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Torelli theorem for moduli stacks of vector bundles and principal G-bundles

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ABSTRACT

Given any irreducible smooth complex projective curve X, of genus at least 2, consider the moduli stack of vector bundles on X of fixed rank and determinant. It is proved that the isomorphism class of the stack uniquely determines the isomorphism class of the curve X and the rank of the vector bundles. The case of trivial determinant, rank 2 and genus 2 is specially interesting: the curve can be recovered from the moduli stack, but not from the moduli space (since this moduli space is $\mathbb{P}^3$ thus independently of the curve).
We also prove a Torelli theorem for moduli stacks of principal G-bundles on a curve of genus at least 3, where G is any non-abelian reductive group.

1. Introduction

Let X be a smooth complex projective curve of genus g, with $g \geq 2$. The classical Torelli theorem states that the isomorphism class of the canonically polarized Jacobian variety $J(X)$ determines uniquely the isomorphism class of the curve. Natural generalizations of this problem to moduli spaces of vector bundles of higher rank have been studied extensively, addressing the question whether the geometry of the curve X can be recovered from the isomorphism class of a certain moduli space of rank r vector bundles on X.

Fix a line bundle ξ on the curve X, and let $M^{\text{ss-vb}}(X, r, \xi)$ denote the moduli space of semistable vector bundles E of rank r on X such that $\det(E) \cong \xi$.

Mumford and Newstead [21] and Tyurin [28] proved that if X and X' have genus at least 2 and the degree of the determinant is odd, then $M^{\text{ss-vb}}(X, 2, \xi) \cong M^{\text{ss-vb}}(X, 2, \xi)$ implies that $X \cong X'$. This Torelli type result was then extended to moduli spaces of vector bundles of rank r when the degree of the determinant is coprime with r by Tyurin [29] and by Narasimhan and Ramanan [23].

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Kouvidakis and Pantev [20] obtained a Torelli theorem for genus at least three and any rank $r \geq 2$ that did not need the coprimality condition on rank and degree; they proved that $M^{\text{ss-vb}}(X, r, \xi) \cong M^{\text{ss-vb}}(X', r, \xi')$ implies that $X \cong X'$. Other proofs for this Torelli theorem for genus at least 4 have been given with different techniques by Hwang and Ramanan [19], by Sun [27] and by Biswas, Gómez and Muñoz [7].

Using the techniques in [7], a 2-birational version of the Torelli was found in [1] for curves of genus at least 4; it was shown there that both the pair $(r, \pm\deg(\xi) \pmod r)$ and the curve can be recovered from the geometry of the moduli scheme. Another Torelli type theorem for moduli spaces of rank three vector bundles with trivial determinant over genus 2 curves was found by Nguyen [24]. On the other hand, a Torelli theorem for the moduli spaces of principal G-bundles over curves of genus at least 3, where G is a complex reductive group, was also proven by Biswas and Hoffmann [9].

In this work we study the moduli stack $\mathcal{M}(X, r, \xi)$ of vector bundles over X of rank r with fixed determinant ξ, in particular, we prove a Torelli type theorem for this moduli stack. The objects of this moduli stack are pairs (E, φ) where E is a vector bundle of rank r on X and $\varphi : \det E \longrightarrow \xi$ is an isomorphism. An isomorphism between two objects (E, φ) and (E', φ') is an isomorphism α between E and E' with $\varphi = \varphi' \circ \det\alpha$. In particular, the automorphism group of (E, φ) is the finite group of $\mathbb{Z}/r\mathbb{Z}$ when E is a simple vector bundle.

Theorem 1.1 *(Theorem 5.2). Let $r, r' \geq 2$. If X and X' are curves of genus $g, g' \geq 2$ and $\mathcal{M}(X, r, \xi) \cong \mathcal{M}(X', r', \xi')$, then $X \cong X'$ and $r = r'$.*

This theorem applies to all instances where the rank and genus are both at least 2 without any additional coprimality conditions. Interestingly, this includes a case where the Torelli theorem for the moduli scheme fails. Narasimhan and Ramanan [22] proved that the moduli scheme $M^{\text{ss-vb}}(X, 2, \mathcal{O}_X)$ is isomorphic to $\mathbb{P}^3$ for every genus 2 curve X. We show that that X can be recovered from the complete moduli stack $\mathcal{M}(X, 2, \mathcal{O}_X)$ and even from the substack $\mathcal{M}^{\text{ss-vb}}(X, r, \mathcal{O}_X)$ of semistable vector bundles (see Theorem 4.2 and Remark 4.3). In the case of the moduli scheme, in Remark 5.3 we see that the case $(g, r) = (2, 2)$ is the only exception to the Torelli theorem.

Theorem 1.1 has been composed combining three different Torelli theorems for stacks which have been proven through three different strategies and using different techniques. Each of these theorems is valid for certain combinations of the genus of the curve and the rank of the bundle which do not cover the entire set of possibilities considered by Theorem 1.1. An additional argument has been made based on computations of the dimension and the Brauer class of the moduli space which allows us to apply selectively the appropriate version of the Torelli in each case and to combine them to obtain the global result summarized by Theorem 1.1 (see Theorem 5.2).

The first proof is based on studying the cotangent bundle to the substack of simple points and identifying it with a moduli stack of Higgs bundles. Working analogously to [7], it is proven that the Hitchin map can be recovered from the geometry of this substack, and the Torelli theorem follows from a study of the geometry of the discriminant locus inside the Hitchin base. Due to a constraint on the codimension of a certain subvariety, the result works for all pairs of genus and rank (g, r) such that $g \geq 2$ and $r \geq 2$ except for the three cases $(2, 2)$, $(2, 3)$ and $(3, 2)$ (which correspond to the moduli schemes of the lowest dimensions 3, 8 and 6 respectively). The details are presented in Section 2.

The second proof uses “beyond GIT” techniques based on the work of Alper, Halpern-Leistner and Heinloth [2,17,16,4] to recover the moduli space of semistable vector bundles from the moduli stack and then reduces the problem to the study of the Torelli theorem for the corresponding moduli space. The limitation of this technique is that it can only be used in cases where the Torelli theorem is known for the corresponding moduli scheme. Combined with the results in [1], we use it to obtain a Torelli theorem for curves of genus $g \geq 4$ in which we recover the pair $(r, \pm\deg(\xi) \pmod r)$ in addition to the curve (see Theorem 3.6). We can also use it to prove a Torelli theorem in genus $g = 3$, but it is limited to certain cases in $g = 2$. This is studied in Section 3.

This technique also allows us to prove a Torelli theorem for moduli stacks of G-bundles, where G is any algebraic connected reductive complex group. Given a curve X, let $\mathcal{M}_G^d(X)$ denote the component of the moduli stack of principal G-bundles on X corresponding to a fixed $d \in \pi_1(G)$. We prove the following.

Theorem 1.2 *(Corollary 3.8). Let X and X' be smooth projective complex curves of genus at least 3, and let G and G' be algebraic connected reductive complex groups. If a moduli stack $\mathcal{M}_G^d(X)$ of principal G-bundles over X is isomorphic to a stack $\mathcal{M}_{G'}^{d'}(X')$ of principal G'-bundles over X', then $X \cong X'$.*

In Section 4, a proof for the Torelli theorem for stacks of rank 2 vector bundles with trivial determinant is obtained by showing that the projection of the substack of simple semistable bundles onto the moduli space of semistable vector bundles coincides with the quotient of the Jacobian of X by the involution $L \longmapsto L^{-1}$. This is used in studying the earlier mentioned special case of rank 2 vector bundles with trivial determinant over a genus 2 curve.

Finally, all these results are combined in Section 5 to prove Theorem 1.1.

2. A Torelli theorem using the Hitchin map

Let X be an irreducible smooth complex projective curve of genus g, with $g \geq 2$. Fix a line bundle ξ on X. Let $\mathcal{M} = \mathcal{M}(X, r, \xi)$ be the moduli stack parametrizing the vector bundles E on X of rank r equipped with an isomorphism

$$\det(E) := \wedge^r E \xrightarrow{\cong} \xi.$$

Let $\mathcal{M}^{\text{simp}}(X, r, \xi) \subset \mathcal{M}$ be the substack of simple points in $\mathcal{M}$, i.e., the locus of vector bundles E with isomorphism $\det(E) \xrightarrow{\cong} \xi$ whose automorphism group is the group of r-th roots of $1 \in \mathbb{C}$.

Recall that a vector bundle E is said to be *stable* (respectively, *semistable*) if for any proper subbundle $0 \neq F \subsetneq E$

$$\frac{\deg(F)}{\operatorname{rk}(F)} < \frac{\deg(E)}{\operatorname{rk}(E)} \quad (\text{ respectively, } \leq)$$

Let $\mathcal{M}^{\text{s-vb}}(X, r, \xi) \subset \mathcal{M}(X, r, \xi)$ be the substack of stable vector bundles with fixed determinant ξ, and denote by $M^{\text{s-vb}}(X, r, \xi)$ the corresponding moduli scheme of rank r stable vector bundles on X with fixed determinant ξ. Clearly, we have a quotient map

$$\mathcal{M}^{\text{s-vb}}(X, r, \xi) \longrightarrow M^{\text{s-vb}}(X, r, \xi).$$

The zero part of the cotangent complex of $\mathcal{M}^{\text{simp}}(X, r, \xi)$ over a vector bundle E is isomorphic, through Serre duality, to $H^0(X, \operatorname{End}_0(E) \otimes K_X)$, where K_X is the canonical line bundle of X and $\operatorname{End}_0(E) \subset \operatorname{End}(E)$ is the subbundle of corank one defined by the sheaf of endomorphisms of trace zero. Thus, we can interpret the total space of that sheaf as the moduli stack $\mathcal{N}^{\text{simp}}(X, r, \xi)$ of pairs (E, φ), where

$$\varphi \in H^0(X, \operatorname{End}_0(E) \otimes K_X)$$

and E is equipped with an isomorphism $\det(E) \xrightarrow{\cong} \xi$, such that E is simple. Such a pair (E, φ) is called a *Higgs bundle*, and φ is called its *Higgs field* on E. Denote by $\mathcal{N}^{\text{s-vb}}(X, r, \xi)$ the substack of pairs (E, φ) with E being a stable vector bundle. Then we have a natural morphism

$$\mathcal{N}^{\text{s-vb}}(X, r, \xi) \longrightarrow T^* M^{\text{s-vb}}(X, r, \xi). \tag{2.1}$$

A Higgs bundle (E, φ) is said to be *semistable* (respectively, *stable*) if for any proper subbundle $0 \neq F \subsetneq E$, such that $\varphi(F) \subseteq F \otimes K_X$, we have

$$\frac{\deg(F)}{\operatorname{rk}(F)} \leq \frac{\deg(E)}{\operatorname{rk}(E)} \quad (\text{ respectively, } <).$$

Let $N(X, r, \xi)$ (respectively, $N^s(X, r, \xi)$) denote the moduli space of semistable (respectively, stable) Higgs bundles on X. We will use the following lemma which is a consequence of the proof of [13, Theorem II.6.(iii)] or [6, Proposition 5.4].

Lemma 2.1. *Let X be a curve of genus $g \geq 2$, and suppose that $r \geq 2$. Then the codimension of $N(X, r, \xi) \backslash T^* M^{\text{s-vb}}(X, r, \xi)$ in $N(X, r, \xi)$ is at least $(g-1)(r-1)$. In particular, if*

$$(g, r) \notin \{(2, 2), (2, 3), (3, 2)\},$$

then this codimension is at least 3.

Given a stack $\mathcal{X}$, let $\Gamma(\mathcal{X})$ denote the algebra of complex algebraic functions on $\mathcal{X}$, i.e., we have $\Gamma(\mathcal{X}) = \operatorname{Hom}_{(\text{Stacks})}(\mathcal{X}, \mathbb{C}) = H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

Lemma 2.2. *The equality*

$$\Gamma(\mathcal{N}^{\text{s-vb}}(X, r, \xi)) = \Gamma(T^* M^{\text{s-vb}}(X, r, \xi))$$

holds.

Proof. It follows immediately from the fact that the morphism (2.1) is a good moduli space (in the sense of J. Alper [2]) and hence the global functions are the same.

We prove that the morphism (2.1) is a good moduli space space as follows. The morphism $\mathcal{M}^{\text{s-vb}}(X, r, d) \longrightarrow M^{\text{s-vb}}(X, r, d)$ from the moduli stack of stable vector bundles to its moduli space is a $B\mathbb{G}_m$-gerbe, meaning that it is locally a product $U \times B\mathbb{G}_m$ where U is an étale covering of $M^s(X, r, d)$. Using this and the following Cartesian diagram (where $\mathcal{P}ic(X)$ is the algebraic stack parametrizing line bundles)

$$\begin{tikzcd}
\mathcal{M}^{\text{s-vb}}(X, r, \xi) \arrow[r] \arrow[d] & \mathcal{M}^{\text{s-vb}}(X, r, d) \arrow[d] \\
\operatorname{Spec} \mathbb{C} \arrow[r, "\xi"] & \mathcal{P}ic(X)
\end{tikzcd}$$

it is easy to see that $\mathcal{M}^{\text{s-vb}}(X, r, \xi) \longrightarrow M^{\text{s-vb}}(X, r, \xi)$ is a $\mathbb{Z}/r\mathbb{Z}$-gerbe. The moduli space $N^{\text{s-vb}}(X, r, \xi)$ of Higgs bundles whose underlying vector bundle is stable is actually a vector bundle over $M^{\text{s-vb}}(X, r, \xi)$. Therefore, this Cartesian diagram

$$\begin{tikzcd} \mathcal{N}^{\text{s-vb}}(X, r, \xi) \arrow[r] \arrow[d] & N^{\text{s-vb}}(X, r, \xi) \arrow[d] \\ \mathcal{M}^{\text{s-vb}}(X, r, \xi) \arrow[r] & M^{\text{s-vb}}(X, r, \xi) \end{tikzcd}$$

shows that

$$\mathcal{N}^{\text{s-vb}}(X, r, \xi) \longrightarrow N^{\text{s-vb}}(X, r, \xi) = T^*M^{\text{s-vb}}(X, r, \xi)$$

is a $\mathbb{Z}/r\mathbb{Z}$-gerbe, and therefore it is a good moduli space. $\square$

Denote the Hitchin base as $W := \bigoplus_{k=2}^r H^0(X, K_X^{\otimes k})$, and write $W_k = H^0(X, K_X^{\otimes k})$ for $k = 2, \cdots, r$. Let

$$H : N(X, r, \xi) \longrightarrow W$$

be the Hitchin map. We also define the Hitchin map for the moduli stack

$$\mathcal{H} : \mathcal{N}^{\text{simp}}(X, r, \xi) \longrightarrow W$$

sending a family $(\mathcal{E}, \Phi)$ over $X \times T$ to the map

$$\sum_{k=2}^r (-1)^k \operatorname{tr}(\wedge^k \Phi) : T \longrightarrow W.$$

For each $s = (s_2, \ldots, s_r) \in W = \bigoplus_{k=2}^r H^0(X, K_X^k)$, the equation

$$t^r + \sum_{k=2}^r s_k t^k = 0$$

defines a spectral curve X_s in the total space of the line bundle T_X^*.

Lemma 2.3. *If $(g, r) \neq (2, 2)$, then the equality*

$$\Gamma(\mathcal{N}^{\text{simp}}(X, r, \xi)) = \Gamma(T^*M^{\text{s-vb}}(X, r, \xi))$$

holds.

Proof. We will show that the restriction of global functions from $\mathcal{N}^{\text{simp}}(X, r, \xi)$ to $\mathcal{N}^{\text{s-vb}}(X, r, \xi)$ is an isomorphism when $(g, r) \neq (2, 2)$, and then the result follows from Lemma 2.2.

The first step is to show that any global function on $\mathcal{N}^{\text{s-vb}}(X, r, \xi)$ can be extended to $\mathcal{N}^{\text{simp}}(X, r, \xi)$. By Lemma 2.1, the codimension of the complement of $T^*M^{\text{s-vb}}(X, r, \xi)$ in $N(X, r, \xi)$ is at least 2, so Hartogs' Theorem implies that $\Gamma(T^*M^{\text{s-vb}}(X, r, \xi)) = \Gamma(N(X, r, \xi))$. The algebra of functions $\Gamma(N(X, r, \xi))$ is generated by the components of the Hitchin map [18], so the algebra of functions on $T^*M^{\text{s-vb}}(X, r, \xi)$ is also generated by the components of the Hitchin map. Using Lemma 2.2 we conclude that the algebra of global functions on $\mathcal{N}^{\text{s-vb}}(X, r, \xi)$ is generated by the components of the Hitchin map. These functions are clearly well defined over arbitrary families of Higgs fields over vector bundles on X, so they extend to algebraic functions on $\mathcal{N}^{\text{simp}}(X, r, \xi)$.

Finally, the extensions are unique because $\mathcal{N}^{\text{simp}}(X, r, \xi)$ is integral. Indeed, it is a vector bundle over $\mathcal{M}^{\text{s-vb}}(X, r, \xi)$, which is integral because it is actually an open substack of the integral stack $\mathcal{M}(X, r, \xi)$. $\square$

Corollary 2.4. *There exists an algebraic isomorphism*

$$\operatorname{Spec}(\Gamma(\mathcal{N}^{\text{simp}}(X, r, \xi))) \xrightarrow{\cong} W$$

such that the composition of maps

$$\mathcal{N}^{\text{simp}}(X, r, \xi) \longrightarrow \operatorname{Spec}(\Gamma(\mathcal{N}^{\text{simp}}(X, r, \xi))) \xrightarrow{\cong} W$$

coincides with the Hitchin map $\mathcal{H} : \mathcal{N}^{\text{simp}}(X, r, \xi) \longrightarrow W$.

Let $\mathcal{D} \subset W$ denote the discriminant locus, i.e., the locus of all $s = (s_i) \in W$ such that the corresponding spectral curve $X_s \subset \mathrm{Tot}(T_X^*)$ is singular.

Lemma 2.5. *Let (E, φ) be a Higgs bundle whose spectral curve is integral. Then (E, φ) does not have any nontrivial invariant subbundle and, in particular, it is a stable Higgs bundle.*

Proof. Let X_s be the spectral curve associated to (E, φ). Then (E, φ) is the pushforward of a rank 1 torsion-free sheaf L on X_s. Assume that F is a nonzero subbundle preserved by φ. Since F is invariant, the characteristic polynomial of the restriction $\varphi|_F : F \longrightarrow F \otimes K_X$ divides the characteristic polynomial of φ. Consequently, the spectral curve associated to $(F, \varphi|_F)$ is a closed subscheme of X_s. Since X_s is integral, the spectral curve for $(F, \varphi|_F)$ must be the entire X_s. But this implies that $\mathrm{rk}(F) = \mathrm{rk}(E)$ and, thus, we have $F = E$. $\square$

Lemma 2.6. *Let $\gamma : \mathbb{P}^1 \longrightarrow \mathcal{N}^{\mathrm{simp}}(X, r, \xi)$ be a map whose image contains at least two non-isomorphic points. Then the image of $\mathcal{H} \circ \gamma$ is a point in the discriminant locus $\mathcal{D}$.*

Proof. First of all, as $\mathcal{H} \circ \gamma : \mathbb{P}^1 \longrightarrow W$ is a map from $\mathbb{P}^1$ to an affine space, its image must be a point $s \in W$. Suppose that

$$s \notin \mathcal{D}.$$

Then the curve X_s is smooth. By Lemma 2.5, the map γ factors though the substack $\mathcal{N}' \hookrightarrow \mathcal{N}^{\mathrm{simp}}(X, r, \xi)$ of Higgs bundles (E, φ) such that (E, φ) is stable and E is simple. Furthermore, the composition $\mathcal{H} \circ \gamma$ factors through the moduli scheme of stable Higgs bundles:

$$\mathbb{P}^1 \xrightarrow{\gamma} \mathcal{N}' \longrightarrow N^s(X, r, \xi) \longrightarrow W.$$

The preimage of $s \in W$ in $N^s(X, r, \xi)$ is isomorphic to the Prym variety of line bundles over X_s whose pushforward has determinant ξ, so it is an abelian variety. Since there is no nonconstant map from $\mathbb{P}^1$ to an abelian variety, the image of γ in the moduli space $N^s(X, r, \xi)$ is a single point. Thus, all the points in the image of γ in the stack $\mathcal{N}'$ must be isomorphic. This contradicts the hypothesis that its image contains at least two non-isomorphic points. This completes the proof of the lemma. $\square$

Lemma 2.7. *Assume that $g, r \geq 2$, and $(g, r) \notin \{(2, 2), (2, 3), (3, 2)\}$. For a general point $s \in \mathcal{D}$ there exists a non-constant morphism (given explicitly in the proof below)*

$$\gamma' : \mathbb{P}^1 \longrightarrow T^* M^{\text{s-vb}}(X, r, \xi)$$

such that $\mathrm{Im}(H \circ \gamma') = s$.

Proof. We can follow the same proof as in [7, Proposition 3.1], incorporating the codimension bound given by Lemma 2.1. By [20, Remark 1.7], there exists a Zariski open subset $\mathcal{D}^0 \subset \mathcal{D}$ such that each point $s \in \mathcal{D}^0$ corresponds to a spectral curve X_s which is an irreducible nodal curve with a single node. Moreover, as a consequence of [7, Proposition 3.2], the Hitchin discriminant $\mathcal{D}$ is irreducible, so the open subset $\mathcal{D}^0$ is actually dense. Let

$$H : N(X, r, \xi) \longrightarrow W$$

denote the Hitchin map for the moduli space of semistable Higgs bundles.

Let $\pi : X_s \longrightarrow X$ be the projection from the spectral curve. The fiber $H^{-1}(s)$ parametrizes torsion free sheaves L on X_s such that $\pi_* L$ is a vector bundle on X of determinant ξ. Torsion free sheaves of rank 1 on nodal curves have been studied by Usha Bhosle, using the notion of generalized parabolic bundles. We will now recall the results that we will need (for details, see the proof of [11, Proposition 2.2]). The results of Bhosle show that the fiber $H^{-1}(s)$ is a fibration over a closed subscheme of the Jacobian $J(\widetilde{X}_s)$ of the normalization $p : \widetilde{X}_s \longrightarrow X_s$ with fiber isomorphic to a rational curve with one node. To see this, we first consider a line bundle on X_s. It can be described by a line bundle $\widetilde{L} \in J(\widetilde{X}_s)$ and an isomorphism between the fibers over the two points $x_1,\ x_2 \in \widetilde{X}_s$ mapping to the node of X_s. This isomorphism can be given by its graph $\Gamma \subset \widetilde{L}_{x_1} \oplus \widetilde{L}_{x_2}$. The corresponding line bundle L on X_s fits in a short exact sequence

$$0 \longrightarrow L_\Gamma \longrightarrow p_* \widetilde{L} \longrightarrow (\widetilde{L}_{x_1} \oplus \widetilde{L}_{x_2})/\Gamma \longrightarrow 0.$$

Note that Γ is a line in $\widetilde{L}_{x_1} \oplus \widetilde{L}_{x_2}$ which projects isomorphically to both $\widetilde{L}_{x_1}$ and $\widetilde{L}_{x_2}$. If we allow Γ to become $\widetilde{L}_{x_1}$ or $\widetilde{L}_{x_2}$, then L_Γ is no longer a line bundle, but it is torsion free.

We thus obtain, for each $s \in \mathcal{D}^0$ and line bundle $\widetilde{L}$ on $\widetilde{X}_s$, a family of torsion free sheaves on X_s parametrized by $\mathbb{P}^1 = \mathbb{P}(\widetilde{L}_{x_1} \oplus \widetilde{L}_{x_2})$

$$0 \longrightarrow \mathcal{L} \longrightarrow p_{X_s}^* p_* \widetilde{L} \longrightarrow \mathcal{O}_{x_0 \times \mathbb{P}^1}(1) \longrightarrow 0,$$

where p_{X_s} is the projection of $X_s \times \mathbb{P}^1$ to the first factor. As we vary over all possible line bundles on $\widetilde{X}_s$ and points in $\mathbb{P}(\widetilde{L}_{x_1} \oplus \widetilde{L}_{x_2})$ we obtain all possible torsion free sheaves on X_s. The condition that the vector bundle $\pi_* L$ on X has determinant ξ picks a closed subset of $J(\widetilde{X}_s)$. Different points in $\mathbb{P}(\widetilde{L}_{x_1} \oplus \widetilde{L}_{x_2})$ will give different isomorphic classes of torsion free sheaves except that the points corresponding to the two lines $\widetilde{L}_{x_1}$ and $\widetilde{L}_{x_2}$ give isomorphic torsion free sheaves. This is the reason why $H^{-1}(s)$ is a fibration with fibers equal to nodal rational curves. Taking the pushforward of the previous sequence, the family $\mathcal{L}$ of torsion free sheaves becomes a family of Higgs bundles $(\mathcal{E}, \Phi)$ on X with $\mathcal{E}$ given by

$$0 \longrightarrow \mathcal{E} = (\pi \times \mathrm{Id}_{\mathbb{P}^1})_* \mathcal{L} \longrightarrow (\pi \times \mathrm{Id}_{\mathbb{P}^1})_* p_{X_s}^* p_* \widetilde{L} \longrightarrow \mathcal{O}_{\pi(x_0) \times \mathbb{P}^1}(1) \longrightarrow 0 \tag{2.2}$$

Furthermore, it follows from this sequence that $\det(\mathcal{E}) \cong p_X^* \xi$.

Lemma 2.1 implies that the codimension of the complement of $T^*M^{\text{s-vb}}(X,r,\xi)$ in $N(X,r,\xi)$ is at least 3, so, intersecting it with the divisor $H^{-1}(\mathcal{D})$, we obtain that the codimension of the complement of $H^{-1}(\mathcal{D}) \cap T^*M^{\text{s-vb}}(X,r,\xi)$ inside $H^{-1}(\mathcal{D})$ must be at least 2. Since the curves in $\mathcal{D}^0$ are all integral, by Lemma 2.5 we have $H^{-1}(s) \subset N^s(X,r,\xi)$ for all s in the dense subset $\mathcal{D}^0 \subset \mathcal{D}$. By [13, Theorem II.5], the restriction of the Hitchin map to $N^s(X,r,\xi)$ is equidimensional. Thus, for a general $s \in \mathcal{D}^0$, the codimension of the complement of $H^{-1}(s) \cap T^*M^{\text{s-vb}}(X,r,\xi)$ inside $H^{-1}(s)$ is at least 2.

As it was mentioned above, $H^{-1}(s)$ is a fibration by nodal rational curves (dimension 1), therefore, there exist complete rational curves $\gamma' : \mathbb{P}^1 \longrightarrow H^{-1}(s) \cap T^*M^{\text{s-vb}}(X,r,\xi)$. □

Lemma 2.8. *The morphism given in Lemma 2.7 can be lifted to the moduli stack, i.e., to a morphism*

$$\gamma : \mathbb{P}^1 \longrightarrow \mathcal{N}^{\mathrm{simp}}(X,r,\xi)$$

such that the image of the composition $\mathcal{H} \circ \gamma$ is the point s, and the image of γ contains at least two non-isomorphic points.

Proof. The morphism in Lemma 2.7 is given by the explicit family $(\mathcal{E}, \Phi)$ given in (2.2). There is an isomorphism $\det(\mathcal{E}) \cong p_X^* \xi$, and hence a morphism γ to the moduli stack.

By construction, the map γ' is nonconstant, so the above morphism γ has at least two non-isomorphic points in its image. □

Corollary 2.9. *Let X be an irreducible smooth complex projective curve of genus $g \geq 2$. Suppose that $r \geq 2$ and $(g, r) \notin \{(2, 2), (2, 3), (3, 2)\}$. Let Γ be the space of all maps*

$$\gamma : \mathbb{P}^1 \longrightarrow \mathcal{N}^{\mathrm{simp}}(X,r,\xi)$$

whose image contains at least two non-isomorphic points. Then the Hitchin discriminant $\mathcal{D}$ is the algebraic closure of the subset

$$\mathcal{D}_\Gamma = \{\mathrm{Im}(\mathcal{H} \circ \gamma) \mid \gamma \in \Gamma\} \subset W.$$

Proof. By Lemma 2.6 we have $\mathcal{D}_\Gamma \subseteq \mathcal{D}$. Moreover, Lemma 2.8 implies that $\mathcal{D}_\Gamma$ contains a dense open subset of $\mathcal{D}$. Therefore, the closure of $\mathcal{D}_\Gamma$ in W is the entire discriminant $\mathcal{D}$. □

Theorem 2.10. *Let X and X' be two irreducible smooth complex projective curves of genus g and g' respectively, with $g, g' \geq 2$. Let $r, r' \geq 2$ such that $(g, r), (g', r') \notin \{(2, 2), (2, 3), (3, 2)\}$. Fix line bundles ξ and ξ' on X and X' respectively. Let*

$$\Psi : \mathcal{M}(X,r,\xi) \longrightarrow \mathcal{M}(X',r',\xi')$$

be an isomorphism between the corresponding moduli stacks of vector bundles with fixed determinant. Then $r = r'$ and $X \cong X'$.

Proof. Let $\Psi : \mathcal{M}(X,r,\xi) \longrightarrow \mathcal{M}(X',r',\xi')$ be an isomorphism of stacks. Then it preserves the locus of objects with zero-dimensional stabilizers. Any vector bundle of rank r with fixed determinant admits a natural action of the group of r-th roots of unity by dilation. Thus, the size of the stabilizer of any object in the moduli stack $\mathcal{M}(X,r,\xi)$ with a zero-dimensional stabilizer is at least r, and objects whose stabilizer has the minimum possible size r are simple. Since these exist (for instance, stable objects are simple), we can characterize the locus $\mathcal{M}^{\mathrm{simp}}(X,r,\xi)$ inside $\mathcal{M}(X,r,\xi)$ as the locus of objects with minimal stabilizer. As this property is preserved through the isomorphism Ψ, the map Ψ restricts to an isomorphism

$$\Psi^{\mathrm{simp}} : \mathcal{M}^{\mathrm{simp}}(X,r,\xi) \longrightarrow \mathcal{M}^{\mathrm{simp}}(X',r',\xi')$$

between the corresponding loci of simple objects. This map Ψ^{simp} induces an isomorphism between the corresponding cotangent complexes. As the moduli stack of bundles is smooth, both complexes are concentrated in orders 0 and 1 and there is an isomorphism

$$d^1((\Psi^{\mathrm{simp}})^{-1}) : \mathcal{N}^{\mathrm{simp}}(X, r, \xi) \longrightarrow \mathcal{N}^{\mathrm{simp}}(X', r', \xi').$$

Let

$$W = \bigoplus_{k=2}^{r} H^0(X,\ K_X^{\otimes k}), \quad W' = \bigoplus_{k=2}^{r'} H^0(X',\ K_{X'}^{\otimes k}).$$

By Corollary 2.4, there exists an isomorphism $f : W \xrightarrow{\cong} W'$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{N}^{\mathrm{simp}}(X, r, \xi) \arrow[r, "d^1((\Psi^{\mathrm{simp}})^{-1})"] \arrow[d, "\mathcal{H}"] & \mathcal{N}^{\mathrm{simp}}(X', r', \xi') \arrow[d, "\mathcal{H}'"] \\ W \arrow[r, "f"] & W' \end{tikzcd} \tag{2.3}$$

As the map $d^1((\Psi^{\mathrm{simp}})^{-1})$ is $\mathbb{C}$-linear, the map f in (2.3) is $\mathbb{C}^*$-equivariant for the $\mathbb{C}^*$ actions on W and W' making the Hitchin maps $\mathbb{C}^*$-equivariant; more precisely, the $\mathbb{C}^*$ action is the diagonal weighted action

$$\lambda \cdot (s_2, \cdots, s_r) = (\lambda^2 s_2, \cdots, \lambda^r s_r).$$

In particular, f preserves the filtrations of subspaces of W and W' in terms of the asymptotic decay of the corresponding $\mathbb{C}^*$-actions:

$$W = W_{\geq 2} \supsetneq W_{\geq 3} \supsetneq \cdots \supsetneq W_{\geq r} \supsetneq 0,$$
$$W' = W'_{\geq 2} \supsetneq W'_{\geq 3} \supsetneq \cdots \supsetneq W'_{\geq r'} \supsetneq 0,$$

where $W_{\geq k} = \bigoplus_{j=k}^{r} H^0(X,\ K_X^k) = \bigoplus_{j=k}^{r} W_k$ and $W'_{\geq k} = \bigoplus_{j=k}^{r'} H^0(X',\ K_{X'}^k) = \bigoplus_{j=k}^{r'} W'_k$.

Observe that the length of the filtrations of W and W' are $r-1$ and $r'-1$ respectively, and so we conclude that $r = r'$. Moreover, f sends $W_r \subset W$ to $W'_{r'} = W'_r \subset W'$ and, as the $\mathbb{C}^*$-action is homogeneous of degree r in W_r and W'_r, and f is $\mathbb{C}^*$-equivariant, we conclude that f restricts to a linear map $f_r : W_r \longrightarrow W'_r$.

On the other hand, as $d^1((\Psi^{\mathrm{simp}})^{-1})$ is an isomorphism, it induces a bijection between the set Γ of maps $\mathbb{P}^1 \longrightarrow \mathcal{N}^{\mathrm{simp}}(X, r, \xi)$ whose image contains non-isomorphic points and the set Γ' of maps $\mathbb{P}^1 \longrightarrow \mathcal{N}^{\mathrm{simp}}(X', r', \xi')$ whose image contains non-isomorphic points. By Corollary 2.9, this implies that the map $f : W \longrightarrow W'$ sends the Hitchin discriminant $\mathcal{D} \subset W$ to the Hitchin discriminant $\mathcal{D}' \subset W'$.

As $f(\mathcal{D}) = \mathcal{D}'$ and $f(W_r) = W'_r$, we have $f(\mathcal{D} \cap W_r) = \mathcal{D}' \cap W'_r$. Let

$$\mathcal{C} = \mathbb{P}(\mathcal{D} \cap W_r) \subset \mathbb{P}(W_r) \ \text{ and } \ \mathcal{C}' = \mathbb{P}(\mathcal{D}' \cap W'_r) \subset \mathbb{P}(W'_r).$$

Since $f_r : W_r \longrightarrow W'_r$ is linear and $f_r(\mathcal{D} \cap W_r) = \mathcal{D}' \cap W'_r$, we conclude that f_r induces an isomorphism between $\mathbb{P}(W_r)$ and $\mathbb{P}(W'_r)$ sending $\mathcal{C}$ to $\mathcal{C}'$. Then, it induces an isomorphism between the corresponding dual varieties $\mathcal{C}^\vee \subset \mathbb{P}(W_r^\vee)$ and $(\mathcal{C}')^\vee \subset \mathbb{P}((W'_r)^\vee)$. By [7, Proposition 4.2], we have $\mathcal{C}^\vee \cong X \subset \mathbb{P}(W_r^\vee)$ and $(\mathcal{C}')^\vee \cong X' \subset \mathbb{P}((W'_r)^\vee)$. This completes the proof. □

3. "Beyond GIT" techniques

As before, let $\mathcal{M}^{\text{ss-vb}}(X, r, \xi) \subset \mathcal{M}(X, r, \xi)$ be the substack of semistable vector bundles with fixed determinant ξ, and denote by $M^{\text{ss-vb}}(X, r, \xi)$ the corresponding projective moduli scheme of rank r semistable vector bundles on X with fixed determinant ξ. As mentioned in the introduction, there exist multiple Torelli type theorems for the moduli scheme of vector bundles of rank $r \geq 2$ [21,28,29,23,19,27,7,1,9,24] showing that if $M^{\text{ss-vb}}(X, r, \xi)$ is isomorphic to $M^{\text{ss-vb}}(X', r', \xi')$ for some irreducible, smooth projective curve X' and a line bundle ξ' on X', then X is isomorphic to X'.

Thus if we want to show that the moduli stack $\mathcal{M}(X, r, \xi)$ uniquely determines X, it is enough to recover the projective moduli scheme $M^{\text{ss-vb}}(X, r, \xi)$ from the stack $\mathcal{M}(X, r, \xi)$ (provided $M^{\text{ss-vb}}(X, r, \xi)$ uniquely determines X).

One way to recover the moduli substack of semistable bundles is to use ideas from "beyond GIT", the theory developed by Alper, Halpern-Leistner and Heinloth [2,17,16,4]. See [3] for an exposition in the case of vector bundles.

In this theory, the notion of $\mathcal{L}$-stability on a stack $\mathcal{M}$ is defined, where $\mathcal{L}$ is a line bundle on $\mathcal{M}$. For this, we first need to introduce the quotient stack $\Theta = [\mathrm{Spec}(\mathbb{C}[t])/\mathbb{G}_m]$, with the standard action of $\mathbb{G}_m$ on the line $\mathrm{Spec}\,\mathbb{C}[t]$. There are two orbits: $t = 0$ and $t \neq 0$ and therefore the stack Θ has two points which we call $t = 0$ (with automorphism group $\mathbb{G}_m$) and $t = 1$ (with trivial automorphism group).

A *filtration* of a point $x \in \mathcal{M}$ is a morphism $f : \Theta \longrightarrow \mathcal{M}$ together with an isomorphism $f(1) \cong x$. We note that the name "filtration" comes from the fact that, if $\mathcal{M}$ is the moduli stack of coherent sheaves then, by the Rees construction, giving such a morphism is equivalent to giving a $\mathbb{Z}$-indexed filtration of the sheaf $f(1)$, and the point $f(0)$ corresponds to the associated graded sheaf.

The line bundle $f^*\mathcal{L}$ on Θ can be thought of as a $\mathbb{G}_m$-equivariant line bundle on $\operatorname{Spec}\mathbb{C}[t]$. Let $\operatorname{wt}(f^*\mathcal{L}|_0)$ be the weight of this equivariant line bundle on the fiber over $t = 0$.

Definition 3.1 *($\mathcal{L}$-semistability [17, Definition 1.2 and Remark 1.3], [16]).* A point $x \in \mathcal{M}$ in an algebraic stack $\mathcal{M}$ is called $\mathcal{L}$-semistable if for all filtrations $f : \Theta \longrightarrow \mathcal{M}$ of x, we have

$$\operatorname{wt}(f^*\mathcal{L}|_0) \le 0.$$

Remark 3.2. Note that the weight $\operatorname{wt}(f^*\mathcal{L}|_0)$ is given by the group homomorphism

$$f^*(\cdot)|_0 \;:\; \operatorname{Pic}(\mathcal{M}) \longrightarrow \operatorname{Pic}(B\mathbb{G}_m) \cong \mathbb{Z}.$$

This implies the following:

- The notion of $\mathcal{L}$-semistability depends only on the class of $\mathcal{L}$ modulo torsion.
- The notion of $\mathcal{L}$-stability only depends on the class of $\mathcal{L}$ in the quotient $\operatorname{Pic}(\mathcal{M})/\operatorname{Pic}^0(\mathcal{M})$, where $\operatorname{Pic}^0(\mathcal{M})$ is the connected component of the identity element.
- Note that $\operatorname{wt}(f^*\mathcal{L}^a|_0) = a\operatorname{wt}(f^*\mathcal{L}|_0)$ and then, if $a > 0$, a point is $\mathcal{L}$-semistable if and only if it is $\mathcal{L}^a$ semistable.
- Therefore, we can define $\mathcal{L}$-semistability for any rational line bundle $\mathcal{L} \in \operatorname{Pic}\mathcal{M} \otimes \mathbb{Q}$, and it depends only on the line $\mathbb{Q}_{>0}\mathcal{L}$.
- If we precompose $f : \Theta \longrightarrow \mathcal{M}$ with the map $\Theta \xrightarrow{[n]} \Theta$ defined by $t \longmapsto t^n$, then the weight $\operatorname{wt}(f^*\mathcal{L}|_0)$ gets multiplied by n, so its sign does not change.

Let $\mathcal{L}_{\det}$ be the determinant line bundle on the moduli stack of vector bundles $\mathcal{M}(X, r, \xi)$ whose fiber over a vector bundle E is $\det(H^0(E))^{-1} \otimes \det(H^1(E))$. More precisely, for any $f : T \longrightarrow \mathcal{M}(X, r, \xi)$ corresponding to a vector bundle $\mathcal{E}$ on $X \times T$, we have $f^*\mathcal{L}_{\det} = \det(Rp_{T*}\mathcal{E})^{-1}$.

Recall that $\operatorname{Pic}(\mathcal{M}(X, r, \xi)) \otimes \mathbb{Q} \cong \mathbb{Q}$ with $\mathcal{L}_{\det}$ being a generator. This was proved for the moduli functor and the moduli scheme in [12]. For a detailed proof in the case of the moduli stack, valid for any genus, see [8, Proposition 4.2.3 and Theorem 4.2.1] (see also [10, Lemma 7.8 and Remark 7.11], [14] and [15]).

Proposition 3.3.

- *If $a < 0$ is an integer, then all points $x \in \mathcal{M}(X, r, \xi)$ are $\mathcal{L}^a_{\det}$-unstable.*
- *If $a = 0$, then all points $x \in \mathcal{M}(X, r, \xi)$ are $\mathcal{L}^a_{\det}$-semistable.*
- *If $a > 0$ is an integer, then $x \in \mathcal{M}(X, r, \xi)$ is $\mathcal{L}^a_{\det}$-semistable if and only if the vector bundle E corresponding to x is semistable in the usual sense.*

Proof. Giving a morphism $\Theta = [\operatorname{Spec}(\mathbb{C}[t])/\mathbb{G}_m] \longrightarrow \mathcal{M}(X, r, \xi)$ is equivalent to giving a $\mathbb{G}_m$-equivariant morphism $\operatorname{Spec}(\mathbb{C}[t]) \longrightarrow \mathcal{M}(X, r, \xi)$, and this is equivalent to giving a vector bundle $\mathcal{E}$ on $\operatorname{Spec}(\mathbb{C}[t]) \times X$ together with a lift of the $\mathbb{G}_m$ action on $\operatorname{Spec}(\mathbb{C}[t])$. By the Rees construction, this is equivalent to giving a $\mathbb{Z}$-indexed filtration $E_\bullet$ of a vector bundle E on X, with

$$E_i \supseteq E_{i+1}$$

for all i such that $E_i = 0$ for $i \gg 0$ and $E_i = E$ for $i \ll 0$. Indeed, given such a filtration, we define an $\mathcal{O}_{X\times\operatorname{Spec}(\mathbb{C}[t])}$-module as $\mathcal{E} := \bigoplus_{i\in\mathbb{Z}} E_i t^{-i}$. Then the restriction of $\mathcal{E}$ to the slice $X \times \{t\}$ is isomorphic to E if $t \neq 0$ and it is isomorphic to the associated graded object $\operatorname{gr} E_\bullet$ if $t = 0$ (see [17, Lemma 1.10] for more details).

A calculation shows the following (see [17, § 1.E.c]):

$$\operatorname{wt}(f^*\mathcal{L}^a_{\det}) = 2a \sum \big(\operatorname{rk}(E)\deg(E_l) - \operatorname{rk}(E_l)\deg(E)\big)$$

and the proposition follows. □

Therefore, the substack $\mathcal{M}^{\text{ss-vb}}(X, r, \xi)$ of semistable vector bundles can be intrinsically recovered from $\mathcal{M}(X, r, \xi)$. More precisely, we have:

Corollary 3.4. *Let X be a smooth projective curve of any genus. Let $\mathcal{L}$ be a line bundle on $\mathcal{M} = \mathcal{M}(X, r, \xi)$ such that the substack of $\mathcal{L}$-semistable points satisfies the condition $\emptyset \subsetneq \mathcal{M}^{\mathcal{L}-ss} \subsetneq \mathcal{M}$. Let $\mathcal{L}'$ be another such line bundle. Then $\mathcal{M}^{\mathcal{L}-ss} = \mathcal{M}^{\mathcal{L}'-ss}$, and this is the substack $\mathcal{M}^{\text{ss-vb}}(X, r, \xi)$ of semistable vector bundles in the usual sense.*

Proof. This follows immediately from Proposition 3.3. □

Alternatively, the substack of semistable vector bundles can also be recovered using a result of Faltings [13, Theorem I.3] (see also [25, Proposition 1.6.2] and [26, Theorem 6.2 and Lemma 8.3 by Nori]) which identifies the complement of $\mathcal{M}^{\text{ss-vb}}(X,r,\xi)$ in $\mathcal{M}(X,r,\xi)$ as the substack of k-points on which all sections of powers of the generator of the determinant of the cohomology line bundle vanish. See also recent works of Weissmann-Zhang for another approach [30].

Once we recover the substack parametrizing the semistable locus, we can apply [3, Theorem 3.12] to construct a *good moduli space* (in the sense of J. Alper [2]) $M^{\text{ss-vb}}(X,r,\xi)$ of $\mathcal{M}^{\text{ss-vb}}(X,r,\xi)$ and a map

$$\mathcal{M}^{\text{ss-vb}}(X,r,\xi) \longrightarrow M^{\text{ss-vb}}(X,r,\xi),$$

which coincides with the usual moduli space of semistable vector bundles.

Proposition 3.5. *Let X and X' be smooth complex projective curves of any genus and $r, r' > 1$. If $\mathcal{M}(X,r,\xi) \cong \mathcal{M}(X',r',\xi')$, then $M^{\text{ss-vb}}(X,r,\xi) \cong M^{\text{ss-vb}}(X',r',\xi')$.*

Proof. Assume that we have an isomorphism

$$\Psi : \mathcal{M}(X,r,\xi) \longrightarrow \mathcal{M}(X',r',\xi').$$

Let $\mathcal{L}' = \mathcal{L}'_{\det}$ be the determinant line bundle on $\mathcal{M}(X',r',\xi')$, and let $\mathcal{L} = \Psi^*\mathcal{L}'$. Using the definition of $\mathcal{L}$-semistability, it is easy to check that Ψ restricts to an isomorphism between $\mathcal{M}^{\mathcal{L}-\text{ss}}(X,r,\xi)$ and $\mathcal{M}^{\mathcal{L}'-\text{ss}}(X',r',\xi') = \mathcal{M}^{\text{ss-vb}}(X',r',\xi')$. By Corollary 3.4 we obtain that $\mathcal{M}^{\mathcal{L}-\text{ss}}(X,r,\xi) = \mathcal{M}^{\text{ss-vb}}(X,r,\xi)$. Therefore, Φ restricts to an isomorphism

$$\Psi^{\text{ss-vb}} : \mathcal{M}^{\text{ss-vb}}(X,r,\xi) \longrightarrow \mathcal{M}^{\text{ss-vb}}(X',r',\xi')\,.$$

Let π and π' be the projections from each of these moduli stacks of semistable bundles to the respective moduli schemes. Consider the composition of maps

$$\pi' \circ \Psi^{\text{ss-vb}} : \mathcal{M}^{\text{ss-vb}}(X,r,\xi) \longrightarrow M^{\text{ss-vb}}(X',r',\xi')\,.$$

By [2, Theorem 6.6], the good quotient $M^{\text{ss-vb}}(X,r,\xi)$ corepresents the moduli stack $\mathcal{M}^{\text{ss-vb}}(X,r,\xi)$. Thus, the map $\pi' \circ \Psi^{\text{ss-vb}}$ factors through the moduli scheme $M^{\text{ss-vb}}(X,r,\xi)$:

$$\begin{tikzcd}
\mathcal{M}^{\text{ss-vb}}(X,r,\xi) \arrow[r, "\Psi^{\text{ss-vb}}"] \arrow[d, "\pi"'] & \mathcal{M}^{\text{ss-vb}}(X',r,\xi') \arrow[d, "\pi'"] \\
M^{\text{ss-vb}}(X,r,\xi) \arrow[r, "\psi"] & M^{\text{ss-vb}}(X',r',\xi')
\end{tikzcd}$$

As the inverse of $\Psi^{\text{ss-vb}}$ also descends, the above map ψ is an isomorphism. $\square$

From Proposition 3.5 we can obtain the Torelli theorem for the moduli stacks applying any of the existing Torelli theorems for the moduli schemes. For instance, the following theorem results by applying [1, Corollary 2.12].

Theorem 3.6. *Let X and X' be smooth complex projective curves of genus at least 4. Suppose that $r, r' \geq 2$. Let ξ and ξ' be line bundles on X and X' respectively. Then $\mathcal{M}(X,r,\xi) \cong \mathcal{M}(X',r',\xi')$ if and only if $X \cong X'$, $r = r'$ and $\deg(\xi) \cong \pm\deg(\xi') \pmod r$.*

Proof. If $\mathcal{M}(X,r,\xi) \cong \mathcal{M}(X',r',\xi')$, then Proposition 3.5 implies that

$$M^{\text{ss-vb}}(X,r,\xi) \cong M^{\text{ss-vb}}(X',r',\xi')$$

and the theorem follows from [1, Corollary 2.12]. $\square$

Observe that the same argument can also be applied to prove a Torelli theorem for the moduli stack of rank r bundles of fixed degree, invoking the appropriate Torelli theorem for moduli spaces.

Similarly, we can consider principal G-bundles for any complex reductive group G. Let $\mathcal{M}^d_G(X)$ be the connected component of moduli stack of principal G-bundles on X corresponding to $d \in \pi_1(G)$ (the connected components of the moduli stack are parametrized by $\pi_1(G)$).

Recall that a principal G-bundle E_G is semistable in the sense of Ramanathan if for any reduction $P^Q \subset E_G$ to a parabolic subgroup $Q \subset G$ and for any dominant character χ of Q, the degree of the associated line bundle $P^Q(\chi)$ satisfies the inequality $\deg(P^Q(\chi)) \leq 0$.

Lemma 3.7. *Let X be a smooth complex projective curve of any genus. Take a (rational) line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{M}_G^d(X)) \otimes \mathbb{Q}$, and let*

$$\mathcal{U}_{\mathcal{L}} = \mathcal{M}_G^d(X)^{\mathcal{L}-\mathrm{ss}} \subset \mathcal{M}_G^d(X)$$

be the substack of $\mathcal{L}$-semistable principal G-bundles on X. Let $\mathcal{U}$ be the intersections of all $\mathcal{U}_{\mathcal{L}}$ which are nonempty. Then $\mathcal{U}$ is the substack of semistable principal G-bundles in the sense of Ramanathan.

Furthermore, there exists a (rational) line bundle $\mathcal{L}$ such that $\mathcal{U} = \mathcal{U}_{\mathcal{L}}$.

Proof. Since the curve X is fixed during this proof, we will drop it entirely from the notation, denoting the moduli stack by just $\mathcal{M}_G^d$. Let Z' be the center of $[G, G]$. It is a finite group. A principal G-bundle is semistable in the sense of Ramanathan if and only if its extension of structure group to a principal G/Z'-bundle is semistable. We are going to see that the same holds for $\mathcal{L}$-semistability in the sense of Definition 3.1.

Consider the morphism

$$p : \mathcal{M}_G^d \longrightarrow \mathcal{M}_{G/Z'}^{d'}$$

which sends a principal G-bundle on X to the associated G/Z'-bundle. Let P be a principal G-bundle on X mapping to a principal G/Z'-bundle P'.

We claim that a morphism $f' : \Theta \longrightarrow \mathcal{M}_{G/Z'}^{d'}$ (with $f'(1) = P'$) can be lifted to

$$f : \Theta \longrightarrow \mathcal{M}_G^d$$

(with $f(1) = P$) after passing to a ramified cover $\Theta \xrightarrow{[n]} \Theta$ given by $t \longmapsto t^n$. Indeed, in [17, 1.F.b] it is proved that there is a bijection between morphisms $f : \Theta \longrightarrow \mathcal{M}_G$ and equivalence classes of pairs $(\lambda : \mathbb{G}_m \to G,\ P_\lambda \subset P)$ consisting of a one parameter subgroup $\lambda : \mathbb{G}_m \longrightarrow G$ and a reduction of structure group $P_\lambda \subset P$ of a principal G-bundle P to the parabolic subgroup

$$\{g \in G \mid \lim_{t\to 0} \lambda(t) g \lambda(t^{-1}) \text{ exists}\} \subset G.$$

Two pairs are equivalent if λ is conjugate by an element of the parabolic subgroup. In this bijection, if the morphism $f(t)$ is replaced by $f(t^n)$, then the one-parameter subgroup $\lambda(t)$ is replaced by $\lambda(t^n)$, and the parabolic subgroup and reduction stay the same. Therefore a morphism $f : \Theta \longrightarrow \mathcal{M}_{G/Z'}^{d'}$ produces a one-parameter subgroup $\lambda' : \mathbb{G}_m \longrightarrow G/Z'$ and a reduction of structure group of the principal G/Z'-bundle P' to the parabolic subgroup associated to λ'. The parabolic subgroups of G are the same as the parabolic subgroups of G/Z', and a reduction of structure group of a principal G-bundle P to a parabolic subgroup of G induces a reduction, to the corresponding parabolic subgroup of G/Z', of the principal G/Z'-bundle corresponding to P. On the other hand, since Z' is a finite abelian group, we have a Cartesian diagram

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, "\lambda"] \arrow[d, "q"'] & G \arrow[d] \\
\mathbb{G}_m \arrow[r, "\lambda'"] & G/Z'
\end{tikzcd}$$

where q is just the cover $t \longmapsto t^n$ for some n. Therefore, λ' can be lifted to G after passing to a cover of order n. This implies that f can be lifted to $\mathcal{M}_G^d$ as claimed.

By [8, Def 5.2.1 and Thm 5.3.1] the morphism p induces an isomorphism $p^* : \mathrm{Pic}(\mathcal{M}_{G/Z'}^{d'}) \otimes \mathbb{Q} \longrightarrow \mathrm{Pic}(\mathcal{M}_G^d) \otimes \mathbb{Q}$.

Therefore, a point in $\mathcal{M}_G^d$ is $\mathcal{L}$-semistable if and only if its image in $\mathcal{M}_{G/Z'}^{d'}$ is $\mathcal{L}'$-semistable, where $p^*(\mathcal{L}') \cong \mathcal{L}$.

Let Z be the center of G. Note that $G/Z' = G/[G, G] \times G/Z$. Therefore, we have

$$\mathcal{M}_{G/Z'}^{d'} = \mathcal{M}_{G/[G,G]}^{d_1} \times \mathcal{M}_{G/Z}^{d_2}. \tag{3.1}$$

The group $G/[G, G]$ is a torus (isomorphic to $\mathbb{G}_m^r$), and the group G/Z is semisimple and of adjoint type. The global functions on $\mathcal{M}_{G/G'}^{d'}$ are just the constant scalars $\mathbb{C}$, and $\mathrm{Pic}(\mathcal{M}_{G/Z}^{d_2})$ is discrete (by [8, Theorem 5.3.1]), so [8, Lemma 2.1.4] gives:

$$\mathrm{Pic}(\mathcal{M}_{G/Z'}^{d'}) = \mathrm{Pic}(\mathcal{M}_{G/[G,G]}^{d_1}) \oplus \mathrm{Pic}(\mathcal{M}_{G/Z}^{d_2}).$$

Therefore, a line bundle $\mathcal{L}$ on $\mathcal{M}_{G/Z'}^{d'}$ is of the form $\mathcal{L}_1 \boxtimes \mathcal{L}_2$, and a point x in $\mathcal{M}_{G/Z'}^{d'}$ is $\mathcal{L}$-semistable if and only if both the projections $x_1 \in \mathcal{M}_{G/[G,G]}^{d_1}$ and $x_2 \in \mathcal{M}_{G/Z}^{d_2}$ are, respectively, $\mathcal{L}_1$-semistable and $\mathcal{L}_2$-semistable. In other words,

$${\mathcal{M}_G^{d'}}^{\mathcal{L}-\mathrm{ss}} = {\mathcal{M}_{G/[G,G]}^{d_1}}^{\mathcal{L}_1-\mathrm{ss}} \times {\mathcal{M}_{G/Z}^{d_2}}^{\mathcal{L}_2-\mathrm{ss}}.$$

The torus $G/[G,G]$ is the product $\mathbb{G}_m^s$. Then

$$\mathcal{M}^{d_1}_{G/[G,G]} \cong B(\mathbb{G}_m{}^{\times s}) \times J^{\times s}$$

where J is the Jacobian scheme of the curve. The scheme $J^{\times r}$ is projective, so the global functions are just the scalars $\mathbb{C}$, and $\mathrm{Pic}(B(\mathbb{G}_m{}^{\times r}) = \mathbb{Z}^r$ is discrete, so applying [8, Lemma 2.1.4] again we get that

$$\mathrm{Pic}(\mathcal{M}^{d_1}_{G/[G,G]}) \cong \mathbb{Z}^r \oplus \mathrm{Pic}(J^{\times r})$$

and then a line bundle on $\mathcal{M}^{d_1}_{G/[G,G]}$ is of the form $\mathcal{L}_1 = \mathcal{L}_{1,1} \boxtimes \mathcal{L}_{1,2}$, where $\mathcal{L}_{1,1} \in \mathrm{Pic}\, B(\mathbb{G}_m{}^{\times r}) = \mathbb{Z}^r$ and $\mathcal{L}_{1,2} \in \mathrm{Pic}(J^{\times r})$, and the point x_1 is $\mathcal{L}_1$-semistable if and only if both $x_{1,1}$ and $x_{1,2}$ are respectively $\mathcal{L}_{1,1}$-semistable and $\mathcal{L}_{1,2}$-semistable. The point $x_{1,2} \in J^{\times s}$ is automatically $\mathcal{L}_{1,2}$-semistable, because $J^{\times s}$ is a scheme, and hence any morphism from Θ into it is trivial.

A line bundle on $B\mathbb{G}_m$ is the same thing as a one dimensional vector space with an action of $\mathbb{G}_m$. Therefore, $\mathrm{Pic}(B\mathbb{G}_m) = \mathbb{Z}$. Let L_a be a line bundle on $B\mathbb{G}_m$. The morphisms from Θ to $B\mathbb{G}_m$ are classified by $\mathbb{Z}$, because such a morphism is equivalent to an equivariant line bundle on $\mathbb{A}^1$, and these are classified by the weight of the action on the fiber over zero. Let $f_b : \Theta \to B\mathbb{G}_m$ be the morphism corresponding to $b \in \mathbb{Z}$. Then $\mathrm{wt}(f_b^* L_a|_0) = ab$. Therefore (Definition 3.1), the point in $B\mathbb{G}_m$ is L_a-semistable if and only a is zero.

It follows that any point $x_{1,1} \in B(\mathbb{G}_m{}^{\times r})$ is $\mathcal{L}_{1,2}$-semistable if and only if all the coordinates of $\mathcal{L}_{1,2} \in \mathbb{Z}^r$ are zero. Therefore, to prove this lemma we may assume that all these coordinates are zero.

Hence, a point x in the stack (3.1) is $\mathcal{L}$-semistable if and only if the projection to $\mathcal{M}_{G/Z}$ is $\mathcal{L}_2$-semistable. The same holds for semistability in the sense of Ramanathan.

Therefore, we may assume that Z is trivial and G is a product of simple groups of adjoint type: $G = G_1 \times \cdots \times G_s$. Using [8, Definition 5.2.1, Remark 4.3.3 and Theorem 5.3.1], $\mathrm{Pic}(\mathcal{M}_{G_1\times\cdots\times G_s}) \otimes \mathbb{Q} \cong \mathbb{Q}^s$, where s is the number of simple factors, and the generators of this group come from pullbacks of line bundles on each factor $\mathcal{M}_{G_i}$. In other words, a line bundle $\mathcal{L}$ in $\mathrm{Pic}(\mathcal{M}_{G/Z}) \otimes \mathbb{Q}$ is of the form $\mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_s$, where $\mathcal{L}_i$ is a (rational) line bundle on $\mathcal{M}_{G_i}$. It is easy to check that a point x in $\mathcal{M}_{G_1\times\cdots\times G_s}$ is $\mathcal{L}$-semistable if and only if all projections $x_i \in \mathcal{M}_{G_i}$ are $\mathcal{L}_i$-semistable:

$$\mathcal{M}^{\mathcal{L}-ss}_{G_1\times\cdots\times G_s} = \mathcal{M}^{\mathcal{L}_1-ss}_{G_1} \times \cdots \times \mathcal{M}^{\mathcal{L}_s-ss}_{G_s}$$

and, again, the same holds for semistability in the sense of Ramanathan.

By [8, Theorem 5.3.1], $\mathrm{Pic}(\mathcal{M}_{G_i}) \otimes \mathbb{Q} \cong \mathbb{Q}$ and the determinant line bundle $\mathcal{L}_{\det}$, whose fiber over P is $(\det H^1(\mathrm{ad}(P))) \otimes (\det H^0(\mathrm{ad}(P)))^{-1}$, is a generator. As in the case of vector bundles, it follows from Remark 3.2, that it is enough to consider three cases: If $\mathcal{L}_i$ is a positive multiple of the determinant bundle, then $\mathcal{L}_i$-semistability is equivalent to the usual notion of semistability defined by Ramanathan (see [17, § 1.F]). If $\mathcal{L}_i$ is a negative multiple of the determinant, then the substack of $\mathcal{L}_i$-semistable points is empty, and if $\mathcal{L}_i$ is trivial, then the substack of $\mathcal{L}_i$-semistable points is the whole moduli stack.

Therefore, $\mathcal{M}^{\mathcal{L}-ss}_G$ is smallest and non-empty when $\mathcal{L}_i$ is a positive multiple of the determinant bundle on $\mathcal{M}_{G_i}$ for all i. Furthermore, for such $\mathcal{L}_i$, a point $x \in \mathcal{M}_G$ is $\mathcal{L} = \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_s$-semistable if and only if it is semistable in the usual sense of Ramanathan. □

Corollary 3.8. *Let X and X' be smooth projective complex curves with genera $g(X),\ g(X') \geq 3$ respectively. Let G and G' be algebraic connected reductive complex groups. If the moduli stacks $\mathcal{M}^d_G(X)$ and $\mathcal{M}^{d'}_{G'}(X')$ are isomorphic as stacks, then the curves X and X' are also isomorphic.*

Proof. Arguing as done for Proposition 3.5 we obtain that the corresponding moduli schemes of principal bundles are isomorphic, and then we apply [9, Theorem 0.1]. □

4. Torelli for moduli stack of rank 2 vector bundles

In this section we prove a Torelli theorem for the moduli stack of rank 2 vector bundles with trivial determinant. Notice that for genus 2 curves the moduli space of rank 2 vector bundles with trivial determinant is isomorphic to $\mathbb{P}^3$, irrespective of the curve. Nevertheless, we will show that the geometry of the moduli stack, contrary to the scheme, does indeed encode the geometry of the curve effectively.

Throughout this section, $M^{\text{s-vb}}(X,2,\mathcal{O}_X) \subset M^{\text{ss-vb}}(X,2,\mathcal{O}_X)$ denotes the subset of stable bundles and $S(X,2,\mathcal{O}_X) := M^{\text{ss-vb}}(X,2,\mathcal{O}_X)\backslash M^{\text{s-vb}}(X,2,\mathcal{O}_X)$ is the subset of strictly semistable vector bundles.

Lemma 4.1. *The image of the set of non-simple points in $\mathcal{M}^{\text{ss-vb}}(X,2,\mathcal{O}_X)$ under the quotient map $\mathcal{M}^{\text{ss-vb}}(X,2,\mathcal{O}_X) \longrightarrow M^{\text{ss-vb}}(X,2,\mathcal{O}_X)$ coincides with the set of strictly semistable vector bundles $S(X,2,\mathcal{O}_X) := M^{\text{ss-vb}}(X,2,\mathcal{O}_X)\backslash M^{\text{s-vb}}(X,2,\mathcal{O}_X)$.*

Proof. Since the preimage of $M^{\text{s-vb}}(X,2,\mathcal{O}_X)$ under the quotient map is the substack of stable vector bundles, and all stable vector bundles are simple, the image of each non-simple vector bundle in $\mathcal{M}^{\text{ss-vb}}(X,2,\mathcal{O}_X)$ must be a strictly semistable vector bundle. Let us prove that the non-simple vector bundles surject onto the strictly semistable ones.

For each strictly semistable vector bundle E in $S(X, 2, \mathcal{O}_X)$ there exists a polystable vector bundle $\widetilde{E} = L \oplus L^{-1}$ which is S-equivalent to E. The map

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} : L \oplus L^{-1} \longrightarrow L \oplus L^{-1}$$

is a nontrivial traceless endomorphism of $\widetilde{E}$, so $\widetilde{E}$ represents a non-simple point in $\mathcal{M}^{\text{ss-vb}}(X, 2, \mathcal{O}_X)$ which projects to E. $\square$

Theorem 4.2. *Let X and X' be two irreducible smooth complex projective curves of genus g and g' respectively with $g, g' \geq 2$. If $\mathcal{M}(X, 2, \mathcal{O}_X) \cong \mathcal{M}(X', 2, \mathcal{O}_{X'})$, then $X \cong X'$.*

Proof. Repeating the argument in the previous section we know that the isomorphism $\mathcal{M}(X, 2, \mathcal{O}_X) \cong \mathcal{M}(X', 2, \mathcal{O}_{X'})$ restricts to an isomorphism of the semistable locus $\mathcal{M}^{\text{ss-vb}}(X, 2, \mathcal{O}_X) \cong \mathcal{M}^{\text{ss-vb}}(X', 2, \mathcal{O}_{X'})$ and that this map descends to an isomorphism

$$\begin{tikzcd} \mathcal{M}^{\text{ss-vb}}(X, 2, \mathcal{O}_X) \arrow[r] \arrow[d, "\pi"'] & \mathcal{M}^{\text{ss-vb}}(X', 2, \mathcal{O}_{X'}) \arrow[d, "\pi'"] \\ M^{\text{ss-vb}}(X, 2, \mathcal{O}_X) \arrow[r, "\psi"] & M^{\text{ss-vb}}(X', 2, \mathcal{O}_{X'}) \end{tikzcd}$$

such that $\psi(S(X, 2, \mathcal{O}_X)) = S(X', 2, \mathcal{O}_{X'})$. Let $K(X) = J(X)/\{\pm 1\}$ denote the quotient of $J(X)$ by the inversion map i defined by $L \longmapsto L^{-1}$. In particular, if the genus of X is two, then $K(X)$ is the Kummer surface associated to the Jacobian. Each S-equivalence class of a bundle E in $S(X, 2, \mathcal{O}_X)$ has a unique representative of the form $E = L \oplus L^{-1}$, so there exists a correspondence between the points of $S(X, 2, \mathcal{O}_X)$ and the points of $K(X)$. By [22, Theorem 2], the moduli space $M^{\text{ss-vb}}(X, 2, \mathcal{O}_X)$ is isomorphic to $\mathbb{P}(H^0(J(X), \mathcal{L}_\theta^2))$, where $\mathcal{L}_\theta$ is the canonical polarization of the Jacobian induced by the natural embedding of X in $J(X)$. Now Proposition 6.3 and the construction from Theorem 2 of [22] prove that the map from $K(X)$ to $\mathbb{P}(H^0(J(X), \mathcal{L}_\theta^2)) \cong M^{\text{ss-vb}}(X, 2, \mathcal{O}_X)$, which sends the class of L to the S-equivalence class of $L \oplus L^{-1}$, gives an embedding $K(X) \hookrightarrow M^{\text{ss-vb}}(X, 2, \mathcal{O}_X)$ whose image is the subvariety $S(X, 2, \mathcal{O}_X)$ preserved by ψ.

From the geometry of $K(X)$ we can reconstruct the map $J(X) \longrightarrow K(X)$ canonically as follows. First, remove the singular points of $K(X)$. Let $K^{\text{sm}}(X)$ be the smooth part. The fundamental group $\pi_1(K^{\text{sm}}(X))$ has a unique maximal torsion free subgroup. This subgroup coincides with the subgroup $\pi_1(J(X) \backslash J(X)[2])$, where $J(X)[2]$ denotes the 2-torsion part of the Jacobian, and the quotient group is $\mathbb{Z}/2\mathbb{Z}$. The corresponding double covering is then $J(X) \backslash J(X)[2] \longrightarrow K^{\text{sm}}(X)$. Now $J(X)$ is the unique abelian compactification of $J(X) \backslash J(X)[2]$, and the map $J(X) \longrightarrow K(X)$ is the unique possible extension to $J(X)$ for the double cover $J(X) \backslash J(X)[2] \longrightarrow K^{\text{sm}}(X)$. Thus, the isomorphism ψ induces an isomorphism $J(X) \cong J(X')$.

Now, consider the compositions of maps

$$j_X : J(X) \longrightarrow K(X) \hookrightarrow M^{\text{ss-vb}}(X, 2, \mathcal{O}_X),$$
$$j_{X'} : J(X') \longrightarrow K(X') \hookrightarrow M^{\text{ss-vb}}(X', 2, \mathcal{O}_{X'}).$$

Let $\mathcal{L}$ be the ample generator of $M^{\text{ss-vb}}(X, 2, \mathcal{O}_X)$. By construction [22], it follows that $j^*\mathcal{L}$ is a multiple of the canonical polarization of $J(X)$. Since $\text{Pic}(M^{\text{ss-vb}}(X, 2, \mathcal{O}_X) \cong \text{Pic}(M^{\text{ss-vb}}(X', 2, \mathcal{O}_{X'}) = \mathbb{Z}$ by [12], $\mathcal{L}' := (\psi^{-1})^*\mathcal{L}$ is a multiple of the ample generator of $M^{\text{ss-vb}}(X', 2, \mathcal{O}_{X'})$, so $j_{X'}^*(\mathcal{L}')$ is also a multiple of the canonical polarization. Thus, the isomorphism $J(X) \cong J(X')$ induced by ψ is an isomorphism of canonically polarized Jacobians. By the classical Torelli Theorem, $X \cong X'$. $\square$

Remark 4.3. The above proof also shows that the Torelli theorem holds for the substack of semistable vector bundles. If $\mathcal{M}^{\text{ss-vb}}(X, 2, \mathcal{O}_X) \cong \mathcal{M}^{\text{ss-vb}}(X', 2, \mathcal{O}_{X'})$, then $X \cong X'$. Thus, the isomorphism class of the curve X cannot be recovered from the moduli scheme $M^{\text{ss-vb}}(X, 2, \mathcal{O}_X)$, when $g = 2$, but it can be recovered from the geometry of the moduli stack $\mathcal{M}^{\text{ss-vb}}(X, 2, \mathcal{O}_X)$.

5. Proof of the Torelli theorem

In this section we will combine the previous cases to obtain a Torelli theorem for the moduli stack of vector bundles for curves of any rank $r \geq 2$ and any genus $g \geq 2$. Let us start with a basic dimensional computation, which will allow us to apply the appropriate Torelli theorems selectively.

Lemma 5.1. *Let M be some variety which is isomorphic to a moduli space of semistable vector bundles of rank $r \geq 2$ and determinant ξ over a smooth complex projective curve X of genus $g \geq 2$. Then, either*

(1) $\dim(M) = 3$, *in which case* $g = 2$ *and* $r = 2$,
(2) $\dim(M) = 6$, *in which case* $g = 3$ *and* $r = 3$,
(3) $\dim(M) = 8$, *in which case* $g = 2$ *and* $r = 3$,
(4) $\dim(M) \geq 9$, *in which case either*
- $g \geq 4$, *or*
- $g = 3$ *and* $r \geq 3$, *or*
- $g = 2$ *and* $r \geq 4$.

Proof. The dimension of a moduli space of a curve of genus $g \geq 2$ and rank $r \geq 2$ is $d_{g,r} = (r^2 - 1)(g - 1)$. Clearly the dimensions

$$d_{2,2} = 3, \qquad d_{2,3} = 8, \quad \text{and} \quad d_{3,2} = 6$$

are distinct and less than 9. Thus, if we prove that the dimension of any other moduli space with $(g, r) \notin \{(2, 2), (2, 3), (3, 2)\}$ is greater or equal to 9, there will exist a unique option for the previous given dimensions and the lemma will follow. Clearly $d_{g,r}$ is increasing in r and g. Thus, if $g \geq 4$ then as $r \geq 2$, we have

$$d_{g,r} = (r^2 - 1)(g - 1) \geq (2^2 - 1)(3 - 1) = 9$$

and, if $r \geq 4$, then as, $g \geq 2$, we have

$$d_{g,r} = (r^2 - 1)(g - 1) \geq (4^2 - 1)(2 - 1) = 15 > 9.$$

Finally, $d_{3,3} = 16 > 9$ and we obtain the following table for the values of $d_{g,r}$ which proves the result:

$g\backslash r$	2	3	≥ 4
2	3	8	≥ 15
3	6	16	≥ 30
≥ 4	≥ 9	≥ 24	≥ 45

This completes the proof. $\square$

Theorem 5.2. *Let X and X' be two irreducible smooth complex projective curves of genus g and g' respectively, with $g, g' \geq 2$. Let $r, r' \geq 2$, and fix line bundles ξ and ξ' on X and X' respectively. Let*

$$\Psi : \mathcal{M}(X, r, \xi) \longrightarrow \mathcal{M}(X', r', \xi')$$

be an isomorphism between the corresponding moduli stacks of vector bundles with fixed determinant. Then $r = r'$ and $X \cong X'$.

Proof. Repeating the argument from Section 3 and applying Proposition 3.5, the above map Ψ induces an isomorphism $M^{\text{ss-vb}}(X, r, \xi) \cong M^{\text{ss-vb}}(X', r', \xi')$ between the moduli spaces. By Lemma 5.1, the dimension of this moduli space is either 3, 6, 8 or at least 9. Let us consider each case individually.

Dimension 3:

By Lemma 5.1, $g = g' = 2$ and $r = r' = 2$. By [22], there are two possible different geometries for these moduli spaces. Either the moduli spaces are both isomorphic to $\mathbb{P}^3$, in which case $\deg(\xi)$ and $\deg(\xi')$ are even, or both the moduli spaces are isomorphic to an intersection of quadrics in $\mathbb{P}^5$, in which case $\deg(\xi)$ and $\deg(\xi')$ are odd.

If $\deg(\xi)$ and $\deg(\xi')$ are both odd then we can apply the Torelli Theorems for rank 2 bundles with fixed determinant with odd degree by Mumford and Newstead [21, Corollary p.1201] or by Tyurin [28, Theorem 1].

Otherwise, if $\deg(\xi)$ and $\deg(\xi')$ are both even, then there exist line bundles L and L' on X and X' respectively such that $L^2 = \xi$ and $(L')^2 = \xi'$. In that case, the maps $E \longmapsto E \otimes L^{-1}$ and $E' \longmapsto E' \otimes L^{-1}$ induce isomorphisms of stacks $\mathcal{M}(X, 2, \xi) \cong \mathcal{M}(X, 2, \mathcal{O}_X)$ and $\mathcal{M}(X', 2, \xi') \cong \mathcal{M}(X', 2, \mathcal{O}_{X'})$ respectively. Thus, we have an isomorphism of stacks $\mathcal{M}(X, 2, \mathcal{O}_X) \cong \mathcal{M}(X', 2, \mathcal{O}_{X'})$ and now we can apply Theorem 4.2 to conclude that $X \cong X'$.

Dimension 6:

By Lemma 5.1, $g = 3 = g'$ and $r = 2 = r'$. So we can apply the Torelli Theorem by Kouvidakis and Pantev for curves of genus $g \geq 3$ of the same rank [20, Theorem E] to conclude that $X \cong X'$.

Dimension 8:

By Lemma 5.1 we have $g = 2 = g'$ and $r = 3 = r'$. By [5, Theorem 1.8], the cohomological Brauer group of the moduli space $M^{\text{ss-vb}}(X, 3, \xi)$ is

$$\operatorname{Br}\left(M^{\text{ss-vb}}(X, 3, \xi)\right) := H^2\left(M^{\text{ss-vb}}(X, 3, \xi)_{et}, \mathbb{G}_m\right) \cong \mathbb{Z}/(\mathrm{g.c.d}(r, \deg(\xi)))\mathbb{Z}$$

so it is either 0 (when $\deg(\xi)$ is coprime to 3) or it is $\mathbb{Z}/3\mathbb{Z}$ (when $\deg(\xi)$ is a multiple of 3). As $\operatorname{Br}(M^{\text{ss-vb}}(X, 3, \xi) \cong \operatorname{Br}(M^{\text{ss-vb}}(X', 3, \xi')$, either $\deg(\xi)$ and $\deg(\xi')$ are both coprime to 3 or they are both multiples of 3.

If $\deg(\xi)$ and $\deg(\xi')$ are both coprime to 3, then we can apply the Torelli theorems by Tyurin [29, Theorem 1] or Narasimhan-Ramanan [23, Theorem 3].

If $\deg(\xi)$ and $\deg(\xi')$ are both multiples of 3, then, as before, there exist line bundles L and L' over X and X' respectively such that $L^3 = \xi$ and $(L')^3 = \xi'$. The maps $E \longmapsto E \otimes L^{-1}$ and $E' \longmapsto E' \otimes L^{-1}$ induce isomorphisms of moduli schemes $M^{\text{ss-vb}}(X, 3, \xi) \cong M^{\text{ss-vb}}(X, 3, \mathcal{O}_X)$ and $M^{\text{ss-vb}}(X', 3, \xi') \cong M^{\text{ss-vb}}(X', 3, \mathcal{O}_{X'})$ respectively. Thus, we have an isomorphism of moduli schemes $M^{\text{ss-vb}}(X, 3, \mathcal{O}_X) \cong M^{\text{ss-vb}}(X', 3, \mathcal{O}_{X'})$. Now we can apply the Torelli theorem for genus 2 curves and rank 3 bundles with trivial determinant by Nguyen [24, Corollary 3.4.4] to obtain that $X \cong X'$.

Dimension at least 9:

By Lemma 5.1, both (g, r) and (g', r') satisfy the conditions on the ranks and genera in Theorem 2.10, and hence we obtain that $X \cong X'$. $\square$

Remark 5.3. Using Lemma 2.7 — instead of Lemma 2.8 — in the proof of Corollary 2.9 we can show that the discriminant $\mathcal{D}$ is the closure of the image under $\mathcal{H}$ of the set of rational curves in $T^*M^{\text{ss-vb}}(X, r, \xi)$ for any curve X of genus $g \geq 2$ and any rank $r \geq 2$ such that $(g, r) \notin \{(2, 2), (2, 3), (3, 2)\}$.

Using this intrinsic characterization of $\mathcal{D}$, the proof of the Torelli Theorem for the moduli scheme of vector bundles by Biswas, Gómez and Muñoz [7, Theorem 4.3] extends to curves of genus $g \geq 2$ and $r \geq 2$ where $(g, r) \notin \{(2, 2), (2, 3), (3, 2)\}$.

Then, the argument of Theorem 5.2 can also be used to prove the following Torelli Theorem for the moduli scheme of vector bundles. Let X and X' be smooth complex projective curves of genus at least 2, and let $r, r' \geq 2$. If $M^{\text{ss-vb}}(X, r, \xi) \cong M^{\text{ss-vb}}(X', r', \xi')$, then $r = r'$ and either

- $X \cong X'$, or
- X and X' are any pair of curves of genus 2 with $r = r' = 2$ and $\deg(\xi)$ and $\deg(\xi')$ being even; in this case we have $M^{\text{ss-vb}}(X, 2, \xi) \cong M^{\text{ss-vb}}(X', 2, \xi') \cong \mathbb{P}^3$.

As a consequence, the unique case for genus and rank at least 2 where the moduli scheme of vector bundles does not admit a Torelli theorem is when the genus is 2, the rank is 2 and the degree of the determinant is even.

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Data availability

No data was used for the research described in the article.

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