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Phase-field model with concentrating-potential terms on the boundary

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ABSTRACT

In this paper we analyze a generalization of the semilinear phase field model from G. Caginalp (1986, 1991) and A. Jiménez-Casas-A. Rodriguez-Bernal (1996, 2005), where we consider a singular term concentrated in a neighborhood of Γ, the boundary of domain. The neighborhood shrinks to Γ as a parameter ϵ approaches zero.

We prove that this family of solutions, of the new semilinear phase field model, converges in suitable spaces when this parameter tends to zero, to the solutions of a semilinear phase field problem where the concentrating potential are transformed into an extra flux condition on Γ.

1. Introduction

There are several previous works about the phase field model (see [1–7]), given by the following semilinear parabolic system, which is known as the "phase-field equations" :

$$\begin{cases} \tau\varphi_t &= \xi^2\Delta\varphi - f(\varphi) + 2u \quad \text{in } (0,T)\times\Omega \\ u_t + \frac{l}{2}\varphi_t &= k\Delta u \quad \text{in } (0,T)\times\Omega \\ \frac{\partial\varphi}{\partial\vec{n}} &= 0 \text{ on } (0,T)\times\partial\Omega \\ \frac{\partial u}{\partial\vec{n}} &= 0 \text{ on } (0,T)\times\partial\Omega \\ \varphi(0,x) &= \varphi_0(x) \text{ in } \Omega \\ u(0,x) &= u_0(x) \text{ in } \Omega \end{cases} \tag{1.1}$$

where Ω is an open bounded set in $I\!R^n, n \geq 1$, with regular boundary, and $f(\varphi)$ is typically $\frac{1}{2}(\varphi^3 - \varphi)$ when considering only two different phases, but here we consider a general sufficiently regular function.

Here $u(t,x)$ represents the temperature of a substance at point x and time t that may appear at least in two different phases, (for example liquid–solid) and $\varphi(t,x)$ the phase-field function or order parameter, is a function depending on time and position, and takes different values in different phases and represents the local phase average.

The positive constants l and k refer to latent heat and diffusivity, whereas τ and ξ (interface width) are positive parameters related to time and length scales [2,3].

Phase field models are used in numerous fields of science. There are many examples of different phase transitions described by an order parameter or phase-field function, but in all of them, we have a physical magnitude that adopts two (or more) different phases, such as vapor–liquid transitions, the concentration of one of the two components of an alloy, or the magnetization (magnetic moment per unit volume) in ferromagnetism, among others.

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In this paper, we assume that the time variation of the phase field also depends on a potential function, $V_\varepsilon(x)$, concentrated in ω_ε, a neighborhood of the boundary of the domain, $\Gamma = \partial\Omega$.

For this, we will consider an open bounded smooth set in $I\!R^N$, Ω, with a C^2 boundary, $\Gamma = \partial\Omega$, and we define, for sufficiently small ε, with $0 < \varepsilon \le \varepsilon_0$, the neighborhood of Γ

$$\omega_\varepsilon = \{x - \sigma\vec{n}(x),\ x \in \Gamma,\ \sigma \in [0,\varepsilon)\} \subset \Omega \tag{1.2}$$

where $\vec{n}(x)$ denotes the outwards normal vector at a point $x \in \Gamma$ and $\mathcal{X}_{\omega_\varepsilon}$ denotes the characteristic function of the set ω_ε. We note that ω_ε collapse to the boundary Γ when ε approaches zero.

In this way, starting of (1.1) we obtain the **"concentrated-potential phase-field model"** by considering the singular term

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x)\varphi^\varepsilon \tag{1.3}$$

in the first equation, i.e. now $\tau\varphi_t^\varepsilon = \xi^2\Delta\varphi^\varepsilon - f(\varphi^\varepsilon) + 2u^\varepsilon - \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x)\varphi^\varepsilon$ and we get:

$$\begin{cases} \tau\varphi_t^\varepsilon &=& \xi^2\Delta\varphi^\varepsilon - f(\varphi^\varepsilon) + 2u^\varepsilon - \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x)\varphi^\varepsilon \quad \text{in } (0,T)\times\Omega \\ u_t^\varepsilon + \frac{l}{2}\varphi_t^\varepsilon &=& k\Delta u^\varepsilon \quad \text{in } (0,T)\times\Omega \\ \frac{\partial\varphi^\varepsilon}{\partial\vec{n}} &=& 0 \text{ on } (0,T)\times\Gamma \\ \frac{\partial u^\varepsilon}{\partial\vec{n}} &=& 0 \text{ on } (0,T)\times\Gamma \\ \varphi^\varepsilon(0,x) &=& \varphi_0^\varepsilon(x) \text{ in } \Omega \\ u^\varepsilon(0,x) &=& u_0^\varepsilon(x) \text{ in } \Omega. \end{cases} \tag{1.4}$$

The first objective of this work, (see Section 2) is to study the well-posedness of (1.4)≡(2.2), using the properties of the singular term (1.3) as in the previous works [8–10].

Next, we will get into the main objective of this work, which is the study of the asymptotic behavior of the solutions of (1.4)≡(2.2) as the parameter ε approaches zero; to show that the family of solutions $(\varphi^\varepsilon(t,x), u^\varepsilon(t,x))$ converges when $\varepsilon \to 0$ in a suitable sense, to the solution $(\varphi^0(t,x), u^0(t,x))$, of the following limit problem:

$$\begin{cases} \tau\varphi_t^0 &=& \xi^2\Delta\varphi^0 - f(\varphi^0) + 2u^0 \quad \text{in } (0,T)\times\Omega \\ u_t^0 + \frac{l}{2}\varphi_t^0 &=& k\Delta u^0 \quad \text{in } (0,T)\times\Omega \\ \xi^2\frac{\partial\varphi^0}{\partial\vec{n}} + V_0(x)\varphi^0 &=& 0 \text{ on } (0,T)\times\Gamma \\ \frac{\partial u^0}{\partial\vec{n}} &=& 0 \text{ on } (0,T)\times\Gamma \\ \varphi^0(0,x) &=& \varphi_0^0(x) \text{ in } \Omega \\ u^0(0,x) &=& u_0^0(x) \text{ in } \Omega \end{cases}$$

where $V_0(x)$ and the initial data $(\varphi_0^0(x), u_0^0(x))$ is given by (3.7) and 3.3 in Lemma 3.4 (see Section 3).

2. Concentrated-potential phase-field model

First, from **"concentrated-potential phase-field model"** (1.4), if we now consider the enthalpy function $v^\varepsilon = u^\varepsilon + \frac{l}{2}\varphi^\varepsilon$, we obtain for the phase-field function $\varphi^\varepsilon(t,x)$ and the enthalpy function $v^\varepsilon(t,x)$, the problem given by:

$$\begin{cases} \tau\varphi_t^\varepsilon &=& \xi^2\Delta\varphi^\varepsilon - f(\varphi^\varepsilon) - l\varphi^\varepsilon + 2v^\varepsilon - \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x)\varphi^\varepsilon \quad \text{in } (0,T)\times\Omega \\ v_t^\varepsilon &=& k_2\Delta v^\varepsilon - c\Delta\varphi^\varepsilon \quad \text{in } (0,T)\times\Omega \\ \frac{\partial\varphi^\varepsilon}{\partial\vec{n}} &=& 0 \text{ on } (0,T)\times\Gamma \\ \frac{\partial v^\varepsilon}{\partial\vec{n}} &=& 0 \text{ on } (0,T)\times\Gamma \\ \varphi^\varepsilon(0,x) &=& \varphi_0^\varepsilon(x) \text{ in } \Omega \\ v^\varepsilon(0,x) &=& v_0^\varepsilon(x) \text{ in } \Omega. \end{cases} \tag{2.1}$$

where $v_0^\varepsilon(x) = u_0^\varepsilon(x) + \frac{l}{2}\varphi_0^\varepsilon(x)$.

Hereafter, in this section we consider (2.1) for $0 < \varepsilon \le \varepsilon_0$, and in order to simplify the notations since ε is fixed, we denoted φ^ε and v^ε by φ and v, respectively, that is

$$(P_\varepsilon) \equiv \begin{cases} \varphi_t &= k_1\Delta\varphi - g(\varphi) - b\varphi + av - \frac{a}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon(x)\varphi \quad \text{in } (0,T)\times\Omega \\ v_t &= k_2\Delta v - c\Delta\varphi \quad \text{in } (0,T)\times\Omega \\ \frac{\partial\varphi}{\partial\vec{n}} &= 0 \text{ on } (0,T)\times\Gamma \\ \frac{\partial v}{\partial\vec{n}} &= 0 \text{ on } (0,T)\times\Gamma \\ \varphi(0,x) &= \varphi_0(x) \text{ in } \Omega \\ v(0,x) &= v_0(x) \text{ in } \Omega. \end{cases} \tag{2.2}$$

where $v_0(x) = u_0(x) + \frac{l}{2}\varphi_0(x)$, with $\mathcal{X}_{\omega_\varepsilon}V_\varepsilon(x) \in L^\rho(\Gamma), \rho > \frac{N}{2}$ and

$$k_1 = \frac{\xi^2}{\tau} > 0, \quad k_2 = k > 0, \quad a = \frac{2}{\tau} > 0, \quad b = \frac{l}{\tau} > 0, \quad c = \frac{kl}{2} > 0, \quad g(\varphi) = \frac{1}{\tau}f(\varphi). \tag{2.3}$$

To understand the local existence and properties of solutions to the given system (2.2), we reformulated it as an evolution equation and analyze it using sectorial operator theory and semigroup techniques [11,12].

Reformulation as an evolution equation

Given the system (2.2), ($0 < \varepsilon \le \varepsilon_0$ fixed, $\varphi = \varphi^\varepsilon$ and $v = v^\varepsilon$), we can rewrite it in a more compact form:

$$U_t + AU = F_\varepsilon(U) \text{ where } U = (\varphi, v)^\perp, F_\varepsilon(U) = F_\varepsilon(\varphi, v)^\perp = \Big(h_\varepsilon(\varphi), 0\Big)^\perp, \tag{2.4}$$

with $h_\varepsilon(\varphi) = -g(\varphi) - b\varphi - \frac{a}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon(x)\varphi$, and

$$A = \begin{pmatrix} -k_1\Delta_N & -aI \\ c\Delta_N & -k_2\Delta_N \end{pmatrix} \tag{2.5}$$

where $-\Delta_N$ represents the laplacian in $L^p(\Omega)$, $\quad 1 < p < \infty$ with homogeneous Neumann boundary conditions defined on $W^{2,p}_N = \{u \in W^{2,p}(\Omega), \frac{\partial u}{\partial n} = 0\}$.

Properties of the operator A

The operator A is decomposed as $A = A_0 + P$ where

$$A_0 = \begin{pmatrix} k_1(-\Delta_N + I) & 0 \\ 0 & k_2(-\Delta_N + I) \end{pmatrix} \text{ and } P\begin{pmatrix} \varphi \\ v \end{pmatrix} = \begin{pmatrix} -k_1\varphi - av \\ c\Delta_N\varphi - k_2 v \end{pmatrix}.$$

Here, A_0 is a sectorial operator in $Y = W^{2\alpha,p}_N \times W^{2\beta,p}_N$ for any $\alpha, \beta \in I\!R$ with $Y^\delta = D(A_0^\delta) = W^{2(\alpha+\delta),p}_N \times W^{2(\beta+\delta),p}_N$ with $1 < p < \infty$ and $\delta \ge 0$, scale of interpolations spaces as constructed in [11,13].

We work in different powers for each component to handle the perturbation P, and using the perturbation results, from Proposition 2.1 in [6] we have A is a sectorial operator in $Y = W^{2\alpha,p}_N \times W^{2\beta,p}_N$ for every $\alpha, \beta \in I\!R$ such that $0 < \alpha - \beta < 1$ and $1 < p < \infty$, with compact resolvent and $D(A) = W^{2(\alpha+1),p}_N \times W^{2(\beta+1),p}_N$.

Now we turn to the evolution Eq. (2.4) governed by a sectorial operator A in $Y = W^{2\alpha,p}_N \times W^{2\beta,p}_N$ and we assumed that the mapping F_ε, defined by $F_\varepsilon(U) = F_\varepsilon(\varphi, v)^\perp = \Big(h_\varepsilon(\varphi), 0\Big)^\perp$, is locally Lipschitz from $Y^\delta = W^{2(\alpha+\delta),p}_N \times W^{2(\beta+\delta),p}_N$ into $Y = W^{2\alpha,p}_N \times W^{2\beta,p}_N$ with $\delta \in [0,1)$, so from Theorem 2.2 in [6] we get the following local existence result of solutions of (2.4).

Lemma 2.1. *For $1 < p < \infty$, $\alpha, \beta \in I\!R$ satisfying $0 < \alpha - \beta < 1$ and $\delta \in [0,1)$ such that the mapping*

$$h_\varepsilon : \varphi \in W^{2(\alpha+\delta),p}_N \longrightarrow h_\varepsilon(\varphi) \in W^{2\alpha,p}_N$$

is locally Lipschitz.

If $(\varphi_0, v_0) \in W^{2(\alpha+\delta),p}_N \times W^{2(\beta+\delta),p}_N$, there exists a unique solution (φ, v) of (2.4) in $[0,T)$, with $T = T(\varphi_0, v_0) > 0$. The solution is given by

$$(\varphi(t), v(t))^\perp = e^{-At}(\varphi_0, v_0)^\perp + \int_0^t e^{-A(t-s)}(h_\varepsilon(\varphi(s)), 0)^\perp ds$$

with A given by (2.5) and satisfies

$$(\varphi, v) \in C([0,T); W^{2(\alpha+\delta),p}_N \times W^{2(\beta+\delta),p}_N) \cap C((0,T); W^{2(\alpha+1),p}_N \times W^{2(\beta+1),p}_N)$$

with

$$(\varphi_t, v_t) \in C((0,T); W^{2(\alpha+\theta),p}_N \times W^{2(\beta+\theta),p}_N)$$

for every $0 \le \theta < 1$, verifying (2.2) as an equality in $W^{2\alpha,p}_N \times W^{2\beta,p}_N$.

Moreover, if h_ε maps bounded sets into bounded sets and we assume that the solution (φ, v) has been extended to a maximal interval of time $[0, T_{max})$, we have that either $T_{max} = \infty$, or the solution blows-up in the $W^{2(\alpha+\delta),p}_N \times W^{2(\beta+\delta),p}_N$ norm as $t \to T_{max}$. $\square$

Next, we will study the solutions of system (2.2) given by (2.4) where

$$h_\varepsilon(\varphi) = -g(\varphi) - b\varphi - \frac{a}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x)\varphi$$

and we will use these results with $p = 2$, in this particular case we have $H^1_N = H^1(\Omega)$ and $H^{-1}_N = (H^1(\Omega))'$ (the dual space), with $H^2_N = \{u \in H^2(\Omega), \frac{\partial u}{\partial n} = 0\}$.

Beside, if $\varphi \in H^2_N$ then we obtain that $-\Delta_N(\varphi) \in H^{-1}$ and for every $\Phi \in H^1(\Omega)$, we have

$$\langle -\Delta_N(\varphi), \Phi\rangle = \int_\Omega -\Delta_N(\varphi)\Phi = \int_\Omega \nabla\varphi\nabla\Phi - \int_{\partial\Omega} \frac{\partial\varphi}{\partial\vec{n}}\Phi = \int_\Omega \nabla\varphi\nabla\Phi.$$

So, the operator $L = -\Delta_N : H^1(\Omega) \mapsto H^{-1}(\Omega)$ defined by:

$$\langle L(\varphi), \Phi\rangle = \int_\Omega \nabla\varphi\nabla\Phi, \quad \varphi, \Phi \in H^1(\Omega), \tag{2.6}$$

is a linear continuous operator.

Moreover, in order to treat terms, like concentrated potentials $\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon$, we also consider the scale of Bessel potentials $H^{s,p}(\Omega)$ [13], incorporating the boundary conditions, since we consider these concentrated terms as convergent sequences in $H^{-s,p}(\Omega)$ for some appropriated s, p. In this case, $H^{-s,p}(\Omega) = (H^{s,p'}(\Omega))'$ with $p' = \frac{p}{p-1}$.

Next, we are going to see properties about the nonlinear term g.

Lemma 2.2. *Let $g : I\!R \to I\!R$ a continuous function such that for some $C > 0$*

$$|g(s)| \le C(1 + |s|^r) \text{ with } 1 \le r \le \frac{N}{(N-2)_+}, \quad s \in I\!R \tag{2.7}$$

and

$$|g(s_1) - g(s_2)| \le C(1 + |s_1|^{r-1} + |s_2|^{r-1})|s_1 - s_2|, \quad s_1, s_2 \in I\!R.$$

Then,

$$\|g(\varphi)\|_{L^2(\Omega)} \le C\|\varphi\|_{H^1(\Omega)} \text{ and}$$
$$\|g(\varphi_1) - g(\varphi_2)\|_{L^2(\Omega)} \le C(1 + \|\varphi_1\|^{r-1}_{H^1(\Omega)} + \|\varphi_2\|^{r-1}_{H^1(\Omega)})\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}.$$

In particular:

$$g : H^1(\Omega) \to L^2(\Omega)$$

is locally Lipschitz and maps bounded sets into bounded sets.

Proof. See Lemma 3.5 in [10]. □

Remark. In order to work with the concentrated functions on ω_ε (see Eq. (1.2)), the neighborhood of boundary Γ, we will use in several times the following result given by Lemma 2.1 in [8].

Assume that $u \in H^{q^*}(\Omega)$ with $\frac{1}{2} < q^* \le 2$ such that $H^{q^*}(\Omega) \subset L^\gamma(\Gamma)$, i.e. $q^* - \frac{N}{2} \ge -\frac{N-1}{\gamma}$, then there exists a positive constant C independent of ε such that for any $0 < \varepsilon \le \varepsilon_0$, we have

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u|^\gamma \le C\|u\|^\gamma_{H^{q*}(\Omega)}. \tag{2.8}$$

Lemma 2.3. *Let $\rho > N - 1$ and $q > \max\{\frac{N-1}{\rho}, \frac{1}{2}\}$, if we assume that the family of functions V_ε satisfies that:*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon|^\rho \le C \tag{2.9}$$

with $C > 0$ independent of ε. Then, the linear operator

$$h^*_\varepsilon : \varphi \in H^1(\Omega) \longrightarrow h^*_\varepsilon(\varphi) = \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon\varphi \in H^{-q}(\Omega)$$

is continuous and uniformly Lipschitz in ε.

Proof. Let $\varphi \in H^1(\Omega)$ and $\phi \in H^q(\Omega)$, from (2.9) and using Hölder inequality with exponents ρ, m, n together with (2.8) (see Lemma 2.1 in [8]), we have

$$|\langle \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon\varphi, \phi\rangle| = \Big|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon\varphi\phi\Big| \le \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon\varphi\phi|$$
$$\le \Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon|^\rho\Big)^{\frac{1}{\rho}}\Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi|^m\Big)^{\frac{1}{m}}\Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\phi|^n\Big)^{\frac{1}{n}} \le C\|\varphi\|_{H^1(\Omega)}\|\phi\|_{H^q(\Omega)} \tag{2.10}$$

where $\frac{1}{\rho}+\frac{1}{m}+\frac{1}{n}=1$ and ρ,m,n are such that $1-\frac{N}{2}\geq-\frac{N-1}{m}$ and $q-\frac{N}{2}\geq-\frac{N-1}{n}$ with $q>\frac{1}{2}$, i.e. $H^1(\Omega)\subset L^m(\Gamma)$ and $H^q(\Omega)\subset L^n(\Gamma)$.

Therefore $\|h^*_\varepsilon(\varphi)\|_{H^{-q}(\Omega)}\leq C\|\varphi\|_{H^1(\Omega)}$, with $C>0$ independent of ε and taking into account that h^*_ε is linear, we conclude. □

Next, we will prove the local existence of solutions for system (2.2) when we consider the initial data $(\varphi_0,v_0)\in H^1(\Omega)\times L^2(\Omega)$ and Neumann boundary conditions.

Proposition 2.4 (*Local Existence of Solutions*). *Under the notations and hypothesis of Lemma 2.3, we also assume that $q<1$, i.e. V_ε satisfies*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|V_\varepsilon|^\rho\leq C,\quad \rho>N-1 \text{ and } \max\left\{\frac{N-1}{\rho},\frac{1}{2}\right\}<q<1 \tag{2.11}$$

where $C>0$ independent of ε. We also assume that $g\in C^1(I\!R)$ and satisfies

$$|g(s)|\leq C(1+|s|^r), |g'(s)|\leq C(1+|s|^{r-1}) \text{ with } 1\leq r\leq\frac{N}{(N-2)_+}, s\in I\!R, C>0. \tag{2.12}$$

If we consider the initial data $(\varphi_0,v_0)\in H^1(\Omega)\times L^2(\Omega)$, then there exists a unique solution (φ,v) of (2.2) in $[0,T)$, with $T=T(\varphi_0,v_0)>0$ given by the variation of constants formula

$$(\varphi(t),v(t))^\perp=e^{-At}(\varphi_0,v_0)^\perp+\int_0^t e^{-A(t-s)}(h_\varepsilon(\varphi(s)),0)^\perp ds$$

with A given by (2.5) and $h_\varepsilon(\varphi)=-g(\varphi)-b\varphi-\frac{a}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon\varphi$.

Moreover, the solution verifies

$$(\varphi,v)\in C([0,T);H^1(\Omega)\times L^2(\Omega))\cap C((0,T);H^{2-q}(\Omega)\times H^{1-q}(\Omega)),$$

with

$$(\varphi_t,v_t)\in C((0,T);H^{-q+2\theta}(\Omega)\times H^{-(1+q)+2\theta}(\Omega)),$$

for every $0\leq\theta<1$ and satisfies (2.2) as an equality in $H^{-q}(\Omega)\times H^{-(1+q)}(\Omega)$.

Proof. We note that if $p=2$, with $\alpha=-\frac{q}{2},\delta=\frac{1+q}{2}$ and $\beta=-\delta$, this is $\alpha-\beta=\frac{1}{2}$, from Proposition 2.1 in [6], A is a sectorial operator in $Y=H^{2\alpha}(\Omega)\times H^{2\beta}(\Omega)=H^{-q}(\Omega)\times H^{-(1+q)}(\Omega)=(H^q(\Omega))'\times(H^{1+q}(\Omega))'$ and $Y^\delta=H^1(\Omega)\times L^2(\Omega)=H^{2(\alpha+\delta)}\times H^{2(\beta+\delta)}$.

Besides, we can consider $F_\varepsilon:Y^\delta=H^1(\Omega)\times L^2(\Omega)\longrightarrow Y=H^{-q}(\Omega)\times H^{-(1+q)}(\Omega)=(H^s(\Omega))'\times(H^{1+s}(\Omega))'$ with $\delta\in[0,1)$ since $0<q<1$.

Moreover, from the above Lemmas 2.2 and 2.3, the function $h_\varepsilon : H^{2(\alpha+\delta)}=H^1(\Omega)\to H^{2\alpha}(\Omega)=H^{-q}(\Omega)$, defined as $h_\varepsilon(\varphi)=-g(\varphi)-b\varphi-\frac{a}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon\varphi$ is locally Lipschitz, since $L^2(\Omega)\subset H^{-q}(\Omega)$, and maps bounded into bounded sets.

Furthermore, if we assume that $\frac{\partial\varphi}{\partial\vec{n}}=0$ on Γ, then $\frac{\partial g(\varphi)}{\partial\vec{n}}=g'(\varphi)\frac{\partial\varphi}{\partial\vec{n}}=0$ on Γ.

Therefore we can apply Lemma 2.1 to conclude. □

We will show that the local solutions of Eqs. (2.2) given by Proposition 2.4 satisfy the energy Eqs. (2.15) and under certain sign conditions are globally defined.

Furthermore, we will obtain estimates on these solutions which are uniform in $\varepsilon>0$.

To achieve the result we can follow a methodology similar to that used in the proof of Proposition 3.1 in Ref. [9], but here we need to account for the new singular term that depends on the parameter ε. The main goal is to obtain uniform estimates on the solutions as ε approaches zero. This analysis provides a understanding of the behavior of the solutions when ε goes to zero, that we will show in the next section (Section 3).

Proposition 2.5 (*Lyapunov Function. Global Existence*). *Under the hypotheses of Proposition 2.4 i.e. V_ε and g satisfy (2.11) and (2.12) respectively, if $(\varphi_0,v_0)\in H^1(\Omega)\times L^2(\Omega)$, let (φ,v) be the local solution of (2.2) given by Proposition 2.4. Also denote by $G(s)=\int_0^s g(r)dr$.*

(I) Then the energy functional defined by

$$E_\varepsilon(\varphi,v)=\frac{k_1}{2}\|\nabla\varphi\|^2+\frac{k_2a}{2c}\|v\|^2+\int_\Omega(G(\varphi)+\frac{b}{2}\varphi^2-av\varphi)+\frac{a}{4\varepsilon}\int_{\omega_\varepsilon}V_\varepsilon|\varphi|^2 \tag{2.13}$$

which can also be written as

$$E_\varepsilon(\varphi,v)=\frac{k_1}{2}\|\nabla\varphi\|^2+\int_\Omega G(\varphi)+\frac{b}{2}\int_\Omega(\frac{a}{b}v-\varphi)^2+\frac{a}{4\varepsilon}\int_{\omega_\varepsilon}V_\varepsilon|\varphi|^2 \tag{2.14}$$

is a Lyapunov functional for the flow of the system (2.2), in the space $H^1(\Omega)\times L^2(\Omega)$.

In particular we have that

$$\frac{d}{dt}E_\varepsilon(\varphi,v)+\|\varphi_t\|^2+\frac{a}{c}\|v_t\|^2_{-1}=0\equiv\int_0^t\|\varphi_t\|^2+\int_0^t\frac{a}{c}\|v_t\|^2_{-1}+E_\varepsilon(\varphi(t),v(t))=E_\varepsilon(\varphi_0,v_0) \tag{2.15}$$

for as long as the solution exists, where $\|.\|$ and $\|.\|_{-1}$ denote the norm in $L^2(\Omega)$ and $H^{-1}(\Omega)=(H^1(\Omega))'$ respectively.

(II) We assume that g verifies $g\in C^1(I\!R)$, (2.12), and also satisfy that:

$$\liminf_{|s|\to\infty}\frac{g(s)}{s}>0. \tag{2.16}$$

Besides, the concentrated-potential function V_ε verifies (2.11) and is no negative, i.e.

$$V_\varepsilon(x) \geq 0 \quad \forall x \in \omega_{\varepsilon_0} \text{ with } 0 < \varepsilon \leq \varepsilon_0. \tag{2.17}$$

Then, there exits $\delta, C(\delta)$ positive constants **independent of** *ε, such that for $(\varphi, v) \in X = H^1(\Omega) \times L^2(\Omega)$, for any $0 < \varepsilon \leq \varepsilon_0$, we have*

$$\frac{k_1}{2}\|\nabla\varphi\|^2 + \delta\|\varphi\|^2 + \frac{b}{2}\|\frac{a}{b}v - \varphi\|^2 - C(\delta)|\Omega| \leq E_\varepsilon(\varphi, v) \leq E_\varepsilon(\varphi_0, v_0). \tag{2.18}$$

Therefore, the solution given by Proposition 2.4 is globally defined.

Moreover, if $K \subset H^1(\Omega) \times L^2(\Omega)$ is a bounded set uniformly on ε, then its orbit, i.e. $\{S_\varepsilon(t)K, t \geq 0\}$, where $S_\varepsilon(t)(\varphi_0^\varepsilon, v_0^\varepsilon) = (\varphi^\varepsilon(t), v^\varepsilon(t))$, is also bounded uniformly on ε, since for any $0 < \varepsilon \leq \varepsilon_0$

$$\|(\varphi^\varepsilon(t), v^\varepsilon(t))\|^2_{H^1(\Omega)\times L^2(\Omega)} \leq C\Big(1 + \|(\varphi^\varepsilon(0), v^\varepsilon(0))\|^2_{H^1(\Omega)\times L^2(\Omega)} + \|\varphi^\varepsilon(0)\|^{r+1}_{H^1(\Omega)}\Big) \tag{2.19}$$

where C is also a positive constant independent of ε.

Proof. As we have already mentioned, the result follows slight variations in the proof of Proposition 3.1 in [9], due to the new singular term concentrated in the first equation.

Since the new term depends on the parameter ε, we need to obtain uniform estimates in ε.

(I) In effect, it is enough to multiply in $L^2(\Omega)$ the first equation in (2.2) by $\frac{\partial\varphi}{\partial t}$, using $\int_\Omega v\varphi_t = \frac{d}{dt}(\int_\Omega v\varphi) - \int_\Omega \varphi v_t$; and the second equation by $\frac{a}{c}(-\Delta)^{-1}v_t$, and to integrate by parts. Thus, by adding the obtained expressions we obtain (2.15), i.e.

$$\frac{d}{dt}E_\varepsilon(\varphi, v) + \|\varphi_t\|^2 + \frac{a}{c}\|v_t\|^2_{-1} = 0$$

with E_ε given by (2.13). It is important to note that $(-\Delta)^{-1}v_t$ is well defined since integrating the second equations in Ω, we have that $\int_\Omega v_t = c\int_\Gamma \frac{\partial\varphi}{\partial\vec{n}} - k_2\int_\Gamma \frac{\partial v}{\partial\vec{n}} = 0$.

Therefore, E_ε is a Lyapunov functional. Finally, taking into account that $\frac{k_2}{c} = \frac{a}{b}$ we get (2.14).

(II) First, working as [9], if g satisfies (2.16) then there exists $\delta > 0$ and $C(\delta) > 0$ such that $G(s) \geq \delta s^2 - C(\delta)$ for every $s \in I\!R$, and hence we have

$$\int_\Omega G(\varphi) \geq \delta\|\varphi\|^2 - C(\delta)|\Omega|. \tag{2.20}$$

Next, using (2.17) and (2.20), from (2.14) we have

$$\frac{k_1}{2}\|\nabla\varphi\|^2 + \delta\|\varphi\|^2 + \frac{b}{2}\|\frac{a}{b}v - \varphi\|^2 \leq E_\varepsilon(\varphi, v) + C(\delta)|\Omega|.$$

Thus, using now $E_\varepsilon(\varphi(t), v(t)) \leq E_\varepsilon(\varphi(0), v(0))$ for $t > 0$, we get (2.18).

Hence, $\|\nabla\varphi\|^2, \|\varphi\|^2$ and $\|\frac{a}{b}v - \varphi\|^2$ remain bounded on finite time intervals. Therefore the solution remains bounded in $X = H^1(\Omega) \times L^2(\Omega)$ and from Lemma 2.1, we obtain that the solution is global.

Moreover, from (2.18) we also have

$$\|(\varphi(t), v(t))\|^2_{H^1(\Omega)\times L^2(\Omega)} \leq c_1 + c_2\Big(\|\nabla\varphi_0\|^2 + \|\varphi_0\|^2 + \|v_0\|^2 + \int_\Omega |G(\varphi_0)| + \frac{a}{4\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon||\varphi_0|^2\Big).$$

Now, from (2.7) we get $\int_\Omega |G(\varphi_0)| \leq c_3\Big(1 + \int_\Omega |\varphi_0|^{r+1}\Big) \leq c_4\Big(1 + \|\varphi_0\|^{r+1}_{H^1(\Omega)}\Big)$ since from hypothesis on r we have $H^1(\Omega) \subset L^{r+1}(\Omega)$.

Furthermore, recording that if $H^1(\Omega) \subset L^p(\Gamma)$, i.e. $1 - \frac{N}{2} \geq -\frac{N-1}{p}$, from Lemma 2.1 in [8], there exists a positive constant C independent of ε such that for any $0 < \varepsilon \leq \varepsilon_0$, we have

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi_0|^p \leq C\|\varphi_0\|^p_{H^1(\Omega)}. \tag{2.21}$$

On the other hand, from (2.11) and using Hölder inequality with $\rho > N - 1$, together with (2.21) for $p = 2\rho'$, we obtain

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon||\varphi_0|^2 \leq \left(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon|^\rho\right)^{\frac{1}{\rho}}\left(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi_0|^{2\rho'}\right)^{\frac{1}{\rho'}} \leq c_5\|\varphi_0\|^2_{H^1(\Omega)} \tag{2.22}$$

since $\rho > N-1$ implies that $\rho' \leq \frac{N-1}{N-2}$ and then $H^1(\Omega) \subset L^{2\rho'}(\Gamma)$. Thus, taking into account that c_5 is a positive constant independent of ε

$$\|(\varphi(t), v(t))\|^2_{H^1(\Omega)\times L^2(\Omega)} \leq c_6\Big(1 + \|(\varphi_0, v_0)\|^2_{H^1(\Omega)\times L^2(\Omega)} + \|\varphi_0\|^{r+1}_{H^1(\Omega)}\Big)$$

where c_6 is also a positive constant independent of ε, and we conclude (2.19). □

3. Limit phase-field model

We consider $(\varphi^\varepsilon(t,x), v^\varepsilon(t,x))$ the solutions of the problem (2.1) with $0 < \varepsilon \leq \varepsilon_0$, this is

$$\begin{cases} \varphi^\varepsilon_t &= k_1\Delta\varphi^\varepsilon - g(\varphi^\varepsilon) - b\varphi^\varepsilon + av^\varepsilon - \frac{a}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon(x)\varphi^\varepsilon \quad \text{in } (0,T)\times\Omega \\ v^\varepsilon_t &= k_2\Delta v^\varepsilon - c\Delta\varphi^\varepsilon \quad \text{in } (0,T)\times\Omega \\ \frac{\partial\varphi^\varepsilon}{\partial\vec{n}} &= 0 \text{ on } (0,T)\times\Gamma \\ \frac{\partial v^\varepsilon}{\partial\vec{n}} &= 0 \text{ on } (0,T)\times\Gamma \\ \varphi^\varepsilon(0,x) &= \varphi^\varepsilon_0(x) \text{ in } \Omega \\ v^\varepsilon(0,x) &= v^\varepsilon_0(x) \text{ in } \Omega. \end{cases}$$

with (2.3), and we will study the limit of this solutions of (2.1), as $\varepsilon \to 0$.

For this we will use the uniform estimates of this solutions in above Proposition 2.5 together with compactness arguments, in order to obtain a limit function $(\varphi^0(t,x), v^0(t,x))$ in $H^1(\Omega)\times L^2(\Omega)$, such that $(\varphi^\varepsilon(t,x), v^\varepsilon(t,x)) \to (\varphi^0(t,x), v^0(t,x))$ as $\varepsilon \to 0$, in "some sense".

Moreover, we will prove that this limit $(\varphi^0(t,x), v^0(t,x))$ is given by the solution of the limit problem (P_0)

$$(P_0) \equiv \begin{cases} \varphi^0_t &= k_1\Delta\varphi^0 - g(\varphi^0) - b\varphi^0 + av^0 \quad \text{in } (0,T)\times\Omega \\ v^0_t &= k_2\Delta v^0 - c\Delta\varphi^0 \quad \text{in } (0,T)\times\Omega \\ k_1\frac{\partial\varphi^0}{\partial\vec{n}} + \frac{a}{2}V_0(x)\varphi^0 &= 0 \text{ on } (0,T)\times\Gamma \\ k_2\frac{\partial v^0}{\partial\vec{n}} + \frac{ac}{2k_1}V_0(x)\varphi^0 &= 0 \text{ on } (0,T)\times\Gamma \\ \varphi^0(0,x) &= \varphi^0_0(x) \text{ in } \Omega \\ v^0(0,x) &= v^0_0(x) \text{ in } \Omega \end{cases} \tag{3.1}$$

where $V_0(x) \in L^\rho(\Gamma)$ is associated to the limit of the family $\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon(x)$, as $\varepsilon \to 0$, in "a suitable sense" (see Lemma 3.4).

First, we will study in the next section the well-posedness of this limit problem (P_0).

3.1. Well-posedness of limit phase-field model (P_0)

In this section we consider the problem (P_0) given by (3.1) and we denote by $(\varphi(t,x), v(t,x))$ the solution of this problem associated to the initial condition $(\varphi(0,x), v(0,x)) = (\varphi^0_0, v^0_0) \in H^1(\Omega)\times L^2(\Omega)$, this is

$$(P_0) \equiv \begin{cases} \varphi_t &= k_1\Delta\varphi - g(\varphi) - b\varphi + av \quad \text{in } (0,T)\times\Omega \\ v_t &= k_2\Delta v - c\Delta\varphi \quad \text{in } (0,T)\times\Omega \\ k_1\frac{\partial\varphi}{\partial\vec{n}} + \frac{a}{2}V_0(x)\varphi &= 0 \text{ on } (0,T)\times\Gamma \\ k_2\frac{\partial v}{\partial\vec{n}} + \frac{ac}{2k_1}V_0(x)\varphi &= 0 \text{ on } (0,T)\times\Gamma \\ \varphi(0,x) &= \varphi^0_0(x) \text{ in } \Omega \\ v(0,x) &= v^0_0(x) \text{ in } \Omega \end{cases}$$

where $V_0(x) \in L^\rho(\Gamma), 1 \le \rho < \infty$.

First, we can provide a suitable weak formulation of (3.1). For this, we note that if (φ, v) is a solution of (3.1)$\equiv (P_0)$ then (φ, v) satisfies the initial and boundary conditions and for every $\phi \in H^1(\Omega)$, we have

$$\begin{cases} 0 = \int_\Omega \varphi_t\phi + \int_\Omega (g(\varphi) + b\varphi - av)\phi + k_1\int_\Omega \nabla\varphi\nabla\phi - k_1\int_\Gamma \frac{\partial\varphi}{\partial\vec{n}}\phi \\ 0 = \int_\Omega v_t\phi + k_2\int_\Omega \nabla v\nabla\phi - c\int_\Omega \nabla\varphi\nabla\phi - k_2\int_\Gamma \frac{\partial v}{\partial\vec{n}}\phi + c\int_\Gamma \frac{\partial\varphi}{\partial\vec{n}}\phi. \end{cases}$$

Now taking into account that the boundary conditions, i.e. $k_1\int_\Gamma \frac{\partial\varphi}{\partial\vec{n}}\phi = -\frac{a}{2}\int_\Gamma V_0(x)\varphi\phi$ and $-k_2\int_\Gamma \frac{\partial v}{\partial\vec{n}}\phi + c\int_\Gamma \frac{\partial\varphi}{\partial\vec{n}}\phi = \frac{ac}{2k_1}\int_\Gamma V_0(x)\varphi\phi - \frac{ac}{2k_1}\int_\Gamma V_0(x)\varphi\phi = 0$, we obtain

$$\begin{cases} 0 = \int_\Omega \varphi_t\phi + \int_\Omega (g(\varphi) + b\varphi - av)\phi + k_1\int_\Omega \nabla\varphi\nabla\phi - \frac{a}{2}\int_\Gamma V_0(x)\varphi\phi \\ 0 = \int_\Omega v_t\phi + k_2\int_\Omega \nabla v\nabla\phi - c\int_\Omega \nabla\varphi\nabla\phi \end{cases} \tag{3.2}$$

for every $\phi \in H^1(\Omega)$.

Next, using $L = -\Delta_N$ defined in (2.6) and the trace operator γ with $\langle V_0\gamma(\varphi), \phi\rangle = \int_\Gamma V_0\varphi\phi$, for $\varphi \in H^1(\Omega)$ and $\phi \in H^q(\Omega)$, with $q > \frac{1}{2}$, we get:

$$\begin{cases} \varphi_t - k_1\Delta_N\varphi - av = -g(\varphi) - b\varphi - \frac{a}{2}V_0\gamma(\varphi) \\ v_t + c\Delta_N\varphi - k_2\Delta v = 0 \end{cases} \tag{3.3}$$

as in equality in $H^{-q}(\Omega)$.

In order to study the local existence of solutions of (P_0), we work as Section 2 and using (3.3) rewriting the system (3.1), as an evolution equation

$$U_t + AU = H(U) \text{ where } U = (\varphi, v)^\perp \tag{3.4}$$

with $A = A_0 + P$ is the sectorial operator given by (2.5), i.e.

$$A = \begin{pmatrix} -k_1\Delta_N & -aI \\ c\Delta_N & -k_2\Delta_N \end{pmatrix}$$

and now

$$H(U) = H(\varphi, v)^{\perp} = \left(-g(\varphi) - b\varphi - \frac{a}{2}V_0\gamma(\varphi), 0\right)^{\perp}, \tag{3.5}$$

where γ denoted the trace operator. We note that γ is well defined in $H^{q,p}(\Omega)$ if $q > \frac{1}{p}$, in particular it is well defined in $H^1(\Omega)$ and also in $H^q(\Omega)$ for $q > \frac{1}{2}$.

First, using again the Proposition 2.1 in [6], we obtain that A is a sectorial operator in $Y = H^{2\alpha} \times H^{2\beta}$ with α, β such that $0 < \alpha - \beta < 1$ and $Y^{\delta} = H^{2(\alpha+\delta)} \times H^{2(\beta+\delta)}$ and $\delta \in [0, 1)$.

Next, working as in Proposition 2.4 if we consider $\alpha = -\frac{q}{2}$, $\beta = -\frac{1+q}{2}$, then A is a sectorial operator in $Y = H^{-q} \times H^{-(1+q)}$ with $Y^{\delta} = H^1 \times L^2$ for $\delta = \frac{1+q}{2} \in [0, 1)$ for $\frac{1}{2} < q < 1$.

Moreover, we will prove now that $H : Y^{\delta} = H^1 \times L^2 \longrightarrow Y = H^{-q} \times H^{-(1+q)}$ is locally Lipschitz for some suitable q, r and ρ.

Lemma 3.1. *Under the above notations. We assume that*

(i) $g \in C^1(I\!R)$ satisfying (2.12), i.e.

$$|g(s)| \leq C(1 + |s|^r), |g'(s)| \leq C(1 + |s|^{r-1}), 1 \leq r \leq \frac{N}{(N-2)_+}, s \in I\!R, C > 0$$

(ii) $V_0 \in L^{\rho}(\Gamma)$ with $\rho > N - 1$ and $\max\{\frac{N-1}{\rho}, \frac{1}{2}\} < q < 1$.

Then,

$$H : Y^{\delta} = H^1 \times L^2 \longrightarrow Y = H^{-q} \times H^{-(1+q)}$$

given by (3.5) is locally Lipschitz and maps bounded sets into bounded sets.

Proof. First, we consider $\gamma^* : \varphi \in H^1 \longrightarrow \gamma^*(\varphi) = V_0\gamma(\varphi) \in H^{-q}$ where γ is the trace operator and we prove that $\|V_0\gamma(\varphi)\|_{H^{-q}(\Omega)} \leq C\|\varphi\|_{H^1(\Omega)}$.

In effect, let $\Phi \in H^q(\Omega)$, using again Lemma 2.1 in [8] working as in Lemma 2.3, we obtain

$$\left|\langle V_0\gamma(\varphi), \Phi\rangle\right| = \left|\int_{\Gamma} V_0(x)\varphi\Phi\right| \leq \int_{\Gamma} |V_0(x)\varphi\Phi| \leq$$

$$\left(\int_{\Gamma} |V_0(x)|^{\rho}\right)^{\frac{1}{\rho}} \left(\int_{\Gamma} |\varphi|^m\right)^{\frac{1}{m}} \left(\int_{\Gamma} |\Phi|^n\right)^{\frac{1}{n}} \leq C\|\varphi\|_{H^1}\|\Phi\|_{H^q}$$

where $\frac{1}{\rho} + \frac{1}{m} + \frac{1}{n} = 1$ and ρ, m, n are such that $1 - \frac{N}{2} \geq -\frac{N-1}{m}$ and $q - \frac{N}{2} \geq -\frac{N-1}{n}$ with $q > \frac{1}{2}$, i.e. $H^1(\Omega) \subset L^m(\Gamma)$ and $H^q(\Omega) \subset L^n(\Gamma)$. Therefore, $\|V_0\gamma(\varphi)\|_{H^{-q}(\Omega)} \leq C\|\varphi\|_{H^1(\Omega)}$. This γ^* is locally Lipschitz and maps bounded into bounded set.

Finally, from Lemma 2.2 $g : H^1(\Omega) \longrightarrow L^2(\Omega) \subset H^{-q}(\Omega)$ is also locally Lipschitz and maps bounded into bounded set, and we conclude. □

Proposition 3.2 (*Local Existence of Solutions*). *Under the above notations and hypotheses of Lemma 3.1, for the initial condition $(\varphi_0^0, v_0^0) \in H^1(\Omega) \times L^2(\Omega)$, there exits a unique solution $(\varphi^0, v^0) \in C([0, T], H^1(\Omega) \times L^2(\Omega))$ of (P_0)(3.1) with $T = T(\varphi_0, v_0) > 0$, given by the variation of constants formula*

$$(\varphi^0(t), v^0(t))^{\perp} = e^{-At}(\varphi_0^0, v_0^0)^{\perp} + \int_0^t e^{-A(t-s)}(H(\varphi^0(s)), v^0(s))^{\perp} ds.$$

with A and H given by (2.5) and (3.5), respectively.

Moreover, for every q such that $max\{\frac{N-1}{\rho}, \frac{1}{2}\} < q < 1$, with $\rho > N - 1$,

$$(\varphi^0, v^0) \in C((0, T); H^{2-q}(\Omega) \times H^{1-q}(\Omega)),$$

$$(\varphi_t^0, v_t^0) \in C((0, T); H^{-q+2\theta}(\Omega) \times H^{-(1+q)+2\theta}(\Omega)),$$

for every $0 \leq \theta < 1$ and satisfies (3.1) as an equality in $H^{-q}(\Omega) \times H^{-(1+q)}(\Omega)$. In particular, $H^{-q}(\Omega) \subset H^{-1}(\Omega)$ and (φ^0, v^0) is a weak solution of (P_0)(3.1).

Besides, if the solution has been extended to a maximal interval of time $[0, T_{max})$, we have either $T_{max} = \infty$, or the solution blows-up in the $H^1(\Omega) \times L^2(\Omega)$ norm as $t \to T_{max}$.

Proof. Under above notations, we consider the evolution Eq. (3.4), i.e.

$$U_t + AU = H(U) \text{ where } U = (\varphi^0, v^0)^{\perp}$$

with A is sectorial operator in $Y = H^{-q}(\Omega) \times H^{-(1+q)}(\Omega)$ and using the above Lemma 3.1, this is $H : Y^{\delta} \longrightarrow Y$ is locally Lipschitz with $\delta \in [0, 1)$ and $(\varphi_0^0, v_0^0) \in H^1(\Omega) \times L^2(\Omega) = Y^{\delta}$ and also maps bounded into bounded sets, so from [11–13] we conclude. □

3.2. Convergence of the concentrated-potential phase-field model

Now, we consider the family of initial data in $H^1(\Omega)\times L^2(\Omega)$ uniformly bounded in ε, and we have the following result.

Lemma 3.3. *Let* $(\varphi^\varepsilon(0,x), v^\varepsilon(0,x)) = (\varphi_0^\varepsilon(x), v_0^\varepsilon(x)) \in H^1(\Omega)\times L^2(\Omega)$ *uniformly bounded in* ε*, this is*

$$\|(\varphi_0^\varepsilon(x), v_0^\varepsilon(x))\|_{H^1(\Omega)\times L^2(\Omega)} \le C, \quad C>0 \text{ independent of } \varepsilon, \tag{3.6}$$

and we assume additionally that G *satisfies* (2.7) *and* V_ε *satisfies* (2.11) .
Then:
(i) There exists $(\varphi_0^0(x), v_0^0(x)) \in H^1(\Omega)\times L^2(\Omega)$ *and a subsequence such that:*

$$(\varphi_0^\varepsilon, v_0^\varepsilon) \to (\varphi_0^0, v_0^0) \text{ weakly in } H^1(\Omega)\times L^2(\Omega) \text{ and strongly in } L^2(\Omega)\times H^{-1}(\Omega) \text{ as } \varepsilon\to 0. \tag{3.7}$$

(ii) There exists $K>0$ *independent of* ε *such that*

$$E_\varepsilon(\varphi_0^\varepsilon(x), v_0^\varepsilon(x)) \le K \tag{3.8}$$

where E_ε *is given by* (2.14)*, i.e.*

$$E_\varepsilon(\varphi, v) = \frac{k_1}{2}\|\nabla\varphi\|^2 + \int_\Omega G(\varphi) + \frac{b}{2}\int_\Omega (\frac{a}{b}v-\varphi)^2 + \frac{a}{4\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon|\varphi|^2.$$

Proof.
(i) It is enough to note that $H^1(\Omega)\times L^2(\Omega)\subset L^2(\Omega)\times H^{-1}(\Omega)$ with compact embedding.
(ii) In effect, $E_\varepsilon(\varphi_0^\varepsilon(x), v_0^\varepsilon(x)) = \frac{k_1}{2}\|\nabla\varphi_0^\varepsilon(x)\|^2 + \int_\Omega G(\varphi_0^\varepsilon(x)) + \frac{b}{2}\int_\Omega(\frac{a}{b}v_0^\varepsilon(x)-\varphi_0^\varepsilon(x))^2 + \frac{a}{4\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon|\varphi_0^\varepsilon(x)|^2 \le c_0\|(\varphi_0^\varepsilon(x), v_0^\varepsilon(x))\|^2_{H^1(\Omega)\times L^2(\Omega)} + \int_\Omega |G(\varphi_0^\varepsilon(x))| + \frac{a}{4\varepsilon}\int_{\omega_\varepsilon}|V_\varepsilon||\varphi_0^\varepsilon(x)|^2$.

Besides, from (2.7) we get $\int_\Omega |G(\varphi_0^\varepsilon)| \le c_1\left(1+\int_\Omega |\varphi_0^\varepsilon|^{r+1}\right) \le c_2\left(1+\|\varphi_0^\varepsilon\|^{r+1}_{H^1(\Omega)}\right)$ since from hypothesis on r we have $H^1(\Omega)\subset L^{r+1}(\Omega)$; and using (2.22) we also have $\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|V_\varepsilon||\varphi_0^\varepsilon|^2 \le c_3\|\varphi_0^\varepsilon\|^2_{H_0^1(\Omega)}$. Finally, taking into account that $c_i>0, i=0,1,2,3$ are independent of ε, from (3.6) we conclude (3.7). □

In order to study the convergence of solutions of (2.1) as $\varepsilon\to 0$, we will prove the next result above the concentrated-potential functions. This result, which we include for completeness, is the consequence of several technical lemmas proved in [8].

Lemma 3.4. *If* V_ε *satisfies* (2.9)*, that is:*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|V_\varepsilon|^\rho \le C, \quad C>0 \text{ independent of } \varepsilon$$

for $1\le\rho<\infty$*, then:*
(I) There exist $V_0\in L^\rho(\Gamma)$ *such that for any smooth function* ϕ *defined in* $\bar\Omega$*, we have*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon\phi = \int_\Gamma V_0\phi. \tag{3.9}$$

(II) Under the hypothesis from Lemma 2.3*, this is* $\rho>N-1$ *and* $q>\max\{\frac{N-1}{\rho},\frac{1}{2}\}$*, taking a function* $\Phi(t,x)\in L^1((0,T),H^q(\Omega))$ *we also have*

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon\Phi \to V_0\gamma(\Phi) \text{ as } \varepsilon\to 0 \text{ in } L^1((0,T),H^{-1}(\Omega)). \tag{3.10}$$

(III) Moreover, under the above hypotheses if we also assume that

$$\varphi^\varepsilon \to \varphi^0 \quad weakly^* \text{ in } L^\infty((0,T),H^1(\Omega)) \text{ as } \varepsilon\to 0. \tag{3.11}$$

Then for every function $\Phi(t,x)\in L^1((0,T),H^q(\Omega))$ *we get*

$$\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} V_\varepsilon(x)\varphi^\varepsilon\Phi \to \int_0^T\int_\Gamma V_0(x)\varphi^0\Phi \text{ as } \varepsilon\to 0. \tag{3.12}$$

Proof. (I) From Lemma 2.2 in [8] we have (3.9).
(II) For fixed $t\in[0,T]$ from Lemma 2.5 in [8], given $\Phi(t,.)\in H^s(\Omega)$ and $\varphi(t,.)\in H^1(\Omega)$ in particular, we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon(x)\Phi(t,.)\varphi(t,.) = \int_\Gamma V_0(x)\Phi(t,.)\varphi(t,.), \text{ a.e. } t\in[0,T],$$

this is

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon(x)\Phi(t,.) \to V_0(x)\gamma(\Phi(t,.)) \text{ as } \varepsilon\to 0 \text{ in } H^{-1}(\Omega), \text{ a.e. } t\in[0,T].$$

Now, working as in Lemma 2.3 i.e. using again Lemma 2.1 in [8] from (2.10) with $\Phi(t,.)\in H^q(\Omega)$ for any $\varphi\in H^1(\Omega)$

$$|\langle\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon\Phi(t,.),\varphi\rangle| \le \frac{1}{\varepsilon}\int_{\omega_\varepsilon}|V_\varepsilon\Phi(t,.)\varphi| \le$$

$$\left(\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|V_\varepsilon|^\rho\right)^{\frac{1}{\rho}}\left(\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|\Phi(t,.)|^m\right)^{\frac{1}{m}}\left(\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|\varphi|^n\right)^{\frac{1}{n}}\leq C\|\Phi(t,.)\|_{H^q(\Omega)}\|\varphi\|_{H^1(\Omega)}$$

where $\frac{1}{\rho}+\frac{1}{m}+\frac{1}{n}=1$ and ρ, m, n are such that $q-\frac{N}{2}\geq -\frac{N-1}{m}$ and $1-\frac{N}{2}\geq -\frac{N-1}{n}$ with $q>\frac{1}{2}$, i.e. $H^q(\Omega)\subset L^m(\Gamma)$ and $H^1(\Omega)\subset L^n(\Gamma)$. Thus, $\|\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon\Phi(t,.)\|_{H^{-1}}\leq C\|\Phi(t,.)\|_{H^q(\Omega)}=h(t)\in L^1(0,T)$, since $\int_0^T h(t)=C\|\Phi\|_{L^1((0,T),H^q(\Omega))}=K$. Therefore, from the Lebesgue dominated converge theorem we conclude.

(III)

First, we note that

$$\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon}V_\varepsilon(x)\varphi^\varepsilon\Phi=\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon}V_\varepsilon(x)\varphi^0\Phi+\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon}V_\varepsilon(x)(\varphi^\varepsilon-\varphi^0)\Phi$$

and from (3.10) in the above part (II), given $\varphi^0\in L^\infty((0,T),H^1(\Omega))$ we have $\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon}V_\varepsilon(x)\varphi^0\Phi=\left\langle\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon\Phi,\varphi^0\right\rangle\to\left\langle V_0\gamma(\Phi),\varphi^0\right\rangle=\int_0^T\int_\Gamma V_0(x)\varphi^0\Phi$ as $\varepsilon\to 0$, where by $\langle .,.\rangle$ we denote the duality between $L^\infty((0,T),H^1(\Omega))$ and $L^1((0,T),H^{-1}(\Omega))$.

Next, we will prove that

$$\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon}V_\varepsilon(x)(\varphi^\varepsilon-\varphi^0)\Phi\to 0 \text{ as } \varepsilon\to 0 \tag{3.13}$$

and we conclude.

In effect, $\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon}V_\varepsilon(x)(\varphi^\varepsilon-\varphi^0)\Phi=\left\langle\varphi^\varepsilon-\varphi^0,\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon(x)\Phi\right\rangle=(1)+(2)$ with $(1)=\left\langle\varphi^\varepsilon-\varphi^0,\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon\Phi-V_0\gamma(\varphi)\right\rangle$ and $(2)=\left\langle\varphi^\varepsilon-\varphi^0,V_0\gamma(\varphi)\right\rangle$.

Besides, $|(1)|\leq\|\varphi^\varepsilon-\varphi^0\|_{L^\infty((0,T),H^1(\Omega))}\|\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon\Phi-V_0\gamma(\Phi)\|_{L^1((0,T),H^{-1}(\Omega))}\leq C\|\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon\Phi-V_0\gamma(\varphi)\|_{L^1((0,T),H^{-1}(\Omega))}$ and from (II) we have $(1)\to 0$ as $\varepsilon\to 0$. On the other hand using again (3.10) in the above part (II), $V_0\gamma(\varphi)\in L^1((0,T),H^{-1}(\Omega))$ and then from (3.11)$(2)\to 0$ as $\varepsilon\to 0$. So we get (3.13) and we conclude. □

Finally, before to prove the main result in this section, Theorem 3.6, we will also show the following result above the non linearity g acting on the solutions of (2.1).

Lemma 3.5. *Let $g: I\!R\to I\!R$ be a continuous function satisfying (2.7) i.e.*

$$|g(s)|\leq C(1+|s|^r), C>0 \text{ with } 1\leq r\leq\frac{N}{(N-2)_+},\quad s\in I\!R.$$

We assume that the family $\varphi^\varepsilon\in L^\infty((0,T),H^1(\Omega))$ is uniformly bounded in ε, i.e.

$$\|\varphi^\varepsilon\|_{L^\infty((0,T),H^1(\Omega))}\leq C$$

with $C>0$ independent of ε, and we also assume that there exists $\varphi^0\in L^\infty((0,T),H^1(\Omega))$ such that

$$\varphi^\varepsilon(t,x)\to\varphi^0(t,x) \text{ a.e. } (t,x)\in(0,T)\times\Omega \text{ as } \varepsilon\to 0.$$

Then, for every $\Phi\in L^1((0,T),L^2(\Omega))$, we have

$$\int_0^T\int_\Omega g(\varphi^\varepsilon)\Phi\to\int_0^T\int_\Omega g(\varphi^0)\Phi \text{ as } \varepsilon\to 0. \tag{3.14}$$

Proof. Let, $\Phi\in L^1((0,T),L^2(\Omega))$, since g is continuous we also have

(i) $g(\varphi^\varepsilon(t,x))\Phi(t,x)\to g(\varphi^0(t,x))\Phi(t,x)$ as $\varepsilon\to 0$ a.e. $(t,x)\in(0,T)\times\Omega$ and from (2.7), we have

(ii) $|g(\varphi^\varepsilon(t,x))\Phi(t,x)|\leq C|\Phi(t,x)|(1+|\varphi^\varepsilon(t,x)|^r)\equiv h(t,x)$.

Finally, we will show that $h(t,x)\in L^1((0,T)\times\Omega)$ and from Lebesgue dominated convergence theorem we conclude.

In effect, $\int_0^T\int_\Omega h(t,x)dxdt=C\int_0^T\int_\Omega|\Phi|+C\int_0^T\int_\Omega|\Phi||\varphi^\varepsilon|^r$ and using now Hölder inequality together with $H^1(\Omega)\subset L^{2r}(\Omega)$ (since $r\leq\frac{N}{(N-2)_+}$), we obtain

$$\int_0^T\int_\Omega|\Phi||\varphi^\varepsilon|^r\leq\int_0^T\left(\int_\Omega|\Phi|^2\right)^{\frac{1}{2}}\left(\int_\Omega|\varphi^\varepsilon|^{2r}\right)^{\frac{1}{2}}\leq C\|\varphi^\varepsilon\|^r_{L^\infty((0,T),H^1(\Omega))}\|\Phi\|_{L^1((0,T),L^2(\Omega))}\leq C$$

and we get (3.14). □

Theorem 3.6. *Assume that the $g\in C^1(I\!R)$ satisfying, (2.12) and (2.16), i.e.*

$$|g(s)|\leq C(1+|s|^r),|g'(s)|\leq C(1+|s|^{r-1}),1\leq r\leq\frac{N}{(N-2)_+},\liminf_{|s|\to\infty}\frac{g(s)}{s}>0,$$

with $C>0$ and $s\in I\!R$.

Besides, the concentrated-potential function V_ε verifies (2.11) and (2.17), i.e.

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|V_\varepsilon|^\rho\leq C,\quad C>0 \text{ independent of } \varepsilon, V_\varepsilon(x)\geq 0\quad\forall x\in\omega_{\varepsilon_0} \text{ with } 0<\varepsilon\leq\varepsilon_0$$

where $\rho>N-1$ and $\max\{\frac{N-1}{\rho},\frac{1}{2}\}<q<1$.

We consider the initial data $(\varphi_0^\varepsilon,v_0^\varepsilon)\in H^1(\Omega)\times L^2(\Omega)$ satisfying (3.6), i.e.

$$\|(\varphi_0^\varepsilon, v_0^\varepsilon)\|_{H^1(\Omega)\times L^2(\Omega)} \leq C$$

with $C > 0$ independent of ε.

Then, for any $T > 0$, we consider $(\varphi^\varepsilon(t), v^\varepsilon(t))$ the solution of (2.2) given by Proposition 2.5 and by taking subsequences if necessary, there exists $\varphi^0(t,x) \in L^\infty((0,T), H^1(\Omega))$ and $v^0(t,x) \in L^\infty((0,T), L^2(\Omega)) \cap L^2((0,T), H^1(\Omega))$ such that

$$(\varphi^\varepsilon, v^\varepsilon) \to (\varphi^0, v^0) \quad weakly^* \text{ in } L^\infty((0,T), H^1(\Omega)\times L^2(\Omega)) \text{ as } \varepsilon \to 0, \tag{3.15}$$

$$(\varphi^\varepsilon, v^\varepsilon) \to (\varphi^0, v^0) \text{ strongly in } C([0,T], L^2(\Omega)\times H^{-1}(\Omega)) \text{ as } \varepsilon \to 0, \tag{3.16}$$

$$(\varphi_t^\varepsilon, v_t^\varepsilon) \to (\varphi_t^0, v_t^0) \quad weakly \text{ in } L^2((0,T), L^2(\Omega)\times H^{-1}(\Omega)) \text{ as } \varepsilon \to 0. \tag{3.17}$$

Moreover, if $\varepsilon \to 0$ then

$$v^\varepsilon \to v^0 \quad weakly \text{ in } L^2((0,T), H^1(\Omega)). \tag{3.18}$$

Finally, taking a smooth enough function $\Phi(t,x)$ we get

$$\int_0^T \int_\Omega g(\varphi^\varepsilon)\Phi \to \int_0^T \int_\Omega g(\varphi^0)\Phi \to \text{ as } \varepsilon \to 0 \tag{3.19}$$

and

$$\frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} V_\varepsilon(x)\varphi^\varepsilon\Phi \to \int_0^T \int_\Gamma V_0(x)\varphi^0\Phi \to \text{ as } \varepsilon \to 0 \tag{3.20}$$

where $V_0(x)$ is the function given in Lemma 3.4 and $\Phi \in L^1((0,T), H^q(\Omega))$ with $1 > q > \max\{\frac{N-1}{\rho}, \frac{1}{2}\}$.

Proof. We split the proof in several steps. Hereafter we will denote by C or K any positive constant which is independent of ε.
Step 1. From (2.15) together with (2.18) in the above Proposition 2.5 we have the global solutions of (2.2) satisfies

$$\int_0^t \|\varphi_t^\varepsilon\|^2 + \int_0^t \frac{a}{c}\|v_t^\varepsilon\|_{-1}^2 + \frac{k_1}{2}\|\nabla\varphi^\varepsilon\|^2 + \delta\|\varphi^\varepsilon\|^2 + \frac{b}{2}\|\frac{a}{b}v^\varepsilon - \varphi^\varepsilon\|^2 - C(\delta)|\Omega| \leq E_\varepsilon(\varphi_0^\varepsilon, v_0^\varepsilon)$$

for any $t > 0$, where $\|.\|$ and $\|.\|_{-1}$ denote the norm in $L^2(\Omega)$ and $H^{-1}(\Omega) = (H_0^1(\Omega))'$ respectively, $\delta, C(\delta)$ are positive constants **independent of** ε and $E_\varepsilon(\varphi_0^\varepsilon, v_0^\varepsilon)$ is given by (2.14).

Moreover, from the assumption on the initial data (3.6) using (3.8) in Lemma 3.3, we also have that $E_\varepsilon(\varphi_0^\varepsilon, v_0^\varepsilon) \leq C$, with $C > 0$ **independent of** ε and thus

$$\int_0^t \|\varphi_t^\varepsilon\|^2 + \int_0^t \frac{a}{c}\|v_t^\varepsilon\|_{-1}^2 + \frac{k_1}{2}\|\nabla\varphi^\varepsilon\|^2 + \delta\|\varphi^\varepsilon\|^2 + \frac{b}{2}\|\frac{a}{b}v^\varepsilon - \varphi^\varepsilon\|^2 \leq K$$

with $K > 0$ **independent of** ε. In particular, for $T > 0$ as in (2.19), we have

$$\sup_{0\leq t\leq T}\|(\varphi^\varepsilon, v^\varepsilon)\|_{H^1(\Omega)\times L^2(\Omega)} = \|(\varphi^\varepsilon, v^\varepsilon)\|_{L^\infty((0,T),H^1(\Omega)\times L^2(\Omega))} \leq K, \tag{3.21}$$

$$\|\varphi_t^\varepsilon\|_{L^2((0,T),L^2(\Omega))} = \|\varphi_t^\varepsilon\|_{L^2((0,T)\times\Omega)} \leq K, \tag{3.22}$$

$$\|v_t^\varepsilon\|_{L^2((0,T),H^{-1}(\Omega))} \leq K. \tag{3.23}$$

From (3.21) there exists $\varphi^0(t,x) \in L^\infty((0,T), H^1(\Omega))$ and $v^0(t,x) \in L^\infty((0,T), L^2(\Omega))$ and subsequences such that

$$(\varphi^\varepsilon, v^\varepsilon) \to (\varphi^0, v^0) \quad weakly^* \text{ in } L^\infty((0,T), H^1(\Omega)\times L^2(\Omega)) \text{ as } \varepsilon \to 0,$$

and we obtain (3.15).
Step 2. In this part we will prove

$$\varphi^\varepsilon \to \varphi^0 \text{ in } C([0,T], L^2(\Omega)), \quad v^\varepsilon \to v^0 \text{ in } C([0,T], H^{-1}(\Omega)) \text{ as } \varepsilon \to 0,$$

and we get (3.16).

In order to prove this, we consider the family $\{(\varphi^\varepsilon, v^\varepsilon)\}_\varepsilon$ with

$$(\varphi^\varepsilon, v^\varepsilon) : [0,T] \to L^2(\Omega)\times H^{-1}(\Omega)$$

and using (3.22) and (3.23), first we will prove that $\{(\varphi^\varepsilon, v^\varepsilon)\}_\varepsilon$ is equicontinuous.

In effect, for $t_i \in [0,T], i = 1,2$ using Hölder inequality together with (3.22), we obtain

$$\|\varphi^\varepsilon(t_1) - \varphi^\varepsilon(t_2)\|_{L^2(\Omega)} \leq \int_{t_1}^{t_2} \|\varphi_t^\varepsilon\|_{L^2(\Omega)} \leq \left(\int_{t_1}^{t_2} \|\varphi_t^\varepsilon\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} |t_2 - t_1|^{\frac{1}{2}} \leq C|t_2 - t_1|^{\frac{1}{2}}$$

and analogously from (3.23)

$$\|v^\varepsilon(t_1) - v^\varepsilon(t_2)\|_{H^{-1}(\Omega)} \leq \int_{t_1}^{t_2} \|v_t^\varepsilon\|_{H^{-1}(\Omega)} \leq \left(\int_{t_1}^{t_2} \|v_t^\varepsilon\|_{H^{-1}}^2\right)^{\frac{1}{2}} |t_2 - t_1|^{\frac{1}{2}} \leq C|t_2 - t_1|^{\frac{1}{2}}.$$

Next, for every $t \in [0,T]$ fixed, from (3.21) we also have $(\varphi^\varepsilon(t,.), v^\varepsilon(t,.))$ is uniformly bounded in $H_0^1(\Omega)\times L^2(\Omega) \subset L^2(\Omega)\times H^{-1}(\Omega)$ with compact embedding.

Therefore, from Ascoli-Arzela's Theorem, there exists a subsequence which converge to a limit function (φ^0, v^0) in $C([0,T], L^2(\Omega)\times H^{-1}(\Omega))$.

In particular, for $t=0$, we have $(\varphi_0^\varepsilon, v_0^\varepsilon) = (\varphi^\varepsilon(0,.), v^\varepsilon(0,.)) \to (\varphi^0(0,.), v^0(0,.)) = (\varphi_0^0, v_0^0)$ as $\varepsilon \to 0$.

Step 3. Here, using standard techniques as in [10] we will prove (3.17), i.e.

$$\varphi_t^\varepsilon \to \varphi_t^0 \text{ weakly in } L^2((0,T), L^2(\Omega)) \text{ and } v_t^\varepsilon \to v_t^0 \text{ weakly in } L^2((0,T), H^{-1}(\Omega)) \text{ as } \varepsilon \to 0.$$

First from (3.22) taking another subsequence if necessary, there exists $w \in L^2((0,T), L^2(\Omega))$ such that $\varphi_t^\varepsilon \to w$ *weakly* in $L^2((0,T), L^2(\Omega))$.

Second, we will prove $w = \varphi_t^0$. In effect, for every $\Phi(t,.)$ smooth such that $\Phi(T,.) = \Phi_t(T,.) = 0$ and integrating by parts, we have

$$\int_0^T \int_\Omega \varphi_t^\varepsilon \Phi = -\int_\Omega \varphi_0^\varepsilon \Phi(0) - \int_0^T \int_\Omega \varphi^\varepsilon \Phi_t.$$

From the convergence of $\varphi_0^\varepsilon, \varphi^\varepsilon$ and φ_t^ε, we obtain

$$\int_0^T \int_\Omega w\Phi = -\int_\Omega \varphi_0^0 \Phi(0) - \int_0^T \int_\Omega \varphi^0 \Phi_t = \int_0^T \int_\Omega \varphi_t^0 \Phi.$$

and we get $w = \varphi_t^0$.

Finally, from (3.23) we obtain a subsequence such that $v_t^\varepsilon \to w^*$ *weakly* in $L^2((0,T), H^{-1}(\Omega))$ and working analogously we conclude $w^* = v_t^0$.

Step 4. Now, we will show (3.18), i.e. $v^\varepsilon \to v^0$ *weakly* in $L^2((0,T), H^1(\Omega))$.

First from (3.21) we also have $\|\Delta\varphi^\varepsilon\|_{L^\infty((0,T),H^{-1}(\Omega))} \le K$, and taking into account that $k_2\Delta v^\varepsilon = v_t^\varepsilon + c\Delta\varphi^\varepsilon$ we have $\|\Delta v^\varepsilon\|_{L^2((0,T),H^{-1}(\Omega))} \le K$.

Next, from Proposition 2.4 taking $2\theta = 1+q$ we note $v_t^\varepsilon = -\Delta(c\varphi^\varepsilon - k_2 v^\varepsilon) \in L^2(\Omega)$, and we use the elliptic regularity together with (3.21) in order to prove that $\|v^\varepsilon\|_{L^2((0,T),H^1(\Omega))} \le K$.

In effect, multiplying this equation, $k_2\Delta v^\varepsilon = v_t^\varepsilon + c\Delta\varphi^\varepsilon$ by v^ε and integrating by parts, we get

$$k_2 \int_\Omega |\nabla v^\varepsilon|^2 = \frac{1}{2}\frac{d}{dt}(\int_\Omega |v^\varepsilon|^2) + c\int_\Omega \nabla\varphi^\varepsilon \nabla v^\varepsilon,$$

thus, integrating in $t \in [0,T]$ we obtain

$$k_2 \int_0^t \int_\Omega |\nabla v^\varepsilon|^2 = \frac{1}{2}\int_\Omega |v^\varepsilon(t)|^2 - \frac{1}{2}\int_\Omega |v^\varepsilon(0)|^2 + c\int_0^t \int_\Omega \nabla\varphi^\varepsilon \nabla v^\varepsilon.$$

Besides, using now Hölder and Young inequality with $p=2$ respectively, there exist a positive constant C depending on k_2 but independent of ε, such that

$$\int_\Omega \nabla\varphi^\varepsilon \nabla v^\varepsilon \le C\|\nabla\varphi^\varepsilon\|\|\nabla v^\varepsilon\| \le \frac{k_2}{2}\|\nabla v^\varepsilon\|^2 + C\|\nabla\varphi^\varepsilon\|^2,$$

and using now (3.21) together with (3.6), we get

$$\frac{k_2}{2}\int_0^T \int_\Omega |\nabla v^\varepsilon|^2 \le K \text{ and } \|v^\varepsilon\|_{L^2((0,T),H^1(\Omega))} \le K.$$

Finally, taking another subsequence if necessary, there exists $v^* \in L^2((0,T), H^1(\Omega))$ such that $v^\varepsilon \to v^*$ *weakly* in $L^2((0,T), H^1(\Omega))$ and working analogously as above steps, we get $v^* = v^0$ and we conclude (3.18).

Step 5.

Using now (3.16), in particular we have $\varphi^\varepsilon(t,x) \to \varphi^0(t,x)$ as $\varepsilon \to 0$ a.e. $(t,x) \in (0,T)\times\Omega$ and we also have $\|\varphi^\varepsilon\|_{L^\infty((0,T),H^1(\Omega))} \le C$, so from Lemma 3.5 we get (3.19) for $\Phi \in L^1((0,T), L^2(\Omega))$.

Finally, we can use Lemma 3.4 to conclude (3.20) for every $\Phi \in L^1((0,T), H^q(\Omega))$ with $q > \max\{\frac{N-1}{\rho}, \frac{1}{2}\}$. $\square$

Now, we identify the limit function above as a weak solution of the limit problem (P_0)(3.1).

Proposition 3.7. *Under the notations and hypotheses of Theorem 3.6, if we consider the limit function (φ^0, v^0) in Theorem 3.6, then we have:*

(I) (φ^0, v^0) satisfies (3.2), i.e.

$$\begin{cases} 0 = \int_\Omega \varphi_t^0\phi + \int_\Omega (g(\varphi^0) + b\varphi^0 - av^0)\phi + k_1\int_\Omega \nabla\varphi^0\nabla\phi - \frac{a}{2}\int_\Gamma V_0(x)\varphi^0\phi \\ 0 = \int_\Omega v_t^0\phi + k_2\int_\Omega \nabla v^0\nabla\phi - c\int_\Omega \nabla\varphi^0\nabla\phi \end{cases}$$

for every $\phi \in H^1(\Omega)$ and it is a weak solution of limit problem (P_0)(3.1).

(II) Moreover, (φ^0, v^0) is the unique solution of limit problem (P_0)(3.1) given by Proposition 3.2 and actually all the family $(\varphi^\varepsilon, v^\varepsilon)$ converges to (φ^0, v^0) (and not only a subsequence).

Proof.

(I) First, we note that from (3.15),(3.16), (3.17) and (3.18) we have

$$(\varphi^0, v^0) \in L^\infty((0,T), H^1(\Omega)\times L^2(\Omega)) \cap C([0,T], L^2(\Omega)\times H^{-1}(\Omega)),$$

$$v^0 \in L^2((0,T), H^1(\Omega)), \quad (\varphi^0_t, v^0_t) \in L^2((0,T), L^2(\Omega)\times H^{-1}(\Omega)) \text{ with } (\varphi^0(0), v^0(0)) = (\varphi^0_0, v^0_0).$$

Second, since $(\varphi^\varepsilon, v^\varepsilon)$ is the solution of (2.2) given by Proposition 2.5, then if $\Phi \in L^2((0,T), H^1(\Omega))$ we have

$$\int_0^T \int_\Omega \varphi^\varepsilon_t \Phi + k_1 \int_0^T \int_\Omega \nabla\varphi^\varepsilon \nabla\Phi + \int_0^T \int_\Omega [g(\varphi^\varepsilon) + b\varphi^\varepsilon - a v^\varepsilon]\Phi + \frac{a}{2\varepsilon}\int_0^T \int_{\omega_\varepsilon} V_\varepsilon(x)\varphi^\varepsilon \Phi = 0 \tag{3.24}$$

and

$$\int_0^T \langle v^\varepsilon_t, \Phi\rangle_{-1,1} + k_2 \int_0^T \int_\Omega \nabla v^\varepsilon \nabla\Phi - c\int_0^T \int_\Omega \nabla\varphi^\varepsilon\nabla\Phi = 0. \tag{3.25}$$

Then, passing to the limit as $\varepsilon \to 0$ in (3.24) and (3.25), from (3.15), (3.16), (3.17), (3.18), (3.19) and (3.20) respectively, we get

$$\int_0^T \int_\Omega \varphi^0_t \Phi + k_1 \int_0^T \int_\Omega \nabla\varphi^0 \nabla\Phi + \int_0^T \int_\Omega [g(\varphi^0) + b\varphi^0 - a v^0]\Phi + \frac{a}{2}\int_0^T \int_\Gamma V_0(x)\varphi^0 \Phi = 0 \tag{3.26}$$

and

$$\int_0^T \langle v^0_t, \Phi\rangle_{-1,1} + k_2 \int_0^T \int_\Omega \nabla v^0 \nabla\Phi - c\int_0^T \int_\Omega \nabla\varphi^0\nabla\Phi = 0. \tag{3.27}$$

Now, we consider $\Phi = \xi(t)\Psi(x) \in L^2((0,T), H^1(\Omega))$ with $\xi(t) \in L^2(0,T)$ and $\Psi(x) \in H^1(\Omega)$ in (3.26) and (3.27), then we obtain:

$$\int_0^T \xi(t)\int_\Omega \varphi^0_t \Psi + k_1 \int_0^T \xi(t)\int_\Omega \nabla\varphi^0 \nabla\Psi + \int_0^T \xi(t)\int_\Omega [g(\varphi^0) + b\varphi^0 - a v^0]\Psi + \frac{a}{2}\int_0^T \xi(t)\int_\Gamma V_0(x)\varphi^0 \Psi = 0 \tag{3.28}$$

$$\int_0^T \xi(t)\langle v^0_t, \Psi\rangle_{-1,1} + k_2 \int_0^T \xi(t)\int_\Omega \nabla v^0 \nabla\Psi - c\int_0^T \xi(t)\int_\Omega \nabla\varphi^0\nabla\Psi = 0. \tag{3.29}$$

This is, (φ^0, v^0) satisfies (3.2) a.e. $t \in [0,T]$.

By this way, in particular we get

$$\begin{cases} \int_\Omega \varphi^0_t \Psi^* + k_1 \int_\Omega \nabla\varphi^0\nabla\Psi^* + \int_\Omega [g(\varphi^0) + b\varphi^0 - a v^0]\Psi^* & = 0 \\ \langle v^0_t, \Psi^*\rangle_{-1,1} + k_2 \int_\Omega \nabla v^0 \nabla\Psi^* - c\int_\Omega \nabla\varphi^0\nabla\Psi^* & = 0 \end{cases}$$

a.e. $t \in [0,T]$ and for every $\Psi^* \in H^1_0(\Omega)$, this is

$$\begin{cases} \varphi^0_t - k_1\Delta\varphi^0 + g(\varphi^0) + b\varphi^0 - a v^0 & = 0 \\ v^0_t - k_2 \Delta v^0 + c\Delta\varphi^0 & = 0. \end{cases} \tag{3.30}$$

Finally, taking into account that for every $\Psi \in H^1(\Omega)$

$$\int_\Omega -\Delta\varphi^0\Psi = \int_\Omega \nabla\varphi^0\nabla\Psi - \int_\Gamma \frac{\partial(\varphi^0)}{\partial(\vec{n})}\Psi \text{ and } \int_\Omega -\Delta v^0\Psi = \int_\Omega \nabla v^0\nabla\Psi - \int_\Gamma \frac{\partial(v^0)}{\partial(\vec{n})}\Psi,$$

using now (3.30) together with (3.28) and (3.29), we also have that for every $\Psi \in H^1(\Omega)$ and a.e. $t \in [0,T]$,

$$\begin{cases} k_1 \int_\Gamma \frac{\partial(\varphi^0)}{\partial(\vec{n})}\Psi + \frac{a}{2}\int_\Gamma V_0(x)\varphi^0\Psi & = 0 \\ k_2 \int_\Gamma \frac{\partial(v^0)}{\partial(\vec{n})}\Psi - c\int_\Gamma \frac{\partial(\varphi^0)}{\partial(\vec{n})}\Psi & = 0, \end{cases}$$

this is

$$\begin{cases} k_1 \frac{\partial(\varphi^0)}{\partial(\vec{n})} + \frac{a}{2}V_0(x)\varphi^0 & = 0 \\ k_2 \frac{\partial(v^0)}{\partial(\vec{n})} - c\frac{\partial(\varphi^0)}{\partial(\vec{n})} & = 0, \end{cases} \equiv \begin{cases} k_1 \frac{\partial(\varphi^0)}{\partial(\vec{n})} + \frac{a}{2}V_0(x)\varphi^0 = 0 \\ k_2 \frac{\partial(v^0)}{\partial(\vec{n})} + \frac{ac}{2k_1}V_0(x)\varphi^0 = 0 \end{cases}$$

a.e. $(t,x) \in [0,T]\times\Gamma$.

Therefore, (φ^0, v^0) is a weak solution of (P_0)(3.1).

(II) First, we note that (3.24) is true for $\Phi = \xi_1(t)\Psi_1(x) \in L^1((0,T), H^q(\Omega))$ with $\xi_1(t) \in L^1(0,T), \Psi_1(x) \in H^q(\Omega)$ and passing to the limit as $\varepsilon \to 0$ we also obtain (3.28), this is:

$$\int_\Omega \varphi^0_t\Psi_1 + k_1\int_\Omega \nabla\varphi^0\nabla\Psi_1 + \int_\Omega [g(\varphi^0) + b\varphi^0 - a v^0]\Psi_1 + \frac{a}{2}\int_\Gamma V_0(x)\varphi^0\Psi_1 = 0$$

a.e. $t \in (0,T)$ and thus

$$\varphi^0_t - k_1\Delta_N\varphi^0 - a v^0 = -g(\varphi^0) - b\varphi^0 - \frac{a}{2}V_0\gamma(\varphi^0) \text{ in } H^{-q}(\Omega).$$

Now, we can take $\Phi = \xi_2(t)\Psi_2(x) \in L^2((0,T), H^{q+1}(\Omega)) \subset L^2((0,T), H^1(\Omega))$ in (3.25) with $\xi_2(t) \in L^2(0,T), \Psi_2(x) \in H^{q+1}(\Omega)$, and passing to the limit as $\varepsilon \to 0$ we also obtain (3.29), this is:

$$\langle \varphi^0_t, \Psi_2\rangle_{-1,1} + k_2\int_\Omega \nabla v^0\nabla\Psi_2 - \int_\Omega \nabla\varphi^0\nabla\Psi_2 = 0$$

a.e. $t \in (0,T)$ and thus

$$v_t^0 - k_2\Delta_N v^0 + c\Delta_N \varphi^0 = 0 \text{ in } H^{-(q+1)}(\Omega).$$

Therefore, $U = (\varphi^0, v^0)^\perp$ satisfies $U_t + AU = H(U)$ in $Y = H^{-q}(\Omega) \times H^{-(q+1)}(\Omega)$, this is $U = (\varphi^0, v^0)^\perp$ is the unique solution given by Proposition 3.2. Since any subsequence of the solutions $(\varphi^\epsilon, v^\epsilon)$ that converges as in Theorem 3.6 has the same limit, then all the family converges to (φ^0, v^0). $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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No data was used for the research described in the article.

References

[1] Caginalp G. An analysis of a phase field model of a free boundary. Arch Ration Mech Anal 1986;92:205–45.
[2] Caginalp G. The dynamics of a conserved phase field system:Stefan-like, Hele-Shaw, and Cahn-Hilliard models as asymptotic limits. IMA J Appl Math 1990;44:77–94.
[3] Caginalp G. Phase field models and sharp interface limits: some differences in subtle situations. Rocky Mountain J Math 1991;21(2):603–16.
[4] Caginalp G, Fife PC. Dynamics of layered interfaces arising from phase boundaries. SIAM J Appl Math 1988;48:506–18.
[5] Jiménez-Casas A, Rodriguez-Bernal A. Dinámica en dimensión finita: Modelos de campos de fase y un termosifón cerrado. (Ph. D. Thesis), U.C.M.; 1996.
[6] Jiménez-Casas A, Rodríguez-Bernal A. Asymptotic behaviour for a phase field model in higher oorder Sobolev spaces. Rev Matem Compl XV 2008;1:213–48.
[7] Jiménez-Casas A, Rodriguez-Bernal A. Linear stability analysis and metastable solutions for a phase-field model. Proc R Soc Edimburgh 1999;129A:571–600.
[8] Arrieta JM, Jiménez-Casas A, Rodríguez-Bernal A. Flux terms and robin boundary conditions as limit of reactions and potentials concentrating at the boundary. Rev Iberoam Matem 2008;24(1):183–211.
[9] Jiménez-Casas A, Rodríguez-Bernal A. PDE problems:with concentrating terms near the boundary. Commun Pure Appl Anal 2020;19(4):2147–95.
[10] Jiménez-Casas A, Rodríguez-Bernal A. A nonlinear damped transmission problem as limit of wave equations with concentrating nonlinear terms away from the boundary. Nonlin Anal-Theory Methods Appl 2024;241(2024):113492.
[11] Amann H. Parabolic evolution equations and nonlinear boundary conditions. J Differential Equations 1988;72:201–69.
[12] Henry D. Geometric theory of semilinear parabolic equations. Lectures notes in mathematics, vol. 840, Springer-Verlag; 1981.
[13] Amann H. Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In: Schmeisser/triebel: function spaces, differential operators and nonlinear analysis. Teubner-Texte Math 1993;133:9–126.