MÁSTER UNIVERSITARIO EN INGENIERÍA INDUSTRIAL

TRABAJO FIN DE MÁSTER

Un enfoque de optimización robusta para el alivio de la pobreza multidimensional

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Caso Aplicado a México

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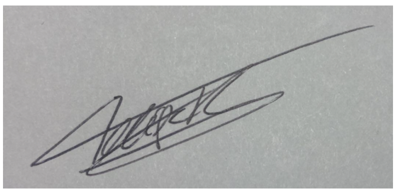

Figure 1: *
Firma del Autor

Figure 2: *
Firma del Director

1 Introducción

Tradicionalmente, la pobreza se ha definido como la falta de dinero. Sin embargo, en los últimos tiempos, se entiende cada vez más como un problema multidimensional, separando a menudo el factor económico de otras dimensiones clave. Para muchas organizaciones, la erradicación de la pobreza es una de las principales prioridades en sus agendas. No obstante, a pesar de estos esfuerzos, las tasas de pobreza están lejos de disminuir.

Nuestra estrategia de alivio de la pobreza, se centra en proporcionar viviendas y talleres de educación financiera a familias en situación de pobreza. El objetivo es crear un modelo autosostenible que minimice la dependencia de donaciones externas y genere un retorno sobre la inversión (ROI, por sus siglas en inglés) suficiente, que luego pueda reinvertirse para reducir aún más la pobreza, el objetivo final. A través de subvenciones de diversas instituciones y préstamos bancarios, adquirimos un capital para invertir de la manera más óptima posible. Aquí, "óptimo" se define como lograr el máximo alivio de la pobreza junto con el ROI más alto. En este marco, trabajamos con familias a través de un programa de cinco años, durante el cual se espera que las familias reembolsen una parte del valor de la vivienda, junto con intereses que se limitan a un tercio de sus ingresos, para garantizar su calidad de vida a lo largo del programa. Al final del programa, las familias no solo serán propietarias de una vivienda adecuada, sino que también habrán construido un historial crediticio, lo que las integra en el sistema financiero formal.

Nuestro objetivo es desarrollar un modelo de asignación de capital que determine en qué comunidades invertir, cuánto y cuándo hacerlo, durante un período de diez años. Para lograr esto, desarrollamos un modelo que cuantifica la pobreza y evalúa el impacto de nuestro proyecto en la reducción de la privación en cada dimensión. Para cuantificar la pobreza, el primer paso es definir qué entendemos por pobreza. En este artículo, la pobreza se define como la falta de acceso a bienes y servicios universales básicos de alta calidad (es decir, buscamos aliviar la pobreza material). Creemos que la pobreza material puede definirse como la falta de acceso a: 1) una vivienda adecuada, 2) electricidad, 3) agua potable, 4) conexión a internet de bajo costo, y 5) un sistema integral de atención médica. Nuestro razonamiento para esta definición se centra en entender quién no es pobre, en lugar de etiquetar a las personas como pobres en función de sus circunstancias personales. Aunque una persona pueda considerarse pobre debido a factores subjetivos, argumentamos que cualquier persona con acceso a estos cinco bienes y servicios esenciales tiene la base necesaria para salir de la pobreza. Nuestro objetivo principal es eliminar las barreras que impiden el acceso a estos recursos esenciales. Una vez que estas necesidades están cubiertas, la responsabilidad de superar la pobreza recae en gran medida en el individuo. En esencia, la provisión de estos bienes crea un entorno donde las personas cuentan con las herramientas necesarias para salir de la pobreza.

Posteriormente, una vez definida y medida la pobreza en cada dimensión, utilizamos la teoría de conjuntos difusos para crear un índice "totalmente relativo" que mide la privación de un individuo en función de su posición dentro de la distribución general de privaciones en la sociedad. Luego, construimos un modelo robusto de asignación de capital que considera la incertidumbre en los parámetros, asegurando que la asignación de capital tome en cuenta cómo la incertidumbre puede modificar los riesgos asociados con la capacidad de repagar la deuda contraída al inicio del proyecto.

En particular, incorporamos una variable que mide la tasa de incumplimiento en cada comunidad. Dada la escasez de datos sobre este tema, la optimización robusta nos permite obtener una solución factible bajo el valor más desfavorable de esta variable sin necesidad de conocer su distribución de probabilidad. Además, proponemos un modelo que permite ajustar el grado de conservadurismo, es decir, la distancia entre la solución robusta y la solución nominal de nuestro modelo. Esta flexibilidad es crucial por dos razones. Primero, las tasas más altas de incumplimiento dificultan la capacidad del proyecto para cumplir con sus obligaciones financieras (es decir, pagar la deuda). Segundo, las tasas más altas de incumplimiento también reducen el número de familias con las que podemos trabajar, disminuyendo así el impacto general en la reducción de la pobreza.

2 Estado de la cuestión

Respecto a la revisión de la literatura en la materia, esta se centra en dos aspectos claves. Los métodos empleados en la cuantificación de la pobreza y las técnicas de optimización robusta.

2.1 Cuantificación de la Pobreza

En primer lugar, revisamos la literatura sobre cuantificación de la pobreza con el objetivo de crear un índice de pobreza multidimensional, abordando los pasos clave como la identificación de los pobres, la selección de dimensiones y pesos, y la agregación de medidas de privación. Este enfoque multidimensional, busca superar las limitaciones del uso exclusivo del ingreso como indicador, integrando factores como acceso a bienes y servicios básicos.

Las medidas de pobreza, ya sean unidimensionales o multidimensionales, buscan crear un índice poblacional basado en niveles individuales de privación. Tradicionalmente, debido a las dificultades de medición y la falta de datos integrales, la pobreza se ha medido unidimensionalmente, utilizando el ingreso como principal indicador. Este enfoque implica dos pasos clave: la identificación de los pobres, basada en un umbral de ingreso, y la agregación, que combina medidas individuales en un índice poblacional.

Sin embargo, este enfoque presenta limitaciones significativas, como ignorar la distribución de bienes y servicios y el acceso gratuito a servicios esenciales provistos por el gobierno (salud, educación, agua potable). En respuesta a estas deficiencias, ha surgido un interés creciente en desarrollar índices de pobreza multidimensionales. Este enfoque evalúa privaciones en múltiples dimensiones, adaptando los pasos tradicionales para incluir: la selección de dimensiones, el cálculo de los niveles de privación, la asignación de pesos y la agregación para construir un índice poblacional. En nuestro estudio, nos enfocamos en los tres primeros pasos para construir un índice de privación a nivel comunitario, ya que contamos con datos promedio por comunidad, no individuales.

Las metodologías más comunes para construir índices de pobreza multidimensional son las siguientes:
1. Dashboards e índices compuestos: Los dashboards miden la privación en cada dimensión de manera separada, proporcionando información detallada pero sin producir

un índice agregado. Por ello, suelen ser un paso preliminar a los índices compuestos, como el Índice de Desarrollo Humano (IDH) o el Índice de Pobreza Humana (IPH). Estos índices combinan medidas de privación por dimensión mediante promedios ponderados. Sin embargo, no consideran cómo las privaciones simultáneas afectan a un individuo, lo que es una limitación importante.
2. Métodos estadísticos: Los métodos estadísticos, como el Análisis de Componentes Principales (PCA) o el Análisis de Factores (FA), permiten reducir la dimensionalidad de las medidas de privación y generar índices específicos por persona o población. Aunque son útiles y complementarios, presentan desafíos, como la falta de claridad sobre su cumplimiento de axiomas de pobreza y el alto grado de discrecionalidad en su implementación. Por estas razones, preferimos metodologías más desarrolladas para nuestro modelo.
3. Método de Alkire y Foster: Este enfoque identifica privaciones en múltiples dimensiones y utiliza técnicas como la matriz de privación ponderada para calcular índices específicos. Luego, clasifica a las personas como pobres según diferentes criterios:

- *Método unidimensional:*Promedio ponderado de privaciones para determinar la pobreza individual.
- *Enfoque de unión:*Una persona es pobre si está privada en al menos una dimensión.
- *Enfoque de intersección:*Una persona es pobre solo si está privada en todas las dimensiones.
- *Técnica de doble umbral* Clasifica como pobres a quienes tienen privaciones en al menos k dimensiones.

4. Métodos de conjuntos difusos: Proponen una alternativa a las clasificaciones binarias tradicionales introduciendo funciones de pertenencia que miden el grado de privación. Por ejemplo, en el método de Cheli y Lemmi (1995), se usa la función de distribución acumulada para calcular cuán privado está un individuo en una dimensión en comparación con otros. Este enfoque es particularmente útil para calcular índices relativos entre comunidades, lo que ayuda a priorizar la asignación de recursos de manera efectiva. Como hemos visto en las técnicas anteriores, un aspecto fundamental de la medición de la pobreza es el cálculo de los pesos de las dimensiones, especialmente en los enfoques difusos y en el método de Alkire-Foster.

Seguidamente, dada la importancia de la selección de las ponderaciones de las dimensiones, pasamos a evaluar las distintas técnicas utilizadas. Valorando que técnica se adapta más a nuestro estudio considerando las dos siguientes propiedades. **Propiedad de discriminación:** El principio de discriminación sugiere que cuando la proporción de una población privada en una dimensión es baja, el impacto de esa privación en un individuo debe ser mayor que cuando la proporción es alta. Por lo tanto, las dimensiones con menor proporción de privados deben tener mayor peso. **Propiedad de no redundancia:** En el cálculo de un índice específico por individuo, es crucial agregar los niveles de privación entre dimensiones minimizando la doble contabilización. Debido a que las dimensiones pueden estar correlacionadas, es importante usar un esquema de ponderación que asigne menores pesos a las dimensiones correlacionadas.

Los principales métodos son los siguientes:

- **Pesos iguales:** Aunque es el método más utilizado, es polémico porque asume que todas las dimensiones son igualmente importantes.

- **Pesos basados en proporciones:** Los pesos se asignan en función de la proporción de la población privada en una dimensión. A pesar de que cumple con el principio de discriminación, no aborda el problema de la doble contabilización.
- **Pesos más favorables:** Los pesos se basan en la importancia que cada dimensión tiene para cada individuo, estableciendo diferentes pesos para cada persona.
- **Pesos estadísticos:** Se utilizan modelos descriptivos como el análisis de componentes principales y modelos explicativos como el análisis factorial. Aunque resuelven el problema de la doble contabilización, estos métodos son complejos y difíciles de explicar para los responsables de políticas.
- **Modelos basados en regresión:** Los pesos se seleccionan como los coeficientes de una regresión lineal, donde las dimensiones son variables independientes y el bienestar es la variable dependiente. Un inconveniente es que, si el bienestar puede capturarse con una sola variable, no es necesario calcular un índice de pobreza multidimensional.
- **Pesos normativos:** Se recogen datos de grupos representativos que clasifican las dimensiones según su importancia. Los resultados se usan para calcular los pesos, aunque la fiabilidad de estos métodos puede verse afectada por la formulación de preguntas y la representatividad de los grupos.

Finalmente, dada la falta de un criterio establecido, elegimos utilizar el siguiente método:
Método de correlación de prevalencia Debido a la falta de consenso sobre el esquema de ponderación, la elección recae en el investigador. En este trabajo, consideramos que las técnicas estadísticas y aquellas comúnmente utilizadas en teoría difusa son las más apropiadas. Sin embargo, dadas las limitaciones de las técnicas estadísticas tradicionales, optamos por el **Método de Correlación de Prevalencia**, adoptado oficialmente por (Eurostat, 2002). En este método, el peso w_k asignado a una dimensión k se calcula como:

$$w_k = w_k^a \cdot w_k^b$$

donde w_k^a se calcula para abordar el problema de discriminación, usando el coeficiente de variación V_k. Mientras que w_k^b aborda el problema de redundancia y se calcula como:

$$w_k^b \propto \left(\frac{1}{1 + \sum_{k'}^{K} \rho_{k,k'} \mid \rho_{k,k'} < \rho_H} \right) \cdot \left(\frac{1}{\sum_{k'}^{K} \rho_{k,k'} \mid \rho_{k,k'} \geq \rho_H} \right)$$

donde $\rho_{k,k'}$ es la correlación entre las dimensiones k y k', y ρ_H es el valor de las correlaciones entre dimensiones con la mayor diferencia en el conjunto de correlaciones ordenadas.

Este modelo reduce el peso w_k^b en proporción al número de dimensiones altamente correlacionadas, evitando la doble contabilización al agregar dimensiones en el cálculo de índices de privación específicos por individuo.

Por otro lado, exploramos técnicas de optimización robusta para diseñar modelos que permitan asignar recursos de manera óptima, maximizando la reducción de la pobreza y el retorno de inversión, al mismo tiempo que valoramos como la incertidumbre puede afectar a los resultados del modelo.

2.2 Métodos de optimización robusta

Uno de los principales enfoques dentro de la comunidad de programación matemática ha sido la incorporación de la incertidumbre en los parámetros de los problemas de optimización. Un enfoque primario para esto es la optimización estocástica, donde la incertidumbre se modela a través de distribuciones de probabilidad. En este enfoque, el objetivo es encontrar una solución que esté protegida contra la incertidumbre estocástica en un sentido probabilístico.

En contraste, la optimización robusta (RO) representa la incertidumbre a través de un conjunto determinista de realizaciones posibles. En lugar de asignar probabilidades a cada estado posible, la RO asume que la incertidumbre puede tomar cualquier valor dentro de un conjunto específico y predefinido. El objetivo en la RO es garantizar que la solución permanezca factible independientemente de cómo se manifieste la incertidumbre, siempre que esta se mantenga dentro de los límites definidos.

El equivalente robusto de un problema de optimización lineal puede expresarse, sin pérdida de generalidad, como:

$$\min \; c^t x$$
$$\text{s.t.} \quad a_i^t x \leq b_i, \quad \forall a_i \in U_{a_i}, \, \forall b_i \in U_{b_i}, \, i = 1, \dots, m$$

donde $U_{a_i} \subseteq \mathbb{R}^n$ y $U_{b_i} \subseteq \mathbb{R}$ son conjuntos de incertidumbre dados.

En la optimización lineal robusta, existen tres métodos principales para definir los conjuntos de incertidumbre:

1. Optimización Lineal Robusta con Incertidumbre Politópica

Este es un caso especial del problema anterior cuando U_{a_i} y U_{b_i} son politópicos, de forma que:

$$U_{a_i} = \{a_i \mid D_i a_i \leq d_i\}$$

donde $D_i \subseteq \mathbb{R}^{k_i \times n}$ y $d_i \subseteq \mathbb{R}^{k_i}$ son datos de entrada. La incertidumbre en b_i se representa como un intervalo, y por lo tanto, podemos prescindir de U_{b_i}, ya que el peor caso es simplemente el valor inferior del intervalo. Esto transforma el problema original al siguiente, donde reutilizamos la notación b_i, aunque realmente significa $\min\{b_{i,\min}, b_{i,\max}\}$.

$$\min \; c^t x$$
$$\text{s.t.} \quad a_i^t x \leq b_i, \quad \forall a_i \in U_{a_i}$$

donde $U_{a_i} = \{a_i \mid D_i a_i \leq d_i\}$.

Además, el problema anterior es equivalente a:

$$\min_x \; c^t x$$
$$\text{s.t.} \quad \begin{bmatrix} \max_{a_i} \{a_i^t x\} \\ D_i a_i \leq d_i \end{bmatrix} \leq b_i, \quad \forall i = 1, \dots, m$$

Intuitivamente, el problema de maximización interno busca encontrar el valor de $a_i^T x$ que hace más difícil satisfacer las restricciones especificadas por el conjunto de incertidumbre U_{a_i} para cada restricción. Luego, debido a la dualidad fuerte, sustituimos el problema de maximización interno por su dual:

$$\min_{x} \; c^t x$$

$$\text{s.t.} \quad \begin{bmatrix} \min_{p_i}\{p_i^t x\} \\ D_i^t p_i = x \\ p_i \geq 0 \end{bmatrix} \leq b_i, \quad \forall i = 1, \ldots, m$$

Lo que es equivalente a:

$$\min_{x,p_i} \; c^t x$$

$$\text{s.t.} \quad p_i^t d_i \leq b_i \quad \forall i = 1, \ldots, m$$

$$D_i^t p_i = x \quad \forall i = 1, \ldots, m$$

$$p_i \geq 0 \quad \forall i = 1, \ldots, m$$

2. Optimización Lineal Robusta con Incertidumbre Elipsoidal

Ahora U_{a_i} es un elipsoide, definido como:

$$U_{a_i} = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}, \quad i = 1, \ldots, m$$

donde $P_i \in \mathbb{R}^{n \times n}$ y $\bar{a}_i \in \mathbb{R}^n$ son datos de entrada. Tal que si $P_i = I$, entonces el conjunto de incertidumbre U_{a_i} se convierte en una esfera y si $P_i = 0$, no hay incertidumbre.

Para incorporar la incertidumbre elipsoidal, reformulamos el problema original como:

$$\min_{x} \; c^T x$$

$$\text{s.t.} \quad \begin{bmatrix} \max_{a_i}\{a_i^T x\} \\ a_i \in U_{a_i} \end{bmatrix} \leq b_i, \quad \forall i = 1, \ldots, m$$

El problema de maximización interno tiene una solución explícita:

$$\max \left\{ a_i^T x \;\middle|\; a_i \in U_{a_i} \right\} = \bar{a}_i^T x + \max \left\{ u^T P_i^T x \;\middle|\; \|u\|_2 \leq 1 \right\} = \bar{a}_i^T x + \|P_i^T x\|_2$$

Por lo tanto, el problema original puede escribirse como:

$$\min_{x} \; c^T x$$

$$\text{s.t.} \quad \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i \quad \forall i = 1, \ldots, m$$

3. Métodos de restricción de cardinalidad

Los métodos de restricción de cardinalidad están basados en (Bertisimas and Sim, 2004) y pueden considerarse un tipo especial de conjuntos poliédricos. Una variable incierta $\tilde{a}_{ij}$ se modela como una variable simétrica y acotada entre: $\tilde{a}_{ij} = [a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$, donde a_{ij} representa el promedio de $\tilde{a}_{ij}$, y $\hat{a}_{ij}$ representa una medida de variación de $\tilde{a}_{ij}$. Como es lógico asumir, es muy poco probable que todos los parámetros inciertos cambien a su peor valor. Por lo tanto, para protegerse contra la violación de cada restricción i, introduciremos el parámetro Γ_i, que toma valores en el intervalo $[0, |\mathcal{J}_i|]$, de manera que nuestro problema se mantendrá factible si hasta $\lfloor \Gamma_i \rfloor$ coeficientes inciertos de la restricción i cambian a su peor valor. Aquí, $\mathcal{J}_i$ representa el conjunto de coeficientes inciertos en la restricción i. Además, si más de $\lfloor \Gamma_i \rfloor$ coeficientes cambian, nuestro problema robusto seguirá siendo factible con alta probabilidad. Por lo tanto, Γ_i controla el equilibrio entre la probabilidad de violación de la restricción y el efecto sobre la función objetivo. A medida que Γ_i aumenta, el problema se vuelve más robusto, pero el resultado obtenido se aleja más de la solución óptima nominal.

Matemáticamente, este enfoque se formula de la siguiente manera:

$$\begin{aligned} \max \quad & c'x \\ \text{s.t.} \quad & \sum_j a_{ij}x_j + \max_{S_i \subseteq [t_i], |S_i| = \lfloor \Gamma_i \rfloor, t_i \in \mathcal{I}_i \setminus S_i} \left\{ \sum_{j \in S_i} \hat{a}_{ij} y_j + (\Gamma_i - \lfloor \Gamma_i \rfloor) \hat{a}_{it_i} y_{t_i} \right\} \leq b_i \\ & -y_j \leq x_j \leq y_j \\ & l \leq x \leq u \\ & y \geq 0 \end{aligned}$$

Después de algunas transformaciones, explicadas con más detalle en (Bertisimas and Sim, 2004), este problema resulta en el siguiente problema de programación lineal (PL):

$$\begin{aligned} \max \quad & c'x \\ \text{s.t.} \quad & \sum_j a_{ij}x_j + z_i\Gamma_i + \sum_{j \in J_i} p_{ij} \leq b_i && \forall i \\ & z_i + p_{ij} \geq \hat{a}_{ij} y_j && \forall i,\ j \in J_i \\ & -y_j \leq x_j \leq y_j && \forall j \\ & l_j \leq x_j \leq u_j && \forall j \\ & p_{ij} \geq 0 && \forall i,\ j \in J_i \\ & y_j \geq 0 && \forall j \\ & z_i \geq 0 && \forall i \end{aligned}$$

3 Motivación del proyecto

Como hemos visto anteriormente, existe una extensa literatura en técnicas de cuantificación de pobreza y de optimización robusta. No obstante, consideramos que no existe un método sistemático para combinar ambos mundos y crear un modelo de alocacion de capital para la eliminación de la pobreza. Además, la gran mayoría de proyectos destinados a reducir la pobreza dependen en gran medida de donaciones externas, pues siguen un modelo de negocio "sin ánimo de lucro". En nuestro caso, buscamos financiar parte del proyecto con deuda bancaria, que más tarde repagaremos con los beneficios extraídos sobre los intereses que pagan las familias participando en el programa. No obstante, cabe destacar que el objetivo fundamental es reducir la pobreza, por ello los intereses cobrados a las familias son mínimos. Respecto al modelo de optimización empleado, utilizamos métodos de optimización robusta, concretamente del tipo de restricción de cardinalidad que nos permiten tener en cuenta la tasa de incumplimiento en el repago de la deuda, sin ni siquiera saber cuál es la distribución de probabilidad de la misma. De esta forma nos aseguramos que la solución es factible, siempre y cuando las tasas de incumplimiento se mantengan dentro de un subconjunto que definiremos más adelante.

4 Metodología del proyecto

Como hemos mencionado, el grueso del proyecto pasa por la elaboración de un modelo de optimización lineal y robusto. En concreto, utilizamos el método explicado en el paper "The price of robustness". Respecto a los índices de pobreza, utilizaremos las técnicas de espacios difusos explicadas en el paper de Cheli y Lemmi de 1995, entre otros.

5 Recursos a emplear

Para obtener las soluciones del modelo utilizamos python. Y en concreto el solver de Gurobipy, muchos más rápido y eficiente que otros solvers.

A Robust Optimization Approach to Multidimensional Poverty Alleviation

Camilo Galvis, Soulaymane Kachani and Felipe Requejo

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Abstract

Traditional poverty alleviation strategies often rely on donations, limiting their scalability and long-term impact. This paper, introduces a self-sustaining investment strategy that, while primarily focused on poverty alleviation, simultaneously seeks to maximize return on investment. These returns are reinvested to extend the reach and effectiveness of the project. Our approach involves constructing a multidimensional poverty index using fuzzy set theory, evaluating the impact of poverty reduction initiatives, and formulating a robust capital allocation optimization model. This model accounts for uncertainty and ensures adaptability to varying conditions. The effectiveness of the model is demonstrated through a case study using data from Mexico.

Keywords: *Multidimensional Poverty, Robust Optimization, Poverty Measurement, Weighting Schemes, Poverty Alleviation, Capital Allocation*

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1. Introduction

As of fairly recently, poverty was defined almost exclusively in monetary terms and primarily in relation to certain income distributions for a given segment of society. However, most recently, poverty is increasingly understood as a multidimensional issue, often disentangling the economic factor from other key dimensions. For many governments and private organizations, poverty eradication is one of the main priorities on their agendas. Yet, despite these efforts, "Global poverty reduction has slowed to a near standstill, with 2020-2030 set to be a lost decade."

Our poverty alleviation strategy, "The Imperative Project", focuses on providing access to financial inclusion and wealth building for families in the poverty spectrum. This is achieved through a multidimensional intervention framework for the provision of financial literacy, along with support for the acquisition of a structural home in a properly developed community. Our goal is to create a self-sustaining model that minimizes reliance on external donations and generates sufficient return on investment (ROI), which can then be reinvested to further reduce poverty,the ultimate objective. Through the combination of public and private sector funding, we source capital to invest as optimally as possible. Here, "optimal" is defined as achieving the maximum poverty alleviation alongside with the highest ROI. In this framework, we work with families through a five-year program, during which, families are expected to repay a portion of the house's value, along with interest, which is capped at one-third of their income to ensure their quality of life throughout the program. By the end of the program, families will not only own a permanent and reliable home with access to basic utilities, goods and services, but they will also build a credit history, which integrates them into the formal financial system.

Our objective in this paper is to develop a capital allocation model that determines, out of a possible universe of communities exposed to material poverty in a specific country, in which communities to invest, how much and when to invest, over a ten-year period. To achieve this, we develop a model that quantifies poverty and also evaluates the impact of our project on reducing deprivation across each dimension. To quantify poverty, the first step is to determine our definition of poverty. In this paper, poverty is defined as the lack of inclusion to a formal economy and the lack of access to basic universal goods and services of high quality (i.e. we seek to alleviate material poverty). We believe material poverty can be defined as the lack of access to: 1) adequate housing, 2) electricity 3) potable water 4) low-cost internet connection 5) a comprehensive healthcare system. Our rationale for this definition is centered around the understanding of who should not be considered materially poor, rather than labeling individuals as poor based on other more subjective and or personal circumstances. That is, we argue, in general, that anyone with access to all five of these essential goods and services has the foundation necessary to grow out of or move away from material poverty. Our primary objective is to eliminate barriers to accessing these essential resources. Once these needs are met, the responsibility for overcoming poverty largely shifts to the individual. In essence, the provision of these goods creates an environment where people are equipped with the necessary tools to not only lift themselves out of poverty but also pursue inter-generational socio-economic development. Afterwards, and once we have defined poverty and measured poverty in each dimension, we utilize fuzzy set theory to create a "totally relative" index that measures an individual's deprivation based on their position within the overall distribution of deprivation in society. Subsequently, we construct a robust capital allocation model that accounts for uncertainty in the parameters, to ensure that the allocation of capital considers how uncertainty can modify the risks associated with the ability to repay the debt incurred at the beginning of the project. In particular, we incorporate a variable that measures the default rate in each community. Given the scarcity of data on this matter, robust optimization allows us to obtain a feasible solution under unfavorable values of this variable without requiring knowledge of its probability distribution. Additionally, we propose a model that allows for the adjustment of the degree of conservatism, that is, the distance from the robust solution to the nominal solution of our model. This flexibility is critical for two reasons. First, higher default rates hinder the project's ability to meet its financial obligations (i.e. repaying the debt). Second, higher default rates also reduce the number of families we can work with, thereby diminishing the overall impact on poverty alleviation.

The paper is structured as follows: Section 2 provides a literature review, which examines and justifies the selection of the techniques later on applied in our model. Here, in order to create a community specific poverty index, we review the existing literature on poverty measurement. Then, we explore various robust optimization techniques, which will be used later to protect the model against uncertainties in some of the parameters. Section 3 presents a detailed explanation of the model, including both the computation of poverty indices and the formulation of the robust optimization model. Finally, in Section 4, we present and analyze the results of our model in order to understand how our model is actually allocating capital.

2. Literature Review

In this section, we review the existing literature on two central themes pertinent to the development of the capital allocation model we aim to construct. These themes are: (1) the construction of a comprehensive poverty index for each community and (2) robust optimization methods. The former focuses on creating a poverty index tailored for community-level analysis, while the latter addresses how to incorporate uncertainty given the stochastic nature of certain variables within the model.

2.1. Poverty Measurement

Poverty measures, whether unidimensional or multidimensional, aim to create a population poverty index based on individual deprivation level assessments. Traditionally, due to the difficulty of measurement and the lack of comprehensive data, poverty has been measured unidimensionally, with income serving as the primary indicator. Under this approach, constructing a population-level poverty index involves two fundamental steps that remain relevant in modern multidimensional poverty calculations. The first step is the *identification of the poor*, which is typically done by establishing an income cut-off, below which individuals are considered to be in poverty. The second step is known as *aggregation*, which involves combining individual poverty measures into a single index that reflects the overall poverty level of the population. The simplest example of this is the Headcount Ratio, which indicates the percentage of individuals classified as poor within a population **(Blackwood and Lynch, 1994)**.

However, relying solely on income as an indicator presents several limitations. Two of the most notable are its failure to account for the distribution of goods and services, and the fact that government-provided goods and services do not require personal income for access (e.g., healthcare, education, and clean water supply). Given these drawbacks, and significantly influenced by **(Sen, 1976)**, there has been a growing shift towards the development of multidimensional poverty indices **(Duclos and Tiberti, 2016)**.

The shift from unidimensional to multidimensional poverty measurement significantly alters traditional methodologies. This transition requires adapting the measurement process to assess deprivation across multiple dimensions to then combine these deprivation measures into a multidimensional population index. As a result, the steps involved in multidimensional methodologies are modified accordingly. Most current approaches follow all or some of the following steps **(Betti et al. 2015)**, **(Duclos and Tiberti, 2016)**, **(Alkire and Foster, 2011)**.

1. Selection of the number and nature of dimensions
2. Calculation of deprivation scores for each dimension and individual
3. Selection of a weighting system for each dimension
4. Aggregation of individual deprivations to compute a population index

For our study, the first three steps are of particular importance. Our goal is to determine the optimal allocation of resources to maximize both poverty reduction and return on investment. Since our data is available at the community level rather than at individual level, we aim to derive a comprehensive community-level deprivation index (i.e. we have data on the average deprivation factors in a community, rather than data on the individuals in the community). Following the notation used so far, our multidimensional community index is equivalent to the multidimensional poverty index for each individual that makes up the population. Additionally, although it is not completely relevant to our current study, we will discuss in detail how different methodologies handle the final aggregation step. We consider understanding this step is crucial to comprehend literature on poverty indices as well as to gather ideas which can be used to compute our community-level indices. Furthermore, this discussion lays the groundwork for future research, as with the availability of individual-level data, our model would need to incorporate all four steps.

In **(Alkire and Foster, 2015)**, researchers conducted an exhaustive examination of poverty measurement methodologies, which serves as a foundation for our own research. Given the lack of standardized procedures and the broad array of techniques available, we select methodologies based on those that best fit our model. Our goal in this section is not to critique any specific technique but to summarize and justify the methodology we choose to use in our model. The most common methodologies are the following.

2.1.1. Aggregation and Dimension Selection Methods

1. Dashboard and Composite Indexes:

Dashboard techniques are considered dimension-specific poverty measures, as their ultimate goal is to compute the level of deprivation in each dimension separately. While dashboards provide a high level of information about different dimensions levels of deprivation, they cannot produce an aggregated measure that relates different dimensions to each other. For this reason, dashboard methods are often considered a preliminary stage for Composite Index methods, which do succeed in summarizing this information into a single value. The most well-known composite indexes are the Human Development Index (HDI), proposed in the UNDP's 1990 Human Development Report, and the Human Poverty Index (HPI), introduced in the UNDP's 1997 Human Development Report. Both methods first calculate population-level indices for each dimension and then aggregate these indices either through: a weighted average of dimension-specific indices, as in the case of the HDI, or through a weighted average of order alpha of the dimension-specific indices, as in the case of HPI. Although composite indices achieve a multidimensional poverty measure at the population level, their main shortcoming is that they treat an individual's deprivation in one dimension as independent of their deprivation in another. Consequently, they do not distinguish the degree of simultaneous deprivation that affects each individual.

2. Statistical Methods:

Statistical methods are widely used in literature as they are considered less problematic than other methods and can also serve as complements to other techniques. The main steps to compute a population poverty index using statistical methods are the following. First, from a population of n individuals and d dimensions, one creates an achievement matrix X of size $n \times d$, where the entry x_{ij} reflects the deprivation level of individual i on dimension j. The next step is to reduce the number of dimensions in this matrix from d to d', where $d' < d$. The most used techniques for this reduction are Principal Component Analysis (PCA), Multiple Correspondence Analysis (MCA), and Factor Analysis (FA). Following the dimensionality reduction, the goal is to create person-specific scores that combine deprivation levels across the d' dimensions into a single value. This can be achieved using various types of weighted averages, as seen in **(Rahman et al. 2011)** and **(Alkire and Foster, 2011)**, or through multivariate methods, as in **(Ballon and Krishnakumar, 2011)**. In the latter approach, authors employ two sequential factor models. First, a first-order factor model that assumes that the deprivation level in each dimension is latent and explained by a series of indicators, which aims to reduce the number of indicators providing information of a dimension. In second place, a second-order factor model assumes that the person-specific deprivation index is latent and explained by the d dimensions previously obtained.

In summary, statistical methods offer several advantages. They aid in dimensionality reduction, they can be used to complement other methods, and they enable the calculation of both individual-specific and population poverty indexes when multidimensional poverty is considered as a factor. However, given the current state of research, it remains unclear how these methods behave concerning the axioms poverty indexes must satisfy, making it difficult to interpret the resulting index. Furthermore, these processes leave considerable room for user discretion, and although thorough research on the impact of such choices on axioms could standardize these methods, such research is yet to be developed. Given these uncertainties, for our model we prefer to use techniques that are more developed, as explained in the following sections.

3. Axiomatic Methods:

These approaches aim to satisfy certain properties or axioms. Among them, the method proposed by **(Alkire and Foster, 2011)**, known as Counting and Multidimensional Poverty Measurement, stands out as it is the most used in literature. Due to its importance and its relevance for understanding the methodology used in this paper, we will now provide a detailed explanation of this method.

Let n represent the number of individuals in a population and $d \geq 2$ the number of dimensions. Let Y denote the $n \times d$ matrix of achievements, where y_{ij} represents the achievement of individual i in dimension j. Let z_j denote the cutoff below which a person is considered deprived in dimension j. If the achievement level y_{ij} for individual i in dimension j falls below the cutoff z_j, the individual is considered deprived in that dimension (i.e. if $y_{ij} < z_j$). Conversely, if the level is at least equal to the cutoff, the individual is not deprived in that dimension.

From the achievement matrix Y and the cutoff vector z, one can create a deprivation matrix g^0, where $g^0_{ij} = 0$ if $y_{ij} \geq z_j$ and $g^0_{ij} = 1$ if $y_{ij} < z_j$. In other words, the matrix g^0 takes the value 1 if individual i is deprived in dimension j, and 0 otherwise. Additionally, it is possible to assign weights w_j to each dimension and thus compute a weighted deprivation matrix g^0.

When all variables in y are cardinal, the associated matrix of normalized gaps or shortfalls can provide additional information for poverty evaluation. For any y_{ij}, let g^1 be the matrix of normalized gaps, where each element is defined as $g^1_{ij} = g^0_{ij} \cdot \frac{z_j - y_{ij}}{z_j}$ and measures the extent to which an individual is deprived in dimension j.

At this point, after determining the level of deprivation for each individual across each dimension, it is necessary to aggregate these measures to obtain a population-level poverty index. Four main techniques exist:

1. **The Unidimensional Method:** After determining the level of deprivation for each individual across each dimension, we calculate a multidimensional poverty score for each individual, usually using a weighted average. Then, we classify each person as poor or not poor based on a general cutoff.
2. **The Union Approach:** Under this method, a person is considered poor if they are deprived in at least one dimension.
3. **The Intersection Approach:** In this method, a person is considered poor if they are deprived in all dimensions.
4. **The Dual Cut-off Technique:** A person is considered poor if they are deprived in at least k dimensions, where $0 < k \leq d$.

Finally, once each person is classified as poor or not poor, the next step is to calculate the population-specific poverty index score. For our paper, the most notable technique is the unidimensional approach, as we aim to compute an index per community, which given the data we have and following the notation used, is equivalent to calculating individual-specific indices.

4. Fuzzy Set Approaches

Fuzzy set approaches emerged as a response to the binary classification inherent in traditional poverty measurement methodologies, where individuals are either classified as poor or not poor. A prominent example of this is the widely used **(Alkire and Foster, 2011)**, which requires setting two cut-offs to determine whether a person is deprived in a particular dimension. One of the main criticisms of such normative methods is the lack of a clear justification for these cut-offs. Moreover, applying cut-offs to continuous variables can be counterintuitive. For instance, consider a cut-off of 60% for access to electricity deprivation factor: a person with 59.9% access would be considered poor, while someone with 60.1% would not. Fuzzy approaches address this issue by introducing a membership function, $m_j(x_{ij})$, which determines the degree to which an individual belongs to a set; in this case, the set of deprived individuals in a given dimension. Mathematically, for an individual i, the membership level in a dimension j is expressed as $m_{ij} : \mathbb{R}^+ \rightarrow [0,1]$ for all j.

The fuzzy approach process **(Cheli and Lemmi, 1995)**, **(Filippone et al. 2001)** involves calculating the membership level for each individual in each dimension, then deriving an individual deprivation score, which is subsequently aggregated to compute a population-specific deprivation index. The key to this technique lies in selecting the most appropriate membership function, which depends heavily on the ultimate goal of the study. As reflected in **(Alkire and Foster, 2015)**, the main types of membership functions are: the linear, the trapezoidal and the sigmoid functions.

For our study, we found of particular interest the method proposed in **(Cheli and Lemmi, 1995)**. In this method, the membership function is defined as the cumulative distribution function of a given dimension j. Accordingly, the membership function for dimension j for individual i is defined as follows:

$$m_j(x_{i,j}) = \begin{cases} 1 & \text{if } x_{i,j} = \min(x_{.j}) \\ 1 - F_j(x_{i,j}) & \text{if } \min(x_{.j}) < x_{i,j} < \max(x_{.j}) \\ 0 & \text{if } x_{i,j} = \max(x_{.j}) \end{cases}$$

Where $F_j(\cdot)$ represents the cumulative density function of dimension j. Intuitively, for continuous dimensions, this approach assesses an individual's level of deprivation as the probability that someone else is more deprived in that dimension. This methodology is particularly valuable for our study, where the ultimate goal is to compute an index for each community in order to allocate resources based on how much poverty can be alleviated. Therefore, it is crucial that our index reflects how much worse off one community is compared to another. For example, if the average access to electricity rate across the population is 50% with a standard deviation of 0.1%, a community with 48% access is significantly worse off than one with 50%. By following this method, we obtain a measure that is relative to the entire distribution.

2.1.2. Overview of Weighting Schemes

As we have seen in the previous techniques, a fundamental aspect of poverty measurement is the calculation of dimension weights, especially in fuzzy approaches and in the Alkire-Foster method. Regarding the different weighting schemes, papers such as **(Decancq and Lugo, 2009)**, **(Belhadj, 2012)**, **(Cavapozzi et al. 2015)**, **(Filippone et al. 2001)** present well-documented sources that we summarize here.

Two major concerns in selecting weights are ensuring they satisfy the properties of discrimination and non-redundancy. First, the principle of discrimination suggests that when the proportion of a population deprived in a particular dimension is significantly low, the impact on an individual who is deprived in that particular dimension should be greater than in a situation where the proportion of deprived population is very high. Consequently, the weight assigned to a dimension with fewer deprived individuals should be higher than the weight assigned to a dimension with more deprived individuals (i.e., dimensions with a strong discriminatory power must have a larger weight). Second, with respect to non-redundancy, when computing an individual-specific index, it is crucial to aggregate the level of deprivation across dimensions while minimizing double counting. Since dimensions are likely to be correlated, it is important to use a weighting scheme that assigns lower weights to correlated dimensions.

According to the previous literature, different weighting schemes can be grouped into the following categories:

- **Equal Weights:** This is the most widely used technique in literature; however, it is also the most controversial because it relies on the unlikely assumption that all dimensions are equally important.
- **Weights based on proportions:** Weights are selected based on the proportion of the population suffering deprivation in that dimension, such that, the smaller the proportion of deprived individuals, the greater the weight. Therefore, although the techniques meet the discriminatory property, they fail to account for double counting.
- **Most favorable weights:** In this case, weights are based on the importance each dimension has for each individual in the population; consequently, this approach sets different weights for each individual.
- **Statistical Weights:** Within statistical weights, there are two main sets of techniques employed to choose weights: descriptive and explanatory models. Descriptive models, such as principal components and cluster analysis, are used to target the problem of double counting, as the new set of dimensions are uncorrelated between them. For explanatory models, also known as latent models, the most popular and simple technique is factor analysis, which assumes the presence of a non-observed variable which is explained by other observed variables. Main drawbacks with statistical weights arise from the fact that weights are not guaranteed to be set according to population priorities and trade-offs between dimensions. At the same time, these methods are complex and are hard to explain from a policy maker's perspective.
- **Regression Based Models:** The technique is based on selecting the weights as the coefficients in a linear regression using the dimensions as independent variables and a certain output of well-being (Y) as an independent variable. As comes logic, a major drawback is that if well-being can be captured by a single variable Y, then there is no need for computing a multidimensional poverty index.
- **Normative Weights:** This approach involves collecting data from various groups that aim to represent the entire population as accurately as possible. These groups typically include random samples, people from different socio-economic areas, or experts in academia and international policy. Each individual will rank and classify each dimension using various methods, then, the resulting outputs are used to compute the weights. For example, in the paper **(Kruijk and Rutten, 2007)**, researchers are able to compute the weights for each dimension by using the average ranking of dimensions and the following expression:

  $$w_j = \frac{1 + n_d - r_j}{\sum_{j=1}^{n_d} \left(1 + n_d - r_j\right)}$$

  where n_d represents the number of dimensions and r_j is the ranking of dimension j, with 1 being the most important dimension and n_d the least.

  Most concerns with these methods stem from the fact that participatory techniques may not always represent the entire population accurately, and the formulation of questions can significantly influence the results. Despite these concerns, the meaning of the weights in this approach closely aligns with their intrinsic purpose; reflecting the importance assigned to each dimension by the population. Thus, with appropriate normative methodologies, this path offers valuable insights. However, due to the lack of authority and a generally accepted method, these techniques are often rejected.

Overall, given the lack of consensus on the weighting scheme, the choice of a weighting scheme ultimately rests with the researcher. For the purpose of our paper, we find that statistical techniques and those commonly used in fuzzy theory are the most appropriate. However, given the drawbacks of traditional statistical techniques as those explained previously, we rely on the method known as the *"Prevalence Correlation Method"*, officially adopted by **(Eurostat, 2002)** and previously outlined in the papers **(Betti and Verma, 1998)** and **(Betti et al. 2015)**. Here, the weight w_k given to a dimension k is computed according to the following expression:

$$w_k = w_k^a \cdot w_k^b$$

where w_k^a is set to target the discriminatory problem, and is calculated as the coefficient of variation V_k. And where w_k^b targets the redundancy issue and is calculated as:

$$w_k^b \propto \left(\frac{1}{1 + \sum_{k'}^{K} \rho_{k,k'} \mid \rho_{k,k'} < \rho_H} \right) \cdot \left(\frac{1}{\sum_{k'}^{K} \rho_{k,k'} \mid \rho_{k,k'} \geq \rho_H} \right)$$

where $\rho_{k,k'}$ is the correlation between dimensions k and k' and where ρ_H is the value of the correlations between dimensions with the largest gap in the set of ordered correlations. As we can infer from the expression above, the left term contains the sum of all correlations with a value less than ρ_H, while the right term contains the sum of all correlations with a value greater or equal to ρ_H. The motivation for this model is that the weight w_k^b is not modified by the introduction of an uncorrelated dimension, but it is reduced in proportion to the number of highly correlated dimensions and therefore avoids double counting in the aggregation of dimensions to compute individual-specific deprivation indices. In **(Filippone et al. 2001)**, we found other important weighting schemes used in fuzzy approaches, these techniques are:

$$w_d = \ln\left(\frac{1}{P_d}\right)$$

$$w_d = 1 - \sqrt{P_d}$$

$$w_d = e^{P_d}$$

where P_d refers to the proportion of population deprived in dimension d. However, although all these methods meet the discriminatory principle, they lack to consider the redundancy impact and therefore, we prefer to use the *Prevalence Correlation Method*, previously explained.

2.2. Robust Optimization Methods

One of the major focuses in the mathematical programming community has been the incorporation of parameter uncertainty into optimization problems. A primary approach to this is stochastic optimization, where uncertainty is modeled through probability distributions. In this approach, the goal is to find a solution which is protected to stochastic uncertainty with some probabilistic sense. In contrast, robust optimization (RO) represents uncertainty through a deterministic set of possible realizations. Instead of assigning probabilities to each possible state, RO assumes that uncertainty can take any value within a specific, predefined set. The objective in RO is to ensure that the solution remains feasible regardless of how the uncertainty manifests, as long as it stays within the defined bounds.

In the works of **(Bertsimas et al. 2011)**, **(Ben-Tal and Nemirovski, 2008)** and in **(Ben-Tal, Ghaoui and Nemirovski, 2009)** , the authors provide a comprehensive study of optimization techniques, which serves as a foundation for our own review. Building on their thorough analysis, we focus on the robust linear optimization techniques that are most relevant and applicable to the problem at hand.

The robust counterpart of a linear optimization problem can be expressed, without loss of generality, as:

$$\min\ c^t x$$
$$\text{s.t.}\quad a_i^t x \le b_i,\quad \forall a_i \in U_{a_i},\ \forall b_i \in U_{b_i},\ i = 1,\dots,m$$

Where $U_{a_i} \subseteq \mathbb{R}^n$ and $U_{b_i} \subseteq \mathbb{R}$ are given uncertainty sets.

In linear robust optimization, there are three main methods to define uncertainty sets.

1. Robust LP with Polytopic Uncertainty

This is a special case of the previous problem when U_{a_i} and U_{b_i} are polyhedral, such that:

$$U_{a_i} = \{a_i \mid D_i a_i \le d_i\}$$

where $D_i \subseteq \mathbb{R}^{k_i \times n}$ and $d_i \subseteq \mathbb{R}^{k_i}$ are given as inputs. Uncertainty on b_i is given as an interval and therefore, we can get rid of U_{b_i} as the worst case is just the lower end value of the interval. This transforms the original problem to the following, where we re-use notation using again b_i although it really means $\min\{b_{i,\min}, b_{i,\max}\}$.

$$\min\ c^t x$$
$$\text{s.t.}\quad a_i^t x \le b_i,\quad \forall a_i \in U_{a_i}$$

Where $U_{a_i} = \{a_i \mid D_i a_i \le d_i\}$. Moreover, the previous LP, is equivalent to:

$$\min_x\ c^t x$$
$$\text{s.t.}\quad \begin{bmatrix} \max_{a_i}\{a_i^t x\} \\ D_i a_i \le d_i \end{bmatrix} \le b_i,\quad \forall i = 1,\dots,m$$

Intuitively, the latter inner optimization problem, seeks to find the value of $a_i^T x$ that makes harder to satisfy the constraints specified by the uncertainty set U_{a_i} for each constraint.Next, because of strong duality, we substitute the inner maximization problem of the previous LP by it´s dual LP.

$$\min_x\ c^t x$$
$$\text{s.t.}\quad \begin{bmatrix} \min_{p_i}\{p_i^t x\} \\ D_i^t p_i = x \\ p_i \ge 0 \end{bmatrix} \le b_i,\quad \forall i = 1,\dots,m$$

Which is equivalent to:

$$\min_{x,p_i}\ c^t x$$
$$\text{s.t.}\quad p_i^t d_i \le b_i \quad \forall i = 1,\dots,m$$
$$D_i^t p_i = x \quad \forall i = 1,\dots,m$$
$$p_i \ge 0 \quad \forall i = 1,\dots,m$$

In our problem, where the variables are independent, matrices D_i and d_i are straightforward. Matrix D_i is designed to capture the individual interval constraints for each variable without considering interactions between variables, therefore each row of D corresponds either to a lower or upper bound of single variable.

2. Robust LP with Ellipsoidal Uncertainty

Now U_{a_i} is an ellipsoid, defined as:

$$U_{a_i} = \left\{\bar{a}_i + P_i u \mid \|u\|_2 \le 1\right\},\quad i = 1,\dots,m$$

where $P_i \in \mathbb{R}^{n\times n}$ and $\bar{a}_i \in \mathbb{R}^n$, are given as input. Such that if $P_i = I$, then the uncertainty set U_{a_i} becomes a sphere and if $P_i = 0$, there is no uncertainty. To account for the ellipsoidal uncertainty, we reformulate the original LP as follows:

$$\min_x\ c^T x$$
$$\text{s.t.}\quad \begin{bmatrix} \max_{a_i}\{a_i^T x\} \\ a_i \in U_{a_i} \end{bmatrix} \le b_i,\quad \forall i = 1,\dots,m$$

Where the inner maximization problem has an explicit solution, such that:

$$\max\left\{a_i^T x \mid a_i \in U_{a_i}\right\} = \bar{a}_i^T x + \max\left\{u^T P_i^T x \mid \|u\|_2 \le 1\right\} = \bar{a}_i^T x + \|P_i^T x\|_2$$

And therefore, the original robust LP, can be written as:

$$\min_x\ c^T x$$
$$\text{s.t.}\quad \bar{a}_i^T x + \|P_i^T x\|_2 \le b_i \quad \forall i = 1,\dots,m$$

Which is an SOCP and consequently has a tractable solution.

3. Cardinality constrained methods

Cardinality constraints methods are based on **(Bertisimas and Sim, 2004)** and can be seen as a special type of polyhedral sets. An uncertain variable $\tilde{a}_{ij}$, is modeled as a symmetric and bounded variable between: $\tilde{a}_{ij} = [a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$, where a_{ij} represents the average of $\tilde{a}_{ij}$, and $\hat{a}_{ij}$ represents some measure of variation of $\tilde{a}_{ij}$. As it is logical to assume, it is very unlikely that all uncertain parameters will change to their worst value, and therefore to protect against the violation of every constraint i, we will introduce the parameter Γ_i, which takes values in the interval $[0, |\mathcal{J}_i|]$, such that our problem will remain feasible if up to $\lfloor\Gamma_i\rfloor$ uncertain coefficients of constraint i change to their worst value. Where, $\mathcal{J}_i$ represents the set of uncertain coefficients in constraint i. Moreover, if more than $\lfloor\Gamma_i\rfloor$ coefficients change, our robust problem will still be feasible with high probability. Therefore, Γ_i controls the trade off between the probability of constraint violation and the effect on the objective function. As Γ_i increases, the problem becomes more robust, but the result obtained is further from the nominal optimal solution. Mathematically, this approach is formulated as follows:

$$\max_x\ c^T x$$

$$\text{s.t.}\quad \sum_j a_{ij}x_j + \max_{S_i \subseteq [t_i], |S_i| = \lfloor\Gamma_i\rfloor, t_i \in \mathcal{J}_i \setminus S_i} \left\{\sum_{j\in S_i} \hat{a}_{ij} y_j + (\Gamma_i - \lfloor\Gamma_i\rfloor)\hat{a}_{it_i} y_{t_i}\right\} \le b_i$$
$$-y_j \le x_j \le y_j$$
$$l \le x \le u$$
$$y \ge 0$$

Which after some transformations, more detailed on **(Bertisimas and Sim, 2004)**, results on the following LP:

$$\max_x\ c' x$$

$$\begin{aligned}
\text{s.t.}\quad & \sum_j a_{ij}x_j + z_i\Gamma_i + \sum_{j\in J_i} p_{ij} \le b_i && \forall i \\
& z_i + p_{ij} \ge \hat{a}_{ij} y_j && \forall i,\ j \in J_i \\
& -y_j \le x_j \le y_j && \forall j \\
& l_j \le x_j \le u_j && \forall j \\
& p_{ij} \ge 0 && \forall i,\ j \in J_i \\
& y_j \ge 0 && \forall j \\
& z_i \ge 0 && \forall i
\end{aligned}$$

3. Model Formulation

In this section, we outline the formulation of the optimization model based on the methodologies identified as most suitable in the preceding section. The goal of our problem is to find an optimal allocation of funds that maximizes the net present value of poverty alleviation and the economic benefits obtained. For this purpose, we develop a robust dual optimization algorithm, which uses the e-constraint method **(Mavrotas, 2009)**, to deal with the dual objective nature of our problem. By the conclusion of the fund's investment horizon of 10 years, the goal is to recuperate the initial $130 million allocated for investment at the program's outset, augmented by the accrued interest on the incurred debt. Each household will make quarterly payments for a period of 5 years; nevertheless, our model will take the simplifying assumption that payments are made at the end of each year.

The section begins with an explanation on the derivation of poverty indices at the community level, which are key for the definition of the objective function of the model at hand and continues with the formulation of the robust optimization model that we use to allocate the available capital for investment.

3.1. Objective Formulation: Maximization of Poverty Alleviation

In order to construct the objective function of the optimization model, we first need to compute community-specific poverty indexes. This process involves the following steps:

1. Selection of the number and nature of dimensions
2. Calculation of deprivation scores for each dimension and individual
3. Selection of a weighting scheme for each dimension
4. Aggregation of dimension-specific deprivations to compute an individual-specific deprivation index

Given the definition of poverty outlined in the introduction chapter, it is not necessary to perform statistical techniques to reduce or select the number of dimensions in step 1. Nevertheless, these techniques also provide uncorrelated dimensions and therefore solve the problem of double counting. As we choose not to use these techniques, we will need to use a weighting scheme with the capability of reducing double counting.

For step 2, we have selected the totally fuzziness method as outlined in **(Cheli and Lemmi, 1995)** and **(Filippone et al. 2001)**. As explained in the literature review section, by using this method we aim to determine how much worse off is one community with respect to the entire data set, while considering the probability distribution of the deprivations levels. Given this objective, we believe it is not necessary to find a parametric fit to our data, and therefore, we compute the empirical cumulative distribution function for each dimension by assigning to each value it´s percentile score. Then, after the computation of each CDF, we determine the level of deprivation for each dimension and for each community according to the following expression:

$$m_d(x_{i,d}) = \begin{cases} 0 & \text{if } x_{i,d} = \min(x_{.,d}) \\ F_d(x_{i,d}) & \text{if } \min(x_{.,d}) < x_{i,d} < \max(x_{.,d}) \\ 1 & \text{if } x_{i,d} = \max(x_{.,d}) \end{cases}$$

Where $F_d(.)$ is the cumulative density function for dimension d.

Step 3 entails the computation of dimension weights, for which we use the Prevalence Correlation Method. With this method, we ensure that: 1) dimensions with a stronger discrimination capacity have a larger weight, and 2) that highly correlated dimensions are not double counted. Using the expressions outlined in the previous chapter to compute the weight for dimension, we obtain the following values.

Table 1. Weights assigned to each dimension

Dimension	Weight
Access to Housing	0.2256
Access to Electricity	0.3027
Access to Water and Sanitation	0.2372
Access to Internet	0.1133
Access to Healthcare	0.1212

Subsequently, we compute the poverty deprivation index for a community i (PDI_i), as a weighted sum of the deprivation levels in each of the 5 dimensions:

$$PDI_i = \sum_{d=1}^{D} w_d * m_{i,d} \quad \forall i \in \{1, \dots, N\}$$

Where w_d is the weight of dimension d, $m_{i,d}$ is the deprivation level of community i in dimension d, and N is the number of communities in our dataset. In addition, given that our model aims to alleviate poverty, we need to determine how our project can reduce the poverty index of a community. To do this, we assess the impact of our project on each poverty dimension and introduce a new term to the previously defined poverty index. This new term accounts for the project's contribution to reducing deprivation in each dimension, modifying the overall index accordingly.The poverty index for each community (PDI_i) is then updated to reflect the project's impact on each dimension. The updated index is defined by the following expression:

$$PDI'_i = \sum_{d=1}^{D} w_d \cdot m_{i,d} \cdot (1 + \alpha_d) \quad \forall i \in \{1, \dots, N\}$$

Where $\alpha_d \in [0, 1]$ reflects the extent to which our project can alleviate the deprivation level of dimension d. Specifically, given the nature of our project, we use the following α_d.

Table 2. Power to alleviate deprivation assumed for each dimension

Dimension	α_d
Access to Housing	1
Access to Electricity	1
Access to Water and Sanitation	0.5
Access to Internet	0.5
Access to Healthcare	0.25

Finally, the objective function is defined as follows:

$$\max \sum_{t=1}^{T} \sum_{i=1}^{N} \frac{PDI'_i \cdot x_{i,t}}{(1 + r)^t}$$

Where, r represents the discount factor employed for poverty alleviation. As will be clear later, the objective function and constraints do not inherently support an early investment bias. To address this, we introduce a discount factor, which like a discount interest rate, assigns higher values to early investments. $x_{i,t}$ is an integer variable representing the number of households constructed in community i in year t. Lastly, T denotes the maximum allowable investment horizon, set at 10 years.

It is important to note this objective function is Pareto efficient, as no agent can be made better off without harming someone else. Relating to fair division literature, our objective function can be seen as a maximization of a linear social welfare function, which is widely known to be Pareto-efficient.

3.2. Optimization Model Formulation

Following the formulation of the objective function, we outline the constraints that define the problem. The problem possesses a dual nature: our primary aim is to maximize poverty alleviation while simultaneously ensuring a return on investment that enables us to repay the debt and reinvest in other communities, thereby further increasing poverty alleviation efforts. It is crucial to emphasize that while maximizing return on investment serves as a tool to achieve our overarching goal of poverty alleviation, it is not our primary focus. Therefore, in alignment with the structure of the e-constraint method, our objective function will prioritize maximizing poverty alleviation while we will incorporate a constraint that specifies the minimum profit required by the end of the program's profitability period. This constraint will increment progressively, delineating a series of pareto efficient solutions. Each household will contribute quarterly payments over a five-year period. Consequently, while investments may continue until year T, the fund will generate passive income until year $T+5$. Thus, through the sequential enhancement of the imposed constraint, we aim to maximize the total benefits accrued during the $T+5$ years of the program.

The capital available at the beginning of year $T+5$, is computed as follows:

$$B_{T+5} = B_{T+4} + Rev_{T+4}$$

Where, B_{T+4} represents the capital available at the beginning of year $T+4$ and Rev_{T+4} reflects the revenue generated throughout the year $T+4$. Based on this, the first constraint of our model will be:

$$B_{T+4} + Rev_{T+4} \geq \beta \cdot (B_{\max} - B_{\min}) + B_{\min} \quad (1.1)$$

Where, $B_{\max}$ represents the maximum potential capital at year $T+5$ achievable from our portfolio. And is computed by solving the model referred to in this paper as LP2. $B_{\min}$ represents the minimum capital achievable at year 15, and is computed by solving the model LP1, but setting right-hand side of equation (1.1) to zero. Finally, β is a coefficient taking values in the interval $[0,1]$, and that we will vary in increments of 1% in order to plot a set of points in between of $B_{\max}$ and $B_{\min}$.

The revenue made through a given year t is calculated as follows:

$$Rev_t = \sum_{i=1}^{N} P_i \cdot m_{i,t} \cdot (1-\pi_i) \quad \forall\, t \in \{1,\dots,T+4\} \quad (1.2)$$

Where, P_i accounts for the amount paid yearly per household on community i, π_i represents the average default rate of community i, and $m_{i,t}$ is an integer variable indicating the number of households that still have outstanding balances with the fund during year t. Following on, we define the capital available at the beginning of period t, according to the next expression:

$$\begin{aligned} B_t &= B_{t-1} + \text{Rev}_{t-1} - \textstyle\sum_{i=1}^{N} C_i \cdot x_{i,t} && \forall t \in \{1,\dots,T\} \quad (1.3)\\ B_t &= B_{t-1} + \text{Rev}_{t-1} && \forall t \in \{T+1,\dots,T+5\} \quad (1.4)\end{aligned}$$

Also note that every year we cannot invest more than the capital have available, therefore, for the first T-1 years, we introduce the following constraints:

$$\begin{aligned} &\textstyle\sum_{i=1}^{N} C_i \cdot x_{i,0} \leq B_0 \quad (1.5)\\ &\textstyle\sum_{i=1}^{N} C_i \cdot x_{i,t} \leq B_{t-1} + \text{Rev}_{t-1} \quad \forall t \in \{2,\dots,T-1\} \quad (1.6)\end{aligned}$$

In year 10, in addition to any investment we decide to make, we are also required to pay the loan received (130M$), plus the accrued interests on debt. Therefore, for year 10 , we add the following constraint:

$$\sum_{i=1}^{N} C_i \cdot x_{i,T} + (B_0 + I) \leq B_{T-1} + \text{Rev}_{T-1} \quad (1.7)$$

Due to practicality reasons during the construction and monitoring phases of the program, we will establish a minimum number of households to be built in case we decide to invest in a certain community i. This, is expressed as follows:

$$\begin{aligned} x_{i,t} &\geq MH \cdot y_{i,t} && \forall i,t \quad (1.8)\\ x_{i,t} &\leq H_i \cdot y_{i,t} && \forall i,t \quad (1.9)\end{aligned}$$

Where, MH is a constant representing the minimum number of households to construct in case of investment, which we have set to 80 and H_i accounts for the total number of households in community i. $y_{i,t}$ is a binary variable, taking value one if there is an investment in community i in year t and zero otherwise.

To invest in as many different communities as possible, we add the following constraint, which will not allow our model to reinvest in the same community.

$$\sum_{t=1}^{T} y_{i,t} \leq 1; \forall i \in \{1,\dots,N\} \quad (1.10)$$

Finally, we recall that to calculate the revenue made through a given year t, we introduced the variable $m_{i,t}$, which reflects the number of communities owing money to the fund on year t. To code for the desire behavior of this variable, we introduce the following constraints.

$$\begin{aligned} m_{i,t} &\geq \textstyle\sum_{l=1}^{4} x_{i,t} && \forall t \in \{1,\dots,4\}, \forall i \in \{1,\dots,N\} \quad (1.11)\\ m_{i,t} &\leq \textstyle\sum_{l=1}^{4} x_{i,t} && \forall t \in \{1,\dots,4\}, \forall i \in \{1,\dots,N\} \quad (1.12)\\ m_{i,t} &\geq \textstyle\sum_{l=1}^{4} x_{i,t-l} && \forall t \in \{5,\dots,T\}, \forall i \in \{1,\dots,N\} \quad (1.13)\\ m_{i,t} &\leq \textstyle\sum_{l=1}^{4} x_{i,t-l} && \forall t \in \{1,\dots,T\}, \forall i \in \{1,\dots,N\} \quad (1.14)\\ m_{i,t} &\geq \textstyle\sum_{l=t-4}^{T} x_{i,t} && \forall t \in \{T+1,\dots,T+4\}, \forall i \in \{1,\dots,N\} \quad (1.15)\\ m_{i,t} &\leq \textstyle\sum_{l=t-4}^{T} x_{i,t} && \forall t \in \{T+1,\dots,T+4\}, \forall i \in \{1,\dots,N\} \quad (1.16)\end{aligned}$$

Finally, putting everything together we obtain the following model (LPI) :

$$\max \sum_{t=1}^{T}\sum_{i=1}^{N} \frac{PDI'_i \cdot x_{i,t}}{(1+r)^t}$$

s.t:

$$\begin{aligned} &B_{T+4} + \text{Rev}_{T+4} \geq \beta \cdot (B_{\max} - B_{\min}) + B_{\min}\\ &\text{Rev}_t = \textstyle\sum_{i=1}^{N} P_i \cdot m_{i,t} \cdot (1-\pi_i) && \forall t\in\{1,\dots,T+4\}\\ &B_t = B_{t-1} + \text{Rev}_{t-1} - \textstyle\sum_{i=1}^{N} C_i \cdot x_{i,t} && \forall t\in\{1,\dots,T\}\\ &B_t = B_{t-1} + \text{Rev}_{t-1} && \forall t\in\{T+1,\dots,T+5\}\\ &\textstyle\sum_{i=1}^{N} C_i \cdot x_{i,0} \leq B_0\\ &\textstyle\sum_{i=1}^{N} C_i \cdot x_{i,t} \leq B_{t-1} + \text{Rev}_{t-1} && \forall t\in\{2,\dots,T-1\}\\ &\textstyle\sum_{i=1}^{N} C_i \cdot x_{i,T} + (B_0 + I) \leq B_{T-1} + \text{Rev}_{T-1}\\ &x_{i,t} \geq MH \cdot y_{i,t} && \forall i,t\\ &x_{i,t} \leq H_i \cdot y_{i,t} && \forall i,t\\ &\textstyle\sum_{t=1}^{T} y_{i,t} \leq 1 && \forall i\in\{1,\dots,N\}\\ &m_{i,t} \geq \textstyle\sum_{l=1}^{4} x_{i,t} && \forall t\in\{1,\dots,4\},\forall i\in\{1,\dots,N\}\\ &m_{i,t} \leq \textstyle\sum_{l=1}^{4} x_{i,t} && \forall t\in\{1,\dots,4\},\forall i\in\{1,\dots,N\}\\ &m_{i,t} \geq \textstyle\sum_{l=1}^{4} x_{i,t-l} && \forall t\in\{5,\dots,T\},\forall i\in\{1,\dots,N\}\\ &m_{i,t} \leq \textstyle\sum_{l=1}^{4} x_{i,t-l} && \forall t\in\{1,\dots,T\},\forall i\in\{1,\dots,N\}\\ &m_{i,t} \geq \textstyle\sum_{l=t-4}^{T} x_{i,t} && \forall t\in\{T+1,\dots,T+4\},\forall i\in\{1,\dots,N\}\\ &m_{i,t} \leq \textstyle\sum_{l=t-4}^{T} x_{i,t} && \forall t\in\{T+1,\dots,T+4\},\forall i\in\{1,\dots,N\}\\ &x_{i,t} \text{ integer}, \quad y_{i,t} \in \{0,1\}, \quad m_{i,t} \text{ integer}\end{aligned}$$

The model ($LP2$) used to compute the maximum capital achievable at year 15, results as follows:

$$\max\{B_{T+4} + \text{Rev}_{T+4}\}$$

s.t:

$$
\begin{aligned}
&Rev_t = \textstyle\sum_{i=1}^{N} P_i \cdot m_{i,t} \cdot (1-\pi_i) && \forall t\in\{1,\dots,T+4\}\\
&B_t = B_{t-1} + \text{Rev}_{t-1} - \textstyle\sum_{i=1}^{N} C_i \cdot x_{i,t} && \forall t\in\{1,\dots,T\}\\
&B_t = B_{t-1} + \text{Rev}_{t-1} && \forall t\in\{T+1,\dots,T+5\}\\
&\textstyle\sum_{i=1}^{N} C_i \cdot x_{i,0} \le B_0 &&\\
&\textstyle\sum_{i=1}^{N} C_i \cdot x_{i,t} \le B_{t-1} + \text{Rev}_{t-1} && \forall t\in\{2,\dots,T-1\}\\
&\textstyle\sum_{i=1}^{N} C_i \cdot x_{i,T} + (B_0 + I) \le B_{T-1} + \text{Rev}_{T-1} &&\\
&x_{i,t} \ge MH \cdot y_{i,t} && \forall i,t\\
&x_{i,t} \le H_i \cdot y_{i,t} && \forall i,t\\
&\textstyle\sum_{t=1}^{T} y_{i,t} \le 1 && \forall i\in\{1,\dots,N\}\\
&m_{i,t} \ge \textstyle\sum_{l=1}^{4} x_{i,t} && \forall t\in\{1,\dots,4\},\forall i\in\{1,\dots,N\}\\
&m_{i,t} \le \textstyle\sum_{l=1}^{4} x_{i,t} && \forall t\in\{1,\dots,4\},\forall i\in\{1,\dots,N\}\\
&m_{i,t} \ge \textstyle\sum_{l=1}^{4} x_{i,t-l} && \forall t\in\{5,\dots,T\},\forall i\in\{1,\dots,N\}\\
&m_{i,t} \le \textstyle\sum_{l=1}^{4} x_{i,t-l} && \forall t\in\{1,\dots,T\},\forall i\in\{1,\dots,N\}\\
&m_{i,t} \ge \textstyle\sum_{l=t-4}^{T} x_{i,t} && \forall t\in\{T+1,\dots,T+4\},\forall i\in\{1,\dots,N\}\\
&m_{i,t} \le \textstyle\sum_{l=t-4}^{T} x_{i,t} && \forall t\in\{T+1,\dots,T+4\},\forall i\in\{1,\dots,N\}
\end{aligned}
$$

$$x_{i,t} \text{ integer}, \quad y_{i,t} \in \{0,1\}, \quad m_{i,t} \text{ integer}$$

3.3. Robust Optimization Model Formulation

The solution obtained in our problem will depend on the average default rate of each community i (π_i). Since this default rate is a random variable, we propose the following robust problem, according to which we are willing to accept a near-optimal solution protected against possible changes in the uncertain parameters.

Following the methodology proposed in **(Bertisimas and Sim, 2004)**, we will assume that $\tilde{\pi}_i$ follows a symmetric distribution bounded between the following values: $\tilde{\pi}_i = [\pi_i - \hat{\pi}_i, \pi_i + \hat{\pi}_i]$, where π_i represents the average of $\tilde{\pi}_i$, and $\hat{\pi}_i$ represents some measure of variation of $\tilde{\pi}_i$. As it is logical to assume, we believe that not all uncertain parameters will change to their worst value and therefore to protect against the violation of every constraint i, we will introduce the parameter Γ_i, which takes values in the interval $[0, |\mathcal{J}_i|]$, such that our problem will remain feasible if up to $\lfloor \Gamma_i \rfloor$ uncertain coefficients of constraint i change to their worst value $[\pi_i + \hat{\pi}_i]$ (highest default rates). And where $\mathcal{J}_i$ represents the set of uncertain coefficients in constraint i. Moreover, if more than $\lfloor \Gamma_i \rfloor$ coefficients change, our robust problem will still be feasible with high probability.

Therefore, Γ_i controls the trade off between the probability of constraint violation and the effect on the objective function. As Γ_i increases, the problem is more robust, however the result obtained is further from the nominal optimal solution.

To simplify the formulation of the robust problem, we have chosen to parameterize the model LP1 beforehand, expressing all constraints in terms of the variables $m_{i,t}$, $x_{i,t}$, $y_{i,t}$, and the original loan received B_0. To this end, we need to plug constraints 1.2, 1.3, and 1.4 into constraints 1.1, 1.6, and 1.7, and then parameterize this last set of constraints. For intuition on the process, please refer to the following figure, where, we represent the cashflows obtained from an investment in year 1.

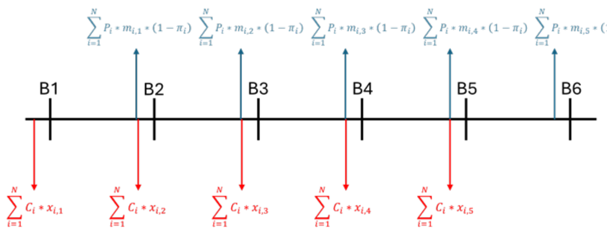

Figure 1. Revenues due to investments in year 1 and costs of future investments

Constraint 1.1 represents the capital available at year 15, consequently it can be computed as the sum of all cash inflows less the sum of all cash outflows up to year 15. Therefore, taking this into account and plugging constraint 1.2 into 1.1 we obtain the following expression:

$$-I + \sum_{t=1}^{T+4}\sum_{i=1}^{N} P_i \cdot m_{i,t} \cdot (1-\pi_i) - \sum_{t=1}^{T}\sum_{i=1}^{N} C_i \cdot x_{i,t} \ge \beta \cdot (B_{\max} - B_{\min}) + B_{\min}$$

Following the same logic, constraints 1.6 and 1.7 result in the following expressions respectively:

$$\textstyle\sum_{i=1}^{N} C_i \cdot x_{i,t} \le B_0 + \sum_{t=1}^{t-1}\sum_{i=1}^{N} \left(P_i \cdot m_{i,t} \cdot (1-\pi_i) - C_i \cdot x_{i,t}\right) \quad \forall t\in\{2,\dots,T-1\}$$

$$B_0 + I + \textstyle\sum_{i=1}^{N} C_i \cdot x_{i,T} \le B_0 + \sum_{t=1}^{T-1}\sum_{i=1}^{N} \left(P_i \cdot m_{i,t} \cdot (1-\pi_i) - C_i \cdot x_{i,t}\right)$$

Finally, the parameterized model ($LP1'$) results as follows:

$$\max \sum_{t=1}^{T}\sum_{i=1}^{N} \frac{SPI_i \cdot x_{i,t}}{(1+r)^t}$$

s.t :

$-I + \sum_{t=1}^{T+4}\sum_{i=1}^{N} P_i \cdot m_{i,t} \cdot (1-\pi_i) - \sum_{t=1}^{T}\sum_{i=1}^{N} C_i \cdot x_{i,t} \ge \beta \cdot (B_{\max} - B_{\min}) + B_{\min}$

$\sum_{i=1}^{N} C_i \cdot x_{i,0} \le B_0$

$\sum_{i=1}^{N} C_i \cdot x_{i,t} \le B_0 + \sum_{t=1}^{T-1}\sum_{i=1}^{N} P_i \cdot m_{i,t} \cdot (1-\pi_i) - C_i \cdot x_{i,t}; \quad \forall t \in \{2,\dots,T-1\}$

$B_0 + I + \sum_{i=1}^{N} C_i \cdot x_{i,T} \le B_0 + \sum_{t=1}^{T-1}\sum_{i=1}^{N} \left(P_i \cdot m_{i,t} \cdot (1-\pi_i) - C_i \cdot x_{i,t}\right)$

$$
\begin{aligned}
&x_{i,t} \ge MH \cdot y_{i,t} && \forall i,t\\
&x_{i,t} \le H_i \cdot y_{i,t} && \forall i,t\\
&\textstyle\sum_{t=1}^{T} y_{i,t} \le 1 && \forall i \in \{1,\dots,N\}\\
&m_{i,t} \ge \textstyle\sum_{l=1}^{4} x_{i,t} && \forall t \in \{1,\dots,4\}, \forall i \in \{1,\dots,N\}\\
&m_{i,t} \le \textstyle\sum_{l=1}^{4} x_{i,t} && \forall t \in \{1,\dots,4\}, \forall i \in \{1,\dots,N\}\\
&m_{i,t} \ge \textstyle\sum_{l=1}^{4} x_{i,t-l} && \forall t \in \{5,\dots,T\}, \forall i \in \{1,\dots,N\}\\
&m_{i,t} \le \textstyle\sum_{l=1}^{4} x_{i,t-l} && \forall t \in \{1,\dots,T\}, \forall i \in \{1,\dots,N\}\\
&m_{i,t} \ge \textstyle\sum_{l=t-4}^{T} x_{i,t} && \forall t \in \{T+1,\dots,T+4\}, \forall i \in \{1,\dots,N\}\\
&m_{i,t} \le \textstyle\sum_{l=t-4}^{T} x_{i,t} && \forall t \in \{T+1,\dots,T+4\}, \forall i \in \{1,\dots,N\}
\end{aligned}
$$

$$x_{i,t} \text{ integer}, \quad y_{i,t} \in \{0,1\}, \quad m_{i,t} \text{ integer}$$

Now that we have a parameterized version of the model, following the methodology on **(Bertisimas and Sim, 2004)**, we move on to formulate the robust version of the model. For this purpose, we need to transform all constraints with an uncertain parameter. The intuition behind the method involves introducing specific variations to each constraint, thereby increasing the difficulty of satisfying the constraint. This variation is defined, as the maximum variation achievable in a given set. This will be defined as follows for constraint 1.1:

$$\sum_{t=1}^{T+4}\sum_{i=1}^{N} P_i m_{i,t}(1-\pi_i) - \sum_{t=1}^{T}\sum_{i=1}^{N} C_i x_{i,t} - \max\{\sum_{j\in J_i} \hat{\pi}_i \cdot P_i \cdot |x_j^*| \cdot z_{ij}\} \ge B_{\min}$$

As default rates increase, the revenue we would make is lower than the revenue initially expected by our model, and therefore meeting constraint on the minimum capital required at year 15 constraint is harder. The same logic applies to constraints 1.6 and 1.7, which will be transformed to the following expressions:

$$\sum_{i=1}^{N} C_i x_{i,t} \le B_0 + \sum_{t=1}^{t-1}\sum_{i=1}^{N} P_i m_{i,t}(1-\pi_i) - C_i x_{i,t} - \max\{\sum_{j\in J_i} \hat{\pi}_i \cdot P_i \cdot |x_j^*| \cdot z_{ij}\} \quad \forall t \in \{2,\dots,T-1\}$$

$$\sum_{i=1}^{N} C_i x_{i,T} \le B_0 + \sum_{t=1}^{T-1}\sum_{i=1}^{N} P_i m_{i,t}(1-\pi_i) - C_i x_{i,t} - \max\{\sum_{j\in J_i} \hat{\pi}_i \cdot P_i \cdot |x_j^*| \cdot z_{ij}\}$$

Finally, we linearize the previous constraints, we obtain the following final robust model (LPR):

$$\max \sum_{t=1}^{T}\sum_{i=1}^{N} \frac{SPI_i \cdot x_{i,t}}{(1+r)^t}$$

$s.t$:

$- \sum_{t=1}^{T+4}\sum_{i=1}^{N}(P_i \cdot m_{i,t} \cdot (1-\pi_i)) + \sum_{t=1}^{T}\sum_{i=1}^{N} C_i \cdot x_{i,t} + z_1 \cdot \Gamma_1 + \sum_{j=1}^{N\cdot(T+4)} p_{1,j} \le -\beta \cdot (B_{\max} - B_{\min}) - B_{\min}$ (Robust Capital at Year $T+5$)

$\sum_{i=1}^{N} C_i \cdot x_{i,0} \le B_0$ (Budget at Year 1)

$\sum_{i=1}^{N} C_i \cdot x_{i,t} - \sum_{t=1}^{t-1}\sum_{i=1}^{N}(P_i \cdot m_{i,t} \cdot (1-\pi_i) - C_i \cdot x_{i,t}) + z_\gamma \cdot \Gamma_\gamma + \sum_{j=1}^{N\cdot(T+4)} p_{\gamma,j} \le B_0;\ \forall t \in \{2,\dots,T-1\}, \forall\gamma \in \{3,\dots,T\}$ (Robust Budget)

$\sum_{i=1}^{N} C_i \cdot x_{i,T} - \sum_{t=1}^{T-1}\sum_{i=1}^{N}(P_i \cdot m_{i,t} \cdot (1-\pi_i) - C_i \cdot x_{i,t}) + z_{11} \cdot \Gamma_{11} + \sum_{j=1}^{N\cdot(T+4)} p_{11,j} \le -I$ (Robust Budget at Year T)

$z_1 + p_{1,j} \ge (P_j \cdot \hat{\pi}_j) \cdot h_j; \quad \forall j \in \{1,\dots,N\cdot(T+4)\}$

$-h_j \le m_{i,j} \le h_j; \quad \forall j \in \{1,\dots,N\cdot(T+4)\}, \forall i \in \{1,\dots,N\}$

$z_\gamma + p_{\gamma,j} \ge (P_j \cdot \hat{\pi}_j) \cdot h_j; \quad \forall j \in \{1,\dots,N\cdot(T-1)\}, \forall\gamma \in \{3,\dots,11\}$

$x_{i,t} \ge MH \cdot y_{i,t} \quad \forall i,t$

$x_{i,t} \le H_i \cdot y_{i,t} \quad \forall i,t$

$\sum_{t=1}^{T} y_{i,t} \le 1 \quad \forall i \in \{1,\dots,N\}$

$m_{i,t} \ge \sum_{l=1}^{4} x_{i,t} \quad \forall t \in \{1,\dots,4\}, \forall i \in \{1,\dots,N\}$

$m_{i,t} \le \sum_{l=1}^{4} x_{i,t} \quad \forall t \in \{1,\dots,4\}, \forall i \in \{1,\dots,N\}$

$m_{i,t} \ge \sum_{l=1}^{4} x_{i,t-l} \quad \forall t \in \{5,\dots,T\}, \forall i \in \{1,\dots,N\}$

$m_{i,t} \le \sum_{l=1}^{4} x_{i,t-l} \quad \forall t \in \{1,\dots,T\}, \forall i \in \{1,\dots,N\}$

$m_{i,t} \ge \sum_{l=t-4}^{T} x_{i,t} \quad \forall t \in \{T+1,\dots,T+4\}, \forall i \in \{1,\dots,N\}$

$m_{i,t} \le \sum_{l=t-4}^{T} x_{i,t} \quad \forall t \in \{T+1,\dots,T+4\}, \forall i \in \{1,\dots,N\}$

$x_{i,t}$ integer, $\quad y_{i,t} \in \{0,1\}, \quad m_{i,t}$ integer $\quad z_i \ge 0, \quad p_{ij} \ge 0, \quad h_j \ge 0$

4. Results: A case study on Mexico

In this chapter, we analyze the results of both robust and non-robust models using a dataset from Mexico. The inputs for our model include the deprivation level across each of the five dimensions within each community, the initial bank debt borrowing of \$130 million, which must be repaid in year 10 along with accrued interest. The interest rate applied is 400 basis points over the 10-year Treasury note. Additionally, based on banking data from the country, we have set the default rate at 5%.

4.1. Non-Robust Results

Using the methodology outlined in the non-robust section, we first calculate the results for the minimum profit generation scenario. In this case, the goal is to maximize poverty while generating just enough cash flow to repay the loan. Additionally, we compute the outcomes for the revenue maximization scenario, where the objective is to maximize total revenues. The results for both scenarios are presented in the following table.

Table 3. Maximum Poverty Alleviation and Maximum Profit Generation Results

Results	Poverty Alleviation	Profit Generation
Profit generated (\$)	1,867,897,426	3,072,589,183
Poverty alleviation	327,661	226,228
Households constructed (#)	419,338	443,604
People involved in the program (#)	1,958,937	2,263,219
Invested Communities (#)	2,234	2,086
Average income per household (\$)	683.58	1010.10
Average poverty index	1.0468	0.6887

Building on this, we compute the outcomes for all intermediate points between the two previous cases by incrementally increasing the parameter β by 1%. The resulting frontier is shown in the following image.

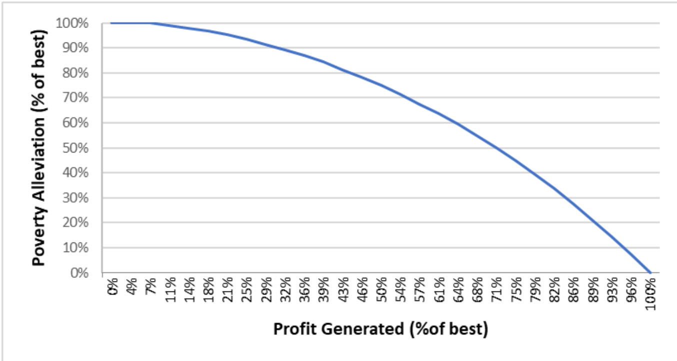

Figure 2. Poverty Alleviation and Profit Generation Frontier

As expected, since the poverty index and household income are negatively correlated, increasing the minimum profit required constraint leads to a significant reduction in poverty alleviation. Given that the primary objective of our strategy is to maximize poverty alleviation, we focus on the point of maximum poverty alleviation, which already yields a 19.44% compound annual growth rate (CAGR).

To further analyze the characteristics of the communities we invest in each year, we conducted two separate linear regressions. In the first regression, we used household income as the dependent variable and the year of investment as a categorical independent variable. In the second regression, we retained the year of investment as the categorical independent variable but used the poverty index as the dependent variable. The results for the base case and the two linear regressions are presented in the following table. Please note that values marked with three asterisks indicate a p-value of 0.001, two asterisks correspond to a p-value less than 0.01, and one asterisk represents a p-value less than 0.05. Therefore, the greater the number of asterisks, the higher the statistical significance of the value.

Table 4. Linear Regression Results: Income per Household ~ Year Invested and Poverty Index ~ Year Invested

Coefficient	Income per Household (\$)	Poverty Index
Intercept	942.692***	1.152***
Year 2	87.470*	-0.116***
Year 3	-30.321	-0.155***
Year 4	-28.986	-0.196***
Year 5	-18.073	-0.237***
Year 6	-24.156	-0.289***
Year 7	-34.373	-0.308***
Year 8	-313.864***	-0.113***
Year 9	-489.005***	0.029***
Year 10	-687.326***	0.094***

Examining both regressions and focusing only on statistically significant coefficients, we can identify the following patterns in the model's investment strategy. In Year 1, resources are allocated to communities that exhibit both high income and high poverty indices. The income regression indicates that in Year 1, the model prioritizes communities with relatively high income levels, as the income coefficient for Year 1 is only smaller than that of Year 2. Similarly, the poverty index regression shows that Year 1 investments target communities with high poverty levels, with this coefficient surpassed only in Years 9 and 10. The underlying rationale for this allocation strategy is tied to the fact that early poverty alleviations yields a greater impact. During the early years, the model seeks to balance poverty alleviation with generating sufficient funds for reinvestment. By prioritizing wealthier communities initially, it ensures higher financial returns, which can then support more targeted poverty alleviation efforts in later years. Thus, the model begins by investing in communities that combine high income and high poverty indices. In the final years, particularly in Years 9 and 10, these funds are directed towards communities with the highest poverty levels, maximizing the effectiveness of the poverty alleviation strategy.

4.2. Robust Results

As discussed in Chapter 2, our robust model is designed to address the default rates within the communities. This variable is critical, as it directly influences both our capacity to repay the debt and the number of individuals we can effectively support. However, accurately modeling default rates is particularly challenging due to the limited availability of data on this parameter. This data scarcity underscores the importance of adopting a robust formulation, which enables us to account for uncertainties in default rates without requiring precise knowledge of their probability distribution.

Moreover, the proposed methodology provides flexibility by allowing us to adjust the level of conservatism in the solution. This is achieved by selecting the maximum allowable variation and determining the maximum number of parameters permitted to deviate, ensuring a tailored balance between robustness and performance.

To show how these variations affect our allocation, we have conducted a sensitivity analysis on the number of parameters we allow to change (Γ) to their worst-case values, as well as the maximum variation permitted for these parameters ($\hat{\pi}$). We ran the model with Γ values ranging from 0 to 10,000, and for $\hat{\pi}$, we considered values of 0.005, 0.05, representing 10% and 100% deviations from the average default rates.

Table 5. Sensitivity analysis on robust parameters

Γ	$\hat{\pi}$	Poverty	Profit ($)	# Communities	Price of Robustness
0	0	327,661	1,867,897,426	2,234	0.00%
100	0.005	323,694	1,840,150,975	2,248	1.21%
500	0.005	318,435	1,812,133,708	2,228	2.81%
1000	0.005	313,961	1,789,812,148	2,194	4.18%
10,000	0.05	279,742	1,589,702,018	1,862	14.89%

When $\Gamma = 0$ or $\hat{\pi} = 0$, the results align exactly with those of the non-robust base case. However, as the number of parameters allowed to change to their worst-case values increases, or as the maximum variation in default rates grows, the allocation changes. Specifically, higher default rates lead to reduced profit generation and, consequently, diminished poverty alleviation. Thus, with each increase in either parameter, the poverty alleviation, profit generation, and the number of communities we work with are all reduced. This divergence from the original LP is commonly referred to in the literature as the *price of robustness*.

5. Conclusion

In conclusion, this paper introduces a novel approach to multidimensional poverty alleviation that integrates financial sustainability and impact maximization. As part of our self-sustaining investment strategy, we presented a step-by-step methodology to compute community-level multidimensional poverty indexes, which employs totally fuzzy and totally relative methods, ensuring that the degree of deprivation in a given dimension reflects an individual's relative position within the overall distribution of that dimension in society. Furthermore, we integrate poverty measurement with poverty alleviation methods by assessing how our poverty alleviation strategy can impact each community. Building upon these insights, we formulated a robust capital allocation optimization model that safeguards the model against parameter uncertainty and allows for flexibility in adjusting the degree of conservatism in the solutions, ensuring practical and reliable outcomes.

Finally, the results obtained in our case study in Mexico highlight the potential of our approach to drive meaningful alleviations in multidimensional poverty while achieving significant financial returns. Overall, we believe this research offers a practical tool for policymakers and practitioners aiming to create long-term, scalable solutions to one of the world's most pressing challenges.

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