Dealing with Non-Reciprocal Matrices in the Additive and Fuzzy Preference Relations Theoretical Frameworks

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Abstract

Many multiple-criteria decision-aiding methods apply the so-called multiplicative pairwise comparisons, where the comparisons have a form of a ratio expressing how many times one entity is more important (or preferred) than another. Besides the multiplicative system, additive and fuzzy preference relations systems have been proposed for pairwise comparisons in recent decades. These systems are appealing for their intuitive use and natural properties, but they are not as intensively studied as their multiplicative counterpart. Namely, studies on inconsistency, and non-reciprocity in particular, in both theoretical frameworks are rather scarce and fragmented. Therefore, our study focuses on the problem of non-reciprocity in both frameworks and fills the current gaps in its understanding and evaluation. We show that when non-reciprocity is allowed, multiplicative, additive, and fuzzy systems do not form an Alo group. However, measures of non-reciprocity in the additive and fuzzy systems corresponding to the existing measure of non-reciprocity in the multiplicative system can be defined and endowed with a set of desirable properties. Furthermore, we perform Monte Carlo simulations on randomly generated non-reciprocal matrices both in additive and fuzzy systems and provide percentile tables allowing decision makers to decide whether a level of non-reciprocity of a given PC matrix is acceptable or not.

Keywords: Additive pairwise comparisons; consistency; fuzzy pairwise comparisons; fuzzy preference relations; multiple-criteria decision making; pairwise comparisons; reciprocity.

1. Introduction

Many multiple-criteria decision-making methods such as the AHP/ANP, BWM, ELECTRE, MACBETH, PAPRIKA, PROMETHE etc., include as one of their major features pairwise comparisons (PC), that is comparisons of only two objects at the same time, e.g. Bana et al. (2005); Brans & Vincke (1985); Govindan & Jepsen (2015); Rezaei (2015); Saaty (1980, 2008); Roy (1968); Vaidya & Kumar (2006). These methods have been applied to most areas of human activity, and are subjects of intense research in terms of their properties, including the aggregation of preferences or their consistency.

One of the features of pairwise comparison, typically neglected, is the potential non-reciprocity of the judgements expressed by the decision makers. For example, if object A is two times more important than object B, then the reciprocal judgement would be that B is only half as important as B. This reciprocity is typically taken for granted in many of the methods mentioned above. This is understandable, given that it seems rational, natural and intuitive (Harker & Vargas (1987)). However, empirical studies have shown (see the next section) that human experts and decision-makers are often not consistent in their judgments. The explanation of this phenomenon may be related to external circumstances affecting the assessment process, such as changes in the context (e.g. Tversky & Simonson (1993)), unobserved variables, asymmetric transaction costs (e.g. non-reciprocal currency exchange rates), or hysteresis. But without a doubt, the main source of non-reciprocal judgments is the human mind, which is known to be imperfect and susceptible to many cognitive biases.

Given this inescapable reality, the question is how to deal with this lack of reciprocity in pairwise comparisons: should it be considered a source of important information, or a flaw of the cognitive process, and hence should it be accepted or eliminated? In case it is accepted, should there be acceptable levels?

In an earlier study of Mazurek & Linares (2023) these questions were addressed and reliable measures of non-reciprocity that satisfy desirable properties for the multiplicative pairwise comparisons (MPC) framework were introduced.

In this paper, the analysis of the previous questions is extended to two alternative pairwise comparisons theoretical frameworks: *additive* and *fuzzy*. Although less used than the MPC framework, additive and fuzzy frameworks are also used for pairwise comparison and hence deserve a proper analysis. Firstly, we show that multiplicative, additive and fuzzy pairwise comparisons do not form Alo groups when non-reciprocity is allowed. Nevertheless, we introduce new measures of non-reciprocity both in the *additive pairwise comparisons* (APC) and *fuzzy preference relations* (FPR) frameworks corresponding to the existing measure of non-reciprocity in the multiplicative system which inherit a set of desirable properties from the multiplicative system. Further on, we perform extensive Monte Carlo simulations to provide percentile tables allowing a decision maker to reject or tolerate a given non-reciprocal APC or FPR matrix with respect to the size of the matrix and applied p-metrics.

The paper is organized as follows: Section 2 reviews the existing literature; Section 3 discusses algebraic structure of pairwise comparisons in the case of non-reciprocity, Sections 4 and 5 are devoted to additive and fuzzy preference relations frameworks respectively, and finally, numerical Sections 6 and 7 provide percentile tables for non-reciprocity measures in both frameworks. Conclusions close the article.

2. Literature review

In the realm of decision-making, as highlighted in the preceding section, the challenge of non-reciprocal matrices often arises. Addressing this challenge is crucial for ensuring accurate and consistent decision outcomes. Essentially, the literature presents four strategies or approaches to manage non-reciprocal matrices:

1. Discarding information in order to consider only reciprocal matrices.
2. Transformations to reciprocal matrices. Some methods transform non-reciprocal matrices into reciprocal ones that are subsequently managed and analyzed by standard optimization methods.
3. Approximations to priority vectors. This approach aims at finding a reciprocal (or more generally a fully consistent) approximation of a given non-reciprocal matrix.
4. Developments of new theoretical framework. This category comprises papers that merge or expand the standard methodology by shifting from the reciprocity paradigm to a more realistic theoretical framework without including non-reciprocity.

An example of the first approach is the study by Saaty (1994), where one of the pairwise comparisons from a non-reciprocal pair are discarded (randomly, as there is no available information to decide which of the two comparisons is the most accurate). As pointed out by Linares et al. (2016), some experts believe that the extra information is unnecessary, because only $(n-1)$ pairwise comparisons are needed (assuming that the decision-maker is perfectly rational and the corresponding directed graph is connected). Indeed, this reciprocity condition is foundational to the AHP, as corroborated by sources like (Harker & Vargas, 1987) and (Saaty, 1986). However, decision-makers might not always behave rationally so, before deciding to discard any information, it is essential to measure how inconsistent the provided preferences are. Thurstone acknowledged in his 1927 Law of Comparative Judgment (Thurstone, 1927) that decision-makers could offer differing comparative judgments on the same stimuli pair over different instances.

Grzybowski (2012) showed by numerical simulations that forcing reciprocity into an MPC matrix may result in poorer estimations of a priority vector. Diaz-Balteiro et al. (2009) discovered inherent non-reciprocity when soliciting identical comparisons at different times, especially in forest management contexts. In their study participants first completed the upper triangle of the MPC matrix and, after a month, the lower triangle. Notably, none of the acquired MPC matrices were reciprocal. Fülöp et al. (2012) added to the list of possible causes for non-reciprocity the fact that several teams might be working in the assessments independently, or the fuzziness in the underlying preference relations. They also provided several real-world examples where the reciprocity condition is violated, including double-blind wine tasting.

From an economic perspective, Hovanov et al. (2008) mentioned the case of transaction costs in exchange rates as a reason for non-reciprocity. In different areas, such as the choice to vote for a political party, it was believed traditionally that preferences stayed constant over time. However, de Andres et al. (2020), demonstrated, with the introduction of local and global decision stability measures, that political preferences might shift with greater frequency than once believed. Additionally, Starczewski (2017) established that while the AHP assumes decision-makers may err in comparing pairs of alternatives, the adopted scale can exacerbate these discrepancies.

Mazurek & Linares (2023) have reviewed the most relevant literature on dealing with this inconsistency in MPC matrices. This literature is large, which is reasonable given its popularity. However, other theoretical frameworks for pairwise comparisons exist, which have not received the same level of attention in terms of their reciprocity properties, namely additive pairwise comparisons (APC) and fuzzy preference relations (FPRs), which constitute the focal point of this paper.

Additive pairwise comparisons were introduced by Barzilai & Golany (1990). The additive formulation of pairwise comparisons allows to use of the mathematical apparatus of linear algebra, see e.g. (Fedrizzi et al., 2020). Unfortunately, we have not found any discussions in the literature about the reciprocity of APC matrices.

Fuzzy preference relations were introduced by Orlowsky (1978) and later elaborated in Tanino (1984) or Herrera et al. (2004). FPRs enable natural pairwise comparisons by dividing a line segment of a unit length into two segments corresponding to two compared entities. Also, FPRs enable the modelling of linguistic preferences, see e.g. Marimin et al. (1998). Traditional methods using fuzzy preference relations (FPRs) are based on the idea that preferences are mutually consistent or reciprocal. However, this assumption does not always align with the intricate and unpredictable situations that experts might face with different opinions in real-world scenarios including business strategy meetings, urban planning, healthcare, public policy, and so on. In this context, Liu et al. (2021 a, b and c) introduced the concept of additively reciprocal property breaking (ARPB), and in (Liu et al., 2022a,b,c; Luo et al., 2023) the concept of non-reciprocal fuzzy preference relations (NrFPRs) was proposed to capture situations where the traditional reciprocal assumptions of FPRs are not met and also constructing optimization models that can elicit priorities from non-reciprocal matrices. Finally, in (Jiang et al., 2021) a non-reciprocal fuzzy preference relation (NrFPR) and a probabilistic linguistic preference relation (PLPR) were combined to obtain a model capable of expressing partial relations of alternatives (indifference, preference, and incomparability relations) providing a practical application in selecting social donation channels during the COVID-19 outbreaks. It should be noted that non-reciprocal pairwise comparison matrices were also studied in the context of incomplete information, see e.g. Khalid & Beg (2018), Huang et al. (2020) and Zhang (2022). Measures of non-reciprocity in the context of (additive) fuzzy preference relations were recently discussed in (Liu et al., 2022c), (Wang & Deng, 2022) and (Yang, 2022).

As mentioned earlier, in this paper our goal is to fill the research gap on the characterization of reciprocity in APC and FPR matrices in a homogeneous way, also compatible with the previous work on MPC (Mazurek & Linares, 2023). In the next sections, we proceed to the evaluation of non-reciprocity in APC and FPR systems.

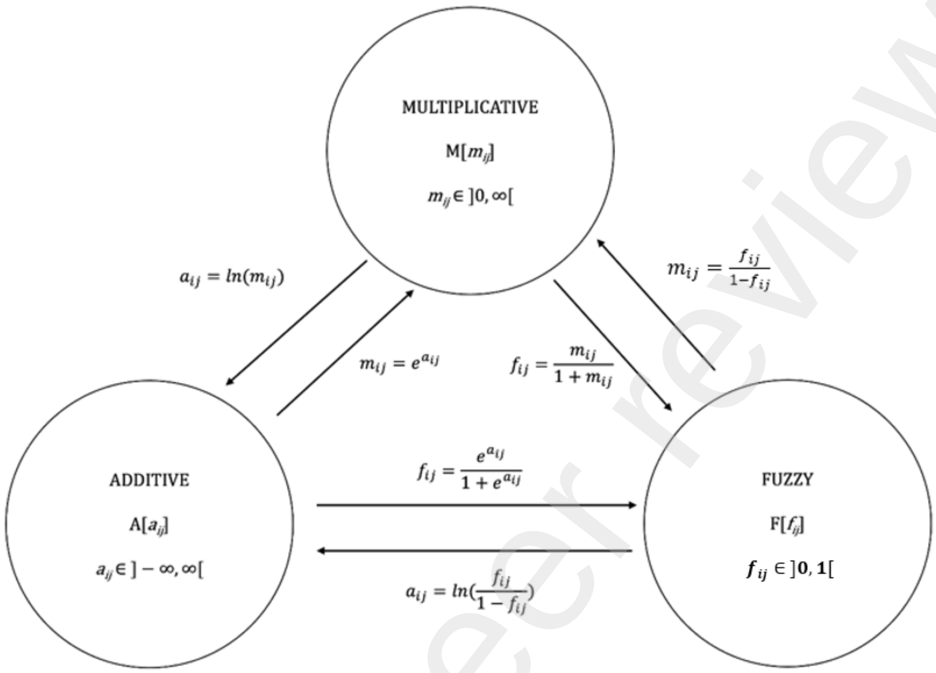

Figure 1: Transformations among the three pairwise comparisons frameworks.

3. Algebraic structure of multiplicative, additive and fuzzy preference relations systems in the case of non-reciprocity

It is well known that multiplicative pairwise comparisons, additive pairwise comparisons and fuzzy preference relations share an identical algebraic structure and form three isomorphic representations of the same *Abelian linearly ordered group (Alo group)*, see e.g. Cavallo & D'Apuzzo (2009), Cavallo & Brunelli (2018), or Kulakowski et al. (2019).

Thereinafter, we show that this is true only when pairwise comparisons are reciprocal.

Definition 1. *Let $\mathbb{G}$ be a non-empty set equiped with a binary operation $\odot : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$. The set $\mathbb{G}$ is a group if the following axioms 1)-3) are satisfied:*

1. *Associativity: for all $a, b, c \in \mathbb{G}$ holds $(a \odot b) \odot c = a \odot (b \odot c)$.*

2. *Existence of a unique neutral element e: for all* $a \in \mathbb{G}$ *holds* $a \odot e = e \odot a = a$.
3. *Existence of a unique inverse element: for all* $a \in \mathbb{G}$*, there exists* a^{-1} *satisfying* $a \odot a^{-1} = a^{-1} \odot a = e$.
4. *Commutativity:* $a \odot b = b \odot a$ *for all* $a, b \in \mathbb{G}$.

If the additional axiom (4) is satisfied, the group is called Abelian or commutative.

The next remarks show that in the case of non-reciprocity, the set of multiplicative pairwise comparisons (with elements denoted by the letter 'm') is not a group.

Remark 1. *Let* $\mathbb{G} = \{m_{ij} \mid m_{ij} \in (0, \infty)\}$ *be the set of multiplicative pairwise comparisons equipped with the operation of standard multiplication '$\cdot$'. If there exists a pair of indices* (i, j) *such that* $m_{ij} \cdot m_{ji} \neq 1$*, where* $m_{ji} \equiv m_{ij}^{-1}$*, then the set* $\mathbb{G}$ *does not form a group under the given operation because it violates group Axiom 3 (this axiom is identical to the reciprocity condition of pairwise comparisons). Conversely, the assumption that the set of multiplicative pairwise comparisons forms an Alo group directly excludes non-reciprocity due to Axiom 3.*

Analogously, the sets of additive pairwise comparisons and fuzzy preference relations are not groups when non-reciprocity is allowed.

Since the multiplicative, additive and fuzzy preference relations frameworks do not form a group under non-reciprocity (in abstract algebra, their structure is called a *monoid*, or a *semigroup with an identity element*), we will refer to them as 'systems' for simplicity thereinafter. Also, we refrain from using the word 'isomorphism' used in the group context to avoid confusion. Nevertheless, the transformations among multiplicative, additive and fuzzy systems shown in Figure 1 are preserved.

In the following sections we introduce measures of non-reciprocity in the additive and fuzzy preference relations systems, which correspond to the measure of non-reciprocity already introduced in Mazurek & Linares (2023) for the multiplicative system.

4. Non-reciprocity in the additive framework

In the additive pairwise comparisons system, a preference of i-th object over a j-th object is expressed by a value $a_{ij} \in \mathbb{R}$, where $a_{ij} > 0$ means that the i-th object is preferred over a j-th object, $a_{ij} = 0$ means indifference between both objects and $a_{ij} < 0$ means that the j-th object is preferred over an i-th object. It is common to understand the value of a_{ij} in the sense of "by how much an object i is better (more preferred) than an object j", see also Figure 2.

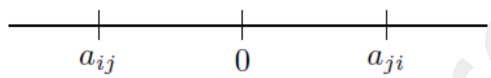

Figure 2: Reciprocal additive pairwise comparisons are antisymmetrical.

Let $A_{n\times n} = [a_{ij}]$ be an additive pairwise comparisons (APC) matrix. The *reciprocity condition* is given as follows:

$$a_{ij} + a_{ji} = 0, \ \ \forall i, j \in \{1, \ldots, n\} \tag{1}$$

The more general *consistency condition* is defined as follows:

$$a_{ij} + a_{jk} = a_{ik}, \ \ \forall i, j, k \in \{1, \ldots, n\} \tag{2}$$

Proposition 1. *Let $A_{n\times n} = [a_{ij}], a_{ij} \in \mathbb{R}$, be an APC matrix. The consistency condition (2) implies: a) $a_{ii} = 0, \forall i \in \{1, 2, ..., n\}$, b) reciprocity condition (1).*

Proof.

i) Let $i = j = k$. Then from relation (2) we get: $a_{ii} + a_{ii} = a_{ii}$, hence $a_{ii} = 0, \forall i$.

ii) Let $i = k \neq j$. Then from (2) we get: $a_{ij} + a_{jk} - a_{ik} = 0, \forall i, j, k$, hence $a_{kj} + a_{jk} - 0 = 0, \forall i, j, k$, thus $a_{kj} + a_{jk} = 0, \forall i, j, k$, as required. $\square$

The next proposition is a straightforward consequence of the reciprocity condition.

Proposition 2. *Let $A_{n\times n} = [a_{ij}], a_{ij} \in \mathbb{R}$, be a reciprocal APC matrix. Then the sum of all its elements is equal to 0.*

Proof. is obvious. $\square$

On the contrary, when the sum of all matrix elements differs from 0, then the matrix is necessarily non-reciprocal:

Proposition 3. *Let $A_{n\times n} = [a_{ij}], a_{ij} \in \mathbb{R}$, be an APC matrix and $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \neq 0$. Then the matrix A is non-reciprocal.*

Proof. (by contradiction) Assume the matrix A is reciprocal, thus $a_{ij} + a_{ji} = 0, \forall i, j$, and, at the same time, $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \neq 0$ holds. Then, after rearranging terms, we have: $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} = \sum_{i<j} a_{ij} + \sum_{i>j} a_{ij} = \sum_{i<j} a_{ij} + \sum_{i<j} a_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij} + a_{ji}) \neq 0$, hence, at least one term $(a_{ij}+a_{ji}) \neq 0$, which is a contradiction with the initial assumption $a_{ij}+a_{ji} = 0, \forall i, j$. Therefore, the matrix is not reciprocal. $\square$

However, the fact that the sum of all matrix elements is equal to 0 does not imply that an APC matrix is reciprocal as shown in the following counterexample:

$$A = \begin{bmatrix} 0 & 8 & -4 \\ -5 & 0 & 5 \\ -8 & 4 & 0 \end{bmatrix}.$$

The proposed measure of non-reciprocity based on the relation (1) and the definition of a $p-$norm is introduced as follows.

Definition 2. *Let $A_{n\times n} = [a_{ij}], a_{ij} \in \mathbb{R}$, be an APC matrix. Let $p \geq 1$. Then the non-reciprocity measure Ξ (for non-diagonal elements) is given as follows:*

$$\Xi_p(A) = (\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} |a_{ij} + a_{ji}|^p)^{\frac{1}{p}} \ . \tag{3}$$

Further on, the measure (3) can be normalized by the factor (the number of reciprocal pairs) $\frac{n(n-1)}{2}$, where n is the matrix order:

$$\Xi_{p,n}(A) = \frac{2}{n(n-1)}(\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} |a_{ij} + a_{ji}|^p)^{\frac{1}{p}} \ . \tag{4}$$

Example 1. *Consider the following additive PC matrix $A = [a_{ij}]$ and evaluate its non-reciprocity (for $p = 1$) via the measure (3)*

$$A = \begin{bmatrix} 0 & 10 & -5 & 15 \\ -8 & 0 & 8 & 22 \\ 4 & -8 & 0 & 13 \\ -15 & -20 & -13 & 0 \end{bmatrix}.$$

Solution:

From (3) we have: $\Xi_1(A) = \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}|a_{ij}+a_{ji}| = |10+(-8)| + |(-5)+4| + |15+(-15)| + |8+(-8)| + |22+(-20)| + |13+(-13)| = 5.$

Proposition 4. *The measure of non-reciprocity in the additive system,* Ξ_p*, given by relation (3), is identical to the measure of non-reciprocity* $\Theta^{(p)}(M) = \left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}|\log(m_{ij}\cdot m_{ji})|^p\right)^{1/p}$ *in the multiplicative system, where* $M = [m_{ij}]$ *denotes the multiplicative pairwise comparison matrix.*

Proof. The transformation from the multiplicative to the additive system (see Figure 1) is given by $m_{ij} = e^{a_{ij}}$. Substituting this into $\Theta^{(p)}(M)$, we get:

$$\Theta^{(p)}(A) = \left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left|\log\left(e^{a_{ij}}\cdot e^{a_{ji}}\right)\right|^p\right)^{1/p} = \left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left|\log\left(e^{a_{ij}}\right)+\log\left(e^{a_{ji}}\right)\right|^p\right)^{1/p}.$$

Assuming the natural logarithm, we obtain:

$$\Theta^{(p)}(A) = \left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}|a_{ij}+a_{ji}|^p\right)^{1/p} = \Xi_p(A),$$

as required. □

It is shown in Mazurek & Linares (2023) that the measure of non-reciprocity in the multiplicative system $\Theta^{(p)}$ satisfies several desirable properties, namely:

- Property 1 (Existence of a unique element representing reciprocity).
- Property 2 (Invariance under permutation of alternatives).
- Property 3 (Monotonicity under reciprocity-preserving mapping).
- Property 4 (Monotonicity on single comparisons).
- Property 5 (Continuity).
- Property 6 (Invariance under inversion of preferences).

Due to simple (*log* and *exp*) transformation between multiplicative and additive systems, also the newly introduced measure $\Xi_{(p)}$ satisfies all properties above.

5. Non-reciprocity in the fuzzy preference relations framework

In the fuzzy preference relations (FPR) system, the preference of object i over object j is represented by a value $f_{ij} \in]0,1[$. When $f_{ij} > 0.5$, it indicates that object i is preferred over object j. If $f_{ij} = 0.5$, there is no preference between the two objects, indicating indifference. Conversely, if $f_{ij} < 0.5$, it implies that object j is preferred over object i. For a visual representation, see also Figure 3.

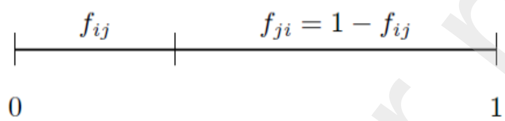

Figure 3: Reciprocal fuzzy preference relations.

Let $F_{n\times n} = [f_{ij}]$ be a fuzzy preference relations (FPR) matrix. The reciprocity condition is defined by

$$f_{ij} + f_{ji} = 1, \ \ \forall i, j \in \{1, \ldots, n\}. \tag{5}$$

The generalized consistency condition for the fuzzy preference relation (FPR) is defined as follows (Tanino, 1984):

$$f_{ij} - f_{ik} - f_{kj} + 0.5 = 0, \quad \forall i, j, k \in \{1, \ldots, n\}. \tag{6}$$

Proposition 5. *Let $F_{n\times n} = [f_{ij}]$, be a FPR matrix, where $f_{ij} \in]0,1[$, satisfying the generalized consistency condition* (6)*. Then:*

(a) $f_{ii} = 0.5$, *for all* $i \in \{1, 2, \ldots, n\}$.

(b) The reciprocity condition (5) *is satisfied.*

Proof. To prove part (a), let $i = j = k$. Then, substituting in (6), we obtain

$$f_{ii} - f_{ii} - f_{ii} + 0.5 = 0,$$

which simplifies to $f_{ii} = 0.5$ for all i.

For part (b), let $i = j \neq k$. Substituting into the consistency condition yields:

$$f_{ii} - f_{ik} - f_{ki} + 0.5 = 0,$$

which simplifies to $f_{ik} + f_{ki} = 1$, for all i, j, k, as required. $\square$

The next proposition is a straightforward consequence of the reciprocity condition.

Proposition 6. *Let $F_{n\times n} = [f_{ij}]$ be a reciprocal fuzzy preference relation (FPR) matrix, where $f_{ij} \in]0,1[$ and $f_{ij} + f_{ji} = 1$ for all $i \neq j$, and $f_{ii} = 0.5$ for all i. Then, the sum of all elements in the matrix is equal to $\frac{n^2}{2}$.*

Proof. The total sum of all elements in the matrix can be written as:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij} = \sum_{i=1}^{n} f_{ii} + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} f_{ij}.$$

The sum of the diagonal elements is:

$$\sum_{i=1}^{n} f_{ii} = 0.5n = \frac{n}{2}.$$

For the off-diagonal elements, there are $\frac{n(n-1)}{2}$ reciprocal pairs, and since each pair sums to 1, the total contribution from all off-diagonal elements is:

$$\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} f_{ij} = \frac{n(n-1)}{2}.$$

Therefore, the total sum is:

$$\frac{n}{2} + \frac{n(n-1)}{2} = \frac{n^2}{2}.$$

$\square$

On the contrary, when the sum of all matrix elements differs from $\frac{n^2}{2}$, then an FPR matrix is necessarily non-reciprocal:

Proposition 7. *Let $F_{n\times n} = [f_{ij}], f_{ij} \in]0,1[$ be a FPR matrix and $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \neq \frac{n^2}{2}$. Then the matrix F is non-reciprocal.*

Proof. is analogous to the proof of Proposition 3.3. $\square$

However, the fact that the sum of all matrix elements is equal to $\frac{n(n-1)}{2}$ does not imply that a matrix is reciprocal as shown in the following counterexample:

$$\begin{bmatrix} 0.5 & 0.8 & 0.4 \\ 0.4 & 0.5 & 0.7 \\ 0.4 & 0.3 & 0.5 \end{bmatrix}.$$

To quantify the extent of non-reciprocity, a suitable function is necessary. The only attempts in this direction are the so-called additive reciprocity property breaking functions proposed in Liu et al. (2021c) and revised in Wang & Deng (2022). However, these functions are defined only for the case $0 \leq f_{ij} + f_{ji} \leq 1; \forall i, j$, thus neglecting the possibility of $f_{ij} + f_{ji} > 1$ for some pair (i, j). Also, the authors do not attempt to investigate the measures' properties or consider transformations (isomorphisms) among multiplicative, additive and fuzzy theoretical frameworks.

Therefore, we propose a new measure of non-reciprocity of FPR matrices without the shortcomings above based on the definition of a $p-$norm as follows.

Definition 3. *Let $F_{n\times n} = [f_{ij}], f_{ij} \in]0,1[$, be a FPR matrix. Let $p \geq 1$. Then the non-reciprocity measure Υ (for non-diagonal elements) is given as follows:*

$$\Upsilon_p(F) = (\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |log(\frac{f_{ij} \cdot f_{ji}}{(1-f_{ij})(1-f_{ji})})|^p)^{\frac{1}{p}} \tag{7}$$

The operation '$\cdot$' in (7) is usual multiplication. Further on, the measure (9) can be normalized by the factor (the number of reciprocal pairs) $\frac{n(n-1)}{2}$, where n is the matrix order:

$$\Upsilon_{p,n}(F) = \frac{2}{n(n-1)}(\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |log(\frac{f_{ij} \cdot f_{ji}}{(1-f_{ij})(1-f_{ji})})|^p)^{\frac{1}{p}} \tag{8}$$

Example 2. *Consider the following fuzzy preference relations matrix $F = [f_{ij}]$ and evaluate its non-reciprocity (for $p = 1$) via the measure (7).*

$$F = \begin{bmatrix} 0.5 & 0.4 & 0.8 & 0.7 \\ 0.6 & 0.5 & 0.6 & 0.3 \\ 0.1 & 0.3 & 0.5 & 0.6 \\ 0.4 & 0.5 & 0.4 & 0.5 \end{bmatrix}.$$

Solution:

From (7) we have: $\Upsilon_1(F) = (|log(\frac{0.4 \cdot 0.6}{(1-0.4)(1-0.6)})| + |log(\frac{0.8 \cdot 0.1}{(1-0.8)(1-0.1)})| + ... = 0 + 0.811 + 0.560 + 0.442 + 0.847 + 0 = 2.660$

Due to the transformations among MPC, APC and FPRs systems shown in Figure 1, the measure $\Upsilon_p(F)$ satisfies the desirable properties listed in the previous section.

The following example illustrates that non-reciprocity evaluation in all three systems leads to identical results.

Example 3. *Consider the following pairwise comparison matrices M, A and F corresponding to multiplicative, additive and fuzzy representations of identical preferences (each matrix can be converted into another one by formulas in Figure 1). We will evaluate their non-reciprocity for $p = 1$ (and natural logarithm).*

$$M = \begin{bmatrix} 1 & 4 & 5 \\ 0.3 & 1 & 0.5 \\ 0.2 & 2 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & log(4) & log(5) \\ log(0.3) & 0 & log(0.5) \\ log(0.2) & log(2) & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 0.5 & 0.8 & 0.833 \\ 0.231 & 0.5 & 0.333 \\ 0.167 & 0.667 & 0.5 \end{bmatrix}.$$

As can be seen, only one pair with indices $(1, 2)$ is non-reciprocal.

For the multiplicative matrix M we get:

$\Theta^{(1)}(M) = \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} |\log(m_{ij} \cdot m_{ji})| = |log(4 \cdot 0.3)| + 0 + 0 = log(1.2) = 0.182$.

For the additive matrix A we obtain:

$\Xi_1(A) = \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} |a_{ij} + a_{ji}| = (|log(4) + log(0.3)| + 0 + 0 = 0.182$.

For the fuzzy preference relations matrix F we get:

$\Upsilon_1(F) = (|log\frac{0.8 \cdot 0.231}{(1-0.8)(1-0.231)}| + 0 + 0 = 0.182$.

The results - the extent of non-reciprocity - are identical in all systems.

In the next two Sections, we provide percentile tables for randomly generated APC and FPRs matrices so that a decision maker can assess the non-reciprocity of a given matrix and decide whether the extent of non-reciprocity is acceptable, or not.

In Mazurek & Linares (2023), percentile tables for non-reciprocity for random multiplicative PC matrices were provided. The applied scale was Saaty's scale from $1/9$ to 9, which is the most common among practitioners.

For APC matrices, we use the scale from -100 to 100, which is also preferred along with ± 10 scale, see e.g. Guh & al. (2009), in practical applications. Since this scale does not correspond to Saaty's scale, both percentile tables naturally differ. The same applies to percentile tables for FPRs matrices, where matrix elements from the interval $]0,1[$ were considered.

6. Assessing the reciprocity of random APC matrices

To provide a decision maker with percentile tables for the measure of non-reciprocity Ξ in the additive framework given by relation (3), we performed Monte Carlo simulations for additive PC matrices of the order $2 \leq n \leq 8$. Matrix elements were drawn randomly from the $[-100, 100]$ interval under the assumption of uniform distribution.

For each matrix size n, a total of 100,000 random matrices was generated. After the set of matrices was generated, each matrix was evaluated for non-reciprocity via relation (3) for $p = 1$, $p = 2$ and $p = \infty$ (the logarithm applied was the decadic one).

Results in the form of percentile tables are summarized in Tables 1-3 and Figure 4. As can be seen, percentile values expectedly grow monotonically with increasing n: the larger the order of the matrix, the more significant the value of non-reciprocity for a corresponding percentile.

Now, let's consider an APC matrix of the order $n = 6$ with $\Xi_1 = 564$. Is this matrix tolerably non-reciprocal? From Table 1 it follows that such a matrix is less non-reciprocal than 99% of random APC matrices of the same order, hence, by applying the 10% rule similarly to the AHP, the matrix can be considered tolerably non-reciprocal.

Table 1: Percentile tables of Ξ defined by relation (3) for random APC matrices, $p = 1; 2 \leq n \leq 8$.

percentile	0.1	1	2	3	4	5	10	20	25	50 (median)
$n = 2$	0	1	2	3	4	5	10.3	21.4	27.2	58.8
$n = 3$	19.4	41.8	53.8	61.5	68.6	74.5	96.7	127.8	140.4	195.8
$n = 4$	104.8	158.3	182.1	197.9	209.1	219.1	254.4	300.8	319.8	397.4
$n = 5$	259.4	348.5	378.7	400.8	417.0	431.7	479.4	541.6	565.6	664.9
$n = 6$	494	602	646	674	694	711	772	848	878	1000
$n = 7$	788	930	980	1012	1037	1058	1132	1224	1258	1403
$n = 8$	1161	1315	1379	1416	1447	1472	1557	1663	1704	1872

Table 2: Percentile tables of Ξ defined by relation (3) for random APC matrices, $p = 2; 2 \leq n \leq 8$.

percentile	0.1	1	2	3	4	5	10	20	25	50 (median)
$n = 2$	0	1	2	3	4	5	10.3	21.4	27.2	58.8
$n = 3$	13.1	28.4	36.3	42.3	47.0	51.1	66.1	87.2	95.9	132.7
$n = 4$	52.9	81.0	92.6	100.1	106.2	111.2	129.4	152.0	160.6	195.5
$n = 5$	104.5	138.6	151.5	159.9	166.3	171.5	189.9	212.8	221.2	255.3
$n = 6$	163.0	198.8	211.8	220.1	226.6	231.8	250.3	272.6	281.0	314.6
$n = 7$	222.3	256.0	270.0	278.9	285.6	290.7	308.9	331.2	339.6	372.9
$n = 8$	280.0	317.1	330.7	339.2	345.4	350.5	368.5	390.2	398.4	432.0

Table 3: Percentile tables of Ξ defined by relation (3) for random APC matrices, $p = \infty; 2 \leq n \leq 8$.

percentile	0.1	1	2	3	4	5	10	20	25	50 (median)
$n = 2$	0	1	2	3	4	5	10.3	21.4	27.2	58.8
$n = 3$	10.5	23.3	29.6	34.2	38.0	41.3	53.9	71.2	78.5	109.5
$n = 4$	35.5	53.8	61.7	67.4	71.4	74.8	87.3	103.5	109.6	134.5
$n = 5$	58.8	79.1	86.6	91.6	95.4	98.7	109.8	123.6	128.7	149
$n = 6$	80.1	97.9	105.1	109.5	113.1	115.6	125.4	136.8	141.2	158.3
$n = 7$	95.6	111.9	117.9	122.2	125.2	127.6	136.2	146.3	150.2	164.8
$n = 8$	106.8	122.6	128.4	132.3	134.8	137	144.5	153.6	156.8	169.6

7. Assessing the reciprocity of random FPR matrices

To provide a decision maker with percentile tables for the measure of non-reciprocity Υ in the fuzzy preference relations framework given by relation (7), we performed Monte Carlo simulations for FPR matrices of the order $2 \leq n \leq 8$. Matrix elements were drawn randomly from the $]-1,1[$ interval under the assumption of uniform distribution.

For each matrix size n, a total of 100,000 random matrices was generated. After the set of matrices was generated, each matrix was evaluated for non-reciprocity via relation (7) for $p = 1$, $p = 2$ and $p = \infty$ (the logarithm applied was the decadic one).

Results in the form of percentile tables are summarized in Tables 4-6 and Figure 4. In general, percentile values expectedly grow monotonically with increasing n: the larger the order of the matrix, the larger is the value of non-reciprocity for a corresponding percentile.

Let us consider a FPR matrix of the order $n = 6$ with $\Upsilon_2 = 8.13$, and $p = 2$. Then this matrix would be considered intolerably non-reciprocal, since the value of non-reciprocity equal to 8.13 is higher than the threshold for the 20^{th} percentile (8.02), see Table 5.

Table 4: Percentile tables of Υ defined by relation (7) for random FPR matrices, $p = 1; 2 \leq n \leq 8$

percentile	0.1	1	2	3	4	5	10	20	25	50 (median)
$n=2$	0.00	0.03	0.06	0.09	0.12	0.15	0.30	0.60	0.77	1.63
$n=3$	0.53	1.18	1.51	1.73	1.93	2.10	2.71	3.58	3.96	5.65
$n=4$	3.07	4.48	5.10	5.57	5.86	6.16	7.21	8.63	9.17	11.65
$n=5$	7.52	9.75	10.73	11.38	11.81	12.27	13.75	15.61	16.37	19.64
$n=6$	13.87	17.11	18.34	19.17	19.85	20.41	22.23	24.66	25.61	29.65
$n=7$	22.36	26.47	28.03	29.08	29.80	30.50	32.73	35.69	36.85	41.58
$n=8$	32.73	37.68	39.63	40.87	41.83	42.56	45.24	48.69	50.04	55.57

Table 5: Percentile tables of Υ defined by relation (7) for random FPR matrices, $p = 2; 2 \leq n \leq 8$

percentile	0.1	1	2	3	4	5	10	20	25	50 (median)
$n = 2$	0.00	0.03	0.06	0.09	0.12	0.15	0.30	0.61	0.77	1.63
$n = 3$	0.35	0.80	1.02	1.19	1.31	1.43	1.84	2.44	2.68	3.80
$n = 4$	1.50	2.26	2.59	2.81	2.98	3.11	3.64	4.32	4.58	5.80
$n = 5$	2.93	3.90	4.27	4.53	4.69	4.85	5.42	6.17	6.44	7.71
$n = 6$	4.57	5.59	6.00	6.27	6.48	6.63	7.24	8.02	8.31	9.60
$n = 7$	6.22	7.30	7.75	8.01	8.26	8.44	9.06	9.86	10.15	11.45
$n = 8$	7,93	9.10	9.54	9.83	10.06	10.24	10.88	11.67	11.99	13.28

Table 6: Percentile tables of Υ defined by relation (7) for random FPR matrices, $p = \infty; 2 \leq n \leq 8$

percentile	0.1	1	2	3	4	5	10	20	25	50 (median)
$n = 2$	0.00	0.03	0.06	0.09	0.12	0.15	0.29	0.61	0.77	1.63
$n = 3$	0.33	0.66	0.83	0.97	1.07	1.15	1.49	1.98	2.19	3.14
$n = 4$	0.97	1.48	1.71	1.85	1.99	2.08	2.45	2.94	3.14	4.07
$n = 5$	1.65	2.18	2.43	2.57	2.69	2.78	3.14	3.63	3.83	4.74
$n = 6$	2.23	2.75	2.97	3.13	3.23	3.34	3.69	4.16	4.36	5.24
$n = 7$	2.64	3.20	3.44	3.58	3.69	3.78	4.13	4.60	4.79	5.65
$n = 8$	3.06	3.60	3.82	3.96	4.08	4.17	4.50	4.96	5.15	6.01

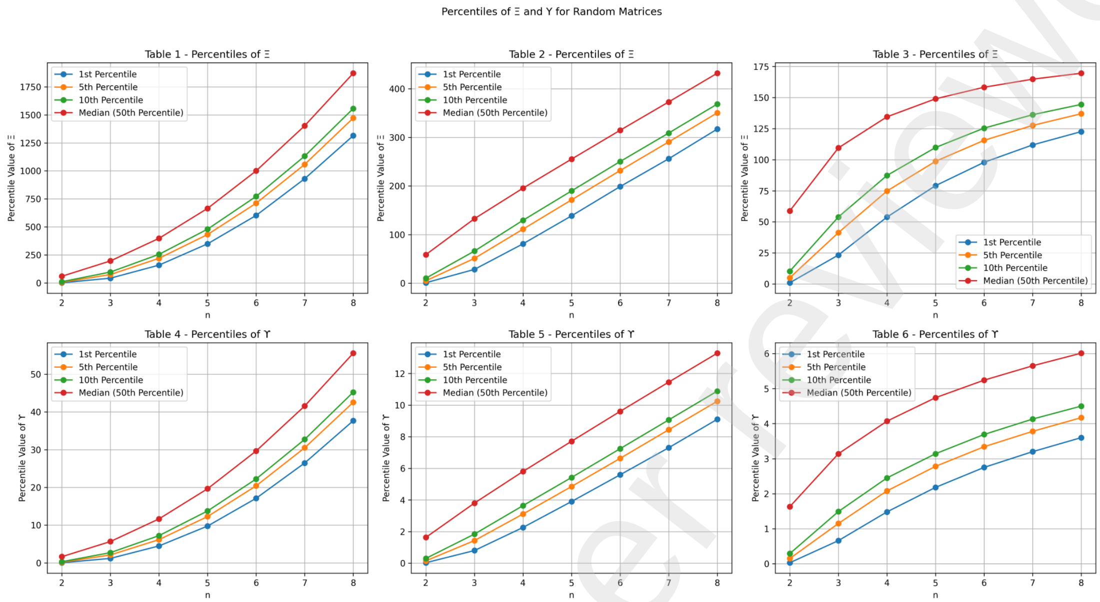

Figure 4: Percentile values for different $p-$metrics. Source: authors.

8. Conclusions

The paradigm of pairwise comparison methods assumes pairwise comparisons are reciprocal (Harker & Vargas, 1987; Saaty, 1977, 1980, 1986). This assumption is based on the notion of rationality and is a considered necessary requirement for consistent judgments. However, recent empirical studies have shown that non-reciprocal matrices appear naturally in many situations and that their root is both subjective (caused by human cognitive bias, lack of knowledge, time pressure, etc.) and objective (Diaz-Balteiro et al., 2009; Fülöp et al., 2012; Hovanov et al., 2008; Linares et al., 2016).

The problem of non-reciprocity in the multiplicative pairwise comparisons framework has been already addressed in the recent study by Mazurek & Linares (2023). The aim of this paper was to extend the study of non-reciprocity in pairwise comparisons to additive and fuzzy preference relations frameworks as well. In both frameworks, new measures of non-reciprocity, Ξ and Υ, satisfying a set of desirable properties, were introduced. These two measures correspond to the previously introduced measure of non-reciprocity in the multiplicative framework in that sense that they provide the same results for the same preferences. However, a unified formula of non-reciprocity based on Alo groups is not feasible since the very existence of non-reciprocity is not compatible with a group structure. Thus, the introduction of measures Ξ and Υ is justified.

Further on, Monte Carlo simulations of randomly-generated non-reciprocal APC and FPR matrices allowed us to provide a decision maker with percentile tables depending on the selected p-metric, the order of the matrix and a given tolerance value, so a decision maker can decide whether a given APC or FPR matrix is acceptably non-reciprocal, or not.

Further research may focus on the analysis of the implications of different p-metrics as well as connections between reciprocity and consistency measured by inconsistency indices proposed for reciprocal PC matrices. Also, generalizations towards pairwise comparisons with uncertainty (e.g. in the form of interval numbers or fuzzy sets), or incomplete pairwise comparisons can be considered.

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Declaration of interest

We have no conflict of interest to declare.

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