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REMARKS ON HIGGS BUNDLES TWISTED BY A VECTOR BUNDLE

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Abstract. For any V–twisted Higgs bundle on a compact Riemann surface X, where V is a holomorphic vector bundle of rank two on X, there are two associated Higgs bundles on X, twisted by line bundles, which are constructed using a Hecke transformation on V. We characterize all such pairs of Higgs bundles (twisted by line bundles) given by V–twisted Higgs bundles. Using this characterization, we provide a spectral correspondence for the moduli space, identifying V–twisted Higgs bundles with the direct images of certain rank one torsionfree Higgs sheaves twisted by a line bundle on a spectral covering of the curve X.

1. Introduction

This work was inspired by [GGN]. In [GGN] the notion of V–twisted Higgs bundles on a compact Riemann surface X, where V is a holomorphic vector bundle on X of rank two, was introduced. Their moduli was constructed in [Si] and [GGN] and some properties of the moduli space were investigated there.

Our aim here is to point out that using a Hecke transformation on V, there is a natural bijective correspondence between the V–twisted Higgs bundles on X and a certain class of pairs of $\mathcal{S}$–twisted Higgs bundles and $\mathcal{L}$–twisted Higgs bundles on X. Here $\mathcal{S}$ and $\mathcal{L}$ are fixed holomorphic line bundles on X such that $\mathcal{S} \oplus \mathcal{L}$ is obtained by a forward Hecke transformation of V. (See Theorem 3.4 and Corollary 3.5.)

From this correspondence an analogue of the Beauville-Narasimhan-Ramanan spectral correspondence for V–twisted Higgs bundles is obtained (see Theorem 4.1). Given a V–twisted Higgs bundle consider the corresponding $\mathcal{S}$–twisted Higgs bundle. This $\mathcal{S}$–twisted Higgs field gives a spectral curve $\mathcal{S} \supset Y \xrightarrow{\phi} X$. The class of V–twisted Higgs bundles such that the spectral curve (for the $\mathcal{S}$–twisted Higgs field) is integral, is identified with the direct image of a $\varphi^*\mathcal{L}$–twisted torsionfree Higgs-sheaf of rank 1 on the corresponding spectral curve.

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2. Decomposition of Higgs fields

Let X be a compact connected Riemann surface. Fix a holomorphic vector bundle V on X of rank two. We will start by showing that we can always construct two holomorphic line bundles $\mathcal{S}$ and $\mathcal{L}$ on X and a torsion sheaf $\mathbb{T}$ supported on a finite set of distinct points of X such that V fits in a short exact sequence, of $\mathcal{O}_X$–coherent sheaves, of the form

$$0 \longrightarrow V \longrightarrow \mathcal{S} \oplus \mathcal{L} \longrightarrow \mathbb{T} \longrightarrow 0; \tag{2.1}$$

in other words, V is obtained from $\mathcal{S} \oplus \mathcal{L}$ by performing a Hecke transformation. To do so, first choose a holomorphic line subbundle

$$S \subset V^*. \tag{2.2}$$

To construct such a line subbundle explicitly, note that the Riemann–Roch theorem says that the vector bundle $V^* \otimes L'$ has a nonzero holomorphic section for any holomorphic line bundle L' with $2 \cdot \text{degree}(L') \geq 2 \cdot \text{genus}(X) + \text{degree}(V) - 1$. Any nonzero section s of $V^* \otimes L'$ generates a line subbundle L^s of $V^* \otimes L'$; in other words, L^s is the inverse image, in $V^* \otimes L'$, of the torsion part of $(V^* \otimes L')/\text{image}(s)$ under the quotient map $V^* \otimes L' \longrightarrow (V^* \otimes L')/\text{image}(s)$. So the line bundle $S = L^s \otimes (L')^*$ is a line subbundle of $V^* \otimes L' \otimes (L')^* = V^*$. Consider the short exact sequence

$$0 \longrightarrow S \longrightarrow V^* \xrightarrow{q} V^*/S \longrightarrow 0, \tag{2.3}$$

where S is the line subbundle in (2.2). Take a reduced effective divisor $\mathbb{D} = \sum_{i=1}^{d} x_i$, where $x_1, \cdots, x_d$ are distinct points of X. Thus, $(V^*/S) \otimes \mathcal{O}_X(-\mathbb{D})$ is a subsheaf of V^*/S. From (2.3) we have the short exact sequence

$$0 \longrightarrow S \longrightarrow W := q^{-1}((V^*/S) \otimes \mathcal{O}_X(-\mathbb{D})) \xrightarrow{q'} (V^*/S) \otimes \mathcal{O}_X(-\mathbb{D}) \longrightarrow 0, \tag{2.4}$$

where q is the quotient map in (2.3) and q' in (2.4) is the restriction of q. If we choose $\mathbb{D}$ such that

$$d > \text{degree}(V^*/S) - \text{degree}(S) + 2(\text{genus}(X) - 1), \tag{2.5}$$

then $\text{degree}(S \otimes ((V^*/S) \otimes \mathcal{O}_X(-\mathbb{D}))^*) > 2(\text{genus}(X) - 1)$, so $H^1(X, S \otimes ((V^*/S) \otimes \mathcal{O}_X(-\mathbb{D}))^*) = 0$; indeed, for any holomorphic line bundle L' on X with $\text{degree}(L') > 2(\text{genus}(X) - 1)$, by Serre duality $H^1(X, L') = H^0(X, (L')^* \otimes K_X)^* = 0$. Hence (2.4) splits holomorphically if (2.5) holds. Assume that $\mathbb{D}$ is chosen so that (2.5) holds. Fix a holomorphic splitting of (2.4). This implies that we have a decomposition

$$W := q^{-1}((V^*/S) \otimes \mathcal{O}_X(-\mathbb{D})) = S \oplus ((V^*/S) \otimes \mathcal{O}_X(-\mathbb{D})). \tag{2.6}$$

Let $L \subset V^*$ be the line subbundle generated by the subsheaf $(V^*/S) \otimes \mathcal{O}_X(-\mathbb{D}) \subset W \subset V^*$ (see (2.6)). In other words, L is the inverse image of the torsion part of $V^*/((V^*/S) \otimes \mathcal{O}_X(-\mathbb{D}))$ under the quotient map $V^* \longrightarrow V^*/((V^*/S) \otimes \mathcal{O}_X(-\mathbb{D}))$. Consequently, we have a short exact sequence

$$0 \longrightarrow S \oplus L \xrightarrow{\Phi^*} V^* \longrightarrow \mathbb{T}' \longrightarrow 0, \tag{2.7}$$

where $\mathbb{T}'$ is a torsion sheaf whose support is contained in the reduced divisor $\mathbb{D}$, and Φ^* is the natural inclusion map (the notation Φ^* is being used because its adjoint will be used more often). Let D denote the support of $\mathbb{T}'$. For each point $y \in D$, the dimension of $\mathbb{T}'_y = \mathbb{T}'/(\mathbb{T}' \otimes \mathcal{O}_X(-y))$ is 1. Taking the dual, from (2.7) we have a short exact sequence

$$0 \longrightarrow V \xrightarrow{\Phi} S^* \oplus L^* \xrightarrow{\xi} \mathbb{T} \longrightarrow 0, \tag{2.8}$$

where $\Phi := (\Phi^*)^*$ is the dual of Φ^*; so $\mathbb{T}$ is a torsion sheaf whose support coincides with the support of $\mathbb{T}'$. In fact, $\mathbb{T}$ and $\mathbb{T}'$ are isomorphic, but there is no natural isomorphism between them.

Taking $\mathcal{S}$ and $\mathcal{L}$ as S^* and L^* respectively, (2.8) becomes the sought exact sequence

$$0 \longrightarrow V \xrightarrow{\Phi} \mathcal{S} \oplus \mathcal{L} \xrightarrow{\xi} \mathbb{T} \longrightarrow 0. \tag{2.9}$$

From the construction of (2.8) it follows immediately that for any point x in the support of the torsion sheaf $\mathbb{T}$, the image of the homomorphism

$$\Phi_x : V_x \longrightarrow (\mathcal{S} \oplus \mathcal{L})_x = \mathcal{S}_x \oplus \mathcal{L}_x$$

has dimension 1 and the restrictions of the map ξ to the fibers $\mathcal{L}_x$ and $\mathcal{S}_x$ are both injective. Thus, the following composition of homomorphisms is an isomorphism:

$$\Phi_x(V_x) \hookrightarrow \mathcal{S}_x \oplus \mathcal{L}_x \longrightarrow \mathcal{S}_x, \tag{2.10}$$

where $\mathcal{S}_x \oplus \mathcal{L}_x \longrightarrow \mathcal{S}_x$ is the natural projection. Denote the isomorphism in (2.10) by $\rho_1^x : \Phi_x(V_x) \longrightarrow \mathcal{S}_x$. Analogously, let $\rho_2^x : \Phi_x(V_x) \longrightarrow \mathcal{L}_x$ denote the composition of homomorphisms

$$\Phi_x(V_x) \hookrightarrow \mathcal{S}_x \oplus \mathcal{L}_x \longrightarrow \mathcal{L}_x, \tag{2.11}$$

where $\mathcal{S}_x \oplus \mathcal{L}_x \longrightarrow \mathcal{L}_x$ is the other natural projection. Then, we have a natural morphism

$$\rho^x := \rho_2^x \circ (\rho_1^x)^{-1} : \mathcal{S}_x \longrightarrow \mathcal{L}_x. \tag{2.12}$$

Let us also consider the homomorphisms

$$\widehat{\rho}_1 := p_1 \circ \Phi : V \longrightarrow \mathcal{S} \quad \text{and} \quad \widehat{\rho}_2 := p_2 \circ \Phi : V \longrightarrow \mathcal{L}, \tag{2.13}$$

where p_1 and p_2 are the projections of $\mathcal{S} \oplus \mathcal{L}$ to $\mathcal{S}$ and $\mathcal{L}$ respectively, and Φ is the homomorphism in (2.9). Note that $(\widehat{\rho}_1)_x = \rho_1^x \circ \Phi_x$ and $(\widehat{\rho}_2)_x = \rho_2^x \circ \Phi_x$, where ρ_1^x and ρ_2^x are the compositions in (2.10) and (2.11) respectively.

Take a holomorphic vector bundle E on X. A *V–twisted Higgs field* on E is a holomorphic section

$$\theta \in H^0(X, \operatorname{End}(E) \otimes V)$$

such that $\theta \bigwedge \theta = 0$ (see [GGN, Definition 2.1]); note that $\theta \bigwedge \theta \in H^0(X, \operatorname{End}(E) \otimes \bigwedge^2 V)$. Using the homomorphisms in (2.13), a V–twisted Higgs field θ on E gives sections

$$\theta_1 := (\mathrm{Id}_E \otimes \widehat{\rho}_1)(\theta) \in H^0(X, \operatorname{End}(E) \otimes \mathcal{S}), \quad \theta_2 := (\mathrm{Id}_E \otimes \widehat{\rho}_2)(\theta) \in H^0(X, \operatorname{End}(E) \otimes \mathcal{L}), \tag{2.14}$$

where $\widehat{\rho}_1$ and $\widehat{\rho}_2$ are constructed in (2.13); here θ is considered as a homomorphism from E to $E \otimes V$. The given condition that $\theta \bigwedge \theta = 0$ is equivalent to the following condition:

$$(\theta_2 \otimes \mathrm{Id}_{\mathcal{S}}) \circ \theta_1 = (\theta_1 \otimes \mathrm{Id}_{\mathcal{L}}) \circ \theta_2 \tag{2.15}$$

as homomorphisms from E to $E \otimes \mathcal{S} \otimes \mathcal{L}$; here θ_1 and θ_2 are considered as homomorphisms from E to $E \otimes \mathcal{S}$ and $E \otimes \mathcal{L}$ respectively (also $E \otimes \mathcal{S} \otimes \mathcal{L}$ is identified with $E \otimes \mathcal{L} \otimes \mathcal{S}$).

Take another V–twisted Higgs field θ' on E. Let θ'_1 (respectively, θ'_2) be the $\mathcal{S}$–twisted (respectively, $\mathcal{L}$–twisted) Higgs field on E given by θ'. If $\theta \neq \theta'$, then either $\theta_1 \neq \theta'_1$ or $\theta_2 \neq \theta'_2$ or both hold. In other words, the above map $\theta \longmapsto (\theta_1,\, \theta_2)$ is injective.

We will characterize all pairs $(\theta_1,\, \theta_2)$, where θ_1 is a $\mathcal{S}$–twisted Higgs field on E and θ_2 is a $\mathcal{L}$–twisted Higgs field on E, that arise from V–twisted Higgs fields on E.

3. Twisted Higgs bundles and spectral data

3.1. **Decomposition of a V–twisted Higgs field.** Let

$$D \,=\, \sum_{i=1}^{\ell} x_i \tag{3.1}$$

be the support of the torsion sheaf $\mathbb{T}$ in (2.9); recall that $D \,\leq\, \mathbb{D}$ (see (2.4) for $\mathbb{D}$).

Take a holomorphic vector bundle E on X. Let $\theta \,\in\, H^0(X,\, \mathrm{End}(E)\otimes V)$ be a V–twisted Higgs field on E. It gives the $\mathcal{S}$–twisted Higgs bundle $(E,\, \theta_1)$ and the $\mathcal{L}$–twisted Higgs bundle $(E,\, \theta_2)$ (see (2.14)). Corresponding to $(E,\, \theta_1)$ we have a spectral curve

$$Y \,\subset\, \mathcal{S} \tag{3.2}$$

and a torsionfree coherent sheaf

$$F \,\longrightarrow\, Y \tag{3.3}$$

of rank one [BNR], [Hi]. Let

$$\varphi \,:\, Y \,\longrightarrow\, X \tag{3.4}$$

be the restriction of the natural projection $\mathcal{S} \,\longrightarrow\, X$.

We recall that Y parametrizes the generalized eigenvalues of θ_1, and F is given by the generalized eigenspaces for θ_1 [BNR], [Hi]. In particular, we have

$$F \,\subset\, \varphi^*E, \tag{3.5}$$

and $\varphi^*\theta_1$ sends the subsheaf F in (3.5) to $F\otimes(\varphi^*\mathcal{S}) \,\subset\, (\varphi^*E)\otimes(\varphi^*\mathcal{S})$.

Lemma 3.1.

(1) *The homomorphism* $\varphi^*\theta_2 \,:\, \varphi^*E \,\longrightarrow\, \varphi^*(E\otimes\mathcal{L}) \,=\, (\varphi^*E)\otimes(\varphi^*\mathcal{L})$, *where* θ_2 *is the* $\mathcal{L}$*–twisted Higgs field on* E *in* (2.14), *sends the subsheaf* F *in* (3.5) *to* $F\otimes(\varphi^*\mathcal{L}) \,\subset\, (\varphi^*E)\otimes(\varphi^*\mathcal{L})$.

(2) *The two homomorphisms* $(\varphi^*\theta_1)\big|_F \,:\, F \,\longrightarrow\, F\otimes(\varphi^*\mathcal{S})$ *and* $(\varphi^*\theta_2)\big|_F \,:\, F \,\longrightarrow\, F\otimes(\varphi^*\mathcal{L})$ *commute. In other words,*

$$(\varphi^*(\theta_2\otimes \mathrm{Id}_{\varphi^*\mathcal{S}})\big|_{F\otimes\varphi^*\mathcal{S}})\circ(\varphi^*\theta_1)\big|_F \,=\, (\varphi^*(\theta_1\otimes \mathrm{Id}_{\varphi^*\mathcal{L}})\big|_{F\otimes\varphi^*\mathcal{L}})\circ(\varphi^*\theta_2)\big|_F$$

as homomorphisms from F *to* $F\otimes\varphi^*(\mathcal{S}\otimes\mathcal{L})$.

Proof. In view of (2.15), the first statement follows immediately from the fact that if A and B are two $r\times r$ matrices with complex entries with $AB \,=\, BA$, and λ is a generalized eigenvalue of A, then B preserves the generalized eigenspace for the generalized eigenvalue λ of A.

The second statement also follows from (2.15). ☐

Take any point x in D, which is the support of $\mathbb{T}$ (see (3.1)). Next take any connected component $y \in \varphi^{-1}(x)$ of the fiber, over x, of the map φ in (3.4). Note that the reduced subscheme $y_{\rm red}$ is a single point, but y need not be reduced. From Lemma 3.1(1) we know that the homomorphism

$$(\varphi^*\theta_2)_y \ : \ E_x \,=\, (\varphi^*E)_y \,\longrightarrow\, \varphi^*(E\otimes\mathcal{L})_y \,=\, E_x\otimes\mathcal{L}_x$$

takes F_y in (3.5) to $F_y\otimes(\varphi^*\mathcal{L})_y \,=\, F_y\otimes\mathcal{L}_x$. In other words, $(\theta_2)(x)$ produces a homomorphism

$$\theta_{2,y} \ : \ F_y \,\longrightarrow\, F_y\otimes\mathcal{L}_x. \tag{3.6}$$

Proposition 3.2. *The homomorphism* $\theta_{2,y}$ *in* (3.6) *has exactly one generalized eigenvalue. This unique eigenvalue of* $\theta_{2,y}$ *is* $-\rho^x(y_{\rm red})$, *where* $\rho^x\,:\,\mathcal{S}_x\longrightarrow\mathcal{L}_x$ *is the homomorphism in* (2.12). *(Note that* $y_{\rm red}$ *is a point of the fiber of* $\mathcal{S}$ *over* x *and it lies on the spectral curve* Y*.)*

Proof. Tensoring the exact sequence in (2.9) with $\text{End}(E)$ we obtain the short exact sequence

$$0\,\longrightarrow\,\text{End}(E)\otimes V\,\xrightarrow{\ \text{Id}_{\text{End}(E)}\otimes\Phi\ }\,\text{End}(E)\otimes(\mathcal{S}\oplus\mathcal{L})\,\xrightarrow{\ \Psi\ }\,\text{End}(E)\otimes\mathbb{T}\,\longrightarrow\,0.$$

This gives an exact sequence

$$0\,\longrightarrow\,H^0(X,\,\text{End}(E)\otimes V)\,\xrightarrow{(\text{Id}_{\text{End}(E)}\otimes\Phi)_*}\,H^0(X,\,\text{End}(E)\otimes(\mathcal{S}\oplus\mathcal{L}))$$
$$\xrightarrow{\ \Psi_*\ }\,H^0(X,\,\text{End}(E)\otimes\mathbb{T}), \tag{3.7}$$

where the homomorphisms $(\text{Id}_{\text{End}(E)}\otimes\Phi)_*$ and Ψ_* are induced by $\text{Id}_{\text{End}(E)}\otimes\Phi$ and Ψ respectively. Consider $\theta_1\oplus\theta_2\,\in\,H^0(X,\,\text{End}(E)\otimes(\mathcal{S}\oplus\mathcal{L}))$ (see (2.14)). Since

$$\theta_1\oplus\theta_2\,=\,(\text{Id}_{\text{End}(E)}\otimes\Phi)_*(\theta),$$

it follows from (3.7) that

$$\Psi_*(\theta_1\oplus\theta_2)\,=\,0. \tag{3.8}$$

For any $x\,\in\,D$, we have $\mathbb{T}_x\,=\,(\mathcal{S}_x\oplus\mathcal{L}_x)/\Phi(V_x)$, where Φ is the homomorphism in (2.9). Let

$$\xi_{1,x}\,:\,\mathcal{S}_x\,\longrightarrow\,\mathbb{T}_x\quad\text{ and }\quad\xi_{2,x}\,:\,\mathcal{L}_x\,\longrightarrow\,\mathbb{T}_x \tag{3.9}$$

be the homomorphisms of fibers over x given by ξ in (2.9). Denote the following compositions of homomorphisms

$$V_x\,\xrightarrow{\ \Phi_x\ }\,\mathcal{S}_x\oplus\mathcal{L}_x\,\longrightarrow\,\mathcal{S}_x\,\xrightarrow{\ \xi_{1,x}\ }\,\mathbb{T}_x,\ \text{ and}$$
$$V_x\,\xrightarrow{\Phi_x}\,\mathcal{S}_x\oplus\mathcal{L}_x\,\longrightarrow\,\mathcal{L}_x\,\xrightarrow{\xi_{2,x}}\,\mathbb{T}_x$$

by α^x_1 and α^x_2 respectively. We have

$$\text{Id}_{\text{End}(E)}\otimes\alpha^x_1\,=\,\Psi_*(\theta_1)(x)\quad\text{ and }\quad\text{Id}_{\text{End}(E)}\otimes\alpha^x_2\,=\,\Psi_*(\theta_2)(x) \tag{3.10}$$

as elements of $\text{End}(E_x)\otimes\mathbb{T}_x$, where Ψ_* is the homomorphism in (3.7). Consequently, the proposition follows from (3.8) and the construction (done in (2.12)) of the homomorphism ρ^x. ☐

As before, y is a connected component of the fiber, over $x \in D$, of the map φ in (3.4). Let

$$\theta_{1,y} \,:\, F_y \longrightarrow F_y \otimes (\varphi^*\mathcal{S})_y \,=\, F_y \otimes \mathcal{S}_x$$

be the homomorphism given by $\theta_1(x)$. From (3.10) we have

$$(\mathrm{Id}_{F_y} \otimes \xi_{1,x}) \circ \theta_{1,y} \,:\, F_y \longrightarrow F_y \otimes \mathbb{T}_x \quad \text{and} \quad (\mathrm{Id}_{F_y} \otimes \xi_{2,x}) \circ \theta_{2,y} \,:\, F_y \longrightarrow F_y \otimes \mathbb{T}_x, \tag{3.11}$$

where $\xi_{1,x}$ and $\xi_{2,x}$ are the homomorphisms in (3.9), and $\theta_{2,y}$ is defined in (3.6).

The following is an immediate consequence of (3.8).

Corollary 3.3. *The two homomorphisms* $(\mathrm{Id}_{F_y} \otimes \xi_{1,x}) \circ \theta_{1,y}$ *and* $(\mathrm{Id}_{F_y} \otimes \xi_{2,x}) \circ \theta_{2,y}$ *in* (3.11) *from* F_y *to* $F_y \otimes \mathbb{T}_x$ *satisfy the following equation:*

$$(\mathrm{Id}_{F_y} \otimes \xi_{1,x}) \circ \theta_{1,y} + (\mathrm{Id}_{F_y} \otimes \xi_{2,x}) \circ \theta_{2,y} \,=\, 0.$$

Note that Proposition 3.2 can be deduced from Corollary 3.3.

3.2. Reconstructing V–twisted Higgs fields.

Take a holomorphic vector bundle E on X together with a $\mathcal{S}$–twisted Higgs field

$$\Theta \,\in\, H^0(X,\, \mathrm{End}(E) \otimes \mathcal{S}). \tag{3.12}$$

Let $Y \,\subset\, \mathcal{S}$ be the spectral curve and

$$F \longrightarrow Y \tag{3.13}$$

the sheaf on Y corresponding to $(E,\, \Theta)$. Let

$$\varphi \,:\, Y \longrightarrow X \tag{3.14}$$

be the restriction of the natural projection $\mathcal{S} \longrightarrow X$. So we have $F \,\subset\, \varphi^*E$. For any $x \,\in\, D$ (see (3.1)), and any connected component y of $\varphi^{-1}(x)$, we have the homomorphism

$$(\mathrm{Id}_{F_y} \otimes \xi_{1,x}) \circ \Theta_y \,:\, F_y \longrightarrow F_y \otimes \mathbb{T}_x, \tag{3.15}$$

where $\xi_{1,x}$ is the homomorphism in (3.9).

Let

$$\Theta' \,\in\, H^0(X,\, \mathrm{End}(E) \otimes \mathcal{L}) \tag{3.16}$$

be a $\mathcal{L}$–twisted Higgs field on E. As before, take any point $x \,\in\, D$ and a connected component y of $\varphi^{-1}(x)$. If the homomorphism

$$(\varphi^*\Theta')_y \,:\, E_x \,=\, (\varphi^*E)_y \longrightarrow \varphi^*(E \otimes \mathcal{L})_y \,=\, E_x \otimes \mathcal{L}_x$$

takes $F_y \,\subset\, E_x$ in (3.13) to $F_y \otimes (\varphi^*\mathcal{L})_y \,=\, F_y \otimes \mathcal{L}_x$, then $\Theta'(x)$ produces a homomorphism

$$\Theta'_y \,:\, F_y \longrightarrow F_y \otimes \mathcal{L}_x,$$

in which case we have the homomorphism

$$(\mathrm{Id}_{F_y} \otimes \xi_{2,x}) \circ \Theta'_y \,:\, F_y \longrightarrow F_y \otimes \mathbb{T}_x, \tag{3.17}$$

where $\xi_{2,x}$ is the homomorphism in (3.9).

The following theorem shows that Lemma 3.1 and Corollary 3.3 together characterize the V–twisted Higgs fields on E in terms of the pair consisting of a $\mathcal{S}$–twisted Higgs field and a $\mathcal{L}$–twisted Higgs field on E.

Theorem 3.4. *Take*

$$\Theta \in H^0(X,\, \mathrm{End}(E)\otimes \mathcal{S}) \quad \textit{and} \quad \Theta' \in H^0(X,\, \mathrm{End}(E)\otimes \mathcal{L})$$

satisfying the following three conditions:

1. *The homomorphism* $\varphi^*\Theta' : \varphi^*E \longrightarrow (\varphi^*E)\otimes\varphi^*\mathcal{L}$ *sends the subsheaf* F *in* (3.13) *to* $F\otimes\varphi^*\mathcal{L} \subset (\varphi^*E)\otimes\varphi^*\mathcal{L}$.
2. *The two homomorphisms* $(\varphi^*\Theta)\big|_F : F \longrightarrow F\otimes\varphi^*\mathcal{S}$ *and* $(\varphi^*\Theta')\big|_F : F \longrightarrow F\otimes\varphi^*\mathcal{L}$ *commute. In other words,*
$$(\varphi^*(\Theta'\otimes \mathrm{Id}_{\varphi^*\mathcal{S}})\big|_{F\otimes\varphi^*\mathcal{S}})\circ(\varphi^*\Theta)\big|_F \;=\; (\varphi^*(\Theta\otimes \mathrm{Id}_{\varphi^*\mathcal{L}})\big|_{F\otimes\varphi^*\mathcal{L}})\circ(\varphi^*\Theta')\big|_F$$
as homomorphisms from F *to* $F\otimes\varphi^*(\mathcal{S}\otimes\mathcal{L})$.
3. *For all* x *and* y *as above, the homomorphisms* $(\mathrm{Id}_{F_y}\otimes\xi_{1,x})\circ\Theta_y$ *and* $(\mathrm{Id}_{F_y}\otimes\xi_{2,x})\circ\Theta'_y$ *(see* (3.15) *and* (3.17)*) satisfy the equation*
$$(\mathrm{Id}_{F_y}\otimes\xi_{1,x})\circ\Theta_y + (\mathrm{Id}_{F_y}\otimes\xi_{2,x})\circ\Theta'_y \;=\; 0.$$

Then there is a unique V*–twisted Higgs field* θ *on* E *such that* Θ *and* Θ' *are given by* θ *(as in* (2.14)*).*

Proof. The first two conditions imply that

$$(\Theta'\otimes \mathrm{Id}_{\mathcal{S}})\circ\Theta \;=\; (\Theta\otimes\mathrm{Id}_{\mathcal{L}})\circ\Theta' \tag{3.18}$$

as homomorphisms from E to $E\otimes\mathcal{S}\otimes\mathcal{L}$. In fact, (3.18) is equivalent to the first two conditions. From (3.18) it follows that the homomorphism

$$\Theta\oplus\Theta' \;:\; E\longrightarrow E\otimes(\mathcal{S}\oplus\mathcal{L})$$

satisfies the equation

$$(\Theta\oplus\Theta')\bigwedge(\Theta\oplus\Theta') \;=\; 0. \tag{3.19}$$

The third condition in the theorem implies that $\Psi_*(\Theta\oplus\Theta') = 0$, where Ψ_* is the homomorphism in (3.7). Hence from (3.7) it follows that there is a section

$$\theta \in H^0(X,\, \mathrm{End}(E)\otimes V)$$

such that $(\mathrm{Id}_{\mathrm{End}(E)}\otimes\Phi)_*(\theta) = \Theta\oplus\Theta'$. Note that this condition implies that θ is unique. Thus Θ and Θ' are given by θ (as in (2.14)). From (3.19) it follows that $\theta\bigwedge\theta = 0$ because θ and $\Theta\oplus\Theta'$ coincide on $X\setminus D$. ☐

Lemma 3.1, Corollary 3.3 and Theorem 3.4 together give the following:

Corollary 3.5. *Let* E *be a holomorphic vector bundle on* X *and*

$$\Theta \in H^0(X,\, \mathrm{End}(E)\otimes\mathcal{S}) \quad \textit{and} \quad \Theta' \in H^0(X,\, \mathrm{End}(E)\otimes\mathcal{L}).$$

Then there is a V*–twisted Higgs field* θ *on* E *such that* Θ *and* Θ' *are given by* θ *(as in* (2.14)*) if and only if the following three statements hold:*

1. *The homomorphism* $\varphi^*\Theta' : \varphi^*E \longrightarrow (\varphi^*E)\otimes\varphi^*\mathcal{L}$ *sends the subsheaf* F *in* (3.13) *to* $F\otimes\varphi^*\mathcal{L} \subset (\varphi^*E)\otimes\varphi^*\mathcal{L}$.

(2) *The two homomorphisms* $(\varphi^*\Theta)\big|_F : F \longrightarrow F \otimes \varphi^*\mathcal{S}$ *and* $(\varphi^*\Theta')\big|_F : F \longrightarrow F \otimes \varphi^*\mathcal{L}$ *commute. In other words,*

$$(\varphi^*(\Theta' \otimes \mathrm{Id}_{\varphi^*\mathcal{S}})\big|_{F\otimes\varphi^*\mathcal{S}}) \circ (\varphi^*\Theta)\big|_F \;=\; (\varphi^*(\Theta \otimes \mathrm{Id}_{\varphi^*\mathcal{L}})\big|_{F\otimes\varphi^*\mathcal{L}}) \circ (\varphi^*\Theta')\big|_F$$

as homomorphisms from F *to* $F \otimes \varphi^*(\mathcal{S} \otimes \mathcal{L})$.

(3) *For all* x *and* y *as before, the homomorphisms* $(\mathrm{Id}_{F_y} \otimes \xi_{1,x}) \circ \Theta_y$ *and* $(\mathrm{Id}_{F_y} \otimes \xi_{2,x}) \circ \Theta'_y$ *(see* (3.15) *and* (3.17)*) satisfy the equation*

$$(\mathrm{Id}_{F_y} \otimes \xi_{1,x}) \circ \Theta_y + (\mathrm{Id}_{F_y} \otimes \xi_{2,x}) \circ \Theta'_y \;=\; 0.$$

It was noted in the proof of Theorem 3.4 that (3.18) is equivalent to the first two conditions in Theorem 3.4. Therefore, Corollary 3.5 gives the following:

Corollary 3.6. *Let* E *be a holomorphic vector bundle on* X *and*

$$\Theta \;\in\; H^0(X,\, \mathrm{End}(E) \otimes \mathcal{S}) \quad \text{and} \quad \Theta' \;\in\; H^0(X,\, \mathrm{End}(E) \otimes \mathcal{L}).$$

Then there is a V*–twisted Higgs field* θ *on* E *such that* Θ *and* Θ' *are given by* θ *(as in* (2.14)*) if and only if the following two statements hold:*

(1) Θ *and* Θ' *commute, meaning*

$$(\Theta' \otimes \mathrm{Id}_{\mathcal{S}}) \circ \Theta \;=\; (\Theta \otimes \mathrm{Id}_{\mathcal{L}}) \circ \Theta'.$$

(2) *For all* x *and* y *as before, the homomorphisms* $(\mathrm{Id}_{F_y} \otimes \xi_{1,x}) \circ \Theta_y$ *and* $(\mathrm{Id}_{F_y} \otimes \xi_{2,x}) \circ \Theta'_y$ *(see* (3.15) *and* (3.17)*) satisfy the equation*

$$(\mathrm{Id}_{F_y} \otimes \xi_{1,x}) \circ \Theta_y + (\mathrm{Id}_{F_y} \otimes \xi_{2,x}) \circ \Theta'_y \;=\; 0.$$

4. SPECTRAL CONSTRUCTION

We can use Theorem 3.4 and Corollaries 3.5 and 3.6 to describe a spectral construction for V–twisted Higgs bundles. Take a rank 2 holomorphic vector bundle V on X, and let $\mathcal{S}$ and $\mathcal{L}$ be holomorphic line bundles on X constructed as in Section 2, so that (2.9) holds, in other words, there exists a torsion sheaf $\mathbb{T}$ supported on a reduced divisor D such that

$$0 \longrightarrow V \longrightarrow \mathcal{S} \oplus \mathcal{L} \longrightarrow \mathbb{T} \longrightarrow 0$$

is a short exact sequence.

Associate to each V–twisted rank r Higgs bundle $(E,\, \theta)$ on X the $\mathcal{S}$–twisted and $\mathcal{L}$–twisted Higgs fields $\theta_1 \in H^0(X,\, \mathrm{End}(E) \otimes \mathcal{S})$ and $\theta_2 \in H^0(X,\, \mathrm{End}(E) \otimes \mathcal{L})$ respectively induced by θ (see (2.14)). Following [BNR], let $s_i = \mathrm{tr}(\bigwedge^i \theta_1) \in H^0(X,\, \mathcal{S}^{\otimes i})$, $1 \leq i \leq r = \mathrm{rank}(E)$ be the coefficients of the characteristic polynomial of θ_1, and let $\mathcal{S} \supset X_s \longrightarrow X$ be the spectral curve associated to $s = (s_i)_{i=1}^r$, defined by the characteristic polynomial

$$t^r - \phi^* s_1 t^{r-1} + \ldots + (-1)^r \phi^* s_r \;=\; 0,$$

where $\phi : \mathcal{S} \longrightarrow X$ is the natural projection and t is the tautological section of $\phi^*\mathcal{S}$ on the total space of $\mathcal{S}$.

Theorem 4.1. *Suppose that the curve X_s is integral. Then there is a natural bijective correspondence of the following type:*

$$\left\{\begin{array}{c}\textit{Rank 1 torsion free } \varphi^*\mathcal{L}\textit{–twisted Higgs}\\ \textit{sheaves } (F,\, \theta_2') \textit{ on } X_s\textit{, where}\\ \theta_2' \,:\, F \longrightarrow F\otimes\varphi^*\mathcal{L}\\ \textit{is such that}\\ \theta_{2,y}' \,=\, -\rho^x(y)\,\mathrm{id}\ \ \forall\ y\,\in\,\varphi^{-1}(D).\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{c}V\textit{–twisted Higgs bundles}\\ (E,\,\theta) \textit{ with}\\ \theta \,:\, E \longrightarrow E\otimes V\\ \textit{such that the induced map}\\ \theta_1 \,:\, E \longrightarrow E\otimes\mathcal{S}\\ \textit{has characteristic polynomial } s.\end{array}\right\}$$

Proof. Let $\theta\,:\,E \longrightarrow E\otimes V$ be a V–twisted Higgs field. Denote by θ_1 (respectively, θ_2) the associated $\mathcal{S}$–twisted (respectively, $\mathcal{L}$–twisted) Higgs field on E. Then θ_1 induces a $\mathrm{Sym}(\mathcal{S}^*)$–module structure on E, which factors through a $\varphi_*\mathcal{O}_{X_s} \,\cong\, \mathrm{Sym}(\mathcal{S}^*)/\mathcal{I}$–module structure, where $\mathcal{I}$ is the ideal generated by the characteristic polynomial of θ_1 in $\mathrm{Sym}(\mathcal{S}^*)$. As a consequence, if X_s is integral then there exists a rank 1 torsionfree sheaf F on X_s, satisfying the condition $E \,=\, \varphi_*F$, such that the map θ_1 is induced by the pushforward of the $\mathcal{O}_{X_s}$–module structure on F.

Then, the twisted Higgs fields θ_1 and θ_2 commute if and only if the map $\theta_2\,:\,E \longrightarrow E\otimes\mathcal{L}$ is a map of $\mathrm{Sym}(\mathcal{S}^*)$–modules in addition to being a map of $\mathcal{O}_X$–modules. Since $E \,=\, \varphi_*F$ is the pushforward of an $\mathrm{Sym}(\mathcal{S}^*)$–module supported on X_s, we conclude that θ_1 and θ_2 commute if and only if $\theta_2 \,=\, \varphi_*\theta_2'$ for some homomorphism

$$\theta_2' \,:\, F \longrightarrow F\otimes\varphi^*\mathcal{S}.$$

By construction, this map coincides with the restriction of $\varphi^*\theta_2$ to F described by Theorem 3.4(1). By Corollary 3.6 and Proposition 3.2, the two Higgs commuting fields θ_1 and θ_2 are induced by a V–twisted Higgs field if and only if the unique eigenvalue of $\theta_{2,y}'$ is $-\rho^x(y)$ for each $y\,\in\,\varphi^{-1}(D)$. As θ_2' is a map of rank 1 torsionfree sheaves, and X_s is assumed to be integral, this is equivalent to the condition that $\theta_{2,y}' \,=\, -\rho^x(y)\,\mathrm{id}$. This completes the proof. $\square$

Recall that a Higgs bundle $(E,\,\theta)$ is called *stable* (respectively *semistable*) if and only if for each $0\,\neq\,E' \,\subsetneq\, E$ such that $\theta(E') \,\subseteq\, E'\otimes V$,

$$\frac{\mathrm{rank}(E')}{\mathrm{degree(E')}} < \frac{\mathrm{rank}(E)}{\mathrm{degree}(E)} \quad \left(\text{respectively, } \frac{\mathrm{rank}(E')}{\mathrm{degree(E')}} \leq \frac{\mathrm{rank}(E)}{\mathrm{degree}(E)}\right).$$

Proposition 4.2. *Let $(E,\,\theta)$ be a V–twisted Higgs bundles, and let $(E,\,\theta_1)$ be its associated $\mathcal{S}$–twisted Higgs bundle. If the spectral curve of θ_1 is integral, then $(E,\,\theta)$ does not admit any nontrivial θ–invariant subbunde. Consequently, $(E,\,\theta)$ is stable.*

Proof. Let $\varphi\,:\,X_s \,\longrightarrow\, X$ be the spectral curve of $(E,\,\theta_1)$. Let $0\,\neq\,E' \,\subsetneq\, E$ be a subbundle of the V–twisted Higgs bundle (E,θ). Denote by $(F,\,\theta_2')$ the associated rank 1 torsionfree Higgs sheaf on X_s, given by the correspondence in Theorem 4.1. It is clear by construction that E' is preserved by θ if and only if it is preserved simultaneously by θ_1 and θ_2. Now, E' is preserved by θ_1 if and only if it is a sub–$\mathrm{Sym}(\mathcal{S}^*)$–module of E, so in that case E' must be the pushforward of some nontrivial subsheaf F' of F. Since X_s is integral, and the rank of E' is less than r, this is not possible. $\square$

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