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Extended interval goal programming

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This paper focuses on possible problems associated with the use of penalty functions in Goal Programming (GP). In this sense we illustrate, with the help of numerical examples, how an assumption of separability amongst decision maker's preferences which underlies these approaches, can produce in the corresponding GP models extremely biased results towards certain goals. A new GP variant is proposed to overcome this type of problem.

Keywords: goal programming; penalty functions; preference modelling

Charnes and Collomb,[1] and Ignizio[2] introduced Interval Goal Programming (IGP). Later, this approach was developed by approximating the piecewise linear function that represents the penalty scale as a continuous function.[3,4] Kvanli[5] suggested an extension of GIP based upon penalty functions. Such an extension attempts to minimise the weighted sum of unwanted deviational variables from a specified set of intervals. Several authors have proposed technical improvements and refinements of Kvanli's approach. Among others, Tamiz and Jones,[6] and Jones and Tamiz[7] have proposed the most efficient method from a computational point of view. An additional advantage to this approach is that it can model preferences not only in increasing penalty scenarios but also in decreasing penalty, and in discontinuity preference scenarios.

However in each of these approaches, as in other closely related methodologies such as satisfaction functions,[8] there is a separability assumption underlying the decision-maker's preferences. The separability assumption can produce extremely biased results towards some of the goals, what can lead to unacceptable solutions by the decision-maker. The purpose of this paper is to clarify this problem by offering several modelling solutions capable of resolving this type of problem.

To illustrate our argument, the following slightly modified example used by Romero[9] is adopted. Let us consider a decision-making problem with the following goals and constraints:

$$
\begin{aligned}
&(g_1)\quad 3x_1 + 2.5x_2 + 2.5x_3 + n_1 - p_1 = 100\\
&(g_2)\quad 4x_1 + 3x_2 + 3.5x_3 + n_2 - p_2 = 100\\
&(g_3)\quad 3.5x_1 + 5x_2 + 3.5x_3 + n_3 - p_3 = 100
\end{aligned}
$$

$$
\begin{gathered}
x_2 + x_3 \geqslant 10\\
x_2 \geqslant 4\\
\mathbf{x} \geqslant \mathbf{0} \qquad \mathbf{n} \geqslant \mathbf{0} \qquad \mathbf{p} \geqslant \mathbf{0}
\end{gathered}
$$

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The above goals have been normalised by multiplying the initial goals by 100 and then dividing the equation by the right-hand side coefficient so the deviational variables measure percentages. In this way, all normalising problems are avoided, in the example the following penalty scale is assumed:

	Unit (%)	*Marginal penalty*
First goal	Below 90	2
	90–100	1
	Over 100	0
Second and third goal	Below 110	0
	110–120	1
	Over 120	2

These penalty scales imply the inclusion of functions of the type shown in Figures 1 and 2.

The above three-sided penalty functions can be built in a weighted GP model according to the Jones and Tamiz[7] formulation:

Objective function:

$$\min\ n_1 + (2-1)n_2 + p_3 + (2-1)p_4 + p_5 + (2-1)p_6$$

subject to

$$
\begin{aligned}
3x_1 + 2.5x_2 + 2.5x_3 + n_1 - p_1 &= 100\\
3x_1 + 2.5x_2 + 2.5x_3 + n_2 - p_2 &= 90\\
4x_1 + 3x_2 + 3.5x_3 + n_3 - p_3 &= 110\\
4x_1 + 3x_2 + 3.5x_3 + n_4 - p_4 &= 120\\
3.5x_1 + 5x_2 + 3.5x_3 + n_5 - p_5 &= 110\\
3.5x_1 + 5x_2 + 3.5x_3 + n_6 - p_6 &= 120
\end{aligned}
\tag{1}
$$

$$
\begin{gathered}
x_2 + x_3 \geqslant 10\\
x_2 \geqslant 4\\
\mathbf{x} \geqslant \mathbf{0} \qquad \mathbf{n} \geqslant \mathbf{0} \qquad \mathbf{p} \geqslant \mathbf{0}
\end{gathered}
$$

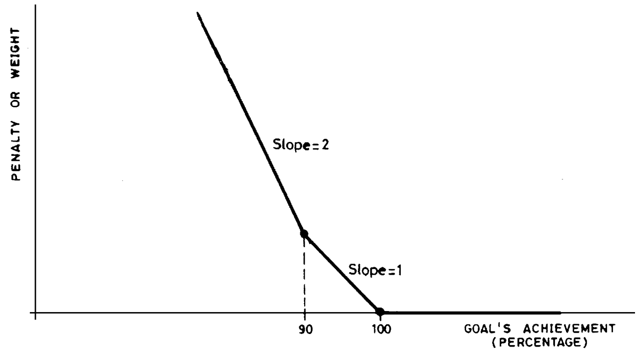

Figure 1 Three-sided penalty function for the first goal.

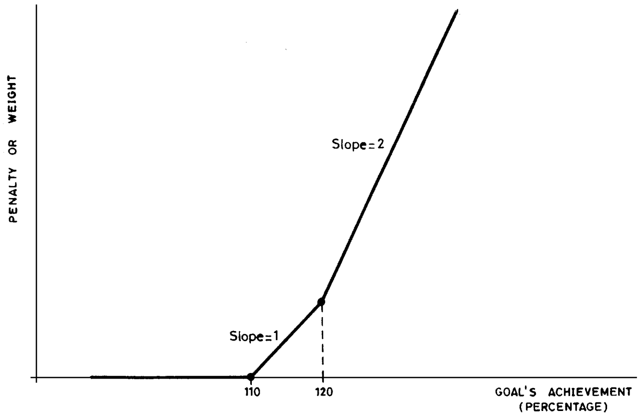

Figure 2 Three-sided penalty function for the second and third goal.

The optimum solution to this problem is:
Decision variables:

$$x_1 = 19.71 \qquad x_2 = 4 \qquad x_3 = 6$$

Unwanted deviational variables:

$$n_1 = 15.86 \qquad n_2 = 5.86 \qquad p_3 = 1.86$$
$$p_4 = p_5 = p_6 = 0$$

The total penalty of this solution is 23.58 units. This solution provides the minimum aggregate deviation for the three goals considered. Underlying this solution is a weighted GP model that implies an assumption of separability between the three goals. Therefore, the solution (the best in aggregate terms) can be extremely biased towards the achievement of some of the goals. To illustrate this point the following Chebyshev Interval GP model is formulated:
Objective function:

$$\min D$$

subject to

$$\begin{aligned} n_1 + (2-1)n_2 - D &\leqslant 0 \\ p_3 + (2-1)p_4 - D &\leqslant 0 \\ p_5 + (2-1)p_6 - D &\leqslant 0 \end{aligned} \tag{2}$$

Goals and constraints of model (1).

The above model minimises the maximum penalty deviation D, consequently, it will provide the most balanced solution between the achievement of the three goals. This is, the decision-maker's preferences have not been aggregated for this model in an additive way, but according to a MAXMIN structure. The optimum solution to this problem is:
Decision variables:

$$x_1 = 21.81 \qquad x_2 = 5.45 \qquad x_3 = 4.54 \qquad D = 9.54$$

Unwanted deviational variables:

$$n_1 = 9.54 \qquad n_2 = 0 \qquad p_3 = 9.54 \qquad p_4 = 0$$
$$p_5 = 9.54 \qquad p_6 = 0$$

The maximum penalty (deviation) D is now 9.54. It is interesting to note that for this solution the total aggregate penalty is 28.63 units, whereas model (1) provides a total aggregate penalty of 23.58 units with a maximum deviation of 21.71 units. This measures the clash between the minimum aggregate penalty and the maximum largest penalty. Although the first solution is the best in aggregate terms, it is also very bad in terms of achievement of the first goal (presenting a deviation of $10 + 2x5.86 = 21.71$ units). On the other hand, the second solution is perfectly balanced (all three goals present a deviation of 9.54 units).

The aforementioned analysis can be generalised with the help of the following Extended Interval GP model:
Objective function:

$$\min (1-\lambda)D + \lambda[n_1 + (2-1)n_2 + p_3 + (2-1)p_4 + p_5 + (2-1)p_6]$$

subject to

Goals and constraints of model (2)

$$-Z + n_1 + (2-1)n_2 + p_3 + (2-1)p_4 + p_5 + (2-1)p_6 = 0 \tag{3}$$

The last constraint is just an accounting row that measures the total aggregate deviation corresponding to each solution. For $\lambda = 0$, the Chebyshev Interval GP model given by (2) is obtained, whereas for $\lambda = 1$ the additive weighted GP model given by (1) is obtained. Other values of λ yield an intermediate solution between the solutions provided by the two GP options mentioned earlier. Solutions provided by model (3) for different values of λ are shown in Table 1. These results underpin the discrepancy between the maximum aggregate achievement and the most balanced solution.

Table 1 Three-sided penalty function. Extended interval GP formulation (***Increasing penalty case***)

Value of parameter λ	*Decision variables*			*Unwanted deviational variables*						*Maximum deviation* D	*Total aggregated deviation* Z
	x_1	x_2	x_3	n_1	n_2	p_3	p_4	p_5	p_6		
[0.82 1]	19.71	4	6	15.86	5.86	1.86	0	0	0	21.71	23.58
[0.42 0.81]	21.67	4	6	10	0	9.67	0	6.83	0	10	26.50
[0.14 0.41]	21.71	4	6	9.86	0	9.86	0	7	0	9.86	26.71
[0 0.13]	21.81	5.45	4.55	9.55	0	9.55	0	9.55	0	9.55	28.63

Let us now illustrate our argument for a decreasing penalty situation. That is a situation where the decision-maker wishes to decrease the value of the marginal penalty, after the deviation exceeds a threshold value. If in the above example, the values of the marginal penalties are switched around, the application of Jones and Tamiz[7] method leads to the following weighted GP model:
Objective function:

$$\min\ 2n_1 + (1-2)n_2 + 2p_3 + (1-2)p_4 + 2p_5 + (1-2)p_6$$

subject to (4)

Goals and constraints of model (1)

If model (4) is solved on a GP solver without the facility for basis restriction (that is, to prevent the positive and negative deviational variable of the *i*th goal from being simultaneously in the basis), then unbounded solutions can be obtained. To avoid this problem, Jones and Tamiz[7] proposed an approach based upon 0–1 variables. For our example, this method implies the incorporation of the following additional constraints:

$$\begin{aligned} n_2 - 100t_1 &\leqslant 0 \\ p_2 + 100t_1 &\leqslant 100 \\ n_4 - 100t_2 &\leqslant 0 \\ p_4 + 100t_2 &\leqslant 100 \\ n_6 - 100t_3 &\leqslant 0 \\ p_6 + 100t_3 &\leqslant 100 \end{aligned} \tag{5}$$

where t_1, t_2 and t_3 are binary variables and 100 represents an arbitrarily large value. The addition of block (5) of auxiliary constraints ensures that the negative and positive deviational variables cannot simultaneously take non-zero values. By solving model (4) augmented with block (5), the following optimum solution is obtained:
Decision variables:

$$x_1 = 19.35 \qquad x_2 = 4.83 \qquad x_3 = 5.16$$

Unwanted deviational variables:

$$n_1 = 16.94 \qquad n_2 = 6.94 \qquad p_3 = p_4 = p_5 = p_6 = 0$$

The total penalty of this solution is 26.94 units. The Extended Interval GP model for this decreasing penalty function has the following structure:

Objective function:

$$\min\,(1-\lambda)D + \lambda[2n_1 + (1-2)n_2 + 2p_3 + (1-2)p_4 + 2p_5 + (1-2)p_6]$$

subject to

$$\begin{aligned} 2n_1 - n_2 - D &\leqslant 0 \\ 2p_3 - p_4 - D &\leqslant 0 \\ 2p_5 - p_6 - D &\leqslant 0 \end{aligned} \tag{6}$$

Goals and constraints of model (1)

Auxiliary constraints given by (5) if necessary

$$-Z + 2n_1 + (1-2)n_2 + 2p_3 + (1-2)p_4 + 2p_5 + (1-2)p_6 = 0$$

The solutions provided by model (6), for different values of parameter λ, are shown in Table 2. These results once again illustrate the discrepancy between the maximum aggregate achievement and the most balanced solution.

The Extended Interval GP model can be easily formalised. Therefore, let us conceive a general setting where q goals are considered and the preferences of the decision-maker are represented by a k-sided penalty function. The structure of the model is as follows:
Objective function:

$$\min\,(1-\lambda)D + \lambda\left[\sum_{i=1}^{q}\alpha_{i1}\bar{n}_{i1} + \sum_{i=1}^{q}\sum_{j=1}^{k}(\alpha_{i,j+1}-\alpha_{ij})\bar{n}_{i,j+1} + \sum_{i=1}^{q}\beta_{i1}p_{i1} + \sum_{i=1}^{q}\sum_{j=1}^{k}(\beta_{i,j+1}-\beta_{ij})p_{ij}\right]$$

Table 2 Three-sided penalty function. Extended interval GP formulation (***Decreasing penalty case***)

Value of parameter λ	*Decision variables*			*Unwanted deviational variables*							*Maximum deviation D*	*Total aggregated deviation Z*
	x_1	x_2	x_3	n_1	n_2	p_3	p_4	p_5	p_6			
[0.30 1]	19.35	4.84	5.16	16.94	6.94	0	0	0	0		26.93	26.93
[0.21 0.29]	19.71	4	6	15.86	5.86	1.86	0	0	0		25.86	29.57
[0.14 0.20]	21.71	4	6	9.86	0	9.86	0	7	0		19.71	53.43
[0 0.13]	21.81	5.45	4.55	9.55	0	9.55	0	9.55	0		19.09	57.27

subject to

$$\left[\sum_{i=1}^{q}\alpha_{i1}\bar{n}_{i1}+\sum_{i=1}^{q}\sum_{j=1}^{k}(\alpha_{i,j+1}-\alpha_{ij})\bar{n}_{i,j+1} + \sum_{i=1}^{q}\beta_{i1}p_{i1}+\sum_{i=1}^{q}\sum_{j=1}^{k}(\beta_{i,j+1}-\beta_{ij})p_{ij}\right]-D\leqslant 0$$

$$\left.\begin{array}{llll} f_i(\mathbf{x}) & +n_{i1} & -p_{i1} & =b_{i1}\\ \vdots & \vdots & & \vdots\\ f_i(\mathbf{x}) & +n_{ik} & -p_{ik} & =b_{ik}\\ f_i(\mathbf{x}) & +\bar{n}_{i1} & -\bar{p}_{i1} & =a_{i1}\\ \vdots & \vdots & & \vdots\\ f_i(\mathbf{x}) & +\bar{n}_{ik} & -\bar{p}_{ik} & =a_{ik} \end{array}\right\} i\in\{1,\ldots,q\} \tag{7}$$

$$\mathbf{x}\in\mathbf{F}$$

$$\mathbf{x}\geqslant\mathbf{0}\qquad\mathbf{n}\geqslant\mathbf{0}\qquad\mathbf{p}\geqslant\mathbf{0}\qquad\bar{\mathbf{n}}\geqslant\mathbf{0}\qquad\bar{\mathbf{p}}\geqslant\mathbf{0}$$

where $f_i(\mathbf{x})$ is the mathematical expression for the ith attribute and $\mathbf{F}$ the feasible set, and $a_{i1} > a_{i2} > \cdots > a_{ik}, b_{i1} < b_{22} < \cdots < b_{IK}$. In an increasing penalty case, coefficients $(\alpha_{i,j+1}-\alpha_{ij})$ and $(\beta_{i,j+1}-\beta_{ij})$ are positive while in a decreasing penalty case these are negative. In the latter case, a set of auxiliary constraints similar to (5) should be introduced to avoid possible unbounded solutions.

We conclude this paper by remarking that all attempts to incorporate the decision-maker's preferences in a GP model through mechanisms like penalty and satisfaction functions found in the literature underly an assumption of preference separability. This fact can generate extremely biased results towards some of the goals, a situation that can be unacceptable to the decision-maker. One possible way to resolve this situation and still maintain the distinguishing properties of the penalty and satisfaction functions, consists of resorting to an Extended Interval GP formulation that can generate the following solutions: (a) the classic solution of minimum aggregate deviation; (b) the solution in which the maximum deviation is minimised; and (c) the best-compromise between both solutions.

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