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On the set covering polytope: Facets with coefficients in {0, 1, 2, 3}*

M. Sánchez-García[a], M.I. Sobrón[a] and B. Vitoriano[b]
[a] *Departamento de Estadística e Investigación Operativa, Universidad Complutense de Madrid, E-28040 Madrid, Spain*
[b] *Departamento Organización Industrial, Universidad Pontificia Comillas, E-28015 Madrid, Spain*

Balas and Ng [1,2] characterized the class of valid inequalities for the set covering polytope with coefficients equal to 0, 1 or 2, and gave necessary and sufficient conditions for such an inequality to be facet defining. We extend this study, characterizing the class of valid inequalities with coefficients equal to 0, 1, 2 or 3, and giving necessary and sufficient conditions for such an inequality to be not dominated, and to be facet defining.

Keywords: set covering, facets, polyhedral combinatorics, combinatorial optimization

1. Introduction

Let $E = \{e_1, \ldots, e_m\}$ be a finite set, let $S = \{S_1, \ldots, S_n\}$ be a given collection of subsets of E, and let $c = (c_1, \ldots, c_n)$ be a vector of costs, where $c_j \geq 0$, $\forall j = 1, \ldots, n$. Let $F \subseteq \{1, \ldots, n\}$ be an index subset, F is said to **cover** E if $\bigcup_{j \in F} S_j = E$.

The set covering problem consists of determining a minimum-cost cover of E, if it exists. Obviously, if $\bigcup_{j=1}^{n} S_j \neq E$, then the problem has no solution.

The set covering problem can be stated as

$$(\mathbf{SC}) \qquad \min\{cx \mid Ax \geq 1,\ x \in \{0,1\}^n\},$$

where $A = (a_{ij})$ is an $m \times n$ matrix with $a_{ij} \in \{0, 1\}$, $\forall i, j$, and 1 is the m-vector of 1's. This is an NP-complete problem for a general 0 – 1 matrix A, and the model has applications such as crew scheduling, facility location, vehicle routing and a host of others.

We denote the set covering polytope

$$P_I(A) := \text{conv}\{x \in \mathbb{R}^n \mid Ax \geq 1,\ 0 \leq x \leq 1,\ x \text{ integer}\},$$

and the polyhedron related to $P_I(A)$

$$P(A) := \{x \in \mathbb{R}^n \mid Ax \geq 1,\ 0 \leq x \leq 1\}.$$

Let M and N be the row and column index sets, respectively, of A.

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An inequality $\alpha x \geq b$ is said to be valid for $P_I(A)$ if and only if, $\forall z \in P_I(A)$, $\alpha z \geq b$. It is said to be a face of $P_I(A)$ if and only if it is a valid inequality and there exists $z \in P_I(A)$, z integer, such that $\alpha z = b$. For an n-dimensional polyhedron, the 0-dimensional faces are its vertices. An inequality $\alpha x \geq \alpha_0$, valid for $P_I(A)$, defines a facet of $P_I(A)$, if and only if, $\alpha x = \alpha_0$ for $n = \dim(P_I(A))$ affinely independent points $x \in P_I(A)$. For an n-dimensional polyhedron, the $(n-1)$-dimensional faces are its facets.

Not much is known about the set covering polytope. Some of the following classic results about it are well known. We assume throughout that A has no zero columns or zero rows.

(1) $P_I(A)$ is full dimensional if and only if $\sum_{j=1}^{n} a_{ij} \geq 2$ for all $i \in M$.

In the following, we assume that $P_I(A)$ is full dimensional.

(2) The inequality $x_k \geq 0$ is a facet defining inequality of $P_I(A)$ if and only if $\sum_{j=1,\, j \neq k}^{n} a_{ij} \geq 2$ for all $i \in M$.

(3) All inequalities $x_j \leq 1$ define facets of $P_I(A)$.

(4) All facet defining inequalities $\alpha x \geq \alpha_0$ for $P_I(A)$ have $\alpha \geq 0$ if $\alpha_0 > 0$.

(5) The inequality

$$\sum_{j=1}^{n} a_{ij} x_j \geq 1$$

defines a facet of $P_I(A)$ if and only if

(a) there exists no $k \in M$ with $a_{kj} \leq a_{ij}, \forall j \in N$, and $\sum_{j=1}^{n} a_{kj} < \sum_{j=1}^{n} a_{ij}$;

(b) for each k such that $a_{ik} = 0$, there exists $j(k)$ such that $a_{ij(k)} = 1$ and $a_{hj(k)} = 1$ for all $h \in M^0(k)$, where $M^0(k) := \{h \in M / a_{hk} = 1$ and $a_{hj} = 0$, $\forall j \neq k$, such that $a_{ij} = 0\}$.

(6) The only minimal valid inequalities (hence the only facet defining inequalities) for $P_I(A)$ with integer coefficients and right-hand side equal to 1 are those of the system $Ax \geq 1$.

Statements (1) through (4) are easily seen to be true, and the proofs of (5) and (6) may be seen in [1].

Therefore, characterizing facet defining inequalities with coefficients in $\{0, 1\}$ is a closed problem.

Other important results about facet defining inequalities of $P_I(A)$ may be seen in [1–5]. In this paper, we are following the research guide developed in [1] and continued in [4].

Balas and Ng [1] studied inequalities of the form $\alpha x \geq 2$, with $\alpha_j = 0$, 1 or 2, $j \in N$. They characterized this class of valid inequalities for the set covering polytope, and gave neccesary and sufficient conditions for such an inequality to be minimal and facet defining. Therefore, the study of this class of inequalities is also complete.

Here, we will study valid inequalities for $P_I(A)$ of the form $\alpha x \geq 3$, with $\alpha_j = 0$, 1, 2 or 3, $j \in N$. These inequalities have been studied in [4], but the study is not complete. We will give necessary and sufficient conditions for such an inequality to be minimal and to be facet defining.

2. Valid and minimal inequalities of $P_I(A)$ with coefficients in {0, 1, 2, 3}

For all $R \subseteq M$ and for all $S \subseteq N$, A_R^S is written as the submatrix of A whose rows and columns are indexed by R and S, and for every $Q \subseteq N$, $M(Q) := \{i \in M \mid a_{ij} = 0, \forall j \in Q\}$, with $M(\emptyset) := M$.

Let $\alpha x \geq 3$ with $\alpha_j = 0, 1, 2$ or 3, $j \in N$, be a valid inequality for $P_I(A)$. We denote

$$J_t(\alpha) = \{j \in N \mid \alpha_j = t\}, \quad t = 0, 1, 2, 3$$

or J_t whenever the meaning is clear from the context.

To each nonempty subset $S \subseteq M$, an inequality $\alpha^S x \geq 3$ is associated, where

$$N_3 = \{j \in N \mid a_{ij} = 1 \quad \forall i \in S\};$$

$$N_0 = \{j \in N \mid a_{ij} = 0 \quad \forall i \in S\};$$

$$N' = N \setminus (N_3 \cup N_0).$$

N_1 is a maximum cardinality set in N' such that $\forall j, h \in N_1\ \exists i \in S$ with $a_{ij} + a_{ih} = 0$; and the inequality is

$$\alpha_j^S = \begin{cases} 0 & \text{if } j \in N_0, \\ 3 & \text{if } j \in N_3, \\ 1 & \text{if } j \in N_1, \\ 2 & \text{otherwise.} \end{cases} \tag{2.1}$$

Notice that this inequality is not unique; with every different subset N_1, different inequalities are obtained.

Example 2.1. Consider the set covering polytope defined by the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The inequalities $\sum_{i=1}^{6} \alpha_i x_i \geq 3$ obtained by (2.1), with the rows $S = \{1, 2, 3, 4\}$ are

$$x_2 + x_3 + x_4 + 2x_5 + 3x_6 + 3x_7 + 3x_8 \geq 3$$

and

$$2x_2 + x_3 + x_4 + x_5 + 3x_6 + 3x_7 + 3x_8 \geq 3.$$

Next, we are going to prove that the inequalities (2.1) are valid for $P_I(A)$.

Theorem 2.1. An inequality $\alpha^S x \geq 3$ obtained by (2.1) is a valid inequality for $P_I(A)$, for any $S \subseteq M$, $S \neq \emptyset$.

Proof. Let $x \in P_I(A)$, then $\forall\, i \in S\ \sum_{j=1}^{n} a_{ij} x_j \geq 1$. Let $J = \{ j / x_j = 1\}$ and $J_i = J \cap N_i$, $i = 0, 1, 2, 3$. Obviously, if $J_3 \neq \emptyset$, then $\alpha^S x \geq 3$. Therefore, we may assume $J_3 = \emptyset$. Then there are three possible situations:

$$\begin{array}{llll} \mathrm{Car}(J_2) \geq 2 & \text{or} & & \\ \mathrm{Car}(J_2) = 1 & \text{and} & \mathrm{Car}(J_1) \geq 1, & \text{or} \\ \mathrm{Car}(J_2) = 0 & \text{and} & \mathrm{Car}(J_1) \geq 3. & \end{array}$$

For every assumption, if $x_0 \in P_I(A)$ satisfies $\alpha^S x_0 \geq 3$, then $\alpha^S x \geq 3$ is a valid inequality for $P_I(A)$. □

In order to compare different inequalities with the right-hand side equal to 3, we must define another concept of dominated inequalities.

Definition 2.1. A valid inequality for $P_I(A)$, $\beta x \geq 3$, is **dominated** by $\gamma x \geq 3$ if and only if

$$J_3(\gamma) \subseteq J_3(\beta), \quad J_0(\gamma) \supseteq J_0(\beta), \quad \mathrm{Car}(J_0(\gamma) \cup J_1(\gamma)) \geq \mathrm{Car}(J_0(\beta) \cup J_1(\beta)).$$

Notice that the above definition does not depend on the subset J_1 itself, but only on its cardinality. Therefore, inequality $\alpha^S x \geq 3$ obtained by (2.1) can be considered as a unique inequality in order to compare it with those with the same right-hand side.

Theorem 2.2. Every valid inequality $\beta x \geq 3$ for $P_I(A)$, with β_j integer, $j \in N$, is dominated by the inequality $\alpha^S x \geq 3$, where $S = M(J_0(\beta))$.

By convention, if $J_0(\beta) = \emptyset$, $S = M$ is considered.

Proof. By contradiction. Without loss of generality, we may assume that $\beta_j \in \{0,1,2,3\}$. If $J_0(\beta) \neq \emptyset$ and $S = \emptyset$, then $\bar{x}$ defined by $\bar{x}_j = 1, j \in J_0(\beta)$, $\bar{x}_j = 0$ otherwise, satisfies $A\bar{x} \geq 1$ but $\beta\bar{x} = 0 < 3$, a contradiction. Therefore, $S \neq \emptyset$, and the inequality $\alpha^S x \geq 3$

is well defined, except by the choice of the subset N_1. From the definition of α^S, $J_0(\alpha^S) \supseteq J_0(\beta)$. Suppose there exists $j_1 \in J_3(\alpha^S)$ and $j_1 \notin J_3(\beta)$; then $\bar{x}$ defined by $\bar{x}_{j_1} = 1$, $\bar{x}_j = 1, j \in J_0(b)$, $\bar{x}_j = 0$ otherwise, satisfies $ax \geq 1$ and violates $\beta x \geq 3$, a new contradiction. Therefore, $J_3(\alpha^S) \subseteq J_3(\beta)$.

Finally, we suppose that $\mathrm{Car}(J_0(\alpha^S) \cup J_1(\alpha^S)) < \mathrm{Car}(J_0(\beta) \cup J_1(\beta))$. This in turn implies that $\mathrm{Car}(J_1(\alpha^S)) < \mathrm{Car}(J_1(\beta))$, but $J_1(\alpha^S)$ is a maximum cardinality set such that $\forall j, k \in J_1(\alpha^S)$ there exists $i \in S$ with $a_{ij} + a_{ik} = 0$; therefore there exist j_1, $k_1 \in J_1(\beta)$ such that $j_1 \in J_2(\alpha^S)$ and $k_1 \in J_1(\alpha^S)$, and $a_{ij_1} + a_{ik_1} \geq 1$, $\forall i \in S$. $\bar{x}$ defined by $\bar{x}_{j_1} = 1$, $\bar{x}_{k_1} = 1$, $\bar{x}_j = 1, j \in J_0(\beta)$, and $\bar{x}_j = 0$ otherwise, satisfies $Ax \geq 1$, but violates $\beta x \geq 3$, a contradiction. □

Next we identify those inequalities $\alpha^S x \geq 3$ that are not strictly dominated by other inequalities of the same form (2.1). Hence, it is clear from the definitions that, among all inequalities $\alpha^S x \geq 3$ with fixed J_0, it is enough to consider those with $S = M(J_0)$.

Definition 2.2. Given any inequality $\alpha^S x \geq 3$ with $S = M(J_0)$, we will say that the set J_0 is **maximal** if and only if $\forall\, T \subset S$ such that $J_0^T \supset J_0^S$ verifies

$$J_3^T \supseteq J_3^S, \tag{2.2}$$

and

$$\mathrm{Car}(J_3^T \cup J_2^T) \geq \mathrm{Car}(J_3^S \cup J_2^S), \tag{2.3}$$

with either (2.2) or (2.3) strictly.

In other words, J_0 is maximal if the transfer of any column from $J_1 \cup J_2$ to J_0 requires either

- transfer some column from $J_1 \cup J_2$ to J_3, or
- keep J_3 the same and increase the number of variables with coefficient equal to 2.

Theorem 2.3. Let $\alpha^S x \geq 3$, with $S = M(J_0(\alpha^S))$, be the valid inequality for $P_I(A)$ associated to S. Then $\alpha^S x \geq 3$ is not dominated if and only if J_0^S is maximal.

Proof. *Necessity*. We prove the necessary condition by contradiction. If J_0^S is not maximal, there exists $T \subset S$ such that $J_0^T \supset J_0^S$, $J_3^T = J_3^S$ and $\mathrm{Car}(J_2^T) = \mathrm{Car}(J_2^S)$, and so $\alpha^S x \geq 3$ is strictly dominated by $\alpha^T x \geq 3$, a contradiction. Therefore, J_0^S is maximal.

Sufficiency. We prove it also by contradiction. Suppose $\alpha^S x \geq 3$ is dominated. There exists $T \subseteq M$ such that $\alpha^T x \geq 3$ dominates $\alpha^S x \geq 3$, which implies that $J_0^T \supseteq J_0^S$, $J_3^T \subseteq J_3^S$ and $\mathrm{Car}(J_0^T \cup J_1^T) \geq \mathrm{Car}(J_0^S \cup J_1^S)$, with some strict relation. But if $J_0^T = J_0^S$ and another inequality is strict, then $\alpha^S x \geq 3$ is not valid, a contradiction. Therefore, $J_0^T \supset J_0^S$, and hence $T \subset S$ strictly. From this we conclude that $J_3^T \supseteq J_3^S$, so $J_3^T = J_3^S$.

Further, since $\mathrm{Car}(J_0^T \cup J_1^T) \geq \mathrm{Car}(J_0^S \cup J_1^S)$, it is clear that $\mathrm{Car}(J_2^T) \leq \mathrm{Car}(J_2^S)$, concluding that J_0^S is not maximal, which is a contradiction. □

Example 2.2. Notice that the hypothesis $S = M(J_0)$ is very important, since in the case this assumption is not verified, theorem 2.3 can be false. In example 2.1, the hypothesis is not verified. In fact, the inequalities obtained with $S = \{1, 2, 3, 4\}$ are dominated by the inequality associated to $S' = M(J_0) = M(\{1\}) = \{1, 2, 3, 4, 5, 6\}$, which is

$$x_2 + x_3 + x_4 + x_5 + 2x_6 + 2x_7 + 3x_8 \geq 3. \tag{2.4}$$

On the other hand, this inequality is not dominated because the subset $J_0 = \{1\}$ is maximal, since no variable can be moved from $J_1 \cup J_2$ to J_0 without violating either conditions (2.2) or (2.3).

An alternative condition for an inequality of the form (2.1) not to be dominated is developed below.

Definition 2.3. Let $\alpha^S x \geq 3$ be a valid inequality associated with S.

- A pair j, h, $j \in J_1$, $h \in J_2$, is called a *2-cover* of $A_{M(J_0)}$, if $a_{ij} + a_{ih} \geq 1 \ \forall\, i \in M(J_0)$.
- A trio $j, h, k \in J_1$ is called a *3-cover* of $A_{M(J_0)}$, if $a_{ij} + a_{ih} + a_{ik} \geq 1 \ \forall\, i \in M(J_0)$.

Corollary 2.1. The inequality $\alpha^S x \geq 3$, where $S = M(J_0)$, is not dominated if and only if every $j \in J_1$ belongs to some 2-cover or 3-cover of $A_{M(J_0)}$.

Proof. By theorem 2.3, $\alpha^S x \geq 3$ is not dominated if and only if J_0 is maximal.

Obviously, if there exists $j \in J_1^S$ such that j does not belong to any 3-cover or any 2-cover of $A_{M(J_0)}$, then J_0^S is not maximal.

Now, suppose J_0^S is not maximal; then there exists $T \subset S$ such that $J_0^T \supset J_0^S$, $J_3^T = J_3^S$ and $\mathrm{Car}(J_2^T) = \mathrm{Car}(J_2^S)$. Let $j \in J_0^T - J_0^S$. It is clear that $j \in J_1^S$ and j does not belong to any 3- or 2-cover of A_S. □

3. Facets of $P_I(A)$ with coefficients in {0, 1, 2, 3}

Next, we address the question of which inequalities $\alpha^S x \geq 3$ are facet inducing for $P_I(A)$. In stating the conditions for this, we will assume that $P_I(A)$ is full dimensional, that is to say,

$$\sum_{j=1}^{n} a_{ij} \geq 2 \quad \forall i \in M. \tag{3.1}$$

Every set covering polytope which is not empty may be transformed into another polytope that satisfies (3.1). If $P_I(A) \neq \varnothing$ and violates (3.1), there exists some $N' \subset N$,

$N' \neq \emptyset$, such that $x \in P_I(A)$ implies $x_j = 1$ for all $j \in N'$. Then setting $x_j = 1$, $j \in N'$, and removing the inequalities satisfied by the assignment, we obtain a set covering polytope that satisfies (3.1).

First, we will show some previous results which are necessary to answer this question. These results study the number of vectors associated to 2-covers and 3-covers which are linearly independent. In these results, we only consider the columns in J_1 and J_2, so the vectors considered have only $|J_1| + |J_2|$ components.

In the following, we denote $n_t = |J_t|$, $t = 0, 1, 2, 3$. We begin with the study of vectors associated to 2-covers of S.

Definition 3.1. The **2-cover graph associated with** $\alpha^S x \geq 3$, G^S, is defined as:

- a bipartite graph $G^S = (J_1 \cup J_2, D^S)$;
- $j, k \in D^S$ if and only if $\{j, k\}$ is a 2-cover of $A_{M(J_0)}$.

Example 3.1. Consider the set covering polytope defined by the next matrix (the same as example 2.1), and the inequality

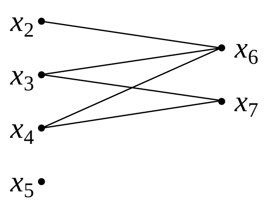
$$x_2 + x_3 + x_4 + x_5 + 2x_6 + 2x_7 + 3x_8 \geq 3$$

associated to $S = \{1, 2, 3, 4, 5, 6\}$, where $J_1 = \{2, 3, 4, 5\}$ and $J_2 = \{6, 7\}$. Then the 2-cover graph associated with this inequality is the following:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$x_2 + x_3 + x_4 + x_5 + 2x_6 + 2x_7 + 3x_8 \geq 3$$

2-cover graph



Theorem 3.1. Let G^S be the 2-cover graph associated with $\alpha^S x \geq 3$, and k the number of components of G^S. The dimension of $V(D^S) = \{I_b / b \in D^S\}$ is $\mathrm{Car}(J_1) + \mathrm{Car}(J_2) - k$.

Proof. Since the vectors belonging to different connected components are independent, we can only prove the theorem for one connected component without loss of generality.

Let $\upsilon_1, \upsilon_2, \ldots, \upsilon_r$ be the elements belonging to J_1^S, ordered such that $\forall\, q = 2, 3, \ldots, r$ there exists $u_{t_q} \in J_{2(q-1)}$ with $(\upsilon_q, u_{t_q}) \in D^S$, where $\forall\, q \geq 2$,

$$J_{2(q-1)} = \{u_i / \exists v_k,\, k \le q-1, \quad \text{with } (v_k, u_i) \in D^S\}$$

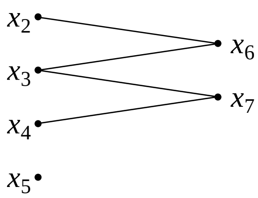
$J_{2(0)} = \emptyset$. The element u_{t_q} exists because the graph is connected.

The vectors $[(v_q, u_{t_q}]_{2 \le q \le r}$ and $\bigcup_{i=1}^{r} \{(v_i, u_j)/j \in J_{2(i)} - J_{2(i-1)}\}$ are linearly independent and there are $p + r - 1$ vectors, with $p = \text{Car}(J_2^S)$.

Furthermore, since the graph is connected, there are no longer any linearly independent vectors. □

Example 3.2. For the graph in the previous example, the following are the edges and their linearly independent vectors obtained in the proof:

Edges



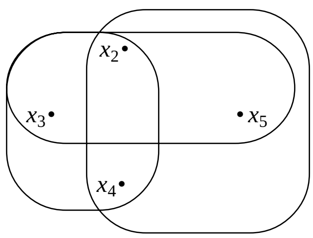
Vectors

(1	0	0	0	1	0)
(0	1	0	0	1	0)
(0	1	0	0	0	1)
(0	0	1	0	0	1)

Next, we obtain results about the number of linearly independent vectors associated to 3- and 2-covers of S, which we will use to characterize the facet-defining inequalities with these coefficients.

Definition 3.2. We define the **3-cover hypergraph associated with** $\alpha^S x \ge 3$, as $H^S = (J_1, C_1)$, with $C \in C_1$ if and only if $\text{Car}(C) = 3$ and $A_{M(J_0)} I_C \ge 1$; i.e. C_1 are the 3-covers of $A_{M(J_0)}$.

Example 3.3. Shown below is the hypergraph associated with the matrix A and the inequality (2.4) for example 2.1.



Theorem 3.2. Let $D_1, \ldots, D_k$ be the components of G^S, the 2-cover graph associated with the inequality $\alpha^S x \ge 3$. There exist $\text{Car}(J_1) + \text{Car}(J_2)$ linearly independent vectors in G^S and H^S if and only if there exist k 3-covers, $C_1, \ldots, C_k \in C_1$ such that the k vectors whose ith components are $\text{Car}(C_j \cap D_i)$, $1 \le i, j \le k$, are linearly independent.

Proof. Let $D_1,\dots,D_{k_1}, D_{k_1+1},\dots,D_k$ be the components of the graph G^S, where D_1, $D_2,\dots,D_{k_1}$ are the components which have elements in J_2, and $D_{k_1+1},\dots,D_k$ are isolated elements.

From theorem 3.1, there exist $n_1 + n_2 - k$ linearly independent vectors associated with 2-covers of S. Let A_{ii} be the matrix whose rows are the linearly independent vectors from theorem 3.1 associated with the component D_i, $i = 1,\dots,k_1$. Without loss of generality, A_{ii} may be considered to be of the form $A_{ii} = [I_i^* \; E_i^*]$, where I_i^* is a diagonal matrix and E_i^* is a column vector whose elements are 1's and -1's: 1 if the element of the diagonal is from a column in J_2 and -1 otherwise.

Using this matrix to transform into zeros the elements of the vectors $\{I_{C_j}\}_{1\le j\le k}$ associated with elements of $\{I_i^*\}_{1\le i\le k_1}$, the vectors defined in the theorem are obtained. Then the proof is easily seen with this transformation. □

Example 3.4. For the example we are studying in this article ($k = 2$), the elements of C_1 and the vectors obtained from them by replacing the components in D_1, D_2 are the following:

$$\begin{array}{ll}
(\overbrace{1\;\;1\;\;1\;\;0\;\;0}^{D_1}\;\;\overbrace{0}^{D_2}) & (3,\;\; 0)\\
(1\;\;1\;\;0\;\;0\;\;0\;\;\;1) & (2,\;\; 1)\\
(1\;\;0\;\;1\;\;0\;\;0\;\;\;1) & (2,\;\; 1)
\end{array}$$

where there are two linearly independent vectors; therefore, there are $n_1 + n_2 = 4 + 2 = 6$ linearly independent vectors 2-covers or 3-covers.

Lemma 3.1. Given $n_1 + n_2$ linearly independent vectors corresponding to 2- and 3-covers of the matrix $A_{M(J_0)}^{J_1\cup J_2}$, any vector associated with a 2-cover or a 3-cover can be written as a linear combination of those, where the sum of the coefficients is equal to 1.

Proof. Let $V = \{\upsilon_1,\dots,\upsilon_{n_1+n_2}\}$ be the linearly independent vectors associated with 2- and 3-covers of $A_{M(J_0)}^{J_1\cup J_2}$. Suppose they are ordered such that $\{\upsilon_1,\dots,\upsilon_{k_0}\}$ are 2-covers and $\{\upsilon_{k_0+1},\dots,\upsilon_{n_1+n_2}\}$ are 3-covers. Also suppose the n_2 first components are indexed by J_2 and the other n_1 components by J_1.

For all $\overline{\upsilon}$, 2-cover or 3-cover may be writen as $\overline{\upsilon} = \sum_{j=1}^{n_1+n_2} \lambda_j \upsilon_j$, for each component i, $\overline{\upsilon}^i = \sum_{j=1}^{n_1+n_2} \lambda_j \upsilon_j^i$. Then

$$\sum_{i=1}^{n_1+n_2} \overline{\upsilon}^i = \sum_{i=1}^{n_1+n_2} \sum_{j=1}^{n_1+n_2} \lambda_j \upsilon_j^i = \sum_{j=1}^{n_1+n_2} \lambda_j \sum_{i=1}^{n_1+n_2} \upsilon_j^i = 2\sum_{j=1}^{k_0} \lambda_j + 3\sum_{j=k_0+1}^{n_1+n_2} \lambda_j. \tag{3.2}$$

Now we study two different cases.

(a) $\overline{\upsilon}$ is a 2-cover.

For each $i \le n_2$ and $k_0 + 1 \le j \le n_1 + n_2$, $\upsilon_j^i = 0$; from υ_j there are 3-covers and the columns different to zero are those in J_1. Therefore, for each $i \le n_2$, $\overline{\upsilon}^i = \sum_{j=1}^{k_0} \lambda_j \upsilon_j^i$.

Since $\overline{\upsilon}$ is a 2-cover and υ_j with $j \le k_0$ are 2-covers, they have exactly one component equal to 1 in the columns in J_2:

$$\sum_{i=1}^{n_2} \overline{\upsilon}_i = 1 = \sum_{i=1}^{n_2} \sum_{j=1}^{k_0} \lambda_j \upsilon_j^i = \sum_{j=1}^{k_0} \sum_{i=1}^{n_2} \upsilon_j^i = \sum_{j=1}^{k_0} \lambda_j .$$

Therefore, substituting into (3.2), we obtain

$$2 = \sum_{i=1}^{n_1+n_2} \overline{\upsilon}_i = 2 \sum_{j=1}^{k_0} \lambda_j + 3 \sum_{j=k_0+1}^{n_1+n_2} \lambda_j = 2 + 3 \sum_{j=k_0+1}^{n_1+n_2} \lambda_j \Rightarrow$$

$$\sum_{j=k_0+1}^{n_1+n_2} \lambda_j = 0 \Rightarrow \sum_{j=1}^{n_1+n_2} \lambda_j = 1.$$

(b) $\overline{\upsilon}$ is a 3-cover. Then $\overline{\upsilon}^i = 0$, $\forall\, i \le n_2$. Further, for all $i \le n_2$ and $k_0 + 1 \le j \le n_1 + n_2$, $\upsilon_j^i = 0$. Therefore,

$$0 = \overline{\upsilon}^i = \sum_{j=1}^{n_1+n_2} \lambda_j \upsilon_j^i = \sum_{j=1}^{k_0} \lambda_j \upsilon_j^i + \sum_{j=k_0+1}^{n_1+n_2} \lambda_j \upsilon_j^i = \sum_{j=1}^{k_0} \lambda_j \upsilon_j^i \quad \forall 1 \le i \le n_2,$$

and since υ_j, $1 \le j \le k_0$, are 2-covers and this implies $\sum_{i=1}^{n_2} \upsilon_j^i = 1$, then

$$0 = \sum_{i=1}^{n_2} \sum_{j=1}^{k_0} \lambda_j \upsilon_j^i = \sum_{j=1}^{k_0} \lambda_j \sum_{i=1}^{n_2} \upsilon_j^i = \sum_{j=1}^{k_0} \lambda_j ,$$

and substituting into (3.2), we obtain

$$3 = \sum_{i=1}^{n_1+n_2} \overline{\upsilon}^i = 3 \sum_{j=k_0+1}^{n_1+n_2} \lambda_j \Rightarrow \sum_{j=k_0+1}^{n_1+n_2} \lambda_j = 1 \Rightarrow$$

$$\sum_{j=1}^{n_1+n_2} \lambda_j = \sum_{j=1}^{k_0} \lambda_j + \sum_{j=k_0+1}^{n_1+n_2} \lambda_j = 1. \qquad \square$$

Now we can conclude the following result, which characterizes those inequalities $\alpha^S x \ge 3$ that are facet inducing for $P_I(A)$.

For every $k \in J_0$, the set $T(k)$ is defined as in [1], i.e.,

$$T(k) = \{i \in M \,|\, a_{ik} = 1, a_{ij} = 0 \text{ for all } j \in J_0 \backslash \{k\}\}.$$

In other words, it is the set of rows such that k is the only column in J_0 to cover $T(k)$. Obviously, $T(k) \subseteq M \backslash \{M(J_0)\}$.

Theorem 3.3. Let $P_I(A)$ be full dimensional and let $\alpha^S x \geq 3$ be a valid inequality for $P_I(A)$, with $S = M(J_0)$. Let $D_1,\dots,D_k$ be the components of the 2-cover graph associated with the inequality $\alpha^S x \geq 3$ and $H^S = (J_1, \mathcal{C}_1)$ the associated 3-cover hypergraph. Then $\alpha^S x \geq 3$ defines a facet of $P_I(A)$ if and only if:

(i) there exist k 3-covers $C_1,\dots,C_k \in \mathcal{C}_1$ such that the k vectors whose ith components are $\mathrm{Car}(C_j \cap D_i)$, $1 \leq i, j \leq k$, are linearly independent;

(ii) for every $k \in J_0$ such that $T(k) \neq \emptyset$, there exists at least one of the following:

(a) some $j(k) \in J_3$ such that $a_{ij(k)} = 1$ for all $i \in T(k)$; or

(b) some pair $j(k)$, $h(k)$, $j(k) \in J_1$, $h(k) \in J_2$, such that $a_{ij(k)} + a_{ih(k)} \geq 1$ for all $i \in T(k) \cup M(J_0)$; or

(c) some trio $j(k)$, $h(k)$, $l(k) \in J_1$ such that $a_{ij(k)} + a_{ih(k)} + a_{il(k)} \geq 1$ for all $i \in T(k) \cup M(J_0)$.

Proof. *Necessity*. Suppose $\alpha^S x \geq 3$ defines a facet of $P_I(A)$. Then there exists a collection of n affinely independent points $\{x^i\}_{1 \leq i \leq n}$ such that $\alpha^S x^i = 3$ for $i = 1,\dots,n$, and $x^i \in P_I(A)$. Let X be the $n \times n$ matrix whose rows are these vectors; then, without loss of generality, X is of the form

$$X = \begin{pmatrix} X_1 & X_2 & 0 \\ X_3 & 0 & X_4 \end{pmatrix},$$

where the columns of X_1, X_3 are indexed by J_0, those of X_2 by $J_1 \cup J_2$, and those of X_4 by J_3. X_4 is the identity matrix of order n_3, and every row of X_2 is a 2-cover or 3-cover of $A^{J_1 \cup J_2}_{M(J_0)}$.

The rows of $(X_3 : 0 : X_4)$ are at most $n_0 + n_4$; then there are at least $n_1 + n_2$ rows in $(X_1 : X_2 : 0)$. Since X is a nonsingular matrix, X_2 is of full column rank, and hence X_2 is of full rank; thus, there exist $n_1 + n_2$ row vectors which are linearly independent. Therefore, from theorem 3.2, condition (i) holds.

To show that (ii) also holds, suppose there exists $k \in J_0$ such that $T(k) \neq \emptyset$ and for which (a), (b) and (c) are not satisfied. Then $x_k = 1$ for every $x \in P_I(A)$ such that $\alpha^S x = 3$, which contradicts the fact that $\alpha^S x \geq 3$ is facet defining.

Sufficiency. Suppose conditions (i) and (ii) hold. We show n linearly independent vectors $\{x^i\}_{1 \leq i \leq n}$ such that $x^i \in P_I(A) \cap \{x / \alpha^S x = 3\}$, $i = 1,\dots,n$.

For $t = 0, 1, 2, 3$, we denote by 1_{n_t} and 0_{n_t} the n_t-vector whose components are all 1 and 0, respectively. For $t = 0, 1, 2, 3$, let $E_j^{n_t}$ be the jth unit vector with n_t components.

Our first n_0 vectors are defined as $x^k = \{(1) \text{ or } (2) \text{ or } (3) \text{ or } (4) \text{ or } (5) \text{ or } (6)\}$, $k \in J_0$, where

(1) $(1_{n_0} - E_k^{n_0}, 0_{n_1+n_2}, E_j^{n_3})$ for some $j \in J_3$, if $T(k) = \emptyset$ and $J_3 \neq \emptyset$.

(2) $(1_{n_0} - E_k^{n_0}, E_j^{n_1+n_2} + E_h^{n_1+n_2}, 0_{n_3})$ for some 2-cover (j, h), $j \in J_1$, $h \in J_2$, if $J_2 \neq \emptyset$ and $T(k) = J_3 = \emptyset$.

(3) $(1_{n_0} - E_k^{n_0}, E_j^{n_1+n_2} + E_h^{n_1+n_2} + E_l^{n_1+n_2}, 0_{n_3})$ for some 3-cover (j, h, l), $j, h, l \in J_1$ if $T(k) = J_3 = J_2 = \varnothing$.

(4) $(1_{n_0} - E_k^{n_0}, 0_{n_1+n_2}, E_{j(k)}^{n_3})$ if $T(k) \neq \varnothing$ and (a) holds (with $j(k)$ as in (a)).

(5) $(1_{n_0} - E_k^{n_0}, E_{j(k)}^{n_1+n_2} + E_{h(k)}^{n_1+n_2}, 0_{n_3})$ if $T(k) \neq \varnothing$ and not (a) but (b) holds (with $j(k)$, $h(k)$ as in (b)).

(6) $(1_{n_0} - E_k^{n_0}, E_{j(k)}^{n_1+n_2} + E_{h(k)}^{n_1+n_2} + E_{l(k)}^{n_1+n_2}, 0_{n_3})$ if $T(k) \neq \varnothing$ and not (a), not (b) but (c) holds (with $j(k)$, $h(k)$, $l(k)$ as in (c)).

By property (ii), these vectors exist and belong to $P_I(A)$.

Our next $n_1 + n_2$ vectors are of the form

$$x^k = (1_{n_0}, E_j^{n_1+n_2} + E_h^{n_1+n_2}, 0_{n_3}) \quad \text{for } (j, h) \text{ a 2-cover of } S,$$

or of the form

$$x^k = (1_{n_0}, E_j^{n_1+n_2} + E_h^{n_1+n_2} + E_l^{n_1+n_2}, 0_{n_3}) \quad \text{for } (j, h, l) \text{ a 3-cover of } S.$$

By condition (i) and theorem 3.2, there exist $n_1 + n_2$ linearly independent vectors in $P_I(A)$ satisfying these conditions.

Finally, the last n_3 vectors are of the form

$$x^k = (1_{n_0}, 0_{n_1+n_2}, E_k^{n_3}), \quad k \in J_3.$$

The vectors $E_k^{n_3}$ form the identity matrix of order n_3. The existence of vectors $x^k \in P_I(A)$ follows from the definition of J_3.

For $t = 0, 1, 2, 3$, we denote by $1_{n_q \times n_t}$ and $0_{n_q \times n_t}$ the $n_q \times n_t$ matrix whose components are all 1 and 0, respectively. Let $E_{n_0 \times n_0}$ be the matrix of order $n_0 \times n_0$ whose diagonal is equal to zero and all other elements are 1. Then the matrix obtained with the previous vectors is

$$B = \begin{pmatrix} E_{n_0 \times n_0} & H_1 & H_2 \\ 1_{(n_1+n_2) \times n_0} & A_{(n_1+n_2) \times (n_1+n_2)} & 0_{(n_1+n_2) \times n_3} \\ 1_{n_3 \times n_0} & 0_{n_3 \times (n_1+n_2)} & 1_{n_3 \times n_3} \end{pmatrix},$$

where the rows of $(H_1 : H_2)$ are of the form

(d1) $(0 : E_j^{n_3})$ for some $j \in J_3$, or

(d2) $(E_j^{n_1+n_2} + E_h^{n_1+n_2} : 0_{n_3})$ for some (j, h) 2-cover of S, or

(d3) $(E_j^{n_1+n_2} + E_h^{n_1+n_2} + E_l^{n_1+n_2} : 0_{n_3})$ for some (j, h, l) 3-cover of S.

Now for every row of $(E_{n_0 \times n_0} : H_1 : H_2)$, one of the following is subtracted:

(s1) if (d1) holds, the similar row of the last n_3 rows $(1_{n_0} : 0_{n_1+n_2} : E_j^{n_3})$, or

(s2) if (d2) or (d3) holds, as the set of the rows vectors of $A_{(n_1+n_2)\times(n_1+n_2)}$ is a base, the linear combination of these rows for obtaining the row of H_1.

By lemma 3.1, the linear combination of (s2) is obtained with coefficients whose sum is equal to 1. Then the matrix obtained is

$$B' = \begin{pmatrix} -1_{n_0 \times n_0} & 0_{n_0 \times (n_1+n_2)} & 0_{n_0 \times n_3} \\ 1_{(n_1+n_2)\times n_0} & A_{(n_1+n_2)\times(n_1+n_2)} & 0_{(n_1+n_2)\times n_3} \\ 1_{n_3 \times n_0} & 0_{n_3 \times (n_1+n_2)} & 1_{n_3 \times n_3} \end{pmatrix},$$

which is a nonsingular one since $A_{(n_1+n_2)\times(n_1+n_2)}$ is nonsingular. Then the matrix B is nonsingular, and the vectors x^k, $k = 1,\dots,n$, belong to $P_I(A)$ and satisfy $\alpha^S x = 3$. Hence, $\alpha^S x \geq 3$ defines a facet of $P_I(A)$. □

Example 3.5. Consider example 2.1 and the inequality associated with $S = \{1, 2, 3, 4, 5, 6\}$, $x_2 + x_3 + x_4 + x_5 + 2x_6 + 2x_7 + 3x_8 \geq 3$. In example 3.4, we have seen that property (i) of theorem 3.3 holds for this inequality.

The only $k \in J_0$ such that $T(k) \neq \emptyset$ is 1, with $T(1) = \{7, 8\}$, and for example 3.5, the pair (3, 6) satisfies $a_{i3} + a_{i6} \geq 1$ for all $i \in M(J_0)$, $T(1) = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Hence, the inequality

$$x_2 + x_3 + x_4 + x_5 + 2x_6 + 2x_7 + 3x_8 \geq 3$$

defines a facet of $P_I(A)$.

4. Conclusions

In this paper, we have studied the set covering polytope, following the guideline initiated in [1]. We characterize the class of valid inequalities for this polytope with coefficients equal to 0, 1, 2 or 3, and give necessary and sufficient conditions for such an inequality to be not dominated and facet defining. These results indeed extend the knowledge about the facial structure of the set covering polytope.

In addition, a procedure to obtain these valid inequalities has been given. This method and other similar methods for inequalities with larger coefficients have been applied to obtain valid cuts for the set covering problem. First numerical results can be seen in [6].

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