New upper bounds for Ramanujan primes

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Abstract

For $n \geq 1$, the n^{th} Ramanujan prime is defined as the smallest positive integer R_n such that for all $x \geq R_n$, the interval $(\frac{x}{2}, x]$ has at least n primes. We show that for every $\epsilon > 0$, there is a positive integer N such that if $\alpha = 2n\left(1 + \dfrac{\log 2 + \epsilon}{\log n + j(n)}\right)$, then $R_n < p_{[\alpha]}$ for all $n > N$, where p_i is the i^{th} prime and $j(n) > 0$ is any function that satisfies $j(n) \to \infty$ and $nj'(n) \to 0$.

1 Introduction

For $n \geq 1$, the n^{th} Ramanujan prime is defined as the smallest positive integer R_n, such that for all $x \geq R_n$, the interval $(\frac{x}{2}, x]$ has at least n primes. Note that by the minimality condition, R_n is prime and the interval $(\frac{R_n}{2}, R_n]$ contains exactly n primes. Let $R_n = p_s$, where p_i denotes the i^{th} prime. Sondow [7] showed that $p_{2n} < R_n < p_{4n}$ for all n, and conjectured that $R_n < p_{3n}$ for all n. This conjecture was proved by Laishram [4], and the upper bound p_{3n} improved by various authors ([1], [8]). Subsequently, Srinivasan [9] and Axler [1] improved these bounds by showing that for every $\epsilon > 0$, there exists an integer N such that

$$R_n < p_{[2n(1+\epsilon)]} \text{ for all } n > N.$$

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Using the method in [9] (outlined below), a further improvement was presented by Srinivasan and Nicholson, who proved that

$$s < 2n\left(1 + \frac{3}{\log n + \log(\log n) - 4}\right)$$

for all $n > 241$. The above result follows from a special case of our main theorem given below. Yang and Togbe [11], also used the method in [9], to give tight upper and lower bounds for R_n for large n (greater than 10^{300}). For some interesting generalizations of Ramanujan primes the reader may refer to [2], [5] and [6].

The main idea in [9] is to define a function $F(x)$ that is decreasing for $x \geq 2n$ and that satisfies $F(s) > 0$. Then, an $\alpha > 2n$ is found such that $F(\alpha) < 0$ for $n > N$, which would imply that $s < \alpha$ for $n > N$ given the decreasing nature of F. We employ a variation of this method, where we first show that $F(\alpha)$ is a decreasing function for $n > N$. Then we find an integer greater than N for which $F(\alpha) < 0$, which leads us to the desired result. Our main result is the following.

Theorem 1.1. *Let $R_n = p_s$ and $\epsilon > 0$. Let $j(n) > 0$ be a function such that $j(n) \to \infty$ and $nj'(n) \to 0$ as $n \to \infty$ and let*

$$g(n) = \frac{\log n + j(n)}{\log 2 + \epsilon}.$$

Then there exists a positive integer N such that for all $n > N$, we have $s < \alpha$, where $\alpha = 2n\left(1 + \frac{1}{g(n)}\right)$.

Let $\log_2 x$ denote $\log \log x$. In the following corollary we record a bound obtained with $\epsilon = 0.5$, where $j(n)$ is chosen so as to minimize the number of calculations. Similar results can be given for smaller values of ϵ (with different $j(n)$) where the determination of N depends solely on computational power.

Corollary 1.1. *Let $R_n = p_s$. Then for $n > 43$ we have $s < 2n\left(1 + \frac{1}{g(n)}\right)$, where*

$$g(n) = \frac{\log n + \log_2 n - \log 2 - 0.5}{\log 2 + 0.5}.$$

2 The basic functions and lemmas

We will use the following bounds for the k^{th} prime given by Dusart.

Lemma 2.1. *The following hold for the* k^{th} *prime* p_k*.*

1. $p_k > k\left(\log k + \log_2 k - 1 + \frac{\log_2 k - 2.1}{\log k}\right)$ *for all* $k \geq 3$.
2. $p_k < k\left(\log k + \log_2 k - 1 + \frac{\log_2 k - 2}{\log k}\right)$ *for all* $k \geq 688383$.

Proof. See [3] □

Let

$$U(k) = k\left(\log k + \log_2 k - 1 + \frac{\log_2 k - 2}{\log k}\right)$$

and

$$L(k) = k\left(\log k + \log_2 k - 1 + \frac{\log_2 k - 2.1}{\log k}\right).$$

Note that $U(x) = L(x) + f(x)$ where $f(x) = \dfrac{0.1x}{\log x}$. We define

$$F(x,n) = U(x) - 2L(x-n) = U(x) - 2U(x-n) + 2f(x-n)$$

and

$$G(n) = F(\alpha, n),$$

where $\alpha = 2n\left(1 + \frac{1}{g(n)}\right)$ and $g(n)$ is a function that satisfies $g(n) \geq 1$ and $g(n) \to \infty$ as $n \to \infty$.

Lemma 2.2. *Let* $R_n = p_s$*. Then the following hold.*

1. $p_{s-n} < \frac{1}{2}p_s$.
2. $2n < s < 2.4n$ *for all* $n > 43$.
3. $F(x,n)$ *is a decreasing function for all* $x \geq 2n$ *and* $F(s,n) > 0$ *for* $n \geq 688383$.

Proof. For parts 1 and 2 see [9, Lemma 2.1] and [9, Remark 2.1] respectively. For part 3 see [11]. □

The following lemma contains useful results that include an expression for the derivative $G'(n)$ in terms of the function $U(x)$.

Lemma 2.3. *Let* $A = U'(\alpha) - U'(\alpha - n)$*. Then the following hold.*

1. $A = A(n) \to \log 2$ *as* $n \to \infty$.

2. $\frac{1}{2}G'(n) = A + f'(\alpha - n) + \left(\frac{n}{g(n)}\right)' (A - U'(\alpha - n) + 2f'(\alpha - n))$.

3. $L'(x) > \log x + \log_2 x$ *for* $x > 20$.

4. $A + f'(\alpha - n) - \log 2 < \log\left(\frac{\log \alpha}{\log(\alpha - n)}\right) + \frac{\log_2 \alpha}{\log \alpha} + \frac{1.1}{\log(\alpha - n)} + \frac{\log_2(\alpha - n)}{\log^2(\alpha - n)}$.

Proof. We have

$$U'(x) = \log x + \log_2 x - \frac{1}{\log x} + \frac{3}{\log^2 x} - \frac{\log_2 x}{\log^2 x} + \frac{\log_2 x}{\log x} \tag{1}$$

and hence

$$A = \log\left(\frac{\alpha}{\alpha - n}\right) + \log\left(\frac{\log(\alpha)}{\log(\alpha - n)}\right) + t(n),$$

where $t(n) \to 0$ as $n \to \infty$. As $\alpha = 2n\left(1 + \frac{1}{g(n)}\right)$ and $g(n) \to \infty$, we have $A \to \log 2$.

For the second part of the lemma, $G(n) = U(\alpha) - 2U(\alpha - n) + 2f(\alpha - n)$, which gives $G'(n) = U'(\alpha)\alpha' - 2U'(\alpha - n)(\alpha' - 1) + 2f'(\alpha - n)(\alpha' - 1)$. As $\alpha' = 2 + 2\left(\frac{n}{g(n)}\right)'$, we have

$$\frac{1}{2}G'(n) = U'(\alpha)\left(1 + \left(\frac{n}{g}\right)'\right) + \left(1 + 2\left(\frac{n}{g}\right)'\right)(f'(\alpha - n) - U'(\alpha - n))$$

and the result follows by the definition of A.

For part 3 we have

$$L'(x) = \log x + \log_2 x + \frac{\log_2 x}{\log x} - \frac{\log_2 x}{\log^2 x} - \frac{1.1}{\log x} + \frac{3.1}{\log^2 x}$$

from which the claim follows as for $n > 20$ we have $\frac{\log_2 x}{\log x} - \frac{\log_2 x}{\log^2 x} - \frac{1.1}{\log x} > 0$.

For the last part, we have

$$\begin{aligned} &A - \log 2 + f'(\alpha - n) \\ =&\log\left(\frac{\log \alpha}{\log(\alpha - n)}\right) + \frac{\log_2 \alpha}{\log \alpha} + \frac{1.1}{\log(\alpha - n)} + \frac{\log_2(\alpha - n)}{\log^2(\alpha - n)} + T, \end{aligned}$$

where

$$T = \log\left(\frac{1 + \frac{1}{g(n)}}{1 + \frac{2}{g(n)}}\right) - \frac{\log_2(\alpha - n)}{\log(\alpha - n)} - \frac{1}{\log \alpha} - \frac{\log_2 \alpha}{\log^2 \alpha} + \frac{3}{\log^2 \alpha} - \frac{3.1}{\log^2(\alpha - n)} < 0$$

as $\frac{3}{\log^2 \alpha} - \frac{3.1}{\log^2(\alpha - n)} < 0$. □

3 Proofs of main results

The following lemma shows that $G'(n)$ is a decreasing function for large n, which is crucial in the proof of Theorem 1.1.

Lemma 3.1. *Let $\epsilon > 0$ and*

$$g(n) = \frac{\log n + j(n)}{\log 2 + \epsilon},$$

where $j(n) > 0$ is a function that satisfies $j(n) \to \infty$ and $nj'(n) \to 0$ as $n \to \infty$. Then $G'(n) \to -2\epsilon$.

Proof. We have $\left(\frac{n}{g(n)}\right)' = \frac{(\log 2+\epsilon)(\log n + j(n) - 1 - nj'(n))}{(\log n + j(n))^2}$ and therefore $\left(\frac{n}{g(n)}\right)' \to 0$ as $n \to \infty$. By our assumption on $j(n)$ it follows (using L'Hôpital's rule) that $\frac{j(n)}{\log n} \to 0$ which gives $\left(\frac{n}{g(n)}\right)' \log(\alpha - n) \to \log 2 + \epsilon$ (as $\frac{\log(\alpha - n)}{\log n} \to 1$). It is easy to see that $\left(\frac{n}{g(n)}\right)' \log_2(\alpha - n) \to 0$. It follows that $\left(\frac{n}{g(n)}\right)' U'(\alpha - n) \to \log 2 + \epsilon$ (see equation (1)). Lastly note that $f'(x) \to 0$ as $x \to \infty$. The result follows now on using all the above and the fact that $A \to \log 2$ (Lemma 2.3 part 1) in part 2 of Lemma 2.3. □

Proof of Theorem 1.1 We will first show that there exists a positive integer N, such that $G(n) < 0$ for $n > N$. We have $G'(n) \to -2\epsilon$ by the lemma above, which means that if $0 < \delta < 2\epsilon$, then there exists an integer M, such that for all $n > M$ we have $|G'(n) + 2\epsilon| < \delta$, that is

$$-2\epsilon - \delta < G'(n) < -2\epsilon + \delta,$$

for all $n > M$. Let a and b be two integers such that $M < a < b$. Then $G(b) - G(a) = \int_a^b G'(n)dn < (b-a)(-2\epsilon + \delta) < 0$. If a is fixed, it follows that $G(b) < G(a) + (b-a)(-2\epsilon+\delta) < 0$ for large b. Therefore there exists a positive integer $N > M$, such that for all $n > N$, we have $G(n) = F(\alpha, n) < 0$.

We may assume that $N > 688383$ so that from Lemma 2.2, part 3 we have $F(s, n) > 0$. Moreover, from the same lemma we have $F(x, n)$ is decreasing for $x \geq 2n$. As s and α are both bigger than $2n$, we have $s < \alpha$ for $n > N$ and the result follows. □

Proof of Corollary 1.1

Let $\epsilon = \epsilon_1 + \epsilon_2 = 0.5$. We will first show that for $n > 688383$ we have $G'(n) < 0$.

Let $\epsilon_1 = 0.1$. It is easy to verify that for $n > 688383$ we have

$$\frac{1+\log n}{\log n(\log n + \log_2 n - \log 2 - \epsilon)} < \frac{\epsilon_1}{\log 2 + \epsilon}.$$

It follows that for all $n > 688383$

$$\frac{ng(n)'}{g(n)^2} = \frac{(\log 2 + \epsilon)(1+\log n)}{\log n(\log n + \log_2 n - \log 2 - \epsilon)^2} < \frac{\epsilon_1}{\log n + \log_2 n - \log 2 - \epsilon}. \quad (2)$$

Next, we will show that $A + f'(\alpha - n) - \log 2 < \epsilon_2$.

Using Lemma 2.3, part 4 and Lemma 2.2 part 2, we have

$$A+f'(\alpha-n)-\log 2 < \log\left(\frac{\log(2.4n)}{\log n}\right)+\frac{\log_2(2.4n)}{\log(2n)}+\frac{1.1}{\log n}+\frac{\log_2(1.4n)}{\log^2 n}. \quad (3)$$

Observe that for $n > 36734$

$$\log\left(\frac{\log(2.4n)}{\log n}\right) < \frac{\epsilon_2}{5} \quad (4)$$

as $\log\left(\frac{\log(2.4n)}{\log n}\right) < \frac{\epsilon_2}{5}$ holds if $\frac{\log(2.4n)}{\log n} < e^{\frac{\epsilon_2}{5}}$, that is if $2.4n < n^{e^{\frac{\epsilon_2}{5}}}$. The above holds if $2.4 < n^{e^{\frac{\epsilon_2}{5}}-1}$ or $n > 36734$.

Computation yields that for $n > 688383$

$$\frac{\log_2(2.4n)}{\log(2n)} + \frac{1.1}{\log n} + \frac{\log_2(1.4n)}{\log^2 n} < \frac{4\epsilon_2}{5}. \quad (5)$$

From equations (3)-(5) we have $A + f'(\alpha - n) - \log 2 < \epsilon_2$. From Lemma 2.3 part 3, $L'(\alpha - n) = U'(\alpha - n) - f'(\alpha - n) > \log(\alpha - n) + \log_2(\alpha - n) > \log n + \log_2 n$ and hence for $n > 688383$ we have

$$\frac{A + f'(\alpha-n)}{-A + U'(\alpha-n) - 2f'(\alpha-n)} < \frac{\log 2 + \epsilon_2}{\log n + \log_2 n - \log 2 - \epsilon_2}. \quad (6)$$

As $\epsilon_1 + \epsilon_2 = \epsilon$, equations (2) and (6) give

$$\frac{A + f'(\alpha-n)}{-A + U'(\alpha-n) - 2f'(\alpha-n)} + \frac{ng(n)'}{g(n)^2} < \frac{\log 2 + \epsilon_1 + \epsilon_2}{\log n + \log_2 n - \log 2 - \epsilon} = \frac{1}{g(n)}. \quad (7)$$

From Lemma 2.3, part 2, noting that $\left(\frac{n}{g(n)}\right)' = \frac{1}{g(n)} - \frac{ng(n)'}{g(n)^2}$, we have $G'(n) < 0$ for all $n > 688383$. Also, $G(688383) < 0$ and hence we conclude that $G(n) < 0$ for $n > 688383$.

From Lemma 2.2, part 3 we have $F(s, n) > 0$ and $F(x, n)$ is decreasing for $x \geq 2n$. As s and α are both bigger than $2n$, it follows that $s < \alpha$ for $n > 688383$. That the result holds for $43 < n \leq 688383$ is a simple calculation. □

Remark 3.1. *Similar results for lower bounds for R_n can be given using $G(x, n) = L(x) - 2U(x - n + 1)$ instead of $F(x, n)$.*

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