Journal of Geometry and Physics 219 (2026) 105693

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Higher-order Euler–Poincaré field equations

Marco Castrillón López [a], Álvaro Rodríguez Abella [b,*]

[a] Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Plaza de las Ciencias, 3, Madrid, 28040, Madrid, Spain
[b] Departamento de Matemática Aplicada, Universidad Pontificia de Comillas, Calle Alberto Aguilera, 23, Madrid, 28015, Madrid, Spain

A R T I C L E I N F O

Article history:
Received 30 January 2025
Accepted 31 October 2025
Available online 5 November 2025

MSC:
primary 70S05, 70S10, 70H50
secondary 53C05, 58E15

Keywords:
Euler–Poincaré equations
Higher-order field theory
Lagrangian density
Multivariate splines
Noether theorem

A B S T R A C T

We develop a reduction theory for G-invariant Lagrangian field theories defined on the higher-order jet bundle of a principal G-bundle, thus obtaining the higher-order Euler–Poincaré field equations. To that end, we transfer the Hamilton's principle to the reduced configuration bundle, which is identified with the bundle of flat connections (up to a certain order) of the principal G-bundle. As a result, the reconstruction condition is always satisfied and, hence, every solution of the reduced field equations locally comes from a solution of the original (unreduced) equations. Furthermore, the reduced equations are shown to be equivalent to the conservation of the Noether current. Lastly, we illustrate the theory by investigating multivariate higher-order splines on Lie groups.

1. Introduction

In the realm of geometric mechanics, reduction by symmetries involves dropping the dynamical equations of a system with symmetry to a lower-dimensional space, which is the family of group orbits of the symmetry action. The modern approach of this procedure was first introduced in [1,32,36,41], and for Lagrangian theories in [12,13,31]. For the latter, the key idea is to transfer the variational principle, typically the Hamilton's principle, to the reduced configuration space. This yields the reduced equations when applied to the reduced Lagrangian.

The reduction results in mechanics for first-order systems, i.e., those whose Lagrangians depend only on the generalized positions and their velocities, have been successfully extended to field theories [8,19], including covariant field theories [7,9,11,16], where the configuration manifold is replaced by a fiber bundle whose base space models the spacetime. The key case for reduction in both Mechanics and field theories is Euler–Poincaré reduction, where the variational variables take values in a Lie group G or are local sections of a G-principal bundle, and G is at the same time the group of symmetries.

Lagrangian reduction in Mechanics (and, in particular, Euler–Poincaré) has been extended to higher-order systems (see for example [20–22]). However, there is not a counterpart in field theory yet, despite the fact that there are interesting systems and situations that fit in this context.

This work addresses higher-order Euler–Poincaré reduction, i.e., reduction for higher-order Lagrangian field theories whose configuration bundle is a principal G-bundle and the group of symmetries is G. We thus extend the results in [7] to higher-order Lagrangian field theories. The analysis of multivariate splines on Lie groups is given at the end of the article to illustrate the main results of it.

* Corresponding author.
E-mail addresses: mcastri@mat.ucm.es (M. Castrillón López), arabella@comillas.edu (Á. Rodríguez Abella).

https://doi.org/10.1016/j.geomphys.2025.105693
0393-0440/

Reduction theory has also been analyzed in the discrete setting for both mechanics [2,3,33,34,38] and field theory [42]. A discretization of the reduction theory introduced here would be desirable in order to build variational integrators for the reduced equations. We leave this as a future work.

The paper is organized as follows. Firstly, some essential facts about higher-order jet bundles are recalled in Section 2, including the total partial derivatives and the jet prolongation of vector fields, and the formulation of higher-order Lagrangian field theories is recalled in 3. Next, in Section 4 the geometry of the reduced configuration space $(J^kP)/G$ for a k-th order Lagrangian theory on a principal bundle is investigated. The understanding of this reduced phase bundle is essential for a correct analysis of the reduced variational principle. For first order theories, this reduced bundle is the bundle of connections $C(P)$, the sections of which are principal connections on P. However, even though one might suspect that $(J^kP)/G \simeq J^{k-1}C(P)$, $k \geq 2$, it turns out that this reduced phase bundle is the space of $(k-1)$-th order pointwise principal connections with vanishing curvature up to $k-2$ order. This fact is already shown for particular cases in [26] and is analyzed in depth in [17]. We follow the strategy of the latter. The main result of the paper is established in Section 5, where we compute the reduced equations, i.e., the higher-order Euler–Poincaré field equations. For $k \geq 2$ the reconstruction process providing solutions of the original problem from solution of the reduced equations is given in detail (at least locally). In this case, the situation $k=1$ and $k \geq 2$ show a different conceptual behavior, since the first order case requires an auxiliary compatibility condition, whereas that condition is implicit in the set of admissible solution for higher-order Lagrangians. Furthermore, the reduced equations are reinterpreted as a Noether conservation law in Section 6. At last, Section 7 is devoted to illustrate the previous theory by computing the reduced equations for multivariate k-splines (brane splines) on Lie groups.

In the following, every manifold or map is assumed to be smooth, meaning C^∞, unless otherwise stated. In addition, every fiber bundle $\pi_{Y,X} : Y \to X$ is assumed to be locally trivial and is denoted by $\pi_{Y,X}$. Given $x \in X$, $Y_x = \pi_{Y,X}^{-1}(\{x\})$ denotes the fiber over x. We assume that $\dim X = n$ and $\dim Y_x = m$. The space of (smooth) global sections of $\pi_{Y,X}$ is denoted by $\Gamma(\pi_{Y,X})$. In particular, vector fields on a manifold X are denoted by $\mathfrak{X}(X) = \Gamma(\pi_{TX,X})$, where TX is the tangent bundle of X. Likewise, k-forms on X are denoted by $\Omega^k(X) = \Gamma(\pi_{T^*X,X})$, where T^*X is the cotangent bundle of X. The space of local sections is denoted by $\Gamma_{loc}(\pi_{Y,X})$, and the same notation stands for local vector fields and forms. In the same vein, given an open set $\mathcal{U} \subset X$, the family of sections of $\pi_{Y,X}$ defined on $\mathcal{U}$ is denoted by $\Gamma(\mathcal{U}, \pi_{Y,X})$, and analogous for the other spaces. The tangent map of a map $f \in C^\infty(X, X')$ between the manifolds X and X' is denoted by $(df)_x : T_xX \to T_{f(x)}X'$ for each $x \in X$. In the same vein, the pull-back of $\alpha \in \Omega^k(X')$ is denoted by $f^*\alpha \in \Omega^k(X)$ and its exterior derivative is denoted by $d\alpha \in \Omega^{k+1}(X')$. When working in local coordinates, indices will be denoted by lowercase letters (μ, α, etc.), and multi-indices by capital letters ($J = (J_1, \dots, J_n) \in \mathbb{N}^n$, etc.). Given two multi-indices $J = (J_1, \dots, J_n)$, $I = (I_1, \dots, I_n) \in \mathbb{N}^n$, we write

$$J! = J_1! \dots J_n!, \qquad \binom{J}{I} = \binom{J_1}{I_1} \cdots \binom{J_n}{I_n}.$$

Besides, we will assume the Einstein summation convention for repeated (multi-)indices.

2. Higher-order jet bundles

We summarize the main results about higher-order jet bundles that we will need in the following (see, for example, [40, Chapter 6]). Let $\pi_{Y,X} : Y \to X$ be a fiber bundle and $k \in \mathbb{Z}^+$. The k-th order jet bundle of $\pi_{Y,X}$ is denoted by

$$\pi_{J^kY,X} : J^kY \longrightarrow X,$$

and its elements by $j_x^k s$. The k-th jet lift of a section $s \in \Gamma(\pi_{Y,X})$ is denoted by $j^k s \in \Gamma\left(\pi_{J^kY,X}\right)$. Recall that the maps $\pi_{k,l} : J^kY \to J^lY$, $j_x^k s \mapsto j_x^l s$, $0 \leq l < k$, are fiber bundles, where we denote $J^0Y = Y$. In addition, $\pi_{k,k-1}$ is an affine bundle modelled on

$$\pi^*_{J^{k-1}Y,X}\left(\bigvee^k T^*X\right) \otimes \pi^*_{k-1,0}(VY),$$

where $VY = \ker(\pi_{Y,X})_*$ is the vertical bundle of $\pi_{Y,X}$.

Let (x^μ, y^α) be bundle coordinates for $\pi_{Y,X}$. The induced coordinates on J^kY are $(x^\mu, y^\alpha; y_J^\alpha)$, $1 \leq |J| \leq k$, where $|J| = J_1 + \dots + J_n$ is the length of J. If, locally, $s(x^\mu) = \left(x^\mu, s^\alpha(x^\mu)\right)$ for some (local) functions $s^\alpha \in C^\infty(X)$, $1 \leq \alpha \leq m$, then $j^k s(x^\mu) = \left(x^\mu, s^\alpha(x^\mu); s_J^\alpha(x^\mu)\right)$ with

$$s_J^\alpha(x^\mu) = \left.\frac{\partial^{|J|}}{\partial x^J}\right|_{x=(x^\mu)} s^\alpha(x) = \left(\frac{\partial}{\partial x^1}\right)^{J_1} \cdots \left.\left(\frac{\partial}{\partial x^n}\right)^{J_n}\right|_{x=(x^\mu)} s^\alpha(x),$$

for each $1 \leq |J| \leq k$.

Definition 2.1. Let $\pi_{Y,X}$ and $\pi_{Y',X'}$ be fiber bundles, $k \in \mathbb{Z}^+$, and $F: Y \to Y'$ be a bundle morphism covering a diffeomorphism $f: X \to X'$. The *k-th order jet lift of F* is the map $j^k F: J^k Y \to J^k Y'$ defined as $j_x^k s \mapsto j_{f(x)}^k \left(F \circ s \circ f^{-1}\right)$.

If $\pi_{Y'',X''}: Y'' \to X''$ is another fiber bundle and $G: Y' \to Y''$ is bundle morphism covering a diffeomorphism $g: X' \to X''$, then $j^k(G \circ F) = j^k G \circ j^k F$. In other words, the following diagram is commutative.

$$
\begin{tikzcd}
J^kY \arrow[r, "j^kF"] \arrow[rr, bend left, "j^k(F\circ G)"] \arrow[d, "\pi_{k,0}"'] & J^kY' \arrow[r, "j^kG"] \arrow[d, "\pi'_{k,0}"] & J^kY'' \arrow[d, "\pi''_{k,0}"] \\
Y \arrow[r, "F"] \arrow[d, "\pi_{Y,X}"'] & Y' \arrow[r, "G"] \arrow[d, "\pi_{Y',X'}"] & Y'' \arrow[d, "\pi_{Y'',X''}"] \\
X \arrow[r, "f"] & X' \arrow[r, "g"] & X''
\end{tikzcd}
\tag{1}
$$

Proposition 2.1. *The following is a canonical decomposition of vector bundles over J^kY,*

$$\pi_{k+1,k}^*\left(T(J^kY)\right) = \pi_{k+1,k}^*\left(V(J^kY)\right) \oplus H(\pi_{k+1,k}),$$

where $V(J^kY) = \ker(\pi_{J^kY,X})_*$ is the vertical bundle of $\pi_{J^kY,X}$ and

$$H(\pi_{k+1,k})_{j_x^{k+1}s} = d(j^ks)_x(T_xX), \qquad j_x^{k+1}s \in J^{k+1}Y.$$

Given $j_x^{k+1}s \in J^{k+1}Y$ and $U_x \in T_xX$, the vector $U^k_{j_x^{k+1}s} = d(j^ks)_x(U_x) \in H(\pi_{k+1,k})$ is called *k-th holonomic lift of U_x by $j_x^{k+1}s$*. In coordinates, if we write $j_x^{k+1}s = \left(x^\mu, y^\alpha; y_J^\alpha\right)$ and $U_x = U^\mu \partial_\mu$, then

$$U^k_{j_x^{k+1}s} = U^\mu \left(\partial_\mu + y^\alpha_{1_\mu}\partial_\alpha + \sum_{|J|=1}^k y^\alpha_{J+1_\mu}\partial_\alpha^J\right),$$

where 1_μ is the multi-index given by $(1_\mu)_\nu = \delta_{\mu\nu}$, and ∂_μ, ∂_α, ∂_α^J are the partial derivatives of the coordinates x^μ, y^α, y_J^α, respectively, $1 \leq \mu, \nu \leq n$, $1 \leq \alpha \leq m$, $1 \leq |J| \leq k$. In particular, the k-th holonomic lifts of the partial vector fields, ∂_μ, $1 \leq \mu \leq n$, are called *coordinate total derivatives* and are given by

$$\frac{d}{dx^\mu} = (\partial_\mu)^k_{j_x^{k+1}s} = \partial_\mu + y^\alpha_{1_\mu}\partial_\alpha + y^\alpha_{J+1_\mu}\partial^J_\alpha.$$

Example 2.1 *(Total time derivative).* Let Q be a smooth manifold and consider the trivial bundle $\pi_{\mathbb{R}\times Q,\mathbb{R}}$ with (local) coordinates (t, q^α). Given $q(t) = \left(t, q^\alpha(t)\right)$, the *total time derivative* is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial q^\alpha}{\partial t}\partial_\alpha + \sum_{j=1}^k \frac{\partial^{j+1} q^\alpha}{\partial t^{j+1}}\partial_\alpha^j.$$

Note that given a (local) function $f \in C^\infty(J^kY)$ and a multi-index $J = (J^1, \ldots, J^n)$, then

$$\frac{d^{|J|}f}{dx^J} \in C^\infty(J^{k+|J|}Y).$$

Furthermore, for a section $s \in \Gamma(\pi_{Y,X})$ we have

$$\frac{d^{|J|}f}{dx^J} \circ j^{k+|J|}s = \frac{\partial^{|J|}\left(f \circ j^ks\right)}{\partial x^J}. \tag{2}$$

Next, we define the prolongation of vector fields.

Definition 2.2. Let $l \in \mathbb{Z}^+$. A *generalized vector field* on J^lY is a section $U \in \Gamma\left(\pi_{l,0}^*TY \to J^lY\right)$. Furthermore, it is said to be vertical if $U(j_x^ls) \in V_{s(x)}Y$ for each $j_x^ls \in J^lY$.

Note that (standard) vector fields on Y would be regarded as generalized vector fields for $l=0$. Locally, generalized vector fields are given by

$$U=U^\mu\partial_\mu+U^\alpha\partial_\alpha,\qquad U^\mu,U^\alpha:J^lY\to\mathbb{R},\ 1\le\mu\le n,\ 1\le\alpha\le m.$$

This way, vertical generalized vector fields are those ones such that $U^\mu=0$ for $1\le\mu\le n$.

Given a generalized vector field, $U\in\Gamma\left(\pi_{l,0}^*TY\to J^lY\right)$, and $k\in\mathbb{Z}^+$, it may be defined the *k-th order prolongation of* U, which is a section $U^{(k)}\in\Gamma\left(\pi_{k+l,k}^*\left(T\left(J^kY\right)\right)\to J^{k+l}Y\right)$ (cf. [40, Definition 6.4.16]). In particular, k-the prolongations of (standard) vector fields on Y are (standard) vector fields on J^kY.

Proposition 2.2. *Let $k\in\mathbb{Z}^+$, $(x^\mu,y^\alpha;y_J^\alpha)$ be bundle coordinates for J^kY and $V=V^\alpha\partial_\alpha\in\mathfrak{X}(Y)$ be a vertical vector field on Y, where $V^\alpha:Y\to\mathbb{R}$, $1\le\alpha\le m$. Then k-th order prolongation of V is given by*

$$V^{(k)}=V^\alpha\partial_\alpha+\frac{d^{|J|}V^\alpha}{dx^J}\partial_\alpha^J\in\mathfrak{X}\left(J^kY\right).$$

Proposition 2.3. *Let $k\in\mathbb{Z}^+$ and $U\in\mathfrak{X}(Y)$ be a $\pi_{Y,X}$-projectable vector field on Y. Then the flow of its k-th order prolongation, $U^{(k)}\in\mathfrak{X}\left(J^kY\right)$, is $\left\{j^k\phi_t\mid t\in(-\epsilon,\epsilon)\right\}$, where $\{\phi_t\mid t\in(-\epsilon,\epsilon)\}$ is the flow of U.*

To conclude this brief overview, we present the Leibniz rule for higher-order, multivariable calculus, which can be straightforwardly proven by induction in the number of factors (see [24, Proposition 6] for two factors).

Lemma 2.1 *(Higher-order Leibniz rule). Let $m\in\mathbb{Z}^+$ and $f_1,\dots,f_m\in C^\infty(\mathbb{R}^n)$. Then*

$$\frac{\partial^{|I|}}{\partial x^I}\prod_{\alpha=1}^m f_\alpha=\sum_{I^{(1)}+\dots+I^{(m)}=I}\frac{I!}{I^{(1)}!\dots I^{(m)}!}\prod_{\alpha=1}^m\frac{\partial^{|I^{(\alpha)}|}f_\alpha}{\partial x^{I^{(\alpha)}}},\qquad |I|\ge 0.$$

3. Calculus of variations for higher-order Lagrangian densities

We now recall higher-order calculus of variations (for a comprehensive exposition see, for example, [18,35,4]). Let X be a compact, $\pi_{Y,X}:Y\to X$ be a fiber bundle, and $k\in\mathbb{Z}^+$. A *k-th order Lagrangian density* on $\pi_{Y,X}$ is a bundle morphism

$$\mathfrak{L}:J^kY\longrightarrow\bigwedge^n T^*X$$

covering the identity on X. If X is oriented by a volume form $v\in\Omega^n(X)$, we will write $\mathfrak{L}=Lv$ for a function $L\in C^\infty\left(J^kY\right)$ known as *Lagrangian*. The *action functional* defined by $\mathfrak{L}$ is

$$\mathbb{S}:\Gamma(\pi_{Y,X})\to\mathbb{R},\qquad s\mapsto\int_X\mathfrak{L}\left(j^ks\right).$$

A *variation* of $s\in\Gamma\left(\pi_{Y,X}\right)$ is a 1-parameter family of sections

$$\left\{s_t\in\Gamma\left(\pi_{Y,X}\right)\mid t\in(-\epsilon,\epsilon),\ s_0=s\right\}.$$

The corresponding *infinitesimal variation* is

$$\delta s=\left.\frac{d}{dt}\right|_{t=0}s_t\in\Gamma\left(\pi_{s^*TY,X}\right).$$

Henceforth, only $\pi_{Y,X}$-vertical variations are considered, that is, those ones satisfying $\delta s(x)\in V_{s(x)}Y$ for each $x\in X$. The variation of the action functional induced by $\{s_t\}$ is defined as

$$\delta\mathbb{S}[s]=\left.\frac{d\,\mathbb{S}[s_t]}{dt}\right|_{t=0},$$

and it only depends on the infinitesimal variation. That is to say, if $\{s_t\}$ and $\{s_t'\}$ are two variations of s such that $\delta s=\delta s'$, then $d\,\mathbb{S}[s_t]/dt\big|_{t=0}=d\,\mathbb{S}[s_t']/dt\big|_{t=0}$.

Definition 3.1. A section $s\in\Gamma\left(\pi_{Y,X}\right)$ is *critical* or *stationary for* $\mathbb{S}$ if the variation of the corresponding action functional vanishes for every vertical variation of s, i.e.,

$$\delta\mathbb{S}[s]=0,\qquad\delta s\in\Gamma\left(\pi_{s^*VY,X}\right).$$

Unlike the first order case, the *covariant Cartan form*,

$$\Theta_{\mathfrak{L}} \in \Omega^n\left(J^{2k-1}Y\right),$$

is not uniquely defined for higher-order Lagrangians if $\dim X = n > 1$. The main reason is that, although there is a canonical embedding $J^kY \hookrightarrow J^1\left(J^{k-1}Y\right)$, there are many different choices for the corresponding projection. Nevertheless, it is always possible to construct a globally defined projection by means of tubular neighborhoods, thus yielding a globally defined covariant Cartan form on J^kY (cf. [40, Theorem 6.5.13]). In any case, there is a unique choice for the *Euler–Lagrange form* associated to $\mathfrak{L}$,

$$\mathcal{EL}(\mathfrak{L}) \in \Omega^n\left(J^{2k}Y, \pi_{2k,0}^*\left(V^*Y\right)\right),$$

where $\pi_{V^*Y,Y}$ is the dual of the vertical bundle $\pi_{VY,Y}$. There exist adapted coordinates $(x^\mu, y^\alpha; y_J^\alpha)$ for $J^{2k}Y$ such that the covariant Cartan form is locally given by[1]

$$\Theta_{\mathfrak{L}} = \sum_{|I|=0}^{k-1}\sum_{|J|=0}^{k-|I|-1}(-1)^{|J|}\frac{d^{|J|}}{dx^J}\left(\frac{\partial L}{\partial y^\alpha_{I+J+1_\mu}}\right)\left(dy_I^\alpha - y^\alpha_{I+1_\nu}dx^\nu\right)\wedge\left(\iota_{\partial_\mu}v\right) + Lv, \tag{3}$$

where $\iota : \mathfrak{X}(X)\times\Omega^{k+1}(X)\to\Omega^k(X)$ denotes the left interior product. Similarly, the Euler–Lagrange form is given by

$$\mathcal{EL}(\mathfrak{L}) = \sum_{|J|=0}^{k}(-1)^{|J|}\frac{d^{|J|}}{dx^J}\left(\frac{\partial L}{\partial y_J^\alpha}\right)v\otimes dy^\alpha. \tag{4}$$

Proposition 3.1. *Let $\delta s \in \Gamma(\pi_{s^*VY,X})$ be a variation of a section $s\in\Gamma(\pi_{Y,X})$. Then the variation of the action functional is given by*

$$\delta\mathbb{S}[s] = \int_X \left\langle \left(j^{2k}s\right)^*\mathcal{EL}(\mathfrak{L}), \delta s\right\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing within the vector bundle $VY\to Y$.

Proof. Since the variation is vertical, we have $s_t = \phi_t\circ s$, $t\in(-\epsilon,\epsilon)$, where $\{\phi_t : Y\to Y \mid t\in(-\epsilon,\epsilon)\}$ is the flow of a vertical, $\pi_{Y,X}$-projectable vector field $V\in\mathfrak{X}(Y)$. Subsequently,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}\mathfrak{L}\left(j^ks_t\right) &= \left.\frac{d}{dt}\right|_{t=0}\left(j^ks\right)^*\left(\mathfrak{L}(j^k\phi_t)\right)\\ &= \left(j^ks\right)^*\left(\left.\frac{d}{dt}\right|_{t=0}\mathfrak{L}\left(j^k\phi_t\right)\right)\\ &= \left(j^ks\right)^*\left(\pounds_{V^{(k)}}\mathfrak{L}\right),\end{aligned} \tag{5}$$

where $\pounds$ denotes the Lie derivative and we have used the commutativity of (1) and Proposition 2.3. Since X is boundaryless, the Stokes theorem leads to

$$\int_X\left(j^ks\right)^*\mathrm{d}\left(\iota_{V^{(k)}}\mathfrak{L}\right) = \int_X\mathrm{d}\left(\left(j^ks\right)^*\left(\iota_{V^{(k)}}\mathfrak{L}\right)\right) = \int_{\partial X}\left(j^ks\right)^*\left(\iota_{V^{(k)}}\mathfrak{L}\right) = 0. \tag{6}$$

Similarly, we pick bundle coordinates $(x^\mu, y^\alpha; y_J^\alpha)$ for J^kY, which allows us to write $V = V^\alpha\partial_\alpha$ for certain (local) functions $V^\alpha : X\to\mathbb{R}$, $1\le\alpha\le m$. Thus, Proposition 2.2 yields

$$\iota_{V^{(k)}}\mathrm{d}\mathfrak{L} = \left(\frac{\partial L}{\partial y^\alpha}V^\alpha + \frac{\partial L}{\partial y_J^\alpha}\frac{d^{|J|}V^\alpha}{dx^J}\right)v. \tag{7}$$

Lastly, the higher-order integration by parts formula for boundaryless manifolds (see, for example, [6, Lemma 4.5]), together with (2), gives

[1] Since the higher-order covariant Cartan form is not uniquely determined, its local expression depends on the choice of bundle coordinates (cf. [40, §5.5], [23, §3B] for first order, and [15, §5], [40, §6.5] for higher orders).

$$\int_X \left(j^k s\right)^* \left(\frac{\partial L}{\partial y_J^\alpha}\frac{d^{|J|}V^\alpha}{dx^J}\, v\right) = \int_X \frac{\partial}{\partial y_J^\alpha}\left(L\circ j^k s\right)\frac{\partial^{|J|}V^\alpha}{\partial x^J}\, v \tag{8}$$
$$= (-1)^{|J|}\int_X \frac{\partial^{|J|}}{\partial x^J}\left(\frac{\partial}{\partial y_J^\alpha}\left(L\circ j^k s\right)\right)V^\alpha\, v$$
$$= (-1)^{|J|}\int_X \left(j^{2k} s\right)^* \left(\frac{d^{|J|}}{dx^J}\left(\frac{\partial L}{\partial y_J^\alpha}\right)V^\alpha\, v\right).$$

On the other hand, it is clear that $\delta s = (d/dt)|_{t=0}(\phi_t\circ s) = V\circ s = V^\alpha \partial_\alpha \in \Gamma\left(\pi_{s^*VY,X}\right)$. Hence, from the local expression of the Euler–Lagrange form (4), we get

$$\left\langle \left(j^{2k}s\right)^* \mathcal{EL}(\mathfrak{L}), \delta s\right\rangle = \left(j^{2k}s\right)^*\left(\sum_{|J|=0}^{k}(-1)^{|J|}\frac{d^{|J|}}{dx^J}\left(\frac{\partial L}{\partial y_J^\alpha}\right)\right)V^\alpha\, v. \tag{9}$$

By using Cartan's formula, i.e., $\pounds = d\circ\iota + \iota\circ d$, and by gathering the previous expressions, we finish

$$\delta\mathbb{S}[s] = \frac{d}{dt}\bigg|_{t=0}\int_X \mathfrak{L}\left(j^k s_t\right) \overset{(5)}{=} \int_X \left(j^k s\right)^*\left(\pounds_{V^{(k)}}\mathfrak{L}\right)$$
$$= \int_X \left(j^k s\right)^*\left(\iota_{V^{(k)}}\mathrm{d}\mathfrak{L}\right) + \int_X \left(j^k s\right)^* \mathrm{d}\left(\iota_{V^{(k)}}\mathfrak{L}\right)$$
$$\overset{(6),(7)}{=} \int_X \left(j^k s\right)^*\left(\frac{\partial L}{\partial y^\alpha}V^\alpha + \frac{\partial L}{\partial y_J^\alpha}\frac{d^{|J|}V^\alpha}{dx^J}\right)v$$
$$\overset{(8)}{=} \int_X \left(j^{2k} s\right)^*\left(\frac{\partial L}{\partial y^\alpha} + (-1)^{|J|}\frac{d^{|J|}}{dx^J}\left(\frac{\partial L}{\partial y_J^\alpha}\right)\right)V^\alpha\, v$$
$$\overset{(9)}{=} \int_X \left\langle \left(j^{2k}s\right)^*\mathcal{EL}(\mathfrak{L}), \delta s\right\rangle. \quad \square$$

The following result is a straightforward consequence of the previous Proposition and it gives the higher-order version of the well-known Euler–Lagrange equations.

Theorem 3.1. *Let $\mathfrak{L} = Lv : J^kY \to \bigwedge^n T^*X$ be a k-th order Lagrangian density, and consider the corresponding Euler–Lagrange form, $\mathcal{EL}(\mathfrak{L})$. Then the following statements for a section $s\in\Gamma\left(\pi_{Y,X}\right)$ are equivalent:*

(i) The variational principle $\delta\mathbb{S}[s] = 0$ holds for arbitrary variations $\delta s \in \Gamma(\pi_{s^*VY,X})$.
(ii) s satisfies the k-th order Euler–Lagrange field equations, *i.e.,*

$$(j^{2k}s)^*\mathcal{EL}(\mathfrak{L}) = 0.$$

From (2) and (4), we obtain the local expression of the higher-order Euler–Lagrange equations:

$$\sum_{|J|=0}^{k}(-1)^{|J|}\frac{\partial^{|J|}}{\partial x^J}\left(\frac{\partial L}{\partial y_J^\alpha}\left(j^k s\right)\right) = 0, \qquad 1\le\alpha\le m.$$

Remark 3.1. The calculus of variations described above is straightforwardly extended to a non-compact base manifold X by considering compactly supported variations. In other words, given a section $s\in\Gamma(\pi_{Y,X})$, the only variations δs allowed are those satisfying

$$j_x^{k-1}\delta s = 0, \qquad x\in\partial\mathcal{U},$$

for some open subset $\mathcal{U}\subset X$ with compact closure, $\overline{\mathcal{U}}$. Locally, this condition ensures that δs as well as its partial derivatives up to order $k-1$ vanish on the boundary, $\partial\mathcal{U}$. As a result, no boundary terms appear when integrating by parts and, thus, Proposition 3.1 is still valid.

4. Geometry of the reduced configuration space

Let G be a Lie group, $\mathfrak{g} = T_e G$ be its Lie algebra, $e \in G$ being the identity element, and $\mathfrak{g}^*$ be the dual of the Lie algebra. The corresponding exponential map is denoted by $\exp : \mathfrak{g} \to G$, and the adjoint representation of G is denoted by $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ for each $g \in G$. In the same vein, the adjoint and coadjoint representations of $\mathfrak{g}$ are denoted by $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ and $\mathrm{ad}_\xi^* : \mathfrak{g}^* \to \mathfrak{g}^*$, respectively, for each $\xi \in \mathfrak{g}$.

Let $\pi_{P,X} : P \to X$ be a principal G-bundle. The corresponding right action is denoted by

$$R : P \times G \to P, \qquad (y, g) \mapsto R_y(g) = R_g(y) = y \cdot g.$$

Recall that the infinitesimal generator of $\xi \in \mathfrak{g}$ is the vertical vector field $\xi^* \in \mathfrak{X}(P)$ given by

$$\xi_y^* = \left.\frac{d}{dt}\right|_{t=0} y \cdot \exp(t\xi) = \left(dR_y\right)_e(\xi), \qquad y \in P.$$

For each $k \in \mathbb{Z}^+$, the k-th jet extension of the action,

$$R^{(k)} : J^k P \times G \to J^k P, \qquad \left(j_x^k s, g\right) \mapsto j_x^k (s \cdot g),$$

is again free and proper, thus yielding a principal G-bundle, $J^k P \to J^k P/G$.

We denote by $C(P) = J^1 P/G \to X$ the bundle of connections of $\pi_{P,X}$, which is an affine bundle modelled on $T^*X \otimes \mathrm{ad}(P) \to X$, being $\mathrm{ad}(P) = (P \times \mathfrak{g})/G$ the adjoint bundle of $\pi_{P,X}$. Recall that there is a bijective correspondence between (local) sections of $\pi_{C(P),X}$ and (local) principal connections on $\pi_{P,X}$, which we denote by

$$\Omega^1(P, \mathfrak{g}) \ni A \stackrel{1:1}{\longleftrightarrow} \sigma_A \in \Gamma(\pi_{C(P),X}).$$

The following result ensures that holonomic sections of the jet bundle yield flat principal connections.

Lemma 4.1. *Let $s \in \Gamma_{loc}(\pi_{P,X})$ and define $\sigma_A = \left[j^1 s\right]_G \in \Gamma_{loc}(\pi_{C(P),X})$. Then the local connection $A \in \Omega^1_{loc}(P, \mathfrak{g})$ is flat.*

Proof. The local section s induces a trivialization of $\pi_{P,X}$ on its domain $\mathcal{U} \subset X$,

$$\varphi : \mathcal{U} \times G \xrightarrow{\sim} P|_{\mathcal{U}}, \qquad (x, g) \mapsto \varphi(x, g) = s(x) \cdot g.$$

Recall that any connection is uniquely determined by its horizontal lift at each point. Given $y = s(x) \cdot g \in P|_{\mathcal{U}}$, we denote by $Hor_y^A : T_x X \to T_y P$ the horizontal lifting of A at y. Since $\sigma_A = \left[j^1 s\right]_G$, we have

$$Hor_y^A(U_x) = d(R_g \circ s)_x(U_x) \in T_y P, \qquad U_x \in T_x X. \tag{10}$$

On the other hand, consider the canonical flat connection $A_0 \in \Omega^1(X \times G, \mathfrak{g})$ on the trivial bundle $\pi_{X \times G, X}$. For each $(x, g) \in X \times G$ it is defined as

$$Hor_{(x,g)}^{A_0}(U_x) = (U_x, 0_g) \in T_{(x,g)}(X \times G), \qquad U_x \in T_x X.$$

It turns out that $\varphi^* A = A_0|_{\mathcal{U} \times G}$, so A is a flat connection. Indeed, let $(x, g) \in \mathcal{U} \times G$ and $U_x \in T_x X$. Since φ covers the identity on $\mathcal{U}$, we have

$$\begin{aligned} Hor_{(x,g)}^{\varphi^* A}(U_x) &= \left(d\varphi^{-1}\right)_{s(x)\cdot g}\left(Hor_{\varphi(x,g)}^A(U_x)\right) = \left(d\varphi^{-1}\right)_{s(x)\cdot g}\left(d(R_g \circ s)_x(U_x)\right) \\ &= d\left(\varphi^{-1} \circ R_g \circ s\right)_x(U_x) = (d\iota_g)_x(U_x) = (U_x, 0_g) = Hor_{(x,g)}^{A_0}(U_x), \end{aligned}$$

where $X \ni x \mapsto \iota_g(x) = (x, g) \in X \times G$. $\square$

The main goal of this section is to study the geometry of the quotient $\left(J^k P\right)/G$, which was first investigated in [17]. For the convenience of the reader, we present a detailed proof of the main result (see Corollary 4.1 below). To begin with, we introduce the following bundle morphisms.

Proposition 4.1. *For each $k \geq 2$, the bundle morphism $\Theta_k : J^k P \to J^{k-1} C(P)$ defined as $j_x^k s \mapsto j_x^{k-1}[j^1 s]_G$ is well-defined and it descends to a map on the quotient, $\Xi_k : \left(J^k P\right)/G \to J^{k-1} C(P)$.*

Proof. To begin with, note that Θ_k is the composition of the natural immersion $J^kP \hookrightarrow J^{k-1}\left(J^1P\right)$ with the $(k-1)$-th jet lift of the canonical projection $\pi_1 : J^1P \to C(P)$. Thus, it is well-defined.

For the second part, let $g \in G$. Then

$$\begin{aligned}\Theta_k\left((j^k_x s)\cdot g\right) &= \Theta_k\left(j^k_x(s\cdot g)\right) = j^{k-1}_x[j^1(s\cdot g)]_G\\ &= j^{k-1}_x[(j^1 s)\cdot g]_G = j^{k-1}_x[j^1 s]_G = \Theta_k\left(j^k_x s\right). \quad \square\end{aligned}$$

The previous result may be summarized in the following commutative diagram.

$$\begin{tikzcd} J^kP \arrow[r, "\Theta_k"] \arrow[d, "\pi_k"'] & J^{k-1}C(P) \\ \left(J^kP\right)/G \arrow[ur, "\Xi_k"'] & \end{tikzcd} \qquad \begin{tikzcd} j^k_x s \arrow[r, mapsto] \arrow[d, mapsto] & j^{k-1}_x[j^1 s]_G \\ \left[j^k_x s\right]_G \arrow[ur, mapsto] & \end{tikzcd} \tag{11}$$

Consider the curvature map, i.e.,

$$\tilde{F} : J^1C(P) \to \bigwedge^2 T^*X \otimes \mathrm{ad}(P), \qquad j^1_x\sigma_A \mapsto \tilde{F}^A(x),$$

where $\tilde{F}^A \in \Omega^2(X, \mathrm{ad}(P))$ is curvature of the (local) principal connection $A \in \Omega^1(P, \mathfrak{g})$ regarded as a 2-from on M with values in the adjoint bundle. By lifting it to the $(k-2)$-jet and by composing it with the natural immersion of $J^{k-1}C(P) \hookrightarrow J^{k-2}\left(J^1C(P)\right)$ we obtain

$$j^{k-2}\tilde{F} : J^{k-1}C(P) \to J^{k-2}\left(\bigwedge^2 T^*X \otimes \mathrm{ad}(P)\right), \qquad j^{k-1}_x\sigma_A \mapsto j^{k-2}_x\tilde{F}^A.$$

Let $\hat{0} : X \to \bigwedge^2 T^*X \otimes \mathrm{ad}(P)$ be the zero section, and consider its $(k-2)$-jet lift, $j^{k-2}\hat{0} : X \to J^{k-2}\left(\bigwedge^2 T^*X \otimes \mathrm{ad}(P)\right)$. We define the kernel of $j^{k-2}\tilde{F}$ as

$$\ker j^{k-2}\tilde{F} = \left\{ j^{k-1}_x\sigma_A \in J^{k-1}C(P) \mid j^{k-2}_x\tilde{F}^A = j^{k-2}_x\hat{0}\right\}.$$

Proposition 4.2. *For each $k \geq 2$, we have*

(i) Ξ_k *is an injective bundle morphism over* X.
(ii) $\mathrm{im}\,\Xi_k \subset \ker j^{k-2}\tilde{F}$.

Proof. To check *(i)*, let $\left[j^k_x s\right]_G, \left[j^k_x s'\right]_G \in \left(J^kP\right)/G$ be two different elements. In other words, $j^k_x s \neq (j^k_x s')\cdot g = j^k_x(s'\cdot g)$ for each $g \in G$. Subsequently, $[j^1_x s]_G \neq [j^1_x s']_G$ and, thus, $\Xi_k\left(\left[j^k_x s\right]_G\right) \neq \Xi_k\left(\left[j^k_x s'\right]_G\right)$. On the other hand, *(ii)* is a straightforward consequence of Lemma 4.1. $\square$

Let us introduce a trivializing chart of $\pi_{P,X}$, i.e., we pick an open subset $\mathcal{U} \subset X$ with $\mathcal{U} \simeq \mathbb{R}^n$ via the local coordinates $x = (x^\mu)$ and such that $P|_{\mathcal{U}} \simeq \mathcal{U} \times G$. To keep the notation simple, we write $\mathcal{U} = X$. Likewise, let $\{B_\alpha \in \mathfrak{g} \mid 1 \leq \alpha \leq m\}$ be a basis of $\mathfrak{g}$ and consider normal coordinates (y^α) in a neighborhood of the identity element $e \in G$, i.e., $g = \exp(y^\alpha B_\alpha)$. This way, we have normal bundle coordinates (x^μ, y^α) for $\pi_{P,X}$. Note that for each $g = \exp(y^\alpha B_\alpha), h = \exp(z^\alpha B_\alpha) \in G$ close enough to the identity, we have $gh = \exp\left(f^\alpha_{CD} y^C z^D B_\alpha\right)$, where $C = (C_1, \ldots, C_m)$ is a multi-index, $y^C = (y^1)^{C_1}\ldots(y^m)^{C_m}$ and f^α_{CD}, $1 \leq \alpha \leq m$, $|C|, |D| \geq 0$, are the constants of the Baker–Campbell–Hausdorff formula (cf. [43, §2.15] and [17, Theorem 4.1]). By using this, it can be checked that

$$P \times \mathfrak{g} \simeq VP, \qquad ((x^\mu, y^\alpha), \xi^\alpha B_\alpha) \mapsto a^\alpha_\beta \xi^\beta \partial_\alpha, \tag{12}$$

where

$$a^\alpha_\beta(y^1, \ldots, y^m) = f^\alpha_{C1_\beta} y^C.$$

Note that for $y = (x, e)$ we have $a^\alpha_\beta(0, \ldots, 0) = \delta^\alpha_\beta$, $1 \leq \alpha \leq m$. Hence, the inverse of the matrix $(a^\alpha_\beta(y^\alpha))$ is well defined in a neighborhood of $y = (x, e)$, and we denote it by $(b^\alpha_\beta(y^1, \ldots, y^m))$. In such case, the inverse of the previous isomorphism is given by

$$VP \simeq P \times \mathfrak{g}, \qquad (y, U^\alpha\partial_\alpha) \mapsto (y, b^\alpha_\beta U^\beta B_\alpha). \tag{13}$$

We pick the following bundle coordinates:

- (x^μ, A_μ^α) for $C(P)$,
- $(x^\mu, F_{\mu\nu}^\alpha)$ for $\bigwedge^2 T^*X \otimes \mathrm{ad}(P)$,
- $(x^\mu, y^\alpha; y_J^\alpha)$ for $J^k P$,
- $(x^\mu, A_\mu^\alpha; A_{\mu,J}^\alpha)$ for $J^{k-1}C(P)$, and
- $(x^\mu, F_{\mu\nu}^\alpha; F_{\mu\nu,J}^\alpha)$ for $J^{k-2}\left(\bigwedge^2 T^*X \otimes \mathrm{ad}(P)\right)$.

In order to keep the notation simple, we will write $y_{J_0}^\alpha = y^\alpha$, $A_{\mu,J_0}^\alpha = A_\mu^\alpha$ and $F_{\mu\nu,J_0}^\alpha = F_{\mu\nu}^\alpha$ for $J_0 = (0,\dots,0)$.

Remark 4.1. Given two multi-indices I, J, the inequality $I \leq J$ means that $I_\mu \leq J_\mu$ for $1 \leq \mu \leq n$. Likewise, we denote $I < J$ when $I \leq J$ and $I \neq J$, that is, $I_\mu \leq J_\mu$ for $1 \leq \mu \leq n$ and there exists $\mu_0 \in \{1,\dots,n\}$ such that $I_{\mu_0} < J_{\mu_0}$.

Let us compute the coordinated expression of Θ_k.

Proposition 4.3. *For each $k \geq 2$ and each $(x^\mu, y^\alpha; y_J^\alpha) \in J^k P$ with $|y^\alpha|$ small enough, $1 \leq \alpha \leq m$, we have $\Theta_k(x^\mu, y^\alpha; y_J^\alpha) = (x^\mu, A_\mu^\alpha; A_{\mu,J}^\alpha)$, where*[2]

$$A_\mu^\alpha = b_\beta^\alpha(y^\alpha) y_{1_\mu}^\beta, \quad A_{\mu,J}^\alpha = \frac{\partial^{|J|}}{\partial x^J}\left(b_\beta^\alpha(y^\alpha) y_{1_\mu}^\beta\right)\Big|_x,$$

for each $1 \leq \mu \leq n$, $1 \leq \alpha \leq m$, and $1 \leq |J| \leq k-1$.

Proof. We denote $j_x^1 s = (x^\mu, y^\alpha; y_J^\alpha)$. Recall from (10) that the horizontal lift given by $[j^1 s]_G \in C(P)$ reads $Hor_{s(x)}^A = dx^\mu \otimes \partial_\mu + y_{1_\mu}^\alpha dx^\mu \otimes \partial_\alpha$. From this and (13), we get

$$A_{s(x)} = b_\beta^\alpha(y^\alpha)\left(-y_{1_\mu}^\beta dx^\mu + dy^\alpha\right) \otimes B_\alpha \in T_{s(x)}^* P \otimes \mathfrak{g}.$$

Then $[j^1 s]_G = (x^\mu, A_\mu^\alpha = b_\beta^\alpha(y^\alpha) y_{1_\mu}^\beta)$. By taking partial derivatives, we conclude. $\square$

Now some technical lemmas are presented.

Lemma 4.2. *Let $k \geq 2$ and $j_x^{k-1}\sigma_A = (x^\mu, A_\mu^\alpha; A_{\mu,J}^\alpha) \in J^{k-1}C(P)$. Then $j_x^{k-2}\tilde{F}^A = \left(x^\mu, F_{\mu\nu}^\alpha; F_{\mu\nu,J}^\alpha\right) \in J^{k-2}\left(\bigwedge^2 T^*X \otimes \mathrm{ad}(P)\right)$ is given by*

$$F_{\mu\nu,J}^\alpha = \frac{1}{2}\left(A_{\nu,J+1_\mu}^\alpha - A_{\mu,J+1_\nu}^\alpha + \sum_{I \leq J}\binom{J}{I} c_{\beta\gamma}^\alpha A_{\mu,I}^\beta A_{\nu,J-I}^\gamma\right), \tag{14}$$

for each $1 \leq \mu,\nu \leq n$, $0 \leq |J| \leq k-2$ and $1 \leq \alpha \leq m$, where we denote by $c_{\beta\gamma}^\alpha = 2 f_{1_\beta 1_\gamma}^\alpha$, $1 \leq \alpha,\beta,\gamma \leq m$, the structure constants of $\mathfrak{g}$.

Proof. We show it by induction in $0 \leq |J| \leq k-2$. The case $|J| = 0$ is straightforward, since the curvature of $j_x^1\sigma_A = (x^\mu, A_\mu^\alpha; A_{\mu,1_\nu}^\alpha)$ is given by (cf. [39, Chapter 1, §4])

$$F_{\mu\nu}^\alpha = \frac{1}{2}\left(A_{\nu,1_\mu}^\alpha - A_{\mu,1_\nu}^\alpha + c_{\beta\gamma}^\alpha A_\mu^\beta A_\nu^\gamma\right).$$

Now assume that (14) holds for $1 \leq |J| (< k-2)$. To conclude, we need to show that it holds for $J' = J + 1_\sigma$ for each $1 \leq \sigma \leq n$. Note that

$$\binom{J'}{I} = \binom{J+1_\sigma}{I} = \binom{J}{I} + \binom{J}{I-1_\sigma}, \qquad 1_\sigma \leq I \leq J.$$

Furthermore, note that

[2] By abusing the notation, for each multi-index J we denote

$$\frac{\partial^J b_\beta^\alpha}{\partial x^J}(y)\Big|_x = \frac{\partial b_\beta^\alpha}{\partial y^\gamma}(y)\Big|_{y^\alpha=0} y_J^\gamma, \qquad 1 \leq \alpha,\beta \leq m.$$

$$\sum_{\substack{I\leq J,\\ I_\sigma=0}}\binom{J'}{I}=\sum_{\substack{I\leq J,\\ I_\sigma=0}}\binom{J}{I},\qquad \sum_{\substack{I\leq J',\\ I_\sigma=J_\sigma+1}}\binom{J'}{I}=\sum_{\substack{I\leq J',\\ I_\sigma=J_\sigma+1}}\binom{J}{I-1_\sigma}.$$

Hence,

$$\begin{aligned}\sum_{I\leq J'}\binom{J'}{I}&=\sum_{\substack{I\leq J,\\ I_\sigma=0}}\binom{J}{I}+\sum_{1_\sigma\leq I\leq J}\left(\binom{J}{I}+\binom{J}{I-1_\sigma}\right)+\sum_{\substack{I\leq J',\\ I_\sigma=J_\sigma+1}}\binom{J}{I-1_\sigma}\\&=\sum_{I\leq J}\binom{J}{I}+\sum_{1_\sigma\leq I\leq J'}\binom{J}{I-1_\sigma}.\end{aligned}$$

By using this and by taking the partial derivative of (14) with respect to x^σ, we finish

$$\begin{aligned}F^\alpha_{\mu\nu,J'}&=\frac{1}{2}\left(A^\alpha_{\nu,J'+1_\mu}-A^\alpha_{\mu,J'+1_\nu}+\sum_{I\leq J}\binom{J}{I}c^\alpha_{\beta\gamma}\left(A^\beta_{\mu,I+1_\sigma}A^\gamma_{\nu,J-I}+A^\beta_{\mu,I}A^\gamma_{\nu,J-I+1_\sigma}\right)\right)\\&=\frac{1}{2}\left(A^\alpha_{\nu,J'+1_\mu}-A^\alpha_{\mu,J'+1_\nu}+\sum_{1_\sigma\leq I\leq J'}\binom{J}{I-1_\sigma}c^\alpha_{\beta\gamma}A^\beta_{\mu,I}A^\gamma_{\nu,J'-I}+\sum_{I\leq J}\binom{J}{I}c^\alpha_{\beta\gamma}A^\beta_{\mu,I}A^\gamma_{\nu,J'-I}\right)\\&=\frac{1}{2}\left(A^\alpha_{\nu,J'+1_\mu}-A^\alpha_{\mu,J'+1_\nu}+\sum_{I\leq J'}\binom{J'}{I}c^\alpha_{\beta\gamma}A^\beta_{\mu,I}A^\gamma_{\nu,J'-I}\right).\quad\square\end{aligned}$$

Lemma 4.3. *Let $k\geq 2$ and $g=\exp(y^\alpha B_\alpha)\in G$ with $|y^\alpha|$ small enough, $1\leq\alpha\leq m$. Then*

$$c^\lambda_{\beta\gamma}b^\beta_\alpha(y)b^\gamma_\epsilon(y)=\frac{\partial b^\lambda_\alpha}{\partial y^\epsilon}(y)-\frac{\partial b^\lambda_\epsilon}{\partial y^\alpha}(y),\qquad 1\leq\alpha,\epsilon,\lambda\leq m.$$

Proof. For each $1\leq\beta,\gamma\leq m$ and $y=(x^\mu,y^\alpha)\in P$, the infinitesimal generator is given in coordinates by $(B_\beta)^*_y=a^\alpha_\beta(y)\partial_\alpha$ and, from the equality $[B_\beta,B_\gamma]=c^\alpha_{\beta\gamma}B_\alpha$, they satisfy

$$c^\alpha_{\beta\gamma}a^\delta_\alpha(t)\partial_\delta=\left[a^\alpha_\beta(y)\partial_\alpha,a^\delta_\gamma(y)\partial_\delta\right]=\left(a^\alpha_\beta(y)\frac{\partial a^\delta_\gamma}{\partial y^\alpha}(y)-a^\alpha_\gamma\frac{\partial a^\delta_\beta(y^\alpha)}{\partial y^\alpha}(y)\right)\partial_\delta.\tag{15}$$

Since $\left(a^\alpha_\beta(y)\right)\left(b^\alpha_\beta(y)\right)=\mathbb{I}$, the identity matrix, we have $a^\delta_\gamma(y)b^\beta_\delta(y)=\delta^\beta_\gamma$ and, thus,

$$\frac{\partial a^\delta_\gamma}{\partial y^\alpha}(y)b^\beta_\delta(y)+a^\delta_\gamma(y)\frac{\partial b^\beta_\delta(y^\alpha)}{\partial y^\alpha}(y)=0,\qquad 1\leq\alpha,\beta,\gamma\leq m.$$

As a result,

$$\frac{\partial a^\delta_\gamma}{\partial y^\alpha}(y)=-a^\delta_\beta(y)a^\epsilon_\gamma(y)\frac{\partial b^\beta_\epsilon}{\partial y^\alpha}(y),\qquad 1\leq\alpha,\gamma,\delta\leq m.$$

By substituting this into (15) and rearranging indices, we get

$$c^\kappa_{\beta\gamma}a^\delta_\kappa(y)=a^\alpha_\beta(y)a^\delta_\lambda(y)a^\epsilon_\gamma(y)\left(-\frac{\partial b^\lambda_\epsilon}{\partial y^\alpha}(y)+\frac{\partial b^\lambda_\alpha}{\partial y^\epsilon}(y)\right),\qquad 1\leq\beta,\gamma,\delta\leq m.$$

Lastly, we multiply each side by $b^{\beta'}_\alpha(y)b^{\lambda'}_\delta(y)b^{\gamma'}_\epsilon(y)$ and we sum over α, δ and ϵ:

$$c^\lambda_{\beta\gamma}b^\beta_\alpha(y)b^\gamma_\epsilon(y)=\frac{\partial b^\lambda_\alpha}{\partial y^\epsilon}(y)-\frac{\partial b^\lambda_\epsilon}{\partial y^\alpha}(y),\qquad 1\leq\alpha,\epsilon,\lambda\leq m.\quad\square$$

We are ready to show the main result about the map Ξ_k.

Theorem 4.1. *For each $k\geq 2$, we have that:*

(i) $\operatorname{im}\Xi_k=\ker j^{k-2}\tilde{\mathrm{F}}\subset J^{k-1}C(P)$ *and,*
(ii) Ξ_k *is a proper map.*

Proof. To begin with, we show *(i)*. Thanks to Proposition 4.2*(ii)*, we only need to prove that $\ker j^{k-2}\tilde{F} \subset \operatorname{im}\Xi_k$. Similarly, the commutativity of (11) yields $\operatorname{im}\Xi_k = \operatorname{im}\Theta_k$.

The fact that the maps are local enables us to work locally. More specifically, we employ the bundle coordinates introduced above. Let $j_x^{k-1}\sigma_A = (x^\mu, A_\mu^\alpha; A_{\mu,J}^\alpha) \in \ker j^{k-2}\tilde{F}$. We need to find $j_x^k s = (x^\mu, y^\alpha; y_J^\alpha) \in J^k P$, such that $\Theta_k(j_x^k s) = j_x^{k-1}\sigma_A$. For that matter, we set $y^\alpha = 0$, $1 \le \alpha \le m$. Besides, we define y_J^α, $1 \le |J| \le k$, by recurrence. Namely, fix $0 \le r \le k-1$, and assume that y_J^α has been defined for $1 \le \alpha \le m$, $0 \le |J| \le r-1$ in such a way that they satisfy

$$\frac{\partial^{|J|}}{\partial x^J}\left(b_\beta^\alpha(y) y_{1_\mu}^\beta\right)\Big|_x = A_{\mu,J}^\alpha, \tag{16}$$

for each $0 \le |J| \le r-2$, $1 \le \mu \le n$ and $1 \le \alpha \le m$. Now pick $J \in \mathbb{N}^n$ with $|J| = r$ and consider $1 \le \mu \le n$ such that $J_\mu > 0$. Then we set

$$y_J^\alpha(\mu) = A_{\mu,J-1_\mu}^\alpha - \sum_{I<J-1_\mu}\binom{J-1_\mu}{I}\frac{\partial^{|J-1_\mu-I|}b_\beta^\alpha(y)}{\partial x^{J-1_\mu-I}}\Big|_x y_{I+1_\mu}^\beta, \qquad 1 \le \alpha \le m.$$

Observe that it makes sense, since the elements y_J^α considered in the RHS are only up to order $r-1$. In addition, for each $1 \le \alpha \le m$ we have

$$\begin{aligned}&\frac{\partial^{|J-1_\mu|}}{\partial x^{J-1_\mu}}\left(b_\beta^\alpha(y) y_{1_\mu}^\beta\right)\Big|_x \\ &\qquad = y_J^\alpha(\mu) + \sum_{I<J-1_\mu}\binom{J-1_\mu}{I}\frac{\partial^{|J-1_\mu-I|}b_\beta^\alpha(y)}{\partial x^{J-1_\mu-I}}\Big|_x y_{I+1_\mu}^\beta = A_{\mu,J-1_\mu}^\alpha,\end{aligned}$$

since $b_\beta^\alpha(y^\alpha = 0) = \delta_\beta^\alpha$. Hence, $y_J^\alpha(\mu)$ keeps satisfying (16).

Let us check that the definition do not depend on the μ chosen, so we can write $y_J^\alpha = y_J^\alpha(\mu)$. Let $1 \le \nu \le n$ be another index such that $J_\nu > 0$. This enables us to write $J = J' + 1_\mu + 1_\nu$ for some multi-index J' with $|J'| = r-2$. Hence, $J - 1_\mu = J' + 1_\nu$ and $J - 1_\nu = J' + 1_\mu$. Therefore,

$$\begin{aligned}&\sum_{I<J'+1_\nu}\binom{J'+1_\nu}{I}\frac{\partial^{|J'+1_\nu-I|}b_\beta^\alpha}{\partial x^{J'+1_\nu-I}}(y)\Big|_x y_{I+1_\mu}^\beta - \sum_{I<J'+1_\mu}\binom{J'+1_\mu}{I}\frac{\partial^{|J'+1_\mu-I|}b_\beta^\alpha}{\partial x^{J'+1_\mu-I}}(y)\Big|_x y_{I+1_\nu}^\beta \\ &\quad = \frac{\partial^{|J'+1_\nu|}}{\partial x^{J'+1_\nu}}\left(b_\beta^\alpha(y) y_{1_\mu}^\beta\right)\Big|_x - \frac{\partial^{|J'+1_\mu|}}{\partial x^{J'+1_\mu}}\left(b_\beta^\alpha(y) y_{1_\nu}^\beta\right)\Big|_x \\ &\quad = \frac{\partial^{|J'|}}{\partial (y) x^{J'}}\left(\frac{\partial b_\beta^\alpha}{\partial y^\gamma}\Big|_x (y)\, y_{1_\nu}^\gamma y_{1_\mu}^\beta + b_\beta^\alpha(y) y_{1_\mu+1_\nu}^\beta - \frac{\partial b_\beta^\alpha}{\partial y^\gamma}\Big|_x (y)\, y_{1_\mu}^\gamma y_{1_\nu}^\beta - b_\beta^\alpha(y) y_{1_\mu+1_\nu}^\beta\right) \\ &\quad = \frac{\partial^{|J'|}}{\partial x^{J'}}\left(\left(\frac{\partial b_\gamma^\alpha}{\partial y^\beta}(y) - \frac{\partial b_\beta^\alpha}{\partial y^\gamma}(y)\right) y_{1_\mu}^\gamma y_{1_\nu}^\beta\right)\Big|_x.\end{aligned} \tag{17}$$

Similarly, by inserting the condition $j_x^{k-1}\sigma_A \in \ker j^{k-2}\tilde{F}$, i.e., $j_x^{k-2}\tilde{F}^A = j_x^{k-2}\hat{0}$, in Lemma 4.2, and by using (16), we obtain

$$\begin{aligned}A_{\mu,J'+1_\nu}^\alpha - A_{\nu,J'+1_\mu}^\alpha &= \sum_{I\le J'}\binom{J'}{I} c_{\lambda\epsilon}^\alpha A_{\mu,I}^\lambda A_{\nu,J'-I}^\epsilon \\ &= \sum_{I\le J'}\binom{J'}{I} c_{\lambda\epsilon}^\alpha \frac{\partial^{|I|}}{\partial x^I}\left(b_\gamma^\lambda(y) y_{1_\mu}^\gamma\right)\Big|_x \frac{\partial^{|J'-I|}}{\partial x^{J'-I}}\left(b_\beta^\epsilon(y) y_{1_\nu}^\beta\right)\Big|_x \\ &= \frac{\partial^{|J'|}}{\partial x^{J'}}\left(c_{\lambda\epsilon}^\alpha b_\gamma^\lambda(y) b_\beta^\epsilon(y) y_{1_\mu}^\gamma y_{1_\nu}^\beta\right)\Big|_x\end{aligned} \tag{18}$$

for each $1 \le \alpha \le m$, and $1 \le \mu, \nu \le n$. By gathering (17) and (18), and by making use of Lemma 4.3, we conclude

$$\begin{aligned}y_J^\alpha(\mu) - y_J^\alpha(\nu) &= A_{\mu,J'+1_\nu}^\alpha - A_{\nu,J'+1_\mu}^\alpha - \frac{\partial^{|J'|}}{\partial x^{J'}}\left(\left(\frac{\partial b_\gamma^\alpha}{\partial y^\beta}(y) - \frac{\partial b_\beta^\alpha}{\partial y^\gamma}(y)\right) y_{1_\mu}^\gamma y_{1_\nu}^\beta\right)\Big|_x \\ &= \frac{\partial^{|J'|}}{\partial x^{J'}}\left(c_{\lambda\epsilon}^\alpha b_\gamma^\lambda(y) b_\beta^\epsilon(y) y_{1_\mu}^\gamma y_{1_\nu}^\beta\right)\Big|_x - \frac{\partial^{|J'|}}{\partial x^{J'}}\left(c_{\lambda\epsilon}^\alpha b_\gamma^\lambda(y) b_\beta^\epsilon(y) y_{1_\mu}^\gamma y_{1_\nu}^\beta\right)\Big|_x\end{aligned}$$

$$= 0.$$

Once we have defined $j_x^k s = (x^\mu, y^\alpha; y_J^\alpha) \in J^k P$, the last step is to check that $\Theta_k(j_x^k s) = j_x^{k-1}\sigma_A$, which is straightforward from (16) and Proposition 4.3.

Lastly, we show that Ξ_k is proper. Let $\left([j_{x_n}^k s_n]_G\right)_{n=1}^\infty$ be a sequence in $\left(J^k P\right)/G$ such that $\left(j_{x_n}^{k-1}\sigma_{A_n} = \Xi_k\left([j_{x_n}^k s_n]_G\right)\right)_{n=1}^\infty$ converges to some $j_x^{k-1}\sigma_A \in J^{k-1}C(P)$. Since $j_{x_n}^{k-2}\tilde{F}_{A_n} = j_{x_n}^{k-2}\hat{0}$ for each $n \in \mathbb{N}$, by continuity we conclude that $j_x^{k-2}\tilde{F}^A = j_x^{k-2}\hat{0}$, that is, $j_x^{k-1}\sigma_A \in \ker j^{k-2}\tilde{F} = \operatorname{im}\Xi_k$. Thus, there exists $j_x^k s \in J^k P$ such that $\Xi_k([j_x^k s]_G) = j_x^{k-1}\sigma_A$. In coordinates, we write $j_{x_n}^k s_n = (x_n^\mu, y_n^\alpha; (y_n)_J^\alpha)$, $j_{x_n}^{k-1}\sigma_{A_n} = (x_n^\mu, (A_n)_\mu^\alpha; (A_n)_{\mu,J}^\alpha)$, $j_x^k s = (x^\mu, y^\alpha; y_J^\alpha)$ and $j_x^{k-1}\sigma_A = (x^\mu, A_\mu^\alpha; A_{\mu,J}^\alpha)$. Without loss of generality, we may assume that $y_n^\alpha = 0$ and $y^\alpha = 0$, $1 \le \alpha \le m$. This way, by recalling Proposition 4.3, we have

$$\lim_n \frac{\partial^{|J|}}{\partial x^J}\left(b_\beta^\alpha(y_n)(y_n)_{1_\mu}^\beta\right)\bigg|_{x_n} = \lim_n (A_n)_{\mu,J}^\alpha = A_{\mu,J}^\alpha = \frac{\partial^{|J|}}{\partial x^J}\left(b_\beta^\alpha(y)y_{1_\mu}^\beta\right)\bigg|_x$$

for each $0 \le |J| \le k-1$, $1 \le \alpha \le m$ and $1 \le \mu \le n$. Subsequently, by recurrence as above, it can be checked that $\lim_n (y_n)_J^\alpha = y_J^\alpha$ for each $0 \le |J| \le k$ and $1 \le \alpha \le m$ and, thus, $\left(j_{x_n}^k s_n\right)_{n=1}^\infty$ converges to $j_x^k s$. $\square$

By gathering the previous results, we obtain the geometry of the reduced configuration bundle.

Corollary 4.1. *For each $k \ge 2$, there is an isomorphism of fiber bundles covering the identity id_X,*

$$\left(J^k P\right)/G \simeq \ker j^{k-2}\tilde{F} = \left\{ j_x^{k-1}\sigma_A \in J^{k-1}C(P) \mid j_x^{k-2}\tilde{F}^A = j_x^{k-2}\hat{0} \right\}.$$

Proof. Let us show that Ξ_k is an immersion, i.e., that $d\Xi_k$ is everywhere injective. Since the property is local we can work in a trivialization of $\pi_{P,X}$, that is, we assume that $P = X \times G$. Given two sections $s = (\mathrm{id}_X, f), s' = (\mathrm{id}_X, f') \in \Gamma(\pi_{P,X})$, we define $\tau : P \to P$ as $\tau(x,g) = \left(x, f'(x)f(x)^{-1}g\right)$ for each $x \in X$, which is a principal bundle automorphism such that $\tau \circ s = s'$. Besides, the jet extension $j^1\tau : J^1 P \to J^1 P$ is G-equivariant, i.e., $j^1\tau(j_x^1 s)\cdot g = j^1\tau\left((j_x^1 s)\cdot g\right)$ for each $g \in G$ and each $j_x^1 s \in J^1 P$. Hence, it induces an automorphism of the bundle of connections, $\hat{\tau} = [j^1\tau]_G : C(P) \to C(P)$. By taking the jet lift of these maps, we arrive at the following commutative diagram

$$\begin{tikzcd}
J^k P \arrow[r, "\Theta_k"] \arrow[d, "j^k\tau"'] & J^{k-1}C(P) \arrow[d, "j^{k-1}\hat{\tau}"] \\
J^k P \arrow[r, "\Theta_k"] & J^{k-1}C(P)
\end{tikzcd}$$

Therefore, Θ_k has constant rank, since τ and $\hat{\tau}$ are isomorphisms and this construction can be done for every pair of sections $s, s' \in \Gamma(\pi_{P,X})$. By using this and the fact that π_k is a submersion, and by recalling (11), we conclude that Ξ_k has constant rank too. Since Ξ_k is injective (recall Proposition 4.2*(i)*), the Global Rank Theorem (see, for example, [29, Theorem 4.14]) ensures that Ξ_k is an immersion.

In short, Ξ_k is an injective immersion and proper (by *(ii)* of Theorem 4.1), whence it is an embedding (cf. [29, Proposition 4.22]). Hence, $\operatorname{im}\Xi_k = \ker j^{k-2}\tilde{F}$ is a submanifold of $J^{k-1}C(P)$ and, again by the Global Rank Theorem, $\Xi_k : \left(J^k P\right)/G \to \ker j^{k-2}\tilde{F}$ is a diffeomorphism. $\square$

Notation. Henceforth, we will denote

$$C_0^{k-1}(P) = \operatorname{im}\Xi_k = \ker j^{k-2}\tilde{F}.$$

5. Higher-order Euler–Poincaré reduction

At this point, we are ready to develop the Euler–Poincaré reduction for higher-order field theories. Given a principal G-bundle, $\pi_{P,X} : P \to X$, over a boundaryless manifold X and $k \ge 2$, we consider a k-th order Lagrangian density, $\mathfrak{L} = Lv : J^k P \to \bigwedge^n T^*X$, that is G-invariant, i.e.,

$$L\left(R_g^{(k)}\left(j_x^k s\right)\right) = L\left(j_x^k s\right), \qquad j_x^k s \in J^k P,\ g \in G.$$

As a result, we have a *dropped* or *reduced Lagrangian*,

$$\mathfrak{l} = lv : \left(J^k P\right)/G \to \bigwedge^n T^*X, \qquad \left[j_x^k s\right]_G \mapsto \mathfrak{l}\left(\left[j_x^k s\right]_G\right) = \mathfrak{L}\left(j_x^k s\right).$$

By means of Corollary 4.1, we may regard the reduced Lagrangian as defined on $C_0^{k-1}(P) = \ker j^{k-2}\tilde{F} \subset J^{k-1}C(P)$. For an open subset $\mathcal{U} \subset X$ with compact closure and $s \in \Gamma\left(\mathcal{U}, \pi_{P,X}\right)$, the corresponding *reduced section* is defined as

$$\sigma_A = \left[j^1 s\right]_G \in \Gamma\left(\mathcal{U}, \pi_{C(P),X}\right).$$

A variation δs of the original section induces a variation $\delta\sigma_A$ of the reduced section. By construction, the original and the reduced variations satisfy $\mathfrak{L}\left(j^k s_t\right) = \mathfrak{l}\left(j^{k-1}\left(\overline{\sigma}_A\right)_t\right)$ for all $t \in (-\epsilon, \epsilon)$ and, thus,

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{U}} \mathfrak{L}\left(j^k s_t\right) = \left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{U}} \mathfrak{l}\left(j^{k-1}\left(\sigma_A\right)_t\right). \tag{19}$$

If we choose a fixed principal connection $A_0 \in \Omega^1(P, \mathfrak{g})$ on $\pi_{P,X}$, we can identify the bundle of connections with the modelling vector bundle,

$$C(P) \ni \sigma_A(x) \overset{1:1}{\longleftrightarrow} \overline{\sigma}_A(x) = \sigma_A(x) - \sigma_{A_0}(x) \in T^*X \otimes \mathrm{ad}(P). \tag{20}$$

By taking the $(k-1)$-jet lift of this map, we can also identify the reduced configuration space with some subbundle of $J^{k-1}\left(T^*X \otimes \mathrm{ad}(P)\right)$, that is,

$$J^{k-1}C(P) \supset C_0^{k-1}(P) \overset{\sim}{\longleftrightarrow} \overline{C_0^{k-1}(P)} \subset J^{k-1}\left(T^*X \otimes \mathrm{ad}(P)\right),$$

so that the reduced Lagrangian could be regarded as

$$\mathfrak{l} : \overline{C_0^{k-1}(P)} \to \bigwedge^n T^*X.$$

The whole situation is summarized in the following commutative diagram:

$$\begin{tikzcd}
J^kP \arrow[dd] \arrow[dr, "\pi_k"] & & & \\
& \left(J^kP\right)/G \arrow[r, "\Xi_k"] \arrow[dd] & C_0^{k-1}(P) \arrow[r, "\sim"] \arrow[dd] & \overline{C_0^{k-1}(P)} \arrow[dd] \\
J^1P \arrow[d] \arrow[dr, "\pi_1"] & & & \\
P \arrow[d] & \left(J^1P\right)/G \arrow[r, "\sim"] \arrow[dl] & C(P) \arrow[r, "\sim"] & T^*X \otimes \mathrm{ad}(P) \\
\mathcal{U}\,(\subset X) \arrow[u, dashed, bend left, "s"] \arrow[uuuu, dashed, bend left=60, "j^k s"] \arrow[urr, dashed, bend right, "\sigma_A"'] \arrow[urrr, dashed, bend right, "\overline{\sigma}_A"'] & & &
\end{tikzcd}$$

Definition 5.1. Let $\nabla^{A_0} : \Gamma\left(\pi_{\mathrm{ad}(P),X}\right) \to \Gamma\left(\pi_{T^*X\otimes\mathrm{ad}(P),X}\right)$ be the linear connection on the adjoint bundle, $\pi_{\mathrm{ad}(P),X}$, induced by A_0. The *divergence of* ∇^{A_0} is minus the adjoint of ∇^{A_0}, i.e., the map $\mathrm{div}^{A_0} : \Gamma\left(\pi_{TX\otimes\mathrm{ad}(P)^*,X}\right) \to \Gamma\left(\pi_{\mathrm{ad}(P)^*,X}\right)$ implicitly defined by

$$\int_X \left\langle \eta, \nabla^{A_0}\xi \right\rangle v = -\int_X \left\langle \mathrm{div}^{A_0}\eta, \xi \right\rangle v,$$

for each $\xi \in \Gamma\left(\pi_{\mathrm{ad}(P),X}\right)$ and $\eta \in \Gamma\left(\pi_{TX\otimes\mathrm{ad}(P)^*,X}\right)$.

Given $\overline{\sigma}_A \in \Gamma\left(\pi_{T^*X\otimes\mathrm{ad}(P),X}\right)$, we define the map

$$\mathrm{ad}^*_{\overline{\sigma}_A} : \Gamma\left(\pi_{TX\otimes\mathrm{ad}(P)^*,X}\right) \to \Gamma\left(\pi_{\mathrm{ad}(P)^*,X}\right)$$

as the coadjoint representation of $\mathrm{ad}(P)^*$ and the dual pairing of TX and T^*X. In the following, since $\pi_{T^*X\otimes\mathrm{ad}(P),X}$ is a vector bundle, we identify its vertical bundle with itself, $V\left(T^*X \otimes \mathrm{ad}(P)\right) \simeq T^*X \otimes \mathrm{ad}(P)$, and analogous for its dual.

Theorem 5.1 *(Higher-order Euler–Poincaré field equations). Let $k \geq 2$, $\pi_{P,X} : P \to X$ be a principal G-bundle and $\mathfrak{L} : J^k P \to \bigwedge^n T^*X$ be a k-th order, G-invariant Lagrangian density. Let $\mathfrak{l} : C_0^{k-1}(P) \to \bigwedge^n T^*X$ be the reduced Lagrangian density, where $C_0^{k-1}(P) \simeq \left(J^k P\right)/G$ is the reduced space, and consider an extension $\hat{\mathfrak{l}} : J^{k-1}(C(P)) \to \bigwedge^n T^*X$ of the reduced Lagrangian $\mathfrak{l}$ to the whole jet. Let $A_0 \in \Omega^1(P, \mathfrak{g})$ be a principal connection on $\pi_{P,X}$, which enables us to identify $C(P) \simeq T^*X \otimes \operatorname{ad}(P)$, and let ∇^{A_0} be the linear connection on $\pi_{\operatorname{ad}(P),X}$ induced by A_0.*

Given a (local) section $s \in \Gamma(\mathcal{U}, \pi_{P,X})$, where $\mathcal{U} \subset X$ is an open subset with compact closure, and the corresponding reduced section $\sigma_A \in \Gamma\left(\mathcal{U}, \pi_{C(P),X}\right)$, the following statements are equivalent:

(i) The variational principle $\delta \int_{\mathcal{U}} \mathfrak{L}\left(j^k s\right) = 0$ holds for arbitrary variations of s such that $j_x^{k-1}\delta s = 0$ for each $x \in \partial\mathcal{U}$.

(ii) The section s satisfies the Euler–Lagrange field equations, i.e.,

$$\left(j^{2k} s\right)^* \mathcal{EL}(\mathfrak{L}) = 0.$$

(iii) The variational principle $\delta \int_{\mathcal{U}} \mathfrak{l}\left(j^{k-1}\sigma_A\right) = 0$ holds for variations of the form

$$\delta\sigma_A = \nabla^{A_0}\xi - [\overline{\sigma}_A, \xi] \in \Gamma\left(\mathcal{U}, \pi_{T^*X\otimes \operatorname{ad}(P),X}\right),$$

where $\xi \in \Gamma(\mathcal{U}, \pi_{\operatorname{ad}(P),X})$ is an arbitrary section such that $j_x^{k-2}\xi = 0$ for each $x \in \partial\mathcal{U}$.

(iv) The reduced section $\overline{\sigma}_A$ satisfies the k-th order Euler–Poincaré field equations*:*

$$\left(\operatorname{div}^{A_0} - \operatorname{ad}^*_{\overline{\sigma}_A}\right)\left(\left(j^{2k-2}\sigma_A\right)^* \mathcal{EL}\left(\hat{\mathfrak{l}}\right)\right) = 0, \tag{21}$$

where $\mathcal{EL}\left(\hat{\mathfrak{l}}\right) \in \Omega^n\left(J^{2k-2}(C(P)), TX \otimes \operatorname{ad}(P)^*\right)$ is the Euler–Lagrange form of $\hat{\mathfrak{l}}$.

Proof. The equivalence between (i) and (ii) is established in Theorem 3.1. In addition, the induced variations of the reduced section were computed in [7, Theorem 1]. Hence, the equivalence between (i) and (iii) is straightforward from this fact and equation (19). To conclude, let us show the equivalence between (iii) and (iv). Firstly, note that given a variation of the original section, together with the induced variation of the reduced section, we have $j_x^{k-1}(\sigma_A)_t \in C_0^{k-1}(P)$ for all $t \in (-\epsilon, \epsilon)$ and $x \in \mathcal{U}$. Hence, $\delta\left(j^{k-1}\sigma_A\right)(x) = (\delta\sigma_A)^{(k-1)}(x) \in T_{j_x^{k-1}\sigma_A}\left(C_0^{k-1}(P)\right)$. By applying the explicit expression of the reduced variations and Proposition 3.1, we obtain

$$\begin{aligned}
\delta \int_{\mathcal{U}} \hat{\mathfrak{l}}\left(j^{k-1}\sigma_A\right) &= \int_{\mathcal{U}} \left\langle \left(j^{2k-2}\sigma_A\right)^* \mathcal{EL}\left(\hat{\mathfrak{l}}\right), \delta\sigma_A \right\rangle \\
&= \int_{\mathcal{U}} \left\langle \left(j^{2k-2}\sigma_A\right)^* \mathcal{EL}\left(\hat{\mathfrak{l}}\right), \nabla^{A_0}\xi - [\overline{\sigma}_A, \xi] \right\rangle \\
&= -\int_{\mathcal{U}} \left\langle \operatorname{div}^{A_0}\left(\left(j^{2k-2}\sigma_A\right)^* \mathcal{EL}\left(\hat{\mathfrak{l}}\right)\right) - \operatorname{ad}^*_{\overline{\sigma}_A}\left(\left(j^{2k-2}\sigma_A\right)^* \mathcal{EL}\left(\hat{\mathfrak{l}}\right)\right), \xi \right\rangle \\
&= 0.
\end{aligned}$$

Since this holds for every $\xi \in \Gamma\left(\mathcal{U}, \pi_{\operatorname{ad}(P),X}\right)$ vanishing at the boundary, we conclude. □

Remark 5.1. Since (the $(k-1)$-th jet lift of) the reduced section lies in $C_0^{k-1}(P)$ and the infinitesimal reduced variations are tangent to this space, the solutions of the reduced equations do not depend on the choice of the extended Lagrangian $\hat{\mathfrak{l}} : J^{k-1}C(P) \to \bigwedge^n T^*X$. That is, if we have two such extensions $\hat{\mathfrak{l}}_1, \hat{\mathfrak{l}}_2$ with $\hat{\mathfrak{l}}_1|_{C_0^{k-1}(P)} = \hat{\mathfrak{l}}_2|_{C_0^{k-1}(P)}$, then a reduced section is critical with respect for $\hat{\mathfrak{l}}_1$ if and only if it is critical for $\hat{\mathfrak{l}}_2$. In other words, $C_0^{k-1}(P) \subset J^{k-1}C(P)$ may be regarded as a kind of holonomic constraint.

Remark 5.2. If we choose the connection $A_0 = A$ itself, that is, the connection defined by the reduced section $\sigma_A = [j^1 s]_G$, the reduced equation (21) has the simpler form

$$\operatorname{div}^A\left(\left(j^{2k-2}\sigma_A\right)^* \mathcal{EL}\left(\hat{\mathfrak{l}}\right)\right) = 0.$$

Let us find the local expression of the reduced equations. As in Section 4, let $\{B_\alpha \mid 1 \leq \alpha \leq m\}$ be a basis of $\mathfrak{g}$, (x^μ, y^α) be normal bundle coordinates for $\pi_{Y,X}$ and (x^μ, A_μ^α) be bundle coordinates for $\pi_{C(P),X}$. Similarly, consider the adapted coordinates $(x^\mu, y^\alpha; y_J^\alpha)$ and $(x^\mu, A_\mu^\alpha; A_{\mu,J}^\alpha)$ for J^kY and $J^{k-1}C(P)$, respectively. For the sake of simplicity, we fix the flat connection on $\pi_{Y,X}$ under these coordinates, i.e., $A_0 = dy^\alpha \otimes B_\alpha$. This way, (x^μ, A_μ^α) may also be regarded as bundle coordinates on $\pi_{T^*X\otimes \mathrm{ad}(P),X}$, and we denote by (x^μ, v_α^μ) the bundle coordinates on the dual bundle, $\pi_{TX\otimes \mathrm{ad}(P)^*,X}$.

Lemma 5.1. *With the above bundle coordinates, let $\eta = \eta_\alpha^\mu\, \partial_\mu \otimes B^\alpha \in \Gamma\left(\pi_{TX\otimes \mathrm{ad}(P)^*,X}\right)$ and $\overline{\sigma}_A = \sigma_\mu^\alpha\, dx^\mu \otimes B_\alpha \in \Gamma\left(\pi_{T^*X\otimes \mathrm{ad}(P),X}\right)$. Then we have*

$$\mathrm{div}^{A_0}(\eta) = \partial_\mu \eta_\alpha^\mu\, B^\alpha, \qquad \mathrm{ad}^*_{\overline{\sigma}_A}(\eta) = -c_{\beta\gamma}^\alpha\, \eta_\alpha^\mu\, \sigma_\mu^\beta\, B^\gamma.$$

Proof. Let $\xi = \xi^\alpha B_\alpha \in \Gamma\left(\pi_{\mathrm{ad}(P),X}\right)$ and note that $\nabla^{A_0}\xi = \partial_\mu \xi^\alpha dx^\mu \otimes B_\alpha$. By definition of divergence and the integration by parts formula [6, Lemma 4.5], we finish

$$\begin{aligned}
\int_X \left\langle \mathrm{div}^{A_0}(\eta), \xi \right\rangle v &= -\int_X \left\langle \eta, \nabla^{A_0}\xi \right\rangle v \\
&= -\int_X \left\langle \eta_\alpha^\mu \partial_\mu \otimes B^\alpha, \partial_\mu \xi^\alpha dx^\mu \otimes B_\alpha \right\rangle v \\
&= -\int_X \left(\eta_\alpha^\mu \partial_\mu \xi^\alpha\right) v = \int_X \left(\partial_\mu \eta_\alpha^\mu \xi^\alpha\right) v.
\end{aligned}$$

For the second part, we have

$$\begin{aligned}
\left\langle \mathrm{ad}^*_{\overline{\sigma}_A}(\eta), \xi \right\rangle &= -\langle \eta, \mathrm{ad}_{\overline{\sigma}_A}(\xi)\rangle = -\left\langle \eta_\alpha^\mu\, \partial_\mu \otimes B^\alpha, \left[\sigma_\mu^\alpha\, dx^\mu \otimes B_\alpha, \xi^\alpha B_\alpha\right]\right\rangle \\
&= -\left\langle \eta_\alpha^\mu\, \partial_\mu \otimes B^\alpha, c_{\beta\gamma}^\alpha\, \sigma_\mu^\beta\, \xi^\gamma\, dx^\mu \otimes B_\alpha \right\rangle = -c_{\beta\gamma}^\alpha\, \eta_\alpha^\mu\, \sigma_\mu^\beta\, \xi^\gamma. \quad \square
\end{aligned}$$

Corollary 5.1 *(Local equations). With the above coordinates, we write*

$$\overline{\sigma}_A = \sigma_\mu^\alpha dx^\mu \otimes B_\alpha \in \Gamma\left(\mathcal{U}, \pi_{T^*X\otimes \mathrm{ad}(P),X}\right)$$

for the reduced section. Then the local expression of the k-th order Euler–Poincaré field equations is given by

$$\sum_{|J|=0}^{k-1} (-1)^{|J|} \left(\frac{\partial^{|J+1_\mu|}}{\partial x^{J+1_\mu}} \left(\frac{\partial \hat{l}}{\partial A_{\mu,J}^\alpha} \left(j^{k-1}\overline{\sigma}_A\right)\right) - c_{\beta\alpha}^\gamma \sigma_\mu^\beta \frac{\partial^{|J|}}{\partial x^J} \left(\frac{\partial \hat{l}}{\partial A_{\mu,J}^\gamma} \left(j^{k-1}\overline{\sigma}_A\right)\right)\right) = 0,$$

for each $1 \leq \alpha \leq m$.

Proof. From (4) and (2), it is clear that

$$\left(j^{2k-2}\overline{\sigma}_A\right)^* \mathcal{EL}\left(\hat{\mathfrak{l}}\right) = \sum_{|J|=0}^{k-1} (-1)^{|J|} \frac{\partial^{|J|}}{\partial x^J} \left(\frac{\partial \hat{l}}{\partial A_{\mu,J}^\alpha} \left(j^{k-1}\overline{\sigma}_A\right)\right) v \otimes dA_\mu^\alpha.$$

Recall that we are identifying $V^*(T^*X \otimes \mathrm{ad}(P)) \simeq TX \otimes \mathrm{ad}(P)^*$, i.e., $dA_\mu^\alpha \simeq \partial_\mu \otimes B^\alpha$. The result is now a straightforward consequence of (iv) of Theorem 5.1 and Lemma 5.1. $\square$

At last, note that a solution of the reduced equations lying in the reduced space corresponds to a flat principal connection and, thus, the reconstruction condition is automatically satisfied.

Theorem 5.2 *(Reconstruction). In the conditions of Theorem 5.1, let $\sigma_A \in \Gamma\left(\mathcal{U}, \pi_{C(P),X}\right)$ be a solution of the k-th order Euler–Poincaré field equations (21) such that $j^{k-1}\sigma_A \in \Gamma\left(\mathcal{U}, \pi_{C_0^{k-1}(P),X}\right)$ and $A \in \Omega^1(P|_{\mathcal{U}}, \mathfrak{g})$ has trivial holonomy on a domain. Then there exists a section $s \in \Gamma\left(\mathcal{U}, \pi_{P,X}\right)$ that is critical for the original variational problem defined by $\mathfrak{L}$ and such that $\sigma_A = \left[j^1 s\right]_G$. Furthermore, any critical section on $\mathcal{U}$ of $\mathfrak{L}$ is of the form $s \cdot g$ for certain $g \in G$.*

Proof. From the condition $j^{k-1}\sigma_A \in \Gamma\left(\mathcal{U}, \pi_{C_0^{k-1}(P),X}\right)$ we know that $A \in \Omega^1(P|_{\mathcal{U}}, \mathfrak{g})$ is a flat connection. Therefore, there exists a foliation of $P|_{\mathcal{U}}$ given by the integral leaves of A. The trivial holonomy ensures that each integral leaf intersects once and only once each fiber of $P|_{\mathcal{U}}$. Subsequently, given an integral leaf of A, it defines a section $s \in \Gamma(\mathcal{U}, \pi_{P,X})$ projecting onto σ_A. This section is critical thanks to Theorem 5.1. Moreover, the remaining integral leaves of A are obtained from s as follows,

$$\mathcal{F}_g = s(\mathcal{U}) \cdot g \subset P|_{\mathcal{U}}, \qquad g \in G.$$

Hence, given another critical section, $\tilde{s} \in \Gamma(\mathcal{U}, \pi_{P,X})$, projecting onto σ_A, we conclude that $\tilde{s} = s \cdot g$ for some $g \in G$, since $\tilde{s}(\mathcal{U})$ must belong to the previous family. □

If X is a simply connected manifold, then every flat connection has trivial holonomy and we have the following equivalence

$$\left(j^{2k}s\right)^* \mathcal{EL}(\mathfrak{L}) = 0 \quad \Longleftrightarrow \quad \begin{cases} \left(\operatorname{div}^{A_0} - \operatorname{ad}^*_{\overline{\sigma}_A}\right)\left(\left(j^{2k-2}\overline{\sigma}_A\right)^* \mathcal{EL}\left(\hat{\mathfrak{l}}\right)\right) = 0, \\ \tilde{F}^A = 0. \end{cases}$$

In an arbitrary manifold the equivalence above is valid locally only.

Remark 5.3. That fact that the reconstruction process requires the flatness of a connection is a characteristic trait in Field Theories that does not show up in Mechanics, i.e., when $\dim X = 1$ (see for example [7,16] for comments on this situation). However, there is a significant difference from first order to higher-order reduction. Indeed, the flatness condition is a compatibility equation that must be incorporated by hand for $k = 1$ theories, whereas this condition comes directly from the geometry of the reduced phase space for $k > 1$. In other words, for higher-order Euler–Poincaré reduction the reconstruction is not inserted ad hoc, but it is intrinsic in the nature of the reduced sections.

6. Noether theorem

The well-known Noether theorem establishes that infinitesimal symmetries of a Lagrangian density lead to conserved quantities for the dynamics of the system. As in the previous section, let $\mathfrak{L} = L v : J^k P \to \bigwedge^n T^*X$ be a G-invariant k-th order Lagrangian on a principal G-bundle, $\pi_{P,X} : P \to X$. The aim of this section is to prove that the conservation laws of $\mathfrak{L}$ arising from its G-invariance are equivalent to the higher-order Euler–Poincaré field equations.

We start by recalling the definition of an infinitesimal symmetry and the Noether theorem for higher-order Lagrangian densities. (cf. [35, §10]).

Definition 6.1. An *infinitesimal symmetry* of $\mathfrak{L}$ is a projectable vector field $U \in \mathfrak{X}(P)$ such that $\pounds_{U^{(k)}}\mathfrak{L} = 0$, where $\pounds$ denotes the Lie derivative and $U^{(k)} \in \mathfrak{X}(J^k P)$ is the k-th order prolongation of U.

For a vertical vector field $V \in \mathfrak{X}(P)$, being an infinitesimal symmetry is equivalent to $\pounds_{V^{(2k-1)}}\Theta_{\mathfrak{L}} = 0$, for any covariant Cartan form, $\Theta_{\mathfrak{L}} \in \Omega^n(J^{2k-1}P)$, of $\mathfrak{L}$.

Theorem 6.1 *(Noether theorem). Let $s \in \Gamma(\pi_{P,X})$ and $U \in \mathfrak{X}(U)$ be a critical section and an infinitesimal symmetry of $\mathfrak{L}$, respectively. Then*

$$\mathrm{d}\left(\left(j^{2k-1}s\right)^* \iota_{U^{(2k-1)}}\Theta_{\mathfrak{L}}\right) = 0,$$

for any covariant Cartan form, $\Theta_{\mathfrak{L}}$, of $\mathfrak{L}$.

Henceforth, we focus on the covariant Cartan form given locally by (3). As our Lagrangian is G-invariant, each infinitesimal generator $\xi^* \in \mathfrak{X}(Y)$ of an element $\xi \in \mathfrak{g}$ is an infinitesimal symmetry of $\mathfrak{L}$. We define the *Noether current* as the $\mathfrak{g}^*$-valued form $\mathcal{J} \in \Omega^{n-1}\left(J^{2k-1}P, \mathfrak{g}^*\right)$ given by

$$\left\langle \mathcal{J}\left(j_x^{2k-1}s\right), \xi\right\rangle = \iota_{(\xi^*)^{(2k-1)}}\Theta_{\mathfrak{L}}\left(j_x^{2k-1}s\right), \qquad j_x^{2k-1}s \in J^{2k-1}P,\ \xi \in \mathfrak{g},$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing, as usual. Observe that the Noether theorem ensures that

$$\mathrm{d}\left(\left(j^{2k-1}s\right)^* \mathcal{J}\right) = 0 \tag{22}$$

for every critical section $s \in \Gamma(\pi_{Y,X})$. We are ready to show the main theorem of this section.

Theorem 6.2. *Let $\pi_{P,X}: P \to X$ be a principal G-bundle, $\mathfrak{L} = L\nu : J^k P \to \bigwedge^n T^*X$ be a G-invariant Lagrangian, $\mathfrak{l} = l\nu : C_0^{k-1}(P) \to \bigwedge^n T^*X$ be the reduced Lagrangian, and $\mathcal{J} \in \Omega^{n-1}\left(J^{2k-1}P, \mathfrak{g}^*\right)$ be the Noether current given by the Cartan form $\Theta_{\mathfrak{L}} \in \Omega^n(J^{2k-1}P)$ locally defined by (3). Consider a section, $s \in \Gamma(\pi_{Y,X})$ and the corresponding reduced section, $\sigma_A \in \Gamma\left(\pi_{C(P),X}\right)$. Then the Noether equation (22) holds if and only if the k-th order Euler–Poincaré field equations hold for the principal connection $A \in \Omega^1(P, \mathfrak{g})$, i.e.,*

$$\operatorname{div}^A \left(\left(j^{2k-2}\sigma_A \right)^* \mathcal{EL}\left(\hat{l}\right) \right) = 0,$$

where $\hat{l}: J^{k-1}(C(P)) \to \bigwedge^n T^*X$ is an extension of the reduced Lagrangian.

Proof. Let $(x^\mu, y^\alpha; y^\alpha_J)$ be adapted coordinates for $J^{2k}P$. Some conditions will be imposed on these coordinates along the proof. Firstly, suppose that (y^α) are normal coordinates on some neighborhood of the identity element. Hence, given a basis $\{B_\alpha \mid 1 \le \alpha \le m\}$ of $\mathfrak{g}$, we may use equation (12) and Proposition 2.2 (recall that $\xi^* \in \mathfrak{X}(P)$ is a vertical vector field) to obtain

$$(\xi^*)^{(2k-1)}_y = \xi^\beta a^\alpha_\beta(y)\partial_\alpha + \xi^\beta \frac{d^{|J|}a^\alpha_\beta(y)}{dx^J}\partial^J_\alpha, \qquad y = (x^\mu, y^\alpha) \in P.$$

If the coordinates are chosen so that the volume form is given by $\nu = d^n x = dx^1 \wedge \cdots \wedge dx^n$, the left interior product of the covariant Cartan form (3) by $(\xi^*)^{(2k-1)}$ reads

$$\iota_{(\xi^*)^{(2k-1)}}\Theta_{\mathfrak{L}} = \sum_{|I|=0}^{k-1}\sum_{|J|=0}^{k-|I|-1}(-1)^{|J|}\xi^\beta \frac{d^{|J|}}{dx^J}\left(\frac{\partial L}{\partial y^\alpha_{I+J+1_\mu}}\right)\frac{d^{|I|}a^\alpha_\beta(y)}{dx^I}d^{n-1}x_\mu,$$

where $d^{n-1}x_\mu = \iota_{\partial_\mu} d^n x$. In coordinates, the critical section is written as $s(x^\mu) = (x^\mu, s^\alpha(x^\mu))$, for some local functions $s^\alpha \in C^\infty(X)$, $1 \le \alpha \le m$. From the previous expression and (2), we get

$$\left(j^{2k-1}s\right)^*\mathcal{J} = \sum_{|I|=0}^{k-1}\sum_{|J|=0}^{k-|I|-1}(-1)^{|J|}\xi^\beta \frac{\partial^{|J|}}{\partial x^J}\left(\frac{\partial L}{\partial y^\alpha_{I+J+1_\mu}}\left(j^k s\right)\right)\frac{\partial^{|I|}a^\alpha_\beta(s^\alpha)}{\partial x^I}d^{n-1}x_\mu. \tag{23}$$

On the other hand, let $(x^\mu, A^\alpha_\mu; A^\alpha_{\mu,J})$ be adapted coordinates for $J^{2k-1}C(P)$. Observe that locally $C(P) \simeq T^*X \otimes \operatorname{ad}(P)$, so we will use these coordinates for both spaces indifferently. For the reduced section, we write $\sigma_A(x^\mu) = \overline{\sigma}_A(x^\mu) = (x^\mu, \sigma^\alpha_\mu(x^\mu))$ for some local functions $\sigma^\alpha_\mu \in C^\infty(X)$, $1 \le \mu \le n$, $1 \le \alpha \le m$. Since $\sigma_A = \left[j^1 s\right]_G$, Proposition 4.3 yields

$$\sigma^\alpha_\mu = b^\alpha_\beta(s^\alpha)\partial_\mu s^\beta, \qquad 1 \le \mu \le n,\ 1 \le \alpha \le m.$$

From this and Lemma 2.1, we get

$$\begin{aligned}
&\frac{\partial}{\partial y^\alpha_{I+J+1_\mu}}\left(\frac{\partial^{|I+J+K|}}{\partial x^{I+J+K}}\left(b^\beta_\gamma(s^\alpha)\partial_\mu s^\gamma\right)\right)\\
&\qquad = \frac{\partial}{\partial y^\alpha_{I+J+1_\mu}}\left(\sum_{I^{(1)}+I^{(2)}=I+J+K}\frac{(I+J+K)!}{I^{(1)}!I^{(2)}!}\frac{\partial^{|I^{(1)}|}b^\beta_\gamma(s^\alpha)}{\partial x^{I^{(1)}}}\frac{\partial^{|I^{(2)}|}\partial_\mu s^\gamma}{\partial x^{I^{(2)}}}\right)\\
&\qquad = \frac{(I+J+K)!}{K!(I+J)!}\frac{\partial^{|K|}b^\beta_\alpha(s^\alpha)}{\partial x^K},
\end{aligned} \tag{24}$$

for each $1 \le \mu \le n$, $1 \le \alpha, \beta \le m$, $0 \le |I| \le k-1$, $0 \le |J| \le k-|I|-1$ and $0 \le |K| \le k-|I|-|J|-1$. Moreover, by definition the extension of the reduced Lagrangian satisfies

$$\hat{l}(x^\mu, \sigma^\alpha_\mu(x^\mu); \partial_J\sigma^\alpha_\mu(x^\mu)) = \hat{l}\left(j^{k-1}_x\overline{\sigma}_A\right) = L\left(j^k_x s\right) = L(x^\mu, s^\alpha(x^\mu); \partial_J s^\alpha(x^\mu)),$$

where $\partial_J = \partial/\partial x^J$. By using this relation and equation (24), we obtain

$$\frac{\partial L}{\partial y^\alpha_{I+J+1_\mu}}\left(j^k s\right) = \sum_{|K|=0}^{k-|I|-|J|-1}\frac{(I+J+K)!}{K!(I+J)!}\frac{\partial \hat{l}}{\partial A^\beta_{\mu,I+J+K}}\left(j^{k-1}\sigma_A\right)\frac{\partial^{|K|}b^\beta_\alpha(s^\alpha)}{\partial x^K},$$

for each $1 \le \mu \le n$, $0 \le |I| \le k-1$ and $0 \le |J| \le k-|I|-1$. For brevity, we have denoted $A^\alpha_{\mu,J_0} = A^\alpha_\mu$ for $J_0 = (0,\ldots,0)$. By introducing this in (23) and by using Lemma 2.1, we obtain

$$\left(j^{2k-1}s\right)^*\mathcal{J}=\sum_{|I|=0}^{k-1}\sum_{|J|=0}^{k-|I|-1}\sum_{|K|=0}^{k-|I|-|J|-1}(-1)^{|J|}\frac{(I+J+K)!}{K!(I+J)!}\xi^\beta \frac{\partial^{|J|}}{\partial x^J}\left(\frac{\partial\hat{l}}{\partial A^\beta_{\mu,I+J+K}}\left(j^{k-1}\overline{\sigma}_A\right)\frac{\partial^{|K|}b^\beta_\alpha(s^\alpha)}{\partial x^K}\right)\frac{\partial^{|I|}a^\alpha_\beta(s^\alpha)}{\partial x^I}d^{n-1}x_\mu$$
$$=\sum_{|I|=0}^{k-1}\sum_{|J|=0}^{k-|I|-1}\sum_{|K|=0}^{k-|I|-|J|-1}\sum_{|L|=0}^{J}(-1)^{|J|}\frac{(I+J+K)!J!}{K!(I+J)!L!(J-L)!}\xi^\beta \frac{\partial^{|L|}}{\partial x^L}\left(\frac{\partial\hat{l}}{\partial A^\beta_{\mu,I+J+K}}\left(j^{k-1}\sigma_A\right)\right)\frac{\partial^{|J+K-L|}b^\beta_\alpha(s^\alpha)}{\partial x^{J+K-L}}\frac{\partial^{|I|}a^\alpha_\beta(s^\alpha)}{\partial x^I}d^{n-1}x_\mu. \tag{25}$$

In addition, we have $b^\beta_\alpha(s^\alpha)a^\alpha_\beta(s^\alpha)=m$. Hence, from Lemma 2.1 we get

$$\frac{\partial^{|M|}}{\partial x^M}\left(b^\beta_\alpha(s^\alpha)a^\alpha_\beta(s^\alpha)\right)=\sum_{M^{(1)}+M^{(2)}=M}\frac{M!}{M^{(1)}!M^{(2)}!}\frac{\partial^{|M^{(1)}|}b^\beta_\alpha(s^\alpha)}{\partial x^{M^{(1)}}}\frac{\partial^{|M^{(2)}|}a^\gamma_\beta(s^\alpha)}{\partial x^{M^{(2)}}}=0,$$

for $|M|>0$. By introducing this in (25), we arrive at

$$\left(j^{2k-1}s\right)^*\mathcal{J}=m\sum_{|J|=0}^{k-1}(-1)^{|J|}\xi^\alpha\frac{\partial^{|J|}}{\partial x^J}\left(\frac{\partial\hat{l}}{\partial A^\alpha_{\mu,J}}\left(j^{k-1}\overline{\sigma}_A\right)\right)d^{n-1}x_\mu.$$

Thence, the Noether conservation law, $\mathrm{d}\left(\left(j^{2k-1}s\right)^*\mathcal{J}\right)=0$, locally reads

$$\sum_{|J|=0}^{k-1}(-1)^{|J|}\frac{\partial^{|J+1_\mu|}}{\partial x^{J+1_\mu}}\left(\frac{\partial\hat{l}}{\partial A^\alpha_{\mu,J}}\left(j^{k-1}\sigma_A\right)\right)=0,\qquad 1\le\alpha\le m, \tag{26}$$

where we have used that $\xi=\xi^\alpha B_\alpha\in\mathfrak{g}$ is arbitrary.

Lastly, fix $x_0=(x_0^\mu)\in X$ and assume that our adapted coordinates are chosen so that A is flat at $y=s(x_0)$, i.e., $A_{s(x_0)}=(dy^\alpha)_{s(x_0)}\otimes B_\alpha$. It is now clear that (26) are exactly the local equations computed in Corollary 5.1 for $\overline{\sigma}_A=\hat{0}$, i.e., $\sigma^\alpha_\mu=0$ for $1\le\mu\le n$ and $1\le\alpha\le m$. $\square$

7. Multivariate *k*-splines on Lie groups

Variational splines are piecewise polynomial functions that allow for interpolating while minimizing some cost functional. For this reason, they have many applications in different areas such as optimal control theory, medical imaging or robotics (cf., for instance, [28]). Here we focus on higher-order splines, i.e., those in which the cost functional depends on higher-order derivatives. Namely, the reduction theory for 1-dimensional k-splines on Lie groups developed in [21, §3.2] (see also [30]) is extended to multivariate functions on Lie groups (cf. [14]).

Let G be a Lie group endowed with a right-invariant, (pseudo-)Riemannian metric, $\mathbf{g}$. The inner product induced on the Lie algebra, $\mathfrak{g}$, is denoted by the same symbol, $\mathbf{g}:\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$, and the induced norms on TG and $\mathfrak{g}$ are denoted by $\|\cdot\|_{\mathbf{g}}:TG\to\mathbb{R}$ and $\|\cdot\|_{\mathfrak{g}}:\mathfrak{g}\to\mathbb{R}$, respectively. Similarly, the adjoint of ad_ξ induced by $\mathbf{g}$ is denoted by $\mathrm{ad}^\dagger_\xi:\mathfrak{g}\to\mathfrak{g}$, i.e.,

$$\mathbf{g}(\mathrm{ad}_\xi(\eta),\zeta)=\mathbf{g}(\eta,\mathrm{ad}^\dagger_\xi(\zeta)),\qquad \xi,\eta,\zeta\in\mathfrak{g}.$$

On the other hand, let $X=\mathbb{R}^n$ (recall Remark 3.1) with the standard (global) coordinates and volume form given by $(x^\mu)=(x^1,\dots,x^n)$ and $d^nx=dx^1\wedge\dots\wedge dx^n$, respectively. The configuration bundle is given by

$$\pi_{P,X}:P=X\times G\to X,\quad (x,g)\mapsto x.$$

The bundle being trivial yields $J^kP=J^k(X,G)$, the family of k-jets of functions $\varsigma:X\to G$, whose elements are denoted by $j^k_x\varsigma$. Let $\nabla:\mathfrak{X}(G)\times\mathfrak{X}(G)\to\mathfrak{X}(G)$ be the Levi–Civita connection of $\mathbf{g}$. Its pull-back by a (local) function $\varsigma:X\to G$, which is a linear connection on $\pi_{\varsigma^*TG,X}:\varsigma^*TG\to X$, is denoted by $\varsigma^*\nabla$ and, given $1\le\mu\le n$, the corresponding covariant derivative is denoted by

$$\frac{\nabla}{\partial x^\mu}=\varsigma^*\nabla_{\partial_\mu}:\Gamma(\pi_{\varsigma^*TG,X})\to\Gamma(\pi_{\varsigma^*TG,X}).$$

Definition 7.1. The Lagrangian for *multivariate k-splines* on G is given by

$$L_s\left(j_x^k\varsigma\right)=\frac{1}{2}\kappa^\mu\left\|\left.\frac{\nabla^{k-1}}{\partial(x^\mu)^{k-1}}\right|_x(\partial_\mu\varsigma)\right\|_{\mathbf{g}}^2,\qquad j_x^k\varsigma\in J^k(X,G),\tag{27}$$

for certain $\kappa^\mu\in\mathbb{R}$, $0\leq\mu\leq n$.

Given a (local) function $\varsigma:X\to G$, we set

$$\sigma=dR_{\varsigma^{-1}}\circ d\varsigma=\left(dR_{\varsigma^{-1}}\circ\partial_\mu\varsigma\right)dx^\mu\in\Gamma(\pi_{T^*X\otimes\mathfrak{g},X})=\Omega^1(X,\mathfrak{g}),$$

where $R_g:G\to G$ denotes the right multiplication by $g\in G$. This induces the following isomorphism,

$$J^1(X,G)\simeq G\ltimes(T^*X\otimes\mathfrak{g}),\quad j_x^1\varsigma\mapsto(\varsigma(x),\sigma(x)),$$

where $\ltimes$ denotes the fibered semidirect product given by the adjoint representation (cf. [10, §4.3.1]). Hence, the isomorphism Ξ_k given in (11) becomes

$$J^k(X,G)/G\ni j_x^k\varsigma\mapsto j_x^{k-1}\sigma\in C_0^{k-1}(P).$$

Observe that for the canonical principal connection on the trivial bundle $\pi_{P,X}$, the isomorphism (20) reduces to the identity and, thus, $\overline{C_0^{k-1}(P)}=C_0^{k-1}(P)$.

Proposition 7.1. *The Lagrangian (27) is right invariant. Furthermore, a (natural) extended reduced Lagrangian is*

$$\hat{l}_s\left(j_x^{k-1}\sigma\right)=\frac{1}{2}\kappa^\mu\left\|\xi_\mu^{k-1}(x)\right\|_{\mathfrak{g}}^2,\qquad j_x^{k-1}\sigma\in J^{k-1}C(P),\tag{28}$$

where $\sigma=\sigma_\mu dx^\mu$ and $\xi_\mu^j:X\to\mathfrak{g}$ are defined recursively as follows,

$$\begin{cases}\xi_\mu^0=\sigma_\mu,\\ \xi_\mu^j=\partial_\mu\xi_\mu^{j-1}+\frac{1}{2}\left(\mathrm{ad}_{\sigma_\mu}^\dagger\left(\xi_\mu^{j-1}\right)+\mathrm{ad}_{\xi_\mu^{j-1}}^\dagger(\sigma_\mu)-\mathrm{ad}_{\sigma_\mu}\left(\xi_\mu^{j-1}\right)\right),\quad 1\leq j\leq k-1.\end{cases}$$

Proof. Given a (local) function $\varsigma:X\to G$, suppose that

$$\frac{\nabla^{k-1}}{\partial(x^\mu)^{k-1}}(\partial_\mu\varsigma)=dR_\varsigma\circ\xi_\mu^{k-1},\qquad 1\leq\mu\leq n.\tag{29}$$

In such case, the Lagrangian (27) may be written as

$$L_s\left(j_x^k\varsigma\right)=\frac{1}{2}\kappa^\mu\left\|\left(dR_{\varsigma(x)}\right)_e\left(\xi_\mu^{k-1}(x)\right)\right\|_{\mathbf{g}}^2=\frac{1}{2}\kappa^\mu\left\|\xi_\mu^{k-1}(x)\right\|_{\mathfrak{g}}^2,$$

where right invariance of $\mathbf{g}$ has been used. It is now clear that L is right invariant, since the maps ξ_μ^{k-1} do not depend on the value of ς, but only on its derivatives (regarded as elements of the Lie algebra). By recalling that $\sigma=dR_{\varsigma^{-1}}\circ d\varsigma$, i.e., $\sigma_\mu=dR_{\varsigma^{-1}}\circ\partial_\mu\varsigma$, the reduced Lagrangian reads

$$l_s\left(j_x^{k-1}\sigma\right)=L_s\left(j_x^k\varsigma\right)=\frac{1}{2}\kappa^\mu\left\|\xi_\mu^{k-1}(x)\right\|_{\mathfrak{g}}^2.$$

Although the reduced Lagrangian is computed for elements $j_x^{k-1}\sigma\in C_0^{k-1}(P)$, it can be extended straightforwardly to the whole jet, $\hat{l}_s:J^{k-1}C(P)\to\mathbb{R}$, by taking arbitrary functions $\sigma_\mu:X\to\mathfrak{g}$, $1\leq\mu\leq n$.

Lastly, let us check (29) by induction. For $k=1$ it is straightforward. Now let $k>1$. The formula for the Levi–Civita covariant derivative of the right-invariant metric $\mathbf{g}$ (cf. [21, Eq. (3.13)] or [27, §46.5]), together with the definition of pull-back connection, yields

$$\begin{aligned}\varsigma^*\nabla_{\partial_\mu}\varsigma^*U&=\varsigma^*(\nabla_{d\varsigma(\partial_\mu)}U)\\&=dR_\varsigma\circ\left(d\phi_U\circ\varsigma_\mu+\frac{1}{2}\mathrm{ad}_{\sigma_\mu}^\dagger(\phi_U)+\frac{1}{2}\mathrm{ad}_{\phi_U}^\dagger(\sigma_\mu)-\frac{1}{2}\mathrm{ad}_{\sigma_\mu}(\phi_U)\right)\circ\varsigma\end{aligned}$$

for each $U\in\mathfrak{X}(G)$, where $\phi_U\in C^\infty(G,\mathfrak{g})$ is given by $\phi_U(g)=(dR_{g^{-1}})_g(U(g))$, $g\in G$. By using this and the induction hypothesis, we finish:

$$
\begin{aligned}
\frac{\nabla^{k-1}}{\partial (x^\mu)^{k-1}}(\partial_\mu \varsigma) &= \varsigma^* \nabla_{\partial_\mu}\left(\frac{\nabla^{k-2}}{\partial (x^\mu)^{k-2}}(\partial_\mu \varsigma)\right) = \varsigma^* \nabla_{\partial_\mu}\left(dR_\varsigma \circ \xi_\mu^{k-2}\right) \\
&= dR_\varsigma \circ \left(\partial_\mu \xi_\mu^{k-2} + \frac{1}{2}\mathrm{ad}^\dagger_{\sigma_\mu}\left(\xi_\mu^{k-2}\right) + \frac{1}{2}\mathrm{ad}^\dagger_{\xi_\mu^{k-2}}(\sigma_\mu) - \frac{1}{2}\mathrm{ad}_{\sigma_\mu}\left(\xi_\mu^{k-2}\right)\right) \\
&= dR_\varsigma \circ \xi_\mu^{k-1}. \quad \square
\end{aligned}
$$

Let us compute explicitly the reduced equations for $k = 2$. This corresponds to the minimization of the accelerations, which in specific applications is directly related with the fuel expenditure of trajectories (cf. [25]). In such case, for each $\sigma = \sigma_\mu dx^\mu \in \Omega^1(X, \mathfrak{g})$, it is easy to check that $\xi_\mu^1 = \partial_\mu \sigma_\mu + \mathrm{ad}^\dagger_{\sigma_\mu}(\sigma_\mu)$, $1 \le \mu \le n$.

Theorem 7.1. *The Euler–Poincaré field equations for* (28) *with* $k = 2$*, i.e.,*

$$
\hat{l}_s\left(j_x^1\sigma\right) = \frac{1}{2}\kappa^\mu \left\|\partial_\mu \sigma_\mu + \mathrm{ad}^\dagger_{\sigma_\mu}(\sigma_\mu)\right\|_{\mathfrak{g}}^2, \qquad j_x^1\sigma \in J^1C(P),
$$

are given by

$$
\kappa^\mu\left(\partial_\mu - \mathrm{ad}^\dagger_{\sigma_\mu}\right)\left(\mathrm{ad}^\dagger_{\eta_\mu}(\sigma_\mu) + \mathrm{ad}_{\eta_\mu}(\sigma_\mu) + \partial_\mu \eta_\mu\right) = 0, \tag{30}
$$

where $\eta_\mu = \partial_\mu \sigma_\mu + \mathrm{ad}^\dagger_{\sigma_\mu}(\sigma_\mu)$ *for each* $1 \le \mu \le n$.

Proof. Let $\sigma = \sigma_\mu dx^\mu \in \Omega^1(X, \mathfrak{g})$. The musical isomorphisms defined by $\mathbf{g}$ are denoted by $\sharp : \mathfrak{g}^* \to \mathfrak{g}$ and $\flat : \mathfrak{g} \to \mathfrak{g}^*$. For each $1 \le \mu, \nu \le n$, it can be easily checked that

$$
\frac{\partial \hat{l}_s}{\partial (\partial_\nu \sigma_\mu)}\left(j^1\sigma\right) = \delta^{\mu\nu}\kappa^\mu \eta_\mu^\flat, \qquad \partial_\nu\left(\frac{\partial \hat{l}_s}{\partial (\partial_\nu \sigma_\mu)}\left(j^1\sigma\right)\right) = \kappa^\mu \partial_\mu \eta_\mu^\flat. \tag{31}
$$

As straightforward computation shows that $\mathrm{ad}^\dagger_\xi(\zeta) = \mathrm{ad}^*_\xi(\zeta^\flat)^\sharp$ for each $\xi, \zeta \in \mathfrak{g}$. Thus, for each $1 \le \mu \le n$ we have

$$
\begin{aligned}
\frac{\partial \hat{l}_s}{\partial \sigma_\mu}\left(j^1\sigma\right) &= \kappa^\mu \mathbf{g}\left(\eta_\mu, \mathrm{ad}^\dagger(\sigma_\mu) + \mathrm{ad}^\dagger_{\sigma_\mu}(\cdot)\right) \\
&= \kappa^\mu\left(\mathbf{g}\left(\mathrm{ad}(\eta_\mu), \sigma_\mu\right) + \mathbf{g}\left(\mathrm{ad}_{\sigma_\mu}(\eta_\mu), \cdot\right)\right) \\
&= \kappa^\mu\left(-\mathbf{g}\left(\mathrm{ad}_{\eta_\mu}(\cdot), \sigma_\mu\right) - \mathbf{g}\left(\mathrm{ad}_{\eta_\mu}(\sigma_\mu), \cdot\right)\right) \\
&= -\kappa^\mu \mathbf{g}\left(\mathrm{ad}^\dagger_{\eta_\mu}(\sigma_\mu) + \mathrm{ad}_{\eta_\mu}(\sigma_\mu), \cdot\right) \\
&= -\kappa^\mu\left(\mathrm{ad}^*_{\eta_\mu}(\sigma_\mu^\flat) + \mathrm{ad}_{\eta_\mu}(\sigma_\mu)^\flat\right).
\end{aligned} \tag{32}
$$

By gathering (31) and (32), the Euler–Lagrange form of $\hat{\mathfrak{l}}_s = \hat{l}_s\, d^n x : J^1C(P) \to \bigwedge^n T^*X$ is readily seen to be

$$
\begin{aligned}
\left(j^2\sigma\right)^* \mathcal{EL}\left(\hat{\mathfrak{l}}_s\right) &= \left(\frac{\partial \hat{l}_s}{\partial \sigma_\mu}\left(j^1\sigma\right) - \partial_\nu\left(\frac{\partial \hat{l}_s}{\partial (\partial_\nu \sigma_\mu)}\left(j^1\sigma\right)\right)\right) d^n x \otimes \partial_\mu \\
&= -\kappa^\mu\left(\mathrm{ad}^*_{\eta_\mu}(\sigma_\mu^\flat) + \mathrm{ad}_{\eta_\mu}(\sigma_\mu)^\flat + \partial_\mu \eta_\mu^\flat\right) d^n x \otimes \partial_\mu \in \Omega^n(X, TX \otimes \mathfrak{g}^*).
\end{aligned} \tag{33}
$$

On the other hand, since we have chosen A_0 as the canonical connection, we obtain the standard divergence, that is,

$$
\mathrm{div}^{A_0} : \Gamma(\pi_{TX\otimes\mathfrak{g}^*,X}) \to C^\infty(X, \mathfrak{g}), \quad \eta_\alpha^\mu\, \partial_\mu \otimes B^\alpha \mapsto (\partial_\mu \eta_\alpha^\mu) B^\alpha,
$$

where $\{B_\alpha \in \mathfrak{g} \mid 1 \le \alpha \le m\}$ is a basis of $\mathfrak{g}$ and $\{B^\alpha \in \mathfrak{g}^* \mid 1 \le \alpha \le m\}$ is its dual basis. In the same fashion, we have

$$
\mathrm{ad}^*_\sigma : \Gamma\left(\pi_{TX\otimes\mathfrak{g}^*,X}\right) \to C^\infty(X, \mathfrak{g}^*), \quad \eta_\alpha^\mu\, \partial_\mu \otimes B^\alpha \mapsto \eta_\alpha^\mu\, \mathrm{ad}^*_{\sigma_\mu}(B^\alpha).
$$

From this and (33), we obtain the Euler–Poincaré field equations for $\hat{l}_s$,

$$
\begin{aligned}
0 &= \kappa^\mu\left(\mathrm{div}^{A_0} - \mathrm{ad}^*_\sigma\right)\left(\mathrm{ad}^*_{\eta_\mu}(\sigma_\mu^\flat) + \mathrm{ad}_{\eta_\mu}(\sigma_\mu)^\flat + \partial_\mu \eta_\mu^\flat\right) d^n x \otimes \partial_\mu \\
&= \kappa^\mu\left(\partial_\mu - \mathrm{ad}^*_{\sigma_\mu}\right)\left(\mathrm{ad}^*_{\eta_\mu}(\sigma_\mu^\flat) + \mathrm{ad}_{\eta_\mu}(\sigma_\mu)^\flat + \partial_\mu \eta_\mu^\flat\right) d^n x.
\end{aligned}
$$

By applying $\sharp$, we conclude. $\square$

Corollary 7.1. *If the metric* $\mathbf{g}$ *is bi-invariant, then the Euler–Poincaré field equations for* 2*-splines* (30) *become*

$$\kappa^\mu \left(\partial_\mu^3 \sigma_\mu + \left[\sigma_\mu, \partial_\mu^2 \sigma_\mu\right]\right) = 0.$$

Proof. For a bi-invariant metric, we have $\mathrm{ad}_\xi^\dagger(\zeta) = -\mathrm{ad}_\xi(\zeta)$ for each $\xi, \eta \in \mathfrak{g}$. Therefore, $\eta_\mu = \partial_\mu \sigma_\mu$ for each $1 \leq \mu \leq n$ and we finish. □

Remark 7.1 *(Elastica).* The incorporation of elastic terms to the Lagrangian for multivariate k-splines is motivated by optimal control problems where the velocities (first order derivatives) have to be minimized together with their accelerations [37,5, 25]. In such case, the Lagrangian is given by

$$L_e\left(j_x^k \varsigma\right) = L_s\left(j_x^k \varsigma\right) + \frac{1}{2}\tau^\mu \left\|\partial_\mu \varsigma\right\|_{\mathbf{g}}^2, \qquad j_x^k \varsigma \in J^k(X, G), \tag{34}$$

for certain $\tau^\mu \in \mathbb{R}$, $1 \leq \mu \leq n$, with $L_s : J^k(X, G) \to \mathbb{R}$ as in (27). The reduced Lagrangian, as well as the reduced equations, is computed by adding the extra term in the above results. Namely, it is clear that

$$\hat{l}_e\left(j_x^{k-1}\sigma\right) = \hat{l}_s\left(j_x^{k-1}\sigma\right) + \frac{1}{2}\tau^\mu \left\|\sigma_\mu\right\|_{\mathfrak{g}}^2, \qquad j_x^{k-1}\sigma \in J^{k-1}C(P),$$

with $\hat{l}_s : J^{k-1}C(P) \to \mathbb{R}$ as in (28). The corresponding Euler–Poincaré field equations for $k = 2$ are given by

$$\kappa^\mu \left(\partial_\mu - \mathrm{ad}_{\sigma_\mu}^\dagger\right)\left(\mathrm{ad}_{\eta_\mu}^\dagger(\sigma_\mu) + \mathrm{ad}_{\eta_\mu}(\sigma_\mu) + \partial_\mu \eta_\mu\right) = \tau^\mu \left(\partial_\mu - \mathrm{ad}_{\sigma_\mu}^\dagger\right)\sigma_\mu.$$

Lastly, when the metric $\mathbf{g}$ is bi-invariant, the previous equations boil down to

$$\kappa^\mu \left(\partial_\mu^3 \sigma_\mu + \left[\sigma_\mu, \partial_\mu^2 \sigma_\mu\right]\right) = \tau^\mu \partial_\mu \sigma_\mu. \tag{35}$$

Funding

MCL and ARA have been partially supported by Ministerio de Ciencia e Innovación (Spain) under grants PID2021-126124NB-I00 and PID2024-156578NB-I00. ARA has been partially supported by Ministerio de Universidades (Spain) under grant FPU18/06036.

Data availability

No data was used for the research described in the article.

References

[1] V.I. Arnold, On the differential geometry of Lie groups of infinite dimension and its applications to the hydrodynamics of perfect fluids, Ann. Inst. Fourier 16 (1) (1966) 319–361.
[2] A. Bloch, L. Colombo, F. Jiménez, The variational discretization of the constrained higher-order Lagrange–Poincaré equations, Discrete Contin. Dyn. Syst. 39 (1) (2019) 309–344.
[3] A.I. Bobenko, Y.B. Suris, Discrete Lagrangian reduction, discrete Euler–Poincaré equations, and semidirect products, Lett. Math. Phys. 49 (1) (1999) 79–93.
[4] J. Brajerčík, D. Krupka, Variational principles for locally variational forms, J. Math. Phys. 46 (5) (2005) 052903, https://doi.org/10.1063/1.1901323.
[5] M. Camarinha, F. Silva Leite, P. Crouch, Splines of class C k on non-Euclidean spaces, IMA J. Math. Control Inf. (ISSN 0265-0754) 12 (4) (1995) 399–410, https://doi.org/10.1093/imamci/12.4.399.
[6] C.M. Campos, M. León, D. Martín de Diego, Constrained variational calculus for higher order classical field theories, J. Phys. A, Math. Theor. 43 (2010) 05, https://doi.org/10.1088/1751-8113/43/45/455206.
[7] M. Castrillón López, P.L. García Pérez, T.S. Ratiu, Euler–Poincaré reduction on principal bundles, Lett. Math. Phys. 58 (2001) 167, https://doi.org/10.1023/A:1013303320765.
[8] M. Castrillón López, J.E. Marsden, Covariant and dynamical reduction for principal bundle field theories, Ann. Glob. Anal. Geom. 34 (2008) 10, https://doi.org/10.1007/s10455-008-9108-x.
[9] M. Castrillón López, T.S. Ratiu, Reduction in principal bundles: covariant Lagrange–Poincaré equations, Commun. Math. Phys. 236 (2003) 01, https://doi.org/10.1007/s00220-003-0797-5.
[10] M. Castrillón López, Á. Rodríguez Abella, Principal bundles and connections modelled by Lie group bundles, Geom. Dedic. 217 (32) (2023) 1–26, https://doi.org/10.1007/s10711-023-00766-2.
[11] M. Castrillón López, P.L. García, C. Rodrigo, Euler–Poincaré reduction in principal bundles by a subgroup of the structure group, J. Geom. Phys. 74 (2013) 352, https://doi.org/10.1016/j.geomphys.2013.08.008.
[12] H. Cendra, J.E. Marsden, T.S. Ratiu, Lagrangian reduction by stages, Mem. Am. Math. Soc. 152 (722) (2001), https://doi.org/10.1090/memo/0722.
[13] H. Cendra, J.E. Marsden, T.S. Ratiu, Geometric Mechanics, Lagrangian Reduction, and Nonholonomic Systems, Springer Berlin Heidelberg, ISBN 978-3-642-56478-9, 2001, pp. 221–273.
[14] T.M. Chin, T.M. Özgökmen, A.J. Mariano, Multivariate-spline and scale-specific solution for variational analyses, J. Atmos. Ocean. Technol. 21 (2) (2004) 379–386, https://doi.org/10.1175/1520-0426(2004)021<0379:MASSFV>2.0.CO;2, https://journals.ametsoc.org/view/journals/atot/21/2/1520-0426_2004_021_0379_massfv_2_0_co_2.xml.

[15] M. de León, P.R. Rodrigues, A contribution to the global formulation of the higher-order Poincaré–Cartan form, Lett. Math. Phys. 14 (1987) 353–362, https://doi.org/10.1007/BF00402145.
[16] David C.P. Ellis, F. Gay-Balmaz, D.D. Holm, T.S. Ratiu, Lagrange–Poincaré field equations, J. Geom. Phys. (ISSN 0393-0440) 61 (11) (2011) 2120–2146, http://www.sciencedirect.com/science/article/pii/S0393044011001483.
[17] F. Etayo Gordejuela, J. Muñoz Masqué, Jets of principal bundles and jets of connections, in: Estudios en honor del profesor José Javier Etayo Miqueo, Contribuciones matemáticas, Editorial Complutense, ISBN 84-7491-510-4, 1994, pp. 41–59.
[18] M. Ferraris, M. Francaviglia, On the global structure of Lagrangian and Hamiltonian formalisms in higher order calculus of variations, in: Proceedings of the IUTAM-ISIMM Symposium on Modern Developments in Analytical Mechanics I, 1983, pp. 43–70.
[19] F. Gay-Balmaz, T. Ratiu, A new Lagrangian dynamic reduction in field theory, Ann. Inst. Fourier 3 (2014) 07, https://doi.org/10.5802/aif.2549.
[20] F. Gay-Balmaz, D.D. Holm, T.S. Ratiu, Higher order Lagrange–Poincaré and Hamilton–Poincaré reductions, Bull. Braz. Math. Soc., New Ser. 42 (2011) 579–606, https://doi.org/10.1007/s00574-011-0030-7.
[21] F. Gay-Balmaz, D.D. Holm, D.M. Meier, T.S. Ratiu, F.-X. Vialard, Invariant higher-order variational problems, Commun. Math. Phys. 309 (2012) 413–458, https://doi.org/10.1007/s00220-011-1313-y.
[22] F. Gay-Balmaz, D.D. Holm, D.M. Meier, T.S. Ratiu, F.-X. Vialard, Invariant higher-order variational problems II, J. Nonlinear Sci. 22 (2012) 553–597, https://doi.org/10.1007/s00332-012-9137-2.
[23] M.J. Gotay, J. Isenberg, J.E. Marsden, R. Montgomery, Momentum maps and classical fields, arXiv:physics/9801019v2 [math-ph], 1997.
[24] M. Hardy, Combinatorics of partial derivatives, Electron. J. Comb. 13 (2006) R1, https://doi.org/10.37236/1027.
[25] I.I. Hussein, A.M. Bloch, Optimal control on Riemannian manifolds with potential fields, in: 2004 43rd IEEE Conference on Decision and Control (CDC) (IEEE Cat. No. 04CH37601), vol. 2, 2004, pp. 1812–1817.
[26] J. Janyška, Higher order valued reduction theorems for general linear connections, Note Mat. 23 (2004) 75–97.
[27] A. Kriegl, P.W. Michor, The Convenient Setting of Global Analysis, Mathematical Surveys., American Mathematical Society, ISBN 9780821807804, 1997.
[28] M.-J. Lai, Multivariate Splines and Their Applications, Springer, New York, New York, NY, ISBN 978-0-387-30440-3, 2009, pp. 5800–5841.
[29] J. Lee, Introduction to Smooth Manifolds, 2nd edition, Springer-Verlag, New York, ISBN 978-1-4419-9981-8, 2012.
[30] L. Machado, F. Silva Leite, K. Krakowski, Higher-order smoothing splines versus least squares problems on Riemannian manifolds, J. Dyn. Control Syst. 16 (2010) 121–148, https://doi.org/10.1007/s10883-010-9080-1.
[31] J.E. Marsden, J. Scheurle, The reduced Euler–Lagrange equations, Fields Inst. Commun. 1 (1993) 07, https://doi.org/10.1090/fic/001/07.
[32] J.E. Marsden, A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys. 5 (1) (1974) 121–130.
[33] J.E. Marsden, S. Pekarsky, S. Shkoller, Discrete Euler–Poincaré and Lie–Poisson equations, Nonlinearity 12 (6) (1999) 1647–1662.
[34] J.E. Marsden, S. Pekarsky, S. Shkoller, Symmetry reduction of discrete Lagrangian mechanics on Lie groups, J. Geom. Phys. 36 (1–2) (2000) 140–151.
[35] J. Muñoz Masqué, Poincaré–Cartan forms in higher order variational calculus on fibred manifolds, Rev. Mat. Iberoam. 1 (1985) 85–126.
[36] K.R. Meyer, Symmetries and integrals in mechanics, in: M.M. Peixoto (Ed.), Dynamical Systems, Academic Press, 1973, pp. 259–272.
[37] L. Noakes, G. Heinzinger, B. Paden, Cubic splines on curved spaces, IMA J. Math. Control Inf. (ISSN 0265-0754) 6 (4) (1989) 465–473, https://doi.org/10.1093/imamci/6.4.465.
[38] Á. Rodríguez Abella, M. Leok, Discrete Dirac reduction of implicit Lagrangian systems with abelian symmetry groups, J. Geom. Mech. (ISSN 1941-4889) 15 (1) (2023) 319–356, https://doi.org/10.3934/jgm.2023013, https://www.aimspress.com/article/doi/10.3934/jgm.2023013.
[39] G. Sardanashvily, Advanced Differential Geometry for Theoreticians, Lap Lambert Academic Publishing GmbH KG, ISBN 9783659378157, 2013.
[40] D.J. Saunders, The Geometry of Jet Bundles, Cambridge University Press, 1989.
[41] S. Smale, Topology and mechanics. I, Invent. Math. 10 (4) (1970) 305–331.
[42] J. Vankerschaver, Euler–Poincaré reduction for discrete field theories, J. Math. Phys. 48 (3) (2007) 032902.
[43] V.S. Varadarajan, Lie Groups, Lie Algebras, and Their Representations, 1 edition, Graduate Texts in Mathematics, Springer, New York, NY, ISBN 978-1-4612-1126-6, 1984.