) = h\mathscr{B}_h w,$$

where $\mathscr{B}_h$ is defined through

$$\mathscr{B}_h\xi \coloneqq \begin{cases} \xi, & \text{on } \Gamma_0, \\ \dfrac{\partial \xi}{\partial \nu} + \beta_h(x)\xi, & \text{on } \Gamma_1, \end{cases} \quad \text{where } \beta_h \coloneqq \frac{\mathscr{B}h}{h}. \tag{3.4}$$

Thus, since

$$\beta_h = \beta + \frac{1}{h}\frac{\partial h}{\partial \nu} = \beta + \mu\frac{\partial \psi}{\partial \nu},$$

for every

$$\mu > \frac{\beta_{M,\Gamma_1}}{(\partial_\nu \psi)_{L,\Gamma_1}} \geq 0, \tag{3.5}$$

where we are denoting

$$\beta_{M,\Gamma_1} \coloneqq \max_{x\in\Gamma_1}|\beta(x)| \geq 0,$$

we have that

$$\beta_h = \beta + \mu\frac{\partial \psi}{\partial \nu} > \beta + \frac{\beta_{M,\Gamma_1}}{(\partial_\nu \psi)_{L,\Gamma_1}}\frac{\partial \psi}{\partial \nu} \geq \beta + \beta_{M,\Gamma_1} \geq 0. \tag{3.6}$$

Hence, choosing μ to satisfy (3.5), we have that $\beta_h(x) > 0$ for all $x \in \Gamma_1$.

Summarizing, for sufficiently large μ, the change of variable (3.1) transforms problem (1.1) into the next one

$$\begin{cases} \partial_t w + \mathrm{d}\kappa(t)\mathscr{L}_h w = m(x,t)w - a_h(x,t)w^p, & \text{in } \Omega \times \mathbb{R}, \\ \mathscr{B}_h w(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ w(x,0) = w(x,T), & \text{in } \Omega, \end{cases} \tag{3.7}$$

where $\mathscr{L}_h$ and $\mathscr{B}_h$ are given by (3.3) and (3.4) with β_h satisfying (3.6). As the regularity of the several coefficients involved in the framework of (3.7) is the same as those imposed in (1.1), in this article, we will work with problem (1.1) assuming, without loss of generality, that condition (1.2) holds.

Suppose, in addition, that $\int_0^T m(x,t)\mathrm{d}t > 0$, for all $x \in \bar{\Omega}$, and that

$$\alpha_{[m,a;\,x]} = \alpha_{[m,a]}(x,\cdot)$$

is a C^2-function in the variable $x \in \bar{\Omega}$, where $\alpha_{[m,a;\,x]}$ is the unique positive solution of (2.1). Then, thanks to Proposition 2.1, $\alpha_{[m,a]}(x,t) > 0$ for all $(x,t) \in \bar{\Omega}_T$. Moreover, setting

$$(\alpha_{[m,a]})_{L,\Gamma_1} \coloneqq \min_{(x,t)\in\Gamma_1\times[0,T]} \alpha_{[m,a]}(x,t) > 0,$$

$$(\partial_\nu \alpha_{[m,a]})_{M,\Gamma_1} \coloneqq \max_{(x,t)\in\Gamma_1\times[0,T]} \left|\frac{\partial \alpha_{[m,a]}}{\partial \nu}(x,t)\right| \geq 0,$$

and enlarging μ so that, instead of (3.5), the next (strongest) condition holds

$$\mu > \frac{\beta_{M,\Gamma_1} + \frac{(\partial_\nu \alpha_{[m,a]})_{M,\Gamma_1}}{(\alpha_{[m,a]})_{L,\Gamma_1}}}{(\partial_\nu \psi)_{L,\Gamma_1}}, \tag{3.8}$$

then, besides (3.6), one can also obtain that

$$\partial_\nu \alpha_{[m,a]} + \beta_h \alpha_{[m,a]} \geq 0, \quad \text{on } \Gamma_1. \tag{3.9}$$

Indeed, along Γ_1, one has that

$$\partial_\nu \alpha_{[m,a]} + \beta_h \alpha_{[m,a]} = \partial_\nu \alpha_{[m,a]} + (\beta + \mu \partial_\nu \psi)\alpha_{[m,a]} \geq \partial_\nu \alpha_{[m,a]} + (\beta + \mu(\partial_\nu \psi)_{L,\Gamma_1})\alpha_{[m,a]}.$$

Thus, as soon as μ satisfies (3.8), we have that

$$\begin{aligned}\partial_\nu \alpha_{[m,a]} + \beta_h \alpha_{[m,a]} &\geq \partial_\nu \alpha_{[m,a]} + \left(\beta + \beta_{M,\Gamma_1} + \frac{(\partial_\nu \alpha_{[m,a]})_{M,\Gamma_1}}{(\alpha_{[m,a]})_{L,\Gamma_1}}\right)\alpha_{[m,a]} \\ &\geq \partial_\nu \alpha_{[m,a]} + \frac{\alpha_{[m,a]}}{(\alpha_{[m,a]})_{L,\Gamma_1}}(\partial_\nu \alpha_{[m,a]})_{M,\Gamma_1} \\ &\geq \partial_\nu \alpha_{[m,a]} + (\partial_\nu \alpha_{[m,a]})_{M,\Gamma_1} \geq 0.\end{aligned}$$

Therefore, (3.8) implies (3.9) if $\alpha_{[m,a]}$ is of class C^2 in $x \in \bar{\Omega}$.

Consequently, throughout this article, besides condition (1.2), we can assume, without loss of generality, that

$$\partial_\nu \alpha_{[m,a]} + \beta \alpha_{[m,a]} \geq 0, \quad \text{on } \Gamma_1, \tag{3.10}$$

when $\alpha_{[m,a]}$ if of class C^2 with respect to $x \in \bar{\Omega}$.

4 Auxiliary eigenvalue problem

In this section, we focus our attention into the eigenvalue problem

$$\begin{cases}(\mathscr{P}_d + V(x,t))\varphi = \lambda\varphi, & \text{in } \Omega_T, \\ \mathscr{B}\varphi = 0, & \text{on } \partial\Omega \times [0,T].\end{cases} \tag{4.1}$$

Thanks to Hess [8] and Antón and López-Gómez [2,3], [4, Sec. 6], problem (4.1) possesses a unique principal eigenvalue, denoted by $\lambda_1[\mathscr{P}_d + V, \mathscr{B}, \Omega_T]$, which is algebraically simple and strictly dominant. To state its main monotonicity properties, we need to introduce some notation. Subsequently, for any proper subdomain $\Omega_0 \subset \Omega$ such that

$$\operatorname{dist}(\partial\Omega_0 \cap \Omega, \Gamma_1) > 0, \tag{4.2}$$

we will denote by $\mathscr{B}_{\Omega_0}$ the boundary operator defined by

$$\mathscr{B}_{\Omega_0}\varphi := \begin{cases}\varphi, & \text{on } \partial\Omega_0 \cap \Omega, \\ \mathscr{B}\varphi, & \text{on } \partial\Omega_0 \cap \partial\Omega.\end{cases}$$

The next result goes back to Antón and López-Gómez [4, Sec. 7]. It collects some important monotonicity properties of $\lambda_1[\mathcal{P}_d + V, \mathscr{B}, \Omega_T]$ that will be used throughout this article.

Proposition 4.1. *Under the general assumptions of this article, the following properties are satisfied*:

(*i*) *Let $V_1, V_2 \in F$ be such that $V_1 < V_2$ in $\bar{\Omega} \times [0,T]$. Then,*

$$\lambda_1[\mathscr{P}_d + V_1, \mathscr{B}, \Omega_T] < \lambda_1[\mathscr{P}_d + V_2, \mathscr{B}, \Omega_T].$$

(*ii*) *Let* Ω_0 *be a proper subdomain of* Ω *of class* $C^{2+\theta}$ *satisfying* (4.2), *and* $V \in F$. *Then,*

$$\lambda_1[\mathscr{P}_d + V, \mathscr{B}, \Omega \times [0,T]] < \lambda_1[\mathscr{P}_d + V, \mathscr{B}_{\Omega_0}, \Omega_0 \times [0,T]].$$

The main result of this section reads as follows. It ascertains the value of $\lambda_1[\mathscr{P}_d + V, \mathscr{B}, \Omega_T]$ and finds from it its asymptotic behavior as $d \downarrow 0$ when $V(x,t) \equiv V(t)$ is independent of $x \in \Omega$.

Theorem 4.1. *Assume that* $V(x,t) \equiv V(t) \in F$ *is independent of* $x \in \bar{\Omega}$. *Then, the principal eigenvalue of the problem*

$$\begin{cases} (\mathscr{P}_d + V(t))\psi = \lambda\psi, & \text{in } \Omega_T, \\ \mathscr{B}\psi = 0, & \text{on } \partial\Omega \times [0,T], \end{cases} \tag{4.3}$$

is given through

$$\lambda_{1,d} \coloneqq \lambda_1[\mathscr{P}_d + V(t), \mathscr{B}, \Omega_T] = \overline{V} + d\sigma_1\, \overline{\kappa}, \tag{4.4}$$

where $\sigma_1 \equiv \sigma_1[\mathscr{L}, \mathscr{B}, \Omega]$ *stands for the principal eigenvalue of the linear elliptic eigenvalue problem*

$$\begin{cases} \mathscr{L}\varphi = \sigma\varphi, & \text{in } \Omega, \\ \mathscr{B}\varphi = 0, & \text{on } \partial\Omega. \end{cases}$$

Moreover, up to a positive multiplicative constant, the principal eigenfunction $\psi_{1,d}(x,t)$ *associated with* $\lambda_{1,d}$ *can be expressed through*

$$\psi_{1,d}(x,t) = e^{-d\sigma_1\int_0^t(\kappa(s)-\overline{\kappa})\mathrm{d}s - \int_0^t (V(s)-\overline{V})\mathrm{d}s}\varphi_1(x), \tag{4.5}$$

where $\varphi_1(x)$ *is the (unique) principal eigenfunction associated with* σ_1 *normalized so that* $\max_{\bar{\Omega}}\varphi_1 = 1$. *Thus,*

$$\lim_{d\downarrow 0} \lambda_1[\mathscr{P}_d + V(t), \mathscr{B}, \Omega_T] = \overline{V}. \tag{4.6}$$

Proof. The existence and the uniqueness of $(\lambda_{1,d}, \psi_{1,d})$ is a direct consequence of Antón and López-Gómez [3, 4]. To prove the theorem, we will search for a T-periodic positive function, $\gamma(t)$, for which

$$\psi_1(x,t) \coloneqq \gamma(t)\varphi_1(x)$$

provides us with a principal eigenfunction of (4.3). By the choice of φ_1,

$$\mathscr{B}\psi_1 = \gamma\mathscr{B}\varphi_1 = 0, \quad \text{on } \partial\Omega.$$

Moreover, inserting ψ_1 into the differential equation of (4.3), we are driven to

$$\gamma'(t)\varphi_1(x) + \gamma(t)\mathrm{d}\kappa(t)\mathscr{L}\varphi_1(x) + V(t)\gamma(t)\varphi_1(x) = \lambda\gamma(t)\varphi_1(x),$$

which can be equivalently expressed as

$$\gamma'(t)\varphi_1(x) = (\lambda - \mathrm{d}\sigma_1\kappa(t) - V(t))\gamma(t)\varphi_1(x).$$

Thus,

$$\gamma'(t) = (\lambda - \mathrm{d}\sigma_1\kappa(t) - V(t))\gamma(t),$$

and hence,

$$\gamma(t) = e^{\lambda t - \mathrm{d}\sigma_1\int_0^t \kappa(s)\mathrm{d}s - \int_0^t V(s)\mathrm{d}s}\gamma(0). \tag{4.7}$$

Since $\gamma(t)$ is T-periodic and positive, we have that $\gamma(T) = \gamma(0) > 0$, and therefore,

$$\lambda T - \mathrm{d}\sigma_1\int_0^T \kappa(s)\mathrm{d}s - \int_0^T V(s)\mathrm{d}s = 0.$$

Consequently, by uniqueness,

$$\lambda = \overline{V} + \mathrm{d}\sigma_1\overline{\kappa}$$

provides us with the principal eigenvalue of (4.3), $\lambda_{1,d}$. Substituting it into (4.7), it readily follows that (4.5) provides us with a principal eigenfunction associated with $\lambda_{1,d}$. Finally, letting $d \downarrow 0$ in (4.4), (4.6) holds. This ends the proof. □

5 Periodic-parabolic problem

Note that, under the general assumptions of this article, $a_{L,\bar{\Omega}_T} > 0$. Thus, since $a(x, t)$ is separated away from zero, problem (1.1) is non-degenerate, though $m(x, t)$ might change of sign. The main existence result for (1.1) can be stated as follows. It is Theorem 6.1 of Aleja et al. [1].

Theorem 5.1. *problem* (1.1) *admits a positive solution if, and only if,*

$$\lambda_1[\mathscr{P}_d - m, \mathscr{B}, \Omega_T] < 0. \tag{5.1}$$

In such case, the positive solution is unique; throughout this article, it will be denoted by $\theta_{[m,a,d]}$, *and the following holds*:

$$\lambda_1[\mathscr{P}_d + a\theta^{p-1}_{[m,a,d]} - m, \mathscr{B}, \Omega_T] = 0.$$

Next result gives some comparison results that will be used later.

Proposition 5.1. *Under condition* (5.1), *the following properties are satisfied*:

(*i*) *For every subsolution*, $\underline{u} \gneq 0$, *of* (1.1), *one has that* $\underline{u} \le \theta_{[m,a,d]}$ *in* Ω_T.

(*ii*) *For every supersolution*, $\bar{u} \gneq 0$, *of* (1.1), *one has that* $\theta_{[m,a,d]} \le \bar{u}$ *in* Ω_T.

(*iii*) *Let* $m_i, a_i \in F$, $i = 1, 2$ *such that* $m_1 \le m_2$ *and* $a_1 \ge a_2$. *Then*,

$$\theta_{[m_1,a_1,d]} \le \theta_{[m_2,a_2,d]}, \quad \text{in } \bar{\Omega}_T. \tag{5.2}$$

Proof. By the uniqueness of the positive solution, $\underline{u} = \theta_{[m,a,d]}$ if $\underline{u}$ is not a strict subsolution of (1.1), i.e., if $\underline{u}$ solves (1.1). Thus, we will assume that $\underline{u} \gneq 0$ is a strict subsolution of (1.1). Then,

$$\begin{aligned}
\mathscr{P}_d(\theta_{[m,a,d]} - \underline{u}) &\ge m(\theta_{[m,a,d]} - \underline{u}) - a(\theta^p_{[m,a,d]} - \underline{u}^p) \\
&= m(\theta_{[m,a,d]} - \underline{u}) - a\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}(s\theta_{[m,a,d]} + (1-s)\underline{u})^p \mathrm{d}s \\
&= \left[m - pa\int_0^1 (s\theta_{[m,a,d]} + (1-s)\underline{u})^{p-1}\mathrm{d}s\right](\theta_{[m,a,d]} - \underline{u}).
\end{aligned}$$

Consequently, setting

$$V \coloneqq pa\int_0^1 (s\theta_{[m,a,d]} + (1-s)\underline{u})^{p-1}\mathrm{d}s,$$

we have that

$$\begin{cases} (\mathscr{P}_d + V - m)(\theta_{[m,a,d]} - \underline{u}) \ge 0, & \text{in } \Omega_T, \\ \mathscr{B}(\theta_{[m,a,d]} - \underline{u}) \ge 0, & \text{on } \partial\Omega \times [0, T], \end{cases} \tag{5.3}$$

with some of these inequalities strict.

On the other hand, since $\underline{u} \gneq 0$, we find that

$$V \gneq pa\int_0^1 (s\theta_{[m,a,d]})^{p-1}\mathrm{d}s = a\theta_{[m,a,d]}^{p-1}.$$

Thus, it follows from Proposition 4.1(i) that

$$\lambda_1[\mathscr{P}_d + V - m, \mathscr{B}, \Omega_T] > \lambda_1[\mathscr{P}_d + a\theta_{[m,a,d]}^{p-1} - m, \mathscr{B}, \Omega_T] = 0.$$

Hence, thanks to Theorem 1.1 of Antón and López-Gómez [3], we find from (5.3) that

$$\theta_{[m,a,d]} - \underline{u} \gg 0,$$

which ends the proof of Part (i). The proof of Part (ii) follows the same general patterns as the proof of Part (i). Thus, we will omit its technical details here.

We now prove Part (iii). Since

$$\begin{aligned}\mathscr{P}_d\theta_{[m_1,a_1,d]} &= m_1(x,t)\theta_{[m_1,a_1,d]} - a_1(x,t)\theta_{[m_1,a_1,d]}^p\\ &\leq m_2(x,t)\theta_{[m_1,a_1,d]} - a_2(x,t)\theta_{[m_1,a_1,d]}^p,\end{aligned}$$

the function $\theta_{[m_1,a_1,d]}$ is a subsolution of

$$\begin{cases}\partial_t u + \mathrm{d}\kappa(t)\mathscr{L}u = m_2(x,t)u - a_2(x,t)u^p, & \text{in } \Omega\times\mathbb{R},\\ \mathscr{B}u(x,t) = 0, & \text{on } \partial\Omega\times\mathbb{R},\\ u(x,0) = u(x,T), & \text{in } \Omega,\end{cases}$$

whose unique positive solution is $\theta_{[m_2,a_2,d]}$. Thus, (5.2) follows from Part (i). This ends the proof. □

The following result gives a sufficient condition for the existence of positive solutions of (1.1) for small diffusions.

Theorem 5.2. *If there exists* $x_0 \in \Omega$ *such that*

$$\int_0^T m(x_0,t)\mathrm{d}t > 0, \tag{5.4}$$

then there exists $d_0 > 0$ *such that*

$$\lambda_1[\mathscr{P}_d - m, \mathscr{B}, \Omega_T] < 0, \quad \text{for all } d \in (0, d_0]. \tag{5.5}$$

Thus, thanks to Theorem 5.1, (1.1) *has a unique positive solution for all* $d \in (0, d_0]$.

Proof. Thanks to (5.4), $\overline{m(x_0,\cdot)} > 0$. Pick any $\varepsilon \in (0, \overline{m(x_0,\cdot)})$. By the uniform continuity of $m(x,t)$ in the compact set $\bar{\Omega}\times[0,T]$, there exists $\delta = \delta(\varepsilon)$ such that

$$|m(x,t) - m(\tilde{x},\tilde{t})| \leq \varepsilon, \quad \text{if } |x-\tilde{x}| + |t-\tilde{t}| \leq \delta,$$

with $(x,t), (\tilde{x},\tilde{t}) \in \bar{\Omega}\times[0,T]$. Thus,

$$|m(x,t) - m(x_0,t)| \leq \varepsilon, \quad \text{if } |x-x_0| \leq \delta$$

for all $t \in [0,T]$. Moreover, δ can be shortened, if necessary, so that $\bar{B}_\delta(x_0) \subset \Omega$. Consequently,

$$m(x_0,t) - \varepsilon \leq m(x,t) \leq m(x_0,t) + \varepsilon, \quad \text{for all } (x,t) \in \bar{B}_\delta(x_0)\times[0,T].$$

Hence,

$$m(x_0,t) - \varepsilon \leq \min_{x\in\bar{B}_\delta(x_0)} m(x,t) \leq m(x_0,t) + \varepsilon, \quad \text{for all } t \in [0,T].$$

Therefore, integrating in $[0, T]$ shows that

$$0 < \int_0^T m(x_0, t)\mathrm{d}t - \varepsilon T \le \int_0^T \min_{x\in\bar{B}_\delta(x_0)} m(x, t)\mathrm{d}t \le \int_0^T m(x_0, t)\mathrm{d}t + \varepsilon T, \tag{5.6}$$

because of the choice of ε.

Thanks to Proposition 4.1, we have that

$$\begin{aligned}\lambda_1[\mathscr{P}_d - m(x, t), \mathscr{B}, \Omega_T] &< \lambda_1[\mathscr{P}_d - m(x, t), \mathscr{D}, \bar{B}_\delta(x_0) \times [0, T]]\\ &\le \lambda_1\left[\mathscr{P}_d - \min_{x\in\bar{B}_\delta(x_0)} m(x, t), \mathscr{D}, \bar{B}_\delta(x_0) \times [0, T]\right],\end{aligned} \tag{5.7}$$

where $\mathscr{D}$ denotes the Dirichlet boundary operator, i.e., $\mathscr{B} \equiv \mathscr{D}$ if $\Gamma_1 = \emptyset$.

On the other hand, owing to Theorem 4.1, it follows from (5.6) that

$$\lim_{d\downarrow 0} \lambda_1\left[\mathscr{P}_d - \min_{x\in\bar{B}_\delta(x_0)} m(x, t), \mathscr{D}, \bar{B}_\delta(x_0) \times [0, T]\right] = -\frac{1}{T}\int_0^T \min_{x\in\bar{B}_\delta(x_0)} m(x, t)\mathrm{d}t < 0. \tag{5.8}$$

Therefore, due to (5.7) and (5.8), there exists $d_0 > 0$ such that (5.5) holds true. This ends the proof. □

Remark 5.1. Under condition (5.4), we already know that there exists $\delta > 0$ such that (5.8) is satisfied. On the other hand, thanks to (4.4), setting

$$m_{L,\bar{B}_\delta(x_0)}(t) \equiv \min_{x\in\bar{B}_\delta(x_0)} m(x, t), \quad \text{for all } t \in [0, T],$$

one has that

$$\lambda_1[\mathscr{P}_d - m_{L,\bar{B}_\delta(x_0)}(t), \mathscr{D}, \bar{B}_\delta(x_0) \times [0, T]] = d\ \bar{\kappa}\ \sigma_1[\mathscr{L}, \mathscr{D}, \bar{B}_\delta(x_0)] - \overline{m_{L,\bar{B}_\delta(x_0)}} < 0, \quad \text{for all } d \in (0, \tilde{d}_0],$$

with

$$\tilde{d}_0 < \frac{\overline{m_{L,\bar{B}_\delta(x_0)}}}{\bar{\kappa}\ \sigma_1[\mathscr{L}, \mathscr{D}, \bar{B}_\delta(x_0)]}.$$

Note that, thanks to (5.7), $0 < \tilde{d}_0 \le d_0$.

The following result gives a sufficient condition for the nonexistence of positive solution of (1.1) for small diffusions.

Proposition 5.2. *If, instead of* (5.4), *the following condition holds*

$$\int_0^T \max_{x\in\bar{\Omega}} m(x, t)\mathrm{d}t < 0, \tag{5.9}$$

then, there exists $d_0 > 0$ such that (1.1) *cannot admit a positive solution if $d \in (0, d_0]$.*

Proof. To prove it, we will argue by contradiction. Let us assume that there exists a sequence of positive real numbers $\{d_n\}_{n\ge 1}$, $d_n > 0$, such that (1.1) possesses a positive solution for each $d \in \{d_n : n \ge 1\}$. Then, thanks to Theorem 5.1,

$$\lambda_1[\mathscr{P}_{d_n} - m(x, t), \mathscr{B}, \Omega_T] < 0, \quad \text{for all } n \ge 1. \tag{5.10}$$

On the other hand, by Proposition 4.1, we have that

$$\lambda_1[\mathscr{P}_{d_n} - m(x, t), \mathscr{B}, \Omega_T] \ge \lambda_1\left[\mathscr{P}_{d_n} - \max_{x\in\bar{\Omega}} m(x, t), \mathscr{B}, \Omega_T\right]. \tag{5.11}$$

Moreover, by Theorem 4.1, it follows from (5.9) that

$$\lim_{n\to\infty} \lambda_1\left[\mathscr{P}_{d_n} - \max_{x\in\bar{\Omega}} m(x,t), \mathscr{B}, \Omega_T\right] = -\overline{\max_{x\in\bar{\Omega}} m(x,t)} = -\frac{1}{T}\int_0^T \max_{x\in\bar{\Omega}} m(x,t)\mathrm{d}t > 0. \tag{5.12}$$

As (5.10) contradicts (5.11) and (5.12), the proof is complete. □

6 Constructing supersolutions

Proposition 6.1. *Assume*

$$\overline{m}(x) := \int_0^T m(x,t)\mathrm{d}t > 0, \quad \textit{for all } x \in \bar{\Omega}. \tag{6.1}$$

Then, for each $\varepsilon > 0$, *there exists* $\tilde{d} = \tilde{d}(\varepsilon) > 0$ *such that*

$$\theta_{[m,a,d]} \le \alpha_{[m,a]} + \varepsilon, \quad \textit{for all } (x,t) \in \bar{\Omega}_T \textit{ and } d \in (0,\tilde{d}). \tag{6.2}$$

Proof. The existence and uniqueness of $\theta_{[m,a,d]}$ for sufficiently small $d > 0$ follows from (6.1) and Theorem 5.2.

First, we will show (6.2) in the special case when $m(\cdot,t)$ and $a(\cdot,t)$ are of class $C^2(\bar{\Omega})$. Then, we will prove (6.2) in the general case. So, assume that $m(\cdot,t)$ and $a(\cdot,t)$ are of class $C^2(\bar{\Omega})$.

Thanks to (6.1), it follows from Proposition 2.1 that

$$\alpha_{[m,a]}(x,t) > 0, \quad \text{for all } (x,t) \in \bar{\Omega} \times [0,T].$$

Moreover, since $m(\cdot,t), a(\cdot,t) \in C^2(\bar{\Omega})$ and $A(x) > 0$ (cf. (2.3)), it follows from (2.3) that $\alpha_{[m,a]} \in C^2(\bar{\Omega}) \cap C^1[0,T]$. Set

$$\alpha_L := \min_{(x,t)\in\bar{\Omega}_T} \alpha_{[m,a]}(x,t) > 0, \quad a_L := \min_{(x,t)\in\bar{\Omega}_T} a(x,t) > 0, \tag{6.3}$$

pick $\varepsilon > 0$ and let $\delta = \delta(\varepsilon) > 0$ be sufficiently small so that

$$\delta\alpha_{[m,a]} < \varepsilon, \quad \text{in } \bar{\Omega}_T. \tag{6.4}$$

Finally, consider the function

$$\bar{u}_\delta := (1+\delta)\alpha_{[m,a]} > 0.$$

Then, it follows from (6.4) that

$$\bar{u}_\delta = (1+\delta)\alpha_{[m,a]} < \alpha_{[m,a]} + \varepsilon, \quad \text{in } \bar{\Omega}_T. \tag{6.5}$$

Subsequently, we will prove that $\bar{u}_\delta$ is a positive strict supersolution of (1.1). Indeed, in Ω_T, we find that

$$\begin{aligned}(\mathscr{P}_d - m(x,t))\bar{u}_\delta + a(x,t)\bar{u}_\delta^p &= (1+\delta)[a(x,t)\alpha_{[m,a]}^p((1+\delta)^{p-1}-1) + \mathrm{d}\kappa(t)\mathscr{L}\alpha_{[m,a]}] \\ &\ge (1+\delta)[a_L\alpha_L^p((1+\delta)^{p-1}-1) + \mathrm{d}\kappa(t)\mathscr{L}\alpha_{[m,a]}].\end{aligned} \tag{6.6}$$

Note that $\mathscr{L}\alpha_{[m,a]} \in C(\bar{\Omega}_T)$ because we are assuming that a and m are of class C^2 in $x \in \bar{\Omega}$. Thus, thanks to (6.3), it becomes apparent from (6.6) that there exists $\tilde{d} := \tilde{d}(\varepsilon) > 0$ such that, for every $d \in (0,\tilde{d})$,

$$(\mathscr{P}_d - m(x,t))\bar{u}_\delta(x,t) + a(x,t)\bar{u}_\delta^p(x,t) > 0, \quad \text{for all } (x,t) \in \Omega_T. \tag{6.7}$$

As for the boundary conditions, since $\alpha_{[m,a]} > 0$ on Γ_0,

$$\bar{u}_\delta = (1+\delta)\alpha_{[m,a]} > 0, \quad \text{on } \Gamma_0. \tag{6.8}$$

Moreover, thanks to the normalization conditions of Section 3, since $\alpha_{[m,a]}$ is of class C^2 with respect to $x \in \bar{\Omega}$, besides $\beta(x) > 0$ for all $x \in \Gamma_1$, we can assume, without loss of generality, that condition (3.10) is satisfied, i.e.,

$$\partial_\nu \alpha_{[m,a]} + \beta \alpha_{[m,a]} \geq 0, \quad \text{on } \Gamma_1. \tag{6.9}$$

Thanks to (6.9), we have that

$$\frac{\partial \bar{u}_\delta}{\partial \nu} + \beta \bar{u}_\delta = (1+\delta)\left(\frac{\partial \alpha_{[m,a]}}{\partial \nu} + \beta \alpha_{[m,a]}\right) \geq 0, \quad \text{on } \Gamma_1. \tag{6.10}$$

Thus, by (6.7), (6.8), and (6.10), $\bar{u}_\delta$ is a positive strict supersolution of (1.1). Therefore, thanks to Proposition 5.1(ii), (6.5) implies that

$$\theta_{[m,a,d]} \leq \bar{u}_\delta < \alpha_{[m,a]} + \varepsilon, \quad \text{in } \Omega_T, \text{ for all } d \in (0, \tilde{d}(\varepsilon)).$$

This ends the proof in the particular case when m and a are of class $C^2(\bar{\Omega})$ in $x \in \bar{\Omega}$.

We now prove the result in the general case when $m, a \in F$. In this case, let us take $m_1(\cdot, t), a_1(\cdot, t) \in C^2(\bar{\Omega})$ such that

$$m \leq m_1 \text{ and } a \geq a_1, \quad \text{in } \bar{\Omega}_T \tag{6.11}$$

and

$$\alpha_{[m,a]} \leq \alpha_{[m_1,a_1]} \leq \alpha_{[m,a]} + \frac{\varepsilon}{2}. \tag{6.12}$$

The first estimate of (6.12) follows from (6.11) and Proposition 2.3. The second estimate holds true from the continuity of $\alpha_{[m,a]}$ with respect to m and a, which is a direct consequence from (2.5) and (2.6). Then, by (6.1), it follows from (6.11) that

$$\int_0^T m_1(x,t)\mathrm{d}t \geq \int_0^T m(x,t)\mathrm{d}t > 0, \quad \text{for all } x \in \bar{\Omega}.$$

Thus, by the previous case, there exists $\tilde{d}(\varepsilon) > 0$ such that

$$\theta_{[m_1,a_1,d]} \leq \alpha_{[m_1,a_1]} + \frac{\varepsilon}{2}, \quad \text{in } \bar{\Omega}_T, \text{ if } 0 < d < \tilde{d}(\varepsilon). \tag{6.13}$$

Moreover, due to Proposition 5.1(iii), it follows from (6.11) that

$$\theta_{[m,a,d]} \leq \theta_{[m_1,a_1,d]}, \quad \text{in } \bar{\Omega}_T. \tag{6.14}$$

Then, thanks to (6.12), (6.13), and (6.14), we find that, for every $d \in (0, \tilde{d}(\varepsilon))$,

$$\theta_{[m,a,d]} \leq \alpha_{[m,a]} + \varepsilon, \quad \text{in } \bar{\Omega}_T.$$

This shows (6.2) and ends the proof. □

7 Constructing subsolutions for m and a autonomous in $x \in \Omega$

Theorem 7.1. *Assume that*

$$m \equiv m(t), \quad a \equiv a(t), \quad t \in [0,T], \tag{7.1}$$

with

$$\int_0^T m(t)\mathrm{d}t > 0. \tag{7.2}$$

Let $K \subset \Omega \cup \Gamma_1$ be a compact set. Then, for every $\varepsilon > 0$, there exists $d(\varepsilon, K) > 0$ such that, for every $d \in (0, d(\varepsilon, K))$,

$$\theta_{[m,a,d]} \geq \alpha_{[m,a]} - \varepsilon, \quad \text{in } K_T. \tag{7.3}$$

Proof. Pick $\varepsilon > 0$. The proof will be distributed into four steps. Thanks to (7.2), it follows from Proposition 2.2(iii) that

$$(\alpha_{[m,a]})_{L,K_T} := \min_{(x,t)\in K_T} \alpha_{[m,a]}(x,t) > 0.$$

Step 1: In this step, we are going to prove that, for every $x_0 \in \Omega$, there exist $R_1 = R_1(x_0) > 0$ and $\bar{d} := \bar{d}(x_0, \varepsilon) > 0$ such that

$$\theta_{[m,a,d]} \geq \alpha_{[m,a]} - \varepsilon, \quad \text{for all } (x,t) \in B_{R_1}(x_0) \times [0,T] \quad \text{and } d \in (0, \bar{d}). \tag{7.4}$$

Indeed, let $x_0 \in \Omega$ and fix $R = R(x_0)$ such that $\bar{B} \subset \Omega$, where $B \equiv B_R(x_0)$. Let $(\sigma_1[\mathscr{L}, \mathscr{D}, B], \varphi_1)$ denote the principal eigenpair of the linear eigenvalue problem

$$\begin{cases} \mathscr{L}\varphi = \sigma\varphi, & \text{in } B, \\ \varphi = 0, & \text{on } \partial B, \end{cases}$$

where $\varphi_1 \gg 0$ is assumed to be normalized so that $\max_{\bar{B}}\varphi_1 = 1$. Now, let us consider the ρ-neighborhood of ∂B in B,

$$\mathcal{N}_\rho := \{x \in B \;:\; 0 < \operatorname{dist}(x, \partial B) < \rho\},$$

with $\rho = \rho(x_0) > 0$ sufficiently small so that

$$0 \leq \varphi_1 \leq \frac{1}{2}, \quad \text{in } \bar{\mathcal{N}}_\rho, \tag{7.5}$$

as well as a function of the type

$$\varphi(x) := \begin{cases} \varphi_1(x), & \text{if } x \in \bar{\mathcal{N}}_\rho, \\ \xi(x), & \text{if } x \in B \backslash \bar{\mathcal{N}}_\rho, \end{cases}$$

where ξ is any regular extension of $\varphi_1|_{\bar{\mathcal{N}}_\rho}$ to B such that

$$\frac{1}{2} \leq \xi \leq 1, \text{in } B\backslash\mathcal{N}_\rho, \quad \max_{\bar{B}} \varphi = \xi(x_0) = 1, \quad 0 \leq \varphi(x) < 1, \text{for all } x \in \bar{B}\backslash\{x_0\}. \tag{7.6}$$

Finally, for every $\delta > 0$, we consider the function

$$u_\delta(x,t) := \delta\varphi(x)\alpha_{[m,a]}(t), \quad (x,t) \in \bar{B}_T = \bar{B} \times [0,T].$$

We are going to show that, for every $\delta \in (0,1)$ and sufficiently small $d > 0$, u_δ provides us with a positive strict subsolution of the problem

$$\begin{cases} \partial_t u + \mathrm{d}\kappa(t)\mathscr{L}u = m(t)u - a(t)u^p, & \text{in } B \times \mathbb{R}, \\ u = 0, & \text{on } \partial B \times \mathbb{R}, \\ u(\cdot, 0) = u(\cdot, T), & \text{in } B. \end{cases} \tag{7.7}$$

Indeed, in the region $\mathcal{N}_\rho \times [0,T]$, since $\varphi = \varphi_1$ in $\bar{\mathcal{N}}_\rho$, $p > 1$, $\sigma_1[\mathscr{L}, \mathscr{D}, B] > 0$, and $\delta \in (0,1)$, which implies $\left(\frac{\delta}{2}\right)^{p-1} < 1$, it follows from the definition of $\alpha_{[m,a]}(t)$ and (7.5) that

$$\begin{aligned} (\mathscr{P}_d - m(t))u_\delta + a(t)u_\delta^p &= \delta\varphi_1(x)\alpha_{[m,a]}(t)[a(t)\alpha_{[m,a]}^{p-1}(t)((\delta\varphi_1)^{p-1} - 1) + \mathrm{d}\kappa(t)\sigma_1[\mathscr{L}, \mathscr{D}, B]] \\ &\leq \delta\varphi_1(x)\alpha_{[m,a]}(t)\left[a(t)\alpha_{[m,a]}^{p-1}(t)\left(\left(\frac{\delta}{2}\right)^{p-1} - 1\right) + \mathrm{d}\kappa(t)\sigma_1[\mathscr{L}, \mathscr{D}, B]\right] \\ &\leq \delta\varphi_1(x)\alpha_{[m,a]}(t)\left[a_L(\alpha_{[m,a]})_L^{p-1}\left(\left(\frac{\delta}{2}\right)^{p-1} - 1\right) + \mathrm{d}\kappa_M\sigma_1[\mathscr{L}, \mathscr{D}, B]\right], \end{aligned}$$

where we are denoting

$$a_L \coloneqq \min_{[0,T]} a > 0, \quad (a_{[m,a]})_L \coloneqq \min_{[0,T]} a_{[m,a]} > 0, \text{ and } \kappa_M \coloneqq \max_{[0,T]} \kappa.$$

Note that $(a_{[m,a]})_L > 0$ by (7.2) and Proposition 2.1. Thus, since $\delta \in (0,1)$, $\varphi_1(x) > 0$ for all $x \in B$, and $a_{[m,a]}(t) > 0$ for all $t \in \mathbb{R}$, there exists $d_1 \coloneqq d_1(\delta) > 0$ such that, for every $d \in (0, d_1)$,

$$(\mathscr{P}_d - m(t))u_\delta + a(t)u_\delta^p \lneq 0, \quad \text{in } \mathcal{N}_\rho \times [0,T]. \tag{7.8}$$

Similarly, since $\delta \in (0,1)$, $p > 1$, and, due to (7.6), $\varphi = \xi \in [\frac{1}{2}, 1]$ in $B \backslash \mathcal{N}_\rho$, we find that, in $B \backslash \mathcal{N}_\rho$,

$$\begin{aligned}(\mathscr{P}_d - m(t))u_\delta + a(t)u_\delta^p &= \delta a_{[m,a]}(t)[\xi(x)a(t)a_{[m,a]}^{p-1}(t)((\delta\xi(x))^{p-1} - 1) + \mathrm{d}\kappa(t)\mathscr{L}\xi]\\ &\leq \delta a_{[m,a]}(t)[\xi(x)a(t)a_{[m,a]}^{p-1}(t)(\delta^{p-1} - 1) + \mathrm{d}\kappa(t)\mathscr{L}\xi]\\ &\leq \delta a_{[m,a]}(t)\left[\frac{a_L}{2}(a_{[m,a]})_L^{p-1}(\delta^{p-1} - 1) + \mathrm{d}\kappa_M \|\mathscr{L}\xi\|_{L^\infty(B\backslash\mathcal{N}_\rho)}\right].\end{aligned}$$

Then, since $\delta^{p-1} - 1 < 0$, there is $d_2 \coloneqq d_2(\delta) > 0$ such that, for every $d \in (0, d_2)$,

$$(\mathscr{P}_d - m(t))u_\delta + a(t)u_\delta^p \lneq 0, \quad \text{in } (B\backslash\mathcal{N}_\rho) \times [0,T]. \tag{7.9}$$

Thus, setting

$$\hat{d} = \hat{d}(\delta) \coloneqq \min\{d_1(\delta), d_2(\delta)\},$$

it follows from (7.8) and (7.9) that, for every $d \in (0, \hat{d})$,

$$(\mathscr{P}_d - m(t))u_\delta + a(t)u_\delta^p \lneq 0, \quad \text{in } B \times [0,T]. \tag{7.10}$$

Moreover, since $\varphi = \varphi_1 = 0$ on ∂B, we also have that

$$u_\delta(x,t) = \delta a_{[m,a]}(t)\varphi_1(x) = 0, \quad \text{for all } (x,t) \in \partial B \times [0,T]. \tag{7.11}$$

Therefore, by (7.10) and (7.11), for every $\delta \in (0,1)$ and $d \in (0, \hat{d}(\delta))$, u_δ provides us with a positive strict subsolution of (7.7).

Thanks to (7.2) and applying Theorem 5.2 in $B = B_R(x_0)$, it becomes apparent that there exists $\tilde{d} = \tilde{d}(x_0) > 0$ such that, for every $d \in (0, \tilde{d})$, problem (7.7) has a unique positive solution, denoted by $\theta_{[m,a,d;\,B]}$. Thus, since u_δ is a positive strict subsolution of (7.7) for each $\delta \in (0,1)$ and $d \in (0, \hat{d}(\delta))$, setting

$$\bar{d}(x_0, \delta) \coloneqq \min\{\hat{d}(\delta), \tilde{d}(x_0)\},$$

it follows from Proposition 5.1(i) applied in B that, for every $\delta \in (0,1)$ and $d \in (0, \bar{d}(x_0, \delta))$,

$$u_\delta \leq \theta_{[m,a,d;\,B]}, \quad \text{in } \bar{B} \times [0,T]. \tag{7.12}$$

On the other hand, by construction, there exist $\delta^* \coloneqq \delta^*(x_0, \varepsilon) \in (0,1)$ and $R_1 = R_1(x_0) < R$ such that $B_{R_1}(x_0) \subset B\backslash\mathcal{N}_\rho \subset \Omega$ and

$$a_{[m,a]}(t)(1 - \delta^*\varphi(x)) < \varepsilon, \quad \text{for all } (x,t) \in B_{R_1}(x_0) \times [0,T]. \tag{7.13}$$

Indeed, since $\max_{\bar{B}}\varphi = \xi(x_0) = 1$, it suffices to take a sufficiently small $R_1 > 0$ and δ^* sufficiently close to 1. For these choices, it follows from (7.13) that

$$u_{\delta^*}(x,t) = \delta^*\varphi(x)a_{[m,a]}(t) > a_{[m,a]}(t) - \varepsilon, \quad \text{for all } (x,t) \in B_{R_1}(x_0) \times [0,T]. \tag{7.14}$$

Fix $\delta^* \in (0,1)$ and $R_1 < R$ satisfying (7.13), and hence (7.14), and set

$$\bar{d}(x_0, \varepsilon) \coloneqq \bar{d}(x_0, \delta^*(x_0, \varepsilon)).$$

Then, from (7.12) and (7.14), it becomes apparent that, for every $d \in (0, \bar{d}(x_0, \varepsilon))$,

$$a_{[m,a]}(t) - \varepsilon \leq u_{\delta^*}(x,t) \leq \theta_{[m,a,d;\,B]}(x,t) \quad \text{for all } (x,t) \in B_{R_1}(x_0) \times [0,T]. \tag{7.15}$$

Finally, taking into account that the unique positive solution $\theta_{[m,a,d]}$ of (1.1) is a positive strict supersolution of (7.7), it follows from Proposition 5.1 that, for every $d \in (0, \bar{d}(x_0, \varepsilon))$,

$$\theta_{[m,a,d;B]} \leq \theta_{[m,a,d]}, \quad \text{for all } (x,t) \in B_{R_1}(x_0) \times [0,T]. \tag{7.16}$$

As (7.4) follows from (7.15) and (7.16), the proof of Step 1 is completed.

Step 2: In this step, we will prove (7.3) in the particular case when $K \subset \Omega$. In such case, according to Step 1, for every $x \in K$, there exist $R_1(x) > 0$ and $\bar{d}(x,\varepsilon) > 0$ such that $\bar{B}_{R_1(x)}(x) \subset \Omega$ and

$$\theta_{[m,a,d]} \geq \alpha_{[m,a]} - \varepsilon, \quad \text{in } B_{R_1(x)}(x) \times [0,T] \text{ and } d \in (0, \bar{d}(x,\varepsilon)). \tag{7.17}$$

By the compactness of K, there are an integer $m \geq 1$ and m points $x_i \in K$, $i \in \{1, 2, \dots, m\}$, such that

$$K \subset \bigcup_{i=1}^{m} B_{R_1(x_i)}(x_i) \subset \Omega.$$

Thus, setting

$$d(\varepsilon, K) \coloneqq \min_{i \in \{1,\dots,m\}} \bar{d}(x_i, \varepsilon),$$

it follows from (7.17) that, for every $d \in (0, d(\varepsilon, K))$,

$$\theta_{[m,a,d]} \geq \alpha_{[m,a]} - \varepsilon, \quad \text{for all } (x,t) \in K_T.$$

As this provides us with (7.3), the proof of the theorem is completed if $K \subset \Omega$.

Step 3: Since $\partial\Omega$ is smooth, Γ_1 consists of finitely many components, say $\Gamma_{1,i}$, $i \in \{1, \dots, q\}$. In this step, we consider one of these components, say $\Gamma_{1,i}$, and, for every

$$R \in (0, \operatorname{dist}(\partial\Omega \setminus \{\Gamma_{1,i}\}, \Gamma_{1,i}))$$

and $\rho \in (0, R)$, we denote

$$\begin{aligned} \tilde{\mathcal{N}}_{R,i} &\coloneqq \{x \in \Omega \ : \ 0 < \operatorname{dist}(x, \Gamma_{1,i}) < R\}, \\ \tilde{\mathcal{N}}_{R-\rho,i} &\coloneqq \{x \in \Omega \ : \ 0 < \operatorname{dist}(x, \Gamma_{1,i}) < R-\rho\}, \\ \tilde{\Gamma}_{R,i} &\coloneqq \partial\tilde{\mathcal{N}}_{R,i} \cap \Omega, \quad \tilde{\Gamma}_{R-\rho,i} \coloneqq \partial\tilde{\mathcal{N}}_{R-\rho,i} \cap \Omega, \\ \tilde{\mathcal{A}}_{R-\rho,R,i} &\coloneqq \{x \in \Omega \ : \ R-\rho < \operatorname{dist}(x, \Gamma_{1,i}) < R\}. \end{aligned} \tag{7.18}$$

By construction,

$$\begin{gathered} \tilde{\mathcal{A}}_{R-\rho,R,i} = \tilde{\mathcal{N}}_{R,i} \setminus \operatorname{clos}(\tilde{\mathcal{N}}_{R-\rho,i}), \quad \partial\tilde{\mathcal{A}}_{R-\rho,R,i} = \tilde{\Gamma}_{R,i} \cup \tilde{\Gamma}_{R-\rho,i}, \\ \partial\tilde{\mathcal{N}}_{R,i} = \tilde{\Gamma}_{R,i} \cup \Gamma_{1,i}, \quad \text{and} \quad \partial\tilde{\mathcal{N}}_{R-\rho,i} = \tilde{\Gamma}_{R-\rho,i} \cup \Gamma_{1,i}. \end{gathered}$$

This step shows that there exist $R_{2,i} \in (0, R)$ and $\bar{d}_i(\varepsilon) > 0$ such that, for every $d \in (0, \bar{d}_i(\varepsilon))$,

$$\theta_{[m,a,d]} \geq \alpha_{[m,a]} - \varepsilon, \quad \text{for all } (x,t) \in \tilde{\mathcal{N}}_{R_{2,i},i} \times [0,T]. \tag{7.19}$$

Indeed, for sufficiently small $R > 0$, let us denote by $(\sigma_1[\mathcal{L}, \mathcal{D}, \tilde{\mathcal{N}}_{R,i}], \tilde{\varphi}_{1,i})$ the principal eigenpair of the linear eigenvalue problem

$$\begin{cases} \mathcal{L}\varphi = \sigma\varphi, & \text{in } \tilde{\mathcal{N}}_{R,i}, \\ \varphi = 0, & \text{on } \partial\tilde{\mathcal{N}}_{R,i} = \tilde{\Gamma}_{R,i} \cup \Gamma_{1,i}, \end{cases}$$

with $\tilde{\varphi}_{1,i} \gg 0$ normalized so that

$$\max_{\tilde{\mathcal{N}}_{R,i}} \tilde{\varphi}_{1,i} = 1.$$

By construction, for sufficiently small $\rho > 0$, we have that

$$0 \leq \tilde{\varphi}_{1,i}(x) \leq \frac{1}{2}, \quad \text{for all } x \in \tilde{\mathcal{A}}_{R-\rho,R,i}. \tag{7.20}$$

Next, we will fix one of those ρ's and consider the function

$$\tilde{\varphi}_i(x) := \begin{cases} \tilde{\varphi}_{1,i}(x), & \text{if } x \in \operatorname{clos}\tilde{\mathcal{A}}_{R-\rho,R,i}, \\ \eta_i(x), & \text{if } x \in \tilde{\mathcal{N}}_{R-\rho,i} = \tilde{\mathcal{N}}_{R,i} \setminus \tilde{\mathcal{A}}_{R-\rho,R,i}, \end{cases}$$

where η_i is a regular extension of $\tilde{\varphi}_{1,i}$ from $\operatorname{clos}\tilde{\mathcal{A}}_{R-\rho,R,i}$ to $\operatorname{clos}\tilde{\mathcal{N}}_{R,i}$ such that

$$\frac{1}{2} \le \eta_i(x) < 1, \quad \text{if } x \in \tilde{\mathcal{N}}_{R-\rho,i}, \quad \max_{\operatorname{clos}\tilde{\mathcal{N}}_{R-\rho,i}} \eta_i = 1, \tag{7.21}$$

and

$$\eta_i(x) = 1, \text{ for all } x \in \Gamma_{1,i}, \quad \frac{\partial \eta_i}{\partial \nu}(x) < -\beta_{M,\Gamma_{1,i}}, \text{ for all } x \in \Gamma_{1,i}. \tag{7.22}$$

In the same vein as in Step 1, for every $\delta > 0$, we consider the function

$$\tilde{u}_{\delta,i}(x,t) = \delta\tilde{\varphi}_i(x)\alpha_{[m,a]}(t), \quad (x,t) \in \operatorname{clos}\tilde{\mathcal{N}}_{R,i} \times [0,T].$$

Similarly, we will show that, for every $\delta \in (0,1)$ and sufficiently small $d > 0$, the function $\tilde{u}_{\delta,i}$ is a positive subsolution of the problem

$$\begin{cases} \partial_t u + \mathrm{d}\kappa(t)\mathcal{L}u = m(t)u - a(t)u^p, & \text{in } \tilde{\mathcal{N}}_{R,i} \times [0,T], \\ u(x,t) = 0, & \text{on } \tilde{\Gamma}_{R,i} \times [0,T], \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0, & \text{on } \Gamma_{1,i} \times [0,T], \\ u(x,0) = u(x,T), & \text{in } \tilde{\mathcal{N}}_{R,i}. \end{cases} \tag{7.23}$$

Indeed, since $\tilde{\varphi}_i = \tilde{\varphi}_{1,i}$ in $\tilde{\mathcal{A}}_{R-\rho,R,i}$, $p > 1$, $\delta \in (0,1)$, and $\sigma_1[\mathcal{L},\mathcal{D},\tilde{\mathcal{N}}_{R,i}] > 0$, it follows from (7.20) that, in the annular cylinder $\tilde{\mathcal{A}}_{R-\rho,R,i} \times [0,T]$, one has that

$$\begin{aligned} (\mathcal{P}_d - m(t))\tilde{u}_{\delta,i} + a(t)\tilde{u}_{\delta,i}^p &= \delta\tilde{\varphi}_{1,i}(x)\alpha_{[m,a]}(t)[a(t)\alpha_{[m,a]}^{p-1}(t)((\delta\tilde{\varphi}_{1,i})^{p-1} - 1) + \mathrm{d}\kappa(t)\sigma_1[\mathcal{L},\mathcal{D},\tilde{\mathcal{N}}_{R,i}]] \\ &\le \delta\tilde{\varphi}_{1,i}(x)\alpha_{[m,a]}(t)\left[a(t)\alpha_{[m,a]}^{p-1}(t)\left(\left(\frac{\delta}{2}\right)^{p-1} - 1\right) + \mathrm{d}\kappa(t)\sigma_1[\mathcal{L},\mathcal{D},\tilde{\mathcal{N}}_{R,i}]\right] \\ &\le \delta\tilde{\varphi}_{1,i}(x)\alpha_{[m,a]}(t)\left[a_L(\alpha_{[m,a]})_L^{p-1}\left(\left(\frac{\delta}{2}\right)^{p-1} - 1\right) + \mathrm{d}\kappa_M\sigma_1[\mathcal{L},\mathcal{D},\tilde{\mathcal{N}}_{R,i}]\right]. \end{aligned} \tag{7.24}$$

Thus, since $\delta \in (0,1)$, $a_L > 0$, $(\alpha_{[m,a]})_L > 0$, and $\tilde{\varphi}_{1,i}(x) > 0$ for all $x \in \tilde{\mathcal{N}}_{R,i}$, it follows from (7.24) that there exists $\tilde{d}_{1,i} := \tilde{d}_{1,i}(\delta) > 0$ such that, for every $d \in (0,\tilde{d}_{1,i})$,

$$(\mathcal{P}_d - m(t))\tilde{u}_{\delta,i} + a(t)\tilde{u}_{\delta,i}^p \le 0, \quad \text{in } \tilde{\mathcal{A}}_{R-\rho,R,i} \times [0,T]. \tag{7.25}$$

Similarly, since $p > 1$ and $0 < \delta < 1$, it follows from (7.21) that

$$\tilde{\varphi}_i = \eta_i \in \left[\frac{1}{2}, 1\right], \quad \text{in } \tilde{\mathcal{N}}_{R-\rho,i},$$

and hence,

$$\begin{aligned} (\mathcal{P}_d - m(t))\tilde{u}_{\delta,i} + a(t)\tilde{u}_{\delta,i}^p &= \delta\alpha_{[m,a]}(t)[\eta_i(x)a(t)\alpha_{[m,a]}^{p-1}(t)((\delta\eta_i(x))^{p-1} - 1) + \mathrm{d}\kappa(t)\mathcal{L}\eta_i] \\ &\le \delta\alpha_{[m,a]}(t)[\eta_i(x)a(t)\alpha_{[m,a]}^{p-1}(t)(\delta^{p-1} - 1) + \mathrm{d}\kappa(t)\mathcal{L}\eta_i] \\ &\le \delta\alpha_{[m,a]}(t)\left[\frac{a_L}{2}(\alpha_{[m,a]})_L^{p-1}(\delta^{p-1} - 1) + \mathrm{d}\kappa_M\|\mathcal{L}\eta_i\|_{L^\infty(\tilde{\mathcal{N}}_{R-\rho,i})}\right]. \end{aligned} \tag{7.26}$$

Thus, since $\delta \in (0,1)$, $a_L > 0$ and $(\alpha_{[m,a]})_L > 0$, it follows from (7.26) that there exists $\tilde{d}_{2,i} := \tilde{d}_{2,i}(\delta) > 0$ such that, for every $d \in (0,\tilde{d}_{2,i})$,

$$(\mathcal{P}_d - m(t))\tilde{u}_{\delta,i} + a(t)\tilde{u}_{\delta,i}^p \le 0, \quad \text{in } \tilde{\mathcal{N}}_{R-\rho,i} \times [0,T]. \tag{7.27}$$

Thus, choosing

$$\tilde{d}_i \equiv \tilde{d}_i(\delta) := \min\{\tilde{d}_{1,i}(\delta), \tilde{d}_{2,i}(\delta)\},$$

we find from (7.25) and (7.27) that, for every $d \in (0, \tilde{d}_i)$,

$$(\mathscr{P}_d - m(t))\tilde{u}_{\delta,i} + a(t)\tilde{u}_{\delta,i}^p \leq 0, \quad \text{in } \tilde{\mathcal{N}}_{R,i} \times [0, T]. \tag{7.28}$$

As to the boundary conditions concerns, by construction, we have that

$$\tilde{u}_{\delta,i}(x, t) = \delta\alpha_{[m,a]}(t)\tilde{\varphi}_{1,i}(x) = 0, \quad \text{for all } (x, t) \in \tilde{\Gamma}_{R,i} \times [0, T]. \tag{7.29}$$

Moreover, thanks to (7.22), on $\Gamma_{1,i} \times [0, T]$, we find that

$$\frac{\partial \tilde{u}_{\delta,i}}{\partial \nu} + \beta\tilde{u}_{\delta,i} = \delta\alpha_{[m,a]}\left(\frac{\partial \eta_i}{\partial \nu} + \beta\eta_i\right) = \delta\alpha_{[m,a]}\left(\frac{\partial \eta_i}{\partial \nu} + \beta\right) \leq \delta\alpha_{[m,a]}(-\beta_{M,\Gamma_{1,i}} + \beta) \leq 0. \tag{7.30}$$

Thanks to (7.28)–(7.30), it becomes apparent that, for every $\delta \in (0, 1)$ and $d \in (0, \tilde{d}_i(\delta))$, the function $\tilde{u}_{\delta,i}$ provides us with a positive subsolution of (7.23).

On the other hand, owing to (7.2), it follows from Theorem 5.2, applied in $\tilde{\mathcal{N}}_{R,i}$, that there exists $\hat{d}_i = \hat{d}_i(R) > 0$ such that (7.23) possesses a unique positive solution, denoted by $\theta_{[m,a,d;\ \tilde{\mathcal{N}}_{R,i}]}$, for each $d \in (0, \hat{d}_i)$. Thus, since $\tilde{u}_{\delta,i}$ is a positive subsolution of (7.23) for every $\delta \in (0, 1)$ and $d \in (0, \tilde{d}_i(\delta))$, setting

$$\bar{d}_i(\delta) := \min\{\hat{d}_i, \tilde{d}_i(\delta)\},$$

applying Proposition 5.1 in $\tilde{\mathcal{N}}_{R,i}$, it becomes apparent that

$$\theta_{[m,a,d;\ \tilde{\mathcal{N}}_{R,i}]} \geq \tilde{u}_{\delta,i}, \quad \text{in } \tilde{\mathcal{N}}_{R,i} \times [0, T], \text{ if } 0 < \delta < 1 \text{ and } 0 < d < \bar{d}_i(\delta). \tag{7.31}$$

On the other hand, by construction, it follows from (7.22) that there exist $\delta_i^*(\varepsilon) \in (0, 1)$ and $R_{2,i} \in (0, R)$ such that

$$\alpha_{[m,a]}(t)(1 - \delta_i^*\tilde{\varphi}_i(x)) < \varepsilon, \quad \text{for all } (x, t) \in \tilde{\mathcal{N}}_{R_{2,i},i} \times [0, T]. \tag{7.32}$$

Therefore,

$$\tilde{u}_{\delta_i^*,i}(x, t) = \delta_i^*\tilde{\varphi}_i(x)\alpha_{[m,a]}(t) > \alpha_{[m,a]}(t) - \varepsilon, \quad \text{for all } (x, t) \in \tilde{\mathcal{N}}_{R_{2,i},i} \times [0, T]. \tag{7.33}$$

Fix $\delta_i^*(\varepsilon) \in (0, 1)$ satisfying (7.32), and consequently, (7.33), and set

$$\bar{d}_i(\varepsilon) := \min\{\hat{d}_i, \tilde{d}_i(\delta_i^*(\varepsilon))\}.$$

Then, according to (7.31) and (7.33), for every $d \in (0, \bar{d}_i(\varepsilon))$, we have that

$$\theta_{[m,a,d;\ \tilde{\mathcal{N}}_{R,i}]}(x, t) \geq \tilde{u}_{\delta_i^*,i}(x, t) \geq \alpha_{[m,a]}(t) - \varepsilon, \quad \text{for all } (x, t) \in \tilde{\mathcal{N}}_{R_{2,i},i} \times [0, T]. \tag{7.34}$$

Moreover, as the unique positive solution $\theta_{[m,a,d]}$ of (1.1) is a positive strict supersolution of (7.23), it follows from Proposition 5.1 that, for every $d \in (0, \bar{d}_i(\varepsilon))$,

$$\theta_{[m,a,d]} \geq \theta_{[m,a,d;\ \tilde{\mathcal{N}}_{R,i}]}, \quad \text{in } \tilde{\mathcal{N}}_{R_{2,i},i} \times [0, T]. \tag{7.35}$$

Finally, combining (7.34) with (7.35), (7.19) holds. This ends the proof of Step 3.

Step 4: Finally, we are going to prove (7.3) for every compact subset K of $\Omega \cup \Gamma_1$. First, we consider

$$\tilde{\mathcal{N}}_{\Gamma_1} := \bigcup_{i=1}^{q} \tilde{\mathcal{N}}_{R_{2,i},i}, \quad \bar{d}(\varepsilon) := \min_{i\in\{1,\dots,q\}} \bar{d}_i(\varepsilon).$$

Then, thanks to Step 3, for every $d \in (0, \bar{d}(\varepsilon))$, we find that

$$\theta_{[m,a,d]} \geq \alpha_{[m,a]} - \varepsilon, \quad \text{for all } (x, t) \in \operatorname{clos}\tilde{\mathcal{N}}_{\Gamma_1} \times [0, T]. \tag{7.36}$$

Now, let K be a compact subset of $\Omega \cup \Gamma_1$. Then,

$$\delta := \operatorname{dist}(K, \Gamma_0) > 0,$$

and hence,

$$K \subset K_\delta := \left\{x \in \Omega \cup \Gamma_1 \ : \ \operatorname{dist}(x, \Gamma_0) \geq \frac{\delta}{2}\right\}.$$

Since

$$K_\delta = \operatorname{clos}\tilde{\mathcal{N}}_{\Gamma_1} \cup (K_\delta \backslash \tilde{\mathcal{N}}_{\Gamma_1}),$$

and $K_\delta \backslash \tilde{\mathcal{N}}_{\Gamma_1} \subset \Omega$, (7.3) follows easily by combining (7.36) with the result of Step 2. This ends the proof of the theorem. □

As a consequence of Theorem 7.1, we obtain the main result of this article in the particular case when $m \equiv m(t)$ and $a \equiv a(t)$.

Theorem 7.2. *Assume* (7.1) *and* (7.2), *and let $K \subset \Omega \cup \Gamma_1$ be a compact set. Then,*

$$\lim_{d \downarrow 0} \theta_{[m,a,d]} = \alpha_{[m,a]}, \quad \textit{uniformly in } K_T.$$

Proof. The existence and uniqueness of $\theta_{[m,a,d]}$ for sufficiently small $d > 0$ are guaranteed by (7.2) and Theorem 5.2. According to (7.1), $\alpha_{[m,a]}(x, t) \equiv \alpha_{[m,a]}(t)$ is autonomous in $x \in \bar{\Omega}$. Thus, $\mathscr{L}\alpha_{[m,a]} = 0$. Hence, in Ω_T, we have that

$$(\mathscr{P}_d - m(t))\alpha_{[m,a]}(t) + a(t)(\alpha_{[m,a]}(t))^p = (\partial_t - m(t))\alpha_{[m,a]}(t) + a(t)(\alpha_{[m,a]}(t))^p = 0.$$

Moreover, $\alpha_{[m,a]} > 0$ on Γ_0, and, due to (1.2),

$$\frac{\partial \alpha_{[m,a]}}{\partial \nu} + \beta\alpha_{[m,a]} = \beta\alpha_{[m,a]} > 0, \quad \text{on } \Gamma_1.$$

Consequently, $\alpha_{[m,a]}(t)$ provides us with a positive supersolution of (1.1) for all $d > 0$, and, thanks to Proposition 5.1(i), we find that

$$\theta_{[m,a,d]} \leq \alpha_{[m,a]}, \quad \text{in } \bar{\Omega}_T. \tag{7.37}$$

Combining (7.37) with Theorem 7.1, the proof is complete. □

8 General non-autonomous case

Throughout this section, we will assume that condition (6.1) holds, i.e.,

$$\int_0^T m(x, t)\mathrm{d}t > 0, \quad \text{for all } x \in \bar{\Omega}. \tag{8.1}$$

In this case, our main result reads as follows. Remember that we have already denoted by $\Gamma_{1,i}$, $i \in \{1, \ldots, q\}$, the components of Γ_1.

Theorem 8.1. *Under condition* (8.1), *for every compact set $K \subset \Omega \cup \Gamma_1$ such that $m \equiv m(t)$ and $a \equiv a(t)$ on a neighborhood in $\bar{\Omega}$ of every component $\Gamma_{1,i}$ of Γ_1 such that $K \cap \Gamma_{1,i} \neq \emptyset$, one has that*

$$\lim_{d \downarrow 0} \theta_{[m,a,d]} = \alpha_{[m,a]}, \quad \textit{uniformly in } K \times \mathbb{R}. \tag{8.2}$$

Proof. Thanks to Proposition 6.1, for any given $\varepsilon > 0$, there exists $\tilde{d} \coloneqq \tilde{d}(\varepsilon) > 0$ such that

$$\theta_{[m,a,d]} \le \alpha_{[m,a]} + \varepsilon, \quad \text{for all } (x,t) \in \bar{\Omega} \times \mathbb{R} \text{ and } d \in (0, \tilde{d}). \tag{8.3}$$

To obtain a lower estimate for $\theta_{[m,a,d]}$ in terms of $\alpha_{[m,a]}$, we will proceed by steps.

Step 1: We claim that, for every $x_0 \in \Omega$ and $\varepsilon > 0$, there exist $R \equiv R(x_0) > 0$ and $d(\varepsilon, x_0) > 0$ such that $\bar{B}_{2R}(x_0) \subset \Omega$ and

$$\alpha_{[m,a]} - \varepsilon \le \theta_{[m,a,d]}, \quad \text{in } \bar{B}_R(x_0) \times \mathbb{R}, \text{ for all } d \in (0, d(\varepsilon, x_0)). \tag{8.4}$$

Indeed, pick $x_0 \in \Omega$, $\varepsilon > 0$, and $R > 0$ sufficiently small so that $\bar{B}_{2R}(x_0) \subset \Omega$ and

$$\alpha_{[m,a]} \le \alpha_{[m_{L,\bar{B}_{2R}(x_0)}, a_{M,\bar{B}_{2R}(x_0)}]} + \frac{\varepsilon}{2}, \quad \text{in } \bar{B}_{2R}(x_0) \times \mathbb{R}. \tag{8.5}$$

Moreover, applying Theorem 7.1 in $\Omega = B_{2R}(x_0)$, with $K \coloneqq \bar{B}_R(x_0)$, it becomes apparent that there exists $d(\varepsilon, x_0)$ such that, for every $d \in (0, d(\varepsilon, x_0))$,

$$\alpha_{[m_{L,\bar{B}_{2R}(x_0)}, a_{M,\bar{B}_{2R}(x_0)}]} - \frac{\varepsilon}{2} \le \theta_{[m_{L,\bar{B}_{2R}(x_0)}, a_{M,\bar{B}_{2R}(x_0)}, d; \bar{B}_{2R}(x_0)]}, \quad \text{in } \bar{B}_R(x_0) \times \mathbb{R}. \tag{8.6}$$

Thus, it follows from (8.5) and (8.6) that, for every $d \in (0, d(\varepsilon, x_0))$,

$$\alpha_{[m,a]} - \varepsilon \le \alpha_{[m_{L,\bar{B}_{2R}(x_0)}, a_{M,\bar{B}_{2R}(x_0)}]} - \frac{\varepsilon}{2} \le \theta_{[m_{L,\bar{B}_{2R}(x_0)}, a_{M,\bar{B}_{2R}(x_0)}, d; B_{2R}(x_0)]}, \quad \text{in } \bar{B}_R(x_0) \times \mathbb{R}. \tag{8.7}$$

On the other hand, since $\theta_{[m,a,d]}$ is a positive strict supersolution of

$$\begin{cases} \partial_t u + \mathrm{d}\kappa(t)\mathscr{L}u = m(x,t)u - a(x,t)u^p, & \text{in } B_{2R}(x_0) \times \mathbb{R}, \\ u(\cdot, t) = 0, & \text{on } \partial B_{2R}(x_0) \times \mathbb{R}, \\ u(\cdot, 0) = u(\cdot, T), & \text{in } B_{2R}(x_0), \end{cases} \tag{8.8}$$

it follows from Proposition 5.1 that

$$\theta_{[m,a,d; B_{2R}(x_0)]} \le \theta_{[m,a,d]} \quad \text{in } B_{2R}(x_0) \times \mathbb{R}, \tag{8.9}$$

where $\theta_{[m,a,d; B_{2R}(x_0)]}$ stands for the unique positive solution of (8.8), whose existence and uniqueness follow from Theorem 5.2 applied in $B_{2R}(x_0) \times \mathbb{R}$. Moreover, applying Proposition 5.1 in $B_{2R}(x_0)$ yields

$$\theta_{[m_{L,\bar{B}_{2R}(x_0)}, a_{M,\bar{B}_{2R}(x_0)}, d; B_{2R}(x_0)]} \le \theta_{[m,a,d; B_{2R}(x_0)]}, \quad \text{in } B_{2R}(x_0) \times \mathbb{R}. \tag{8.10}$$

Finally, combining (8.7) with (8.10) and (8.9), Estimate (8.4) holds, and the proof of Step 1 is complete.

Step 2: In this step, we will prove that, for every compact subset $K \subset \Omega$ and $\varepsilon > 0$, there exists $\tilde{d}(\varepsilon, K) > 0$ such that

$$\alpha_{[m,a]} - \varepsilon \le \theta_{[m,a,d]}, \quad \text{in } K \times \mathbb{R}, \quad \text{for all } d \in (0, \tilde{d}(\varepsilon, K)). \tag{8.11}$$

Indeed, since

$$K \subset \bigcup_{x \in K} B_{R(x)}(x),$$

where $R(x)$ is the radius associated with $x \in K$ constructed in Step 1, by the compactness of K, there is a finite subset of K, say $\{x_1, \dots, x_p\} \subset K$, such that

$$K \subset \bigcup_{i=1}^{p} B_{R(x_i)}(x_i). \tag{8.12}$$

Fix $\varepsilon > 0$. Then, thanks to Step 1, we already know that, for every $i \in \{1, \dots, p\}$, there exists $d_i = d_i(\varepsilon, x_i) > 0$ such that

$$\alpha_{[m,a]} - \varepsilon \le \theta_{[m,a,d]}, \quad \text{in } \bar{B}_{R(x_i)}(x_i) \times \mathbb{R}. \tag{8.13}$$

Therefore, setting

$$\tilde{d}(\varepsilon, K) \coloneqq \min_{i \in \{1, \dots, p\}} d_i(\varepsilon, x_i),$$

it follows from (8.12) and (8.13) that (8.11) holds. This ends the proof of Step 2.

Step 3: For every $i \in \{1, \dots, q\}$ such that $K \cap \Gamma_{1,i} \neq \emptyset$, let $\mathcal{U}_i$ be the neighborhood of $\Gamma_{1,i}$ in $\bar{\Omega}$ such that $m(x,t)$ and $a(x,t)$ are autonomous in x, and consider the open neighborhood $\tilde{\mathcal{N}}_{2R,i}$ defined in (7.18) for a sufficiently small $R > 0$ such that $\tilde{\mathcal{N}}_{2R,i} \subset \mathcal{U}_i$, as well as the compact sets

$$K_{\Gamma_1,i} := \operatorname{clos} \tilde{\mathcal{N}}_{R,i} \subset \tilde{\mathcal{N}}_{2R,i} \cup \Gamma_{1,i}$$

and

$$K_{\Gamma_1} := \bigcup_{\substack{i \in \{1,\dots,q\} \\ K \cap \Gamma_{1,i} \neq \emptyset}} K_{\Gamma_1,i}. \tag{8.14}$$

We claim that, for every $\varepsilon > 0$, there exists $\hat{d} := \hat{d}(\varepsilon) > 0$ such that

$$\alpha_{[m,a]} - \varepsilon \leq \theta_{[m,a,d]}, \quad \text{in } K_{\Gamma_1} \times \mathbb{R}, \text{ for all } d \in (0, \hat{d}). \tag{8.15}$$

Indeed, since $a(x,t)$ and $m(x,t)$ are autonomous in x in $\mathcal{U}_i$, also $\alpha_{[m,a]}$ is autonomous in $x \in \tilde{\mathcal{N}}_{2R,i}$. Thus, applying Theorem 7.1 in the open set $\tilde{\mathcal{N}}_{2R,i}$, with $K := K_{\Gamma_1,i}$, it becomes apparent that, for every $\varepsilon > 0$, there exists $d_i(\varepsilon) > 0$ such that

$$\alpha_{[m,a]} - \varepsilon \leq \theta_{[m,a,d;\,\tilde{\mathcal{N}}_{2R,i}]}, \quad \text{in } K_{\Gamma_1,i} \times \mathbb{R}, \text{ for all } d \in (0, d_i(\varepsilon)). \tag{8.16}$$

On the other hand, since $\theta_{[m,a,d]}$ is a positive strict supersolution of

$$\begin{cases} \partial_t u + \mathrm{d}\kappa(t)\mathscr{L} u = m(x,t)u - a(x,t)u^p, & \text{in } \tilde{\mathcal{N}}_{2R,i} \times \mathbb{R}, \\ \mathscr{B} u(x,t) = 0, & \text{on } \partial\tilde{\mathcal{N}}_{2R,i} \times \mathbb{R}, \\ u(\cdot,0) = u(\cdot,T), & \text{in } \tilde{\mathcal{N}}_{2R,i}, \end{cases} \tag{8.17}$$

it follows from Proposition 5.1 that

$$\theta_{[m,a,d;\,\tilde{\mathcal{N}}_{2R,i}]} \leq \theta_{[m,a,d]}, \quad \text{in } \tilde{\mathcal{N}}_{2R,i} \times \mathbb{R}, \tag{8.18}$$

where $\theta_{[m,a,d;\,\tilde{\mathcal{N}}_{2R,i}]}$ stands for the unique positive solution of (8.17), whose existence and uniqueness follow from Theorem 5.2 applied in $\tilde{\mathcal{N}}_{2R,i} \times \mathbb{R}$. Now, combining (8.16) and (8.18), it is apparent that

$$\alpha_{[m,a]} - \varepsilon \leq \theta_{[m,a,d]}, \quad \text{in } K_{\Gamma_1,i} \times \mathbb{R}, \text{ for all } d \in (0, d_i(\varepsilon)). \tag{8.19}$$

Finally, setting

$$\tilde{d}(\varepsilon) := \min_{\substack{i \in \{1,\dots,q\} \\ K \cap \Gamma_{1,i} \neq \emptyset}} d_i(\varepsilon),$$

Estimate (8.15) follows from (8.14) and (8.19). This ends the proof of Step 3.

Step 4: Since K is a compact subset of $\Omega \cup \Gamma_1$, we have that

$$\delta := \operatorname{dist}(K, \Gamma_K) > 0, \quad \text{where } \Gamma_K \equiv \partial\Omega \setminus \bigcup_{\substack{i \in \{1,\dots,q\} \\ K \cap \Gamma_{1,i} \neq \emptyset}} \Gamma_{1,i}.$$

Thus,

$$K \subset K_\delta := \left\{ x \in \Omega \cup \bigcup_{\substack{i \in \{1,\dots,q\} \\ K \cap \Gamma_{1,i} \neq \emptyset}} \Gamma_{1,i} \;:\; \operatorname{dist}(x, \Gamma_K) \geq \frac{\delta}{2} \right\}.$$

Note that

$$K_\delta = K_{\Gamma_1} \cup (K_\delta \setminus K_{\Gamma_1}), \quad K_\delta \setminus K_{\Gamma_1} \subset \Omega,$$

where K_{Γ_1} is the compact set defined in (8.14).

Applying (8.11) in the compact set $\operatorname{clos}(K_\delta \backslash K_{\Gamma_1}) \subset \Omega$, there exists $\tilde{d} = \tilde{d}(\varepsilon) > 0$ such that

$$\alpha_{[m,a]} - \varepsilon \le \theta_{[m,a,d]}, \quad \text{in } \operatorname{clos}(K_\delta \backslash K_{\Gamma_1}) \times \mathbb{R}, \text{ for all } d \in (0, \tilde{d}). \tag{8.20}$$

Therefore, combining (8.15) and (8.20), it becomes apparent that

$$\alpha_{[m,a]} - \varepsilon \le \theta_{[m,a,d]}, \quad \text{in } K_\delta \times \mathbb{R} \supset K \times \mathbb{R}, \text{ for all } d \in (0, \min\{\tilde{d}, \hat{d}\}). \tag{8.21}$$

Finally, thanks to (8.3) and (8.21), (8.2) holds. This ends the proof. □

9 Main result

The main result of this article reads as follows.

Theorem 9.1. *Assume that there exists* $x_0 \in \Omega$ *such that* $\overline{m}(x_0) > 0$, *and let* $K \subset \Omega \cup \Gamma_1$ *be a compact subset. Then, the following conditions are satisfied*:

(*i*) *If* $\overline{m}(x) \le 0$, *for all* $x \in K$, *then*

$$\lim_{d \downarrow 0} \theta_{[m,a,d]} = 0, \quad \textit{uniformly in } K_T. \tag{9.1}$$

(*ii*) *If* $\overline{m}(x) > 0$, *for all* $x \in K$ *and* $K \subset \Omega$, *then*

$$\lim_{d \downarrow 0} \theta_{[m,a,d]} = \alpha_{[m,a]}, \quad \textit{uniformly in } K_T. \tag{9.2}$$

(*iii*) *If* $\overline{m}(x) > 0$ *for all* $x \in K$ *and there exists a nonempty subset* $\mathscr{I} \subset \{1, \ldots, q\}$ *such that*

$$\partial K \cap \Gamma_1 = \bigcup_{i \in \mathscr{I}} \Gamma_{1,i}, \quad \operatorname{dist}(\partial K \cap \Omega, \Gamma_1) > 0,$$

and $(m, a) = (m(t), a(t))$ *on a neighborhood of* $\partial K \cap \Gamma_1$, *then* (9.2) *holds.*

Proof. The existence and uniqueness of $\theta_{[m,a,d]}$ for sufficiently small $d > 0$ follow from $\overline{m}(x_0) > 0$ and Theorem 5.2. Next, we will prove Part (i). Assume $\overline{m}(x) \le 0$ for all $x \in K$, and consider the auxiliary functions

$$m_0(x, t) \coloneqq m(x, t) + |\overline{m}(x)|.$$

and, for every $\delta > 0$,

$$m_\delta(x, t) \coloneqq m(x, t) + (1 + \delta)|\overline{m}(x)| + \delta.$$

Then,

$$\overline{m}_\delta(x) = \overline{m}(x) + (1 + \delta)|\overline{m}(x)| + \delta,$$

and, since $\delta > 0$, it is apparent that

$$m_\delta > m, \quad \text{in } \bar{\Omega}_T. \tag{9.3}$$

Moreover,

$$\overline{m}_\delta(x) > 0, \quad \text{for all } x \in \bar{\Omega}. \tag{9.4}$$

Indeed, (9.4) is obvious if $\overline{m}(x) > 0$. Suppose $\overline{m}(x) \le 0$. Then, by definition,

$$\overline{m}_\delta(x) = \overline{m}(x) + (1 + \delta)|\overline{m}(x)| + \delta = \overline{m}(x) - (1 + \delta)\overline{m}(x) + \delta = (1 - \overline{m}(x))\delta > 0.$$

Thus, (9.4) holds.

On the other hand, thanks to (9.3), it follows from Proposition 5.1(iii) that

$$\theta_{[m,a,d]} \le \theta_{[m_\delta,a,d]}, \quad \text{in } \bar{\Omega}_T, \tag{9.5}$$

and, owing to (9.4), Proposition 6.1 guarantees that, for every $\varepsilon > 0$, there exists $\tilde{d}(\varepsilon) > 0$ such that

$$\theta_{[m_\delta,a,d]} \leq \alpha_{[m_\delta,a]} + \varepsilon, \quad \text{for all } (x,t) \in \bar{\Omega}_T \text{ and } d \in (0,\tilde{d}(\varepsilon)). \tag{9.6}$$

Thus, combining (9.5) with (9.6), it becomes apparent that, for every $\delta > 0$,

$$0 < \theta_{[m,a,d]} \leq \theta_{[m_\delta,a,d]} \leq \alpha_{[m_\delta,a]} + \varepsilon \quad \text{in } \bar{\Omega}_T, \text{ for all } d \in (0,\tilde{d}(\varepsilon)). \tag{9.7}$$

Hence, letting $d \downarrow 0$ in (9.7), we find that

$$0 \leq \limsup_{d\downarrow 0} \theta_{[m,a,d]} \leq \alpha_{[m_\delta,a]} + \varepsilon, \quad \text{in } \bar{\Omega}_T, \text{ for all } \delta > 0. \tag{9.8}$$

On the other hand, since

$$\lim_{\delta\downarrow 0} \|m_\delta - m_0\| = \|1 + |\overline{m}|\| \lim_{\delta\downarrow 0} \delta = 0,$$

letting $\delta \downarrow 0$ in (9.8) yields

$$0 \leq \limsup_{d\downarrow 0} \theta_{[m,a,d]} \leq \lim_{\delta\downarrow 0} \alpha_{[m_\delta,a]} + \varepsilon = \alpha_{[m_0,a]} + \varepsilon, \quad \text{in } \bar{\Omega}_T. \tag{9.9}$$

As

$$\overline{m}_0(x) = \overline{m}(x) + |\overline{m}(x)| = \overline{m}(x) - \overline{m}(x) = 0, \quad \text{for all } x \in K,$$

it follows from Proposition 2.1 and (2.7) that $\alpha_{[m_0,a]} \equiv 0$ in K_T. Thus, (9.9) implies that, for every $\varepsilon > 0$,

$$0 \leq \limsup_{d\downarrow 0} \theta_{[m,a,d]} \leq \varepsilon, \quad \text{in } K_T.$$

Therefore, (9.1) holds.

Now, we will prove Part (ii). Suppose that $K \subset \Omega$ and $\overline{m}(x) > 0$ for all $x \in K$. By the continuity of $\overline{m}(x)$, there exist open neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of K with smooth boundaries such that

$$K \subset \mathcal{U} \subset \bar{\mathcal{U}} \subset \mathcal{V} \subset \Omega \quad \text{and} \quad \overline{m}(x) > 0, \quad \text{for all } x \in \mathcal{V}. \tag{9.10}$$

Now, let $\eta \in C_0^\infty(\bar{\Omega})$ be such that

$$\eta \equiv 0 \text{ in } \bar{\mathcal{U}}, \quad \eta \equiv 1 \text{ in } \bar{\Omega}\backslash\mathcal{V}, \quad \text{and} \quad \eta(x) \in (0,1), \quad \text{for all } x \in \mathcal{V}\backslash\bar{\mathcal{U}},$$

and, for every $\gamma > 0$, consider the function

$$m_\gamma(x,t) \coloneqq m(x,t) + \gamma\eta(x).$$

By construction,

$$m_\gamma(x,t) = m(x,t), \ \text{ for all } (x,t) \in \mathcal{U}_T, \tag{9.11}$$

and

$$m_\gamma \gneq m, \ \text{ in } \bar{\Omega}_T. \tag{9.12}$$

Note that (9.11) entails that

$$\alpha_{[m_\gamma,a]} \equiv \alpha_{[m,a]}, \quad \text{in } K_T. \tag{9.13}$$

Moreover, according to (9.10), we have that, for every $x \in \mathcal{V}$,

$$\overline{m}_\gamma(x) = \overline{m}(x) + \gamma\eta(x) \geq \overline{m}(x) > 0, \quad \text{for all } x \in \mathcal{V}.$$

Similarly, if $x \in \Omega\backslash\mathcal{V}$, then, for sufficiently large $\gamma > 0$, we have that

$$\overline{m}_\gamma(x) = \overline{m}(x) + \gamma\eta(x) = \overline{m}(x) + \gamma > 0, \quad \text{for all } x \in \Omega\backslash\mathcal{V}.$$

Therefore, for sufficiently large $\gamma > 0$, say $\gamma \geq \gamma_0$, we have that

$$\overline{m}_\gamma(x) > 0, \quad \text{for all } x \in \bar{\Omega}. \tag{9.14}$$

Fix $\gamma \geq \gamma_0$. Then, due to (9.12), Proposition 5.1(iii) implies that

$$\theta_{[m,a,d]} \leq \theta_{[m_\gamma,a,d]}, \quad \text{in } \bar{\Omega}_T. \tag{9.15}$$

Moreover, owing to (9.14), it follows from Proposition 6.1 that, for every $\varepsilon > 0$, there exists $\tilde{d}_1(\varepsilon) > 0$ such that

$$\theta_{[m_\gamma,a,d]} \leq \alpha_{[m_\gamma,a]} + \varepsilon, \quad \text{in } \bar{\Omega}_T, \text{ for all } d \in (0, \tilde{d}_1(\varepsilon)). \tag{9.16}$$

Consequently, by (9.13), (9.15), and (9.16), we find that

$$\theta_{[m,a,d]} \leq \alpha_{[m,a]} + \varepsilon, \quad \text{in } K_T, \text{ for all } d \in (0, \tilde{d}_1(\varepsilon)). \tag{9.17}$$

On the other hand, since $\bar{\mathcal{U}} \subset \Omega$, the solution $\theta_{[m,a,d]}$ is a positive strict supersolution of

$$\begin{cases} \partial_t u + \mathrm{d}\kappa(t)\mathscr{L}u = m(x,t)u - a(x,t)u^p, & \text{in } \mathcal{U} \times \mathbb{R}, \\ u(\cdot,t) = 0, & \text{on } \partial\mathcal{U} \times \mathbb{R}, \\ u(\cdot,0) = u(\cdot,T), & \text{in } \mathcal{U}, \end{cases} \tag{9.18}$$

and hence, shortening $\tilde{d}_1(\varepsilon)$, if necessary, it follows from Proposition 5.1(ii) that

$$\theta_{[m,a,d;\mathcal{U}]} \leq \theta_{[m,a,d]}, \quad \text{in } \mathcal{U} \times [0,T], \text{ for all } d \in (0, \tilde{d}_1(\varepsilon)), \tag{9.19}$$

where $\theta_{[m,a,d;\mathcal{U}]}$ stands for the unique positive solution of (9.18), whose existence and uniqueness follow from Theorem 5.2 applied in $\mathcal{U} \times \mathbb{R}$. Also, by (9.10), it follows from Theorem 8.1 (or Step 2 of Theorem 8.1) applied to (9.18) that there exists $\tilde{d}_2(\varepsilon, K) > 0$ such that

$$\alpha_{[m,a]} - \varepsilon \leq \theta_{[m,a,d,\mathcal{U}]}, \quad \text{in } K_T, \text{ for all } d \in (0, \tilde{d}_2(\varepsilon, K)). \tag{9.20}$$

Thus, setting

$$\tilde{d}(\varepsilon) \coloneqq \min\{\tilde{d}_1(\varepsilon), \tilde{d}_2(\varepsilon, K)\} > 0,$$

it follows from (9.19) and (9.20) that

$$\alpha_{[m,a]} - \varepsilon \leq \theta_{[m,a,d]}, \quad \text{in } K_T, \text{ for all } d \in (0, \tilde{d}(\varepsilon)). \tag{9.21}$$

Since (9.17) and (9.21) imply that

$$\alpha_{[m,a]} - \varepsilon \leq \theta_{[m,a,d]} \leq \alpha_{[m,a]} + \varepsilon, \quad \text{in } K_T, \text{ for all } d \in (0, \tilde{d}(\varepsilon)),$$

the proof of Part (ii) is completed in case $K \subset \Omega$.

The proof of Part (iii) follows the same general patterns as in Part (ii), but choosing $\mathcal{U}$ and $\mathcal{V}$ to satisfy

$$K \subset \mathcal{U} \subset \bar{\mathcal{U}} \subset \mathcal{V} \subset \Omega \cup (\partial K \cap \Gamma_1) \quad \text{and} \quad \partial K \cap \Gamma_1 \subset \partial K \cap \partial\mathcal{U} \cap \partial\mathcal{V}.$$

Now, one should use that $\theta_{[m,a,d]}$ is a positive strict supersolution of

$$\begin{cases} \partial_t u + \mathrm{d}\kappa(t)\mathscr{L}u = m(x,t)u - a(x,t)u^p, & \text{in } \mathcal{U} \times \mathbb{R}, \\ \mathscr{B}u(\cdot,t) = 0, & \text{on } \partial\mathcal{U} \times \mathbb{R}, \\ u(\cdot,0) = u(\cdot,T), & \text{in } \mathcal{U}, \end{cases}$$

and apply Step 4 of Theorem 8.1. So, the technical details will be omitted by repetitive. This ends the proof. □

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