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Bi-infinite Riordan matrices: A matricial approach to multiplication and composition of formal Laurent series

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A B S T R A C T

We propose and investigate a bi-infinite matrix approach to the multiplication and composition of formal Laurent series. We generalize the concept of Riordan matrix to this bi-infinite context, obtaining matrices that are not necessarily lower triangular and are determined, not by a pair of formal power series, but by a pair of formal Laurent series. We extend the First Fundamental Theorem of Riordan Matrices to this setting, as well as the Toeplitz and Lagrange subgroups, that are subgroups of the classical Riordan group. Finally, as an illustrative example, we apply our approach to derive a classical combinatorial identity that cannot be proved using the techniques related to the classical Riordan group, showing that our generalization is not fruitless.

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1. Introduction

Basic notation

Throughout this paper, let $\mathbb{K}$ be a field with identity elements denoted by 0 and 1. We denote by $\mathbb{K}[x]$ the set of polynomials with coefficients in $\mathbb{K}$.

- A **formal power series** with coefficients in $\mathbb{K}$ is an object of the form

$$\gamma = \sum_{k=0}^{\infty} \gamma_k x^k.$$

  Additionally, we denote by $\mathbb{K}[[x]]$ the set of formal power series with coefficients in the field $\mathbb{K}$ (so $\mathbb{K}[x] \subset \mathbb{K}[[x]]$).
- A **formal Laurent series** with coefficients in $\mathbb{K}$ is an object of the form

$$\omega = \sum_{k=-\infty}^{\infty} \omega_k x^k.$$

  We denote by $\mathbb{L}(\mathbb{K})$ (or simply by $\mathbb{L}$) the set of formal Laurent series with coefficients in $\mathbb{K}$ (so, again, $\mathbb{K}[[x]] \subset \mathbb{L}(\mathbb{K})$).
- We also define $\mathbb{K}((x))$ to be the subset of $\mathbb{L}(\mathbb{K})$ consisting in those formal Laurent series of the form

$$\omega = \sum_{k=n}^{\infty} \omega_k x^k,$$

  for some $n \in \mathbb{Z}$, called the **order** of ω.
- In a similar fashion, we define $\mathbb{K}((\frac{1}{x}))$ as the subset of $\mathbb{L}(\mathbb{K})$ consisting in those formal Laurent series of the form

$$\omega = \sum_{k=-\infty}^{n} \omega_k x^k,$$

  for some $n \in \mathbb{Z}$, that we will also call the **order** of ω.

We use the word *formal* to emphasize that the series are not assumed to be convergent, even if this distinction is meaningful for the particular choice of the field $\mathbb{K}$ (see the example in Section 4 for a remark concerning this).

Multiplication and composition in $\mathbb{L}$

The sum of elements in $\mathbb{L}$ is straightforward to define and is not the focus of this paper, so we omit details. However, since we will use it to define another operation below, it is worth mentioning here.

The present paper focuses on two operations in the set $\mathbb{L}$, namely, multiplication and composition. For that reason, both are briefly discussed next.

For two elements $\alpha = \sum_{i=-\infty}^{\infty} \alpha_i x^i$, $\beta = \sum_{j=-\infty}^{\infty} \beta_j x^j$ in $\mathbb{L}$, we can, under certain circumstances, define the product $\mu = \alpha \cdot \beta$ to be the formal Laurent series

$$\mu = \sum_{k=-\infty}^{\infty} \mu_k x^k \text{ given, for all } k \in \mathbb{Z}, \text{ by } \mu_k = \sum_{i+j=k} \alpha_i \beta_j. \tag{1}$$

Note that, for this definition to make sense, for each $k \in \mathbb{Z}$, only a finite number of elements in the previous summation can be nonzero. We have the following:

Lemma 1 *((a) is in [8,14,15] and (b), (c) are just some variations).*

(a) *The set of formal power series $\mathcal{F}_0(\mathbb{K}) \subset \mathbb{K}[[x]]$ (in the following denoted just by $\mathcal{F}_0$) consisting in those elements*

$$\mathcal{F}_0(\mathbb{K}) = \{\omega = \omega_0 + \omega_1 x + \omega_2 x^2 + \omega_3 x^3 + \ldots \in \mathbb{K}[[x]] : \omega_0 \neq 0\}$$

is a group with respect to the multiplication.

(b) *As a consequence of the previous, $\mathbb{K}((x)) \setminus \{0\}$ together with the multiplication is a group.*

(c) *In a similar fashion, $\mathbb{K}((\frac{1}{x})) \setminus \{0\}$ is also a group with respect to the multiplication (see the comment at the end of this subsection).*

On the other hand, for two elements ω and χ in $\mathbb{L}$, we can, under certain circumstances, define the composition $\omega \circ \chi$ to be the formal Laurent series

$$\omega \circ \chi = \sum_{i=-\infty}^{\infty} \omega_i \chi^i. \tag{2}$$

In general, the problem of deciding when the composition of two formal Laurent series is well defined is not immediate. For the series shown in Equation (2) to make sense, we need the power series $\ldots, \omega_{-1}\chi^{-1}, \omega_0\chi^0, \omega_1\chi^1, \ldots$ to be well defined and the corresponding infinite sum to be well defined too. We omit details because we will discuss them later from our matricial point of view and we refer the reader to [2,5].

We have the following:

Lemma 2 *(Section 2.8 in [1]). The set of formal power series $\mathcal{F}_1(\mathbb{K}) \subset \mathbb{K}[[x]]$ (in the following denoted just by $\mathcal{F}_1$) consisting in those elements*

$$\mathcal{F}_1(\mathbb{K}) = \{\omega = \omega_1 x + \omega_2 x^2 + \omega_3 x^3 + \ldots \in \mathbb{K}[[x]] : \omega_1 \neq 0\}$$

is a group with respect to the composition.

Before closing this section, it is worth to mention that there is a bijection $\mathbb{K}((x)) \longmapsto \mathbb{K}((\frac{1}{x}))$ preserving the multiplication and given by

$$\omega = \omega_n x^n + \omega_{n+1} x^{n+1} + \ldots \longmapsto \omega \circ \left(\frac{1}{x}\right) = \omega_n \frac{1}{x^n} + \omega_{n+1} \frac{1}{x^{n+1}} + \ldots$$

Infinite matrices, infinite column vectors and formal series

The relation between elements in $\mathbb{K}[[x]]$ and lower triangular (mono-) infinite matrices has already been explored through the Riordan group. A lower triangular (mono-) infinite matrix is an object of the type

$$[a_{ij}]_{i,j\in\mathbb{N}} = \begin{bmatrix} a_{0,0} & & \\ a_{1,0} & a_{1,1} & \\ \vdots & \vdots & \ddots \end{bmatrix}, \qquad a_{ij} \in \mathbb{K}. \tag{3}$$

The multiplication of these objects is formally well defined. If we have two such lower triangular (mono-) infinite matrices $[a_{ij}]_{i,j\in\mathbb{N}}$, $[b_{ij}]_{i,j\in\mathbb{N}}$ the product is a third lower triangular (mono-) infinite matrix $[c_{ij}]_{i,j\in\mathbb{N}}$ where, for every $i, j \in \mathbb{N}$, $c_{ij} = \sum_{k\in\mathbb{N}} a_{ik} b_{kj}$. Let us remark that, in this infinite sum, there is only a finite number of nonzero terms and convergence issues do not arise. The set of such lower triangular (mono-) infinite matrices with nonzero elements in the main diagonal forms a group with respect to the multiplication [17].

For every element (d, h) in $\mathcal{F}_0 \times \mathcal{F}_1$ the associated **Riordan matrix** $R(d, h)$ is a lower triangular infinite matrix $[a_{ij}]_{i,j\in\mathbb{N}}$, as the one in Equation (3), such that, for $j \in \mathbb{N}$, the column $[a_{j0}, a_{j1}, \ldots]^T$ contains the coefficients of $d \cdot h^j$. We have the following:

Theorem 3 *(1FTRM, Theorem 3.1 in [16]). For every Riordan matrix $R(d, h)$ and for every column vector $\boldsymbol{v} = [v_0, v_1, \ldots]^T$, let $\boldsymbol{w} = R(d, h)\boldsymbol{v}$. If γ is the formal power series whose coefficients are the entries in $\boldsymbol{v}$, then $d \cdot (\gamma \circ h)$ is the formal power series whose coefficients are the entries in $\boldsymbol{w}$.*

As a consequence of the previous theorem, we can see that the set of Riordan matrices, denoted by $\mathcal{R}(\mathbb{K})$ (or simply by $\mathcal{R}$), is a group with respect to the multiplication. Moreover, the multiplication of Riordan matrices is given by

$$R(d, h)R(f, g) = R(d \cdot f(h), g \circ h)$$

and the inverse of a given element is

$$(R(d, h))^{-1} = R\left(\frac{1}{d \circ h^{[-1]}}, h^{[-1]}\right).$$

Both, in the previous equation as well as in the rest of the paper, we use the notation $h^{[n]}$ for the iterated composition and $h^{[-1]}$ for the compositional inverse, to distinguish from the multiplicative power and inverse, denoted by h^n, h^{-1}, respectively.

The Riordan group has two important subgroups that will be recalled later in this paper. The first one is the **Toeplitz subgroup**, denoted by $\mathcal{T}(\mathbb{K})$ (or simply by $\mathcal{T}$), consisting in those elements in $\mathcal{R}$ of the type $R(d,x)$. The second one is the **Lagrange subgroup** that, for convenience, in this paper will be denoted by $\mathcal{B}(\mathbb{K})$ (or simply by $\mathcal{B}$) and consists in those elements in $\mathcal{R}$ of the type $R(1,h)$. Note that every element $R(d,h)$ in $\mathcal{R}$ can be written univocally as product of an element in $\mathcal{T}$ and an element in $\mathcal{B}$:

$$R(d,h) = R(d,x)R(1,h).$$

In this paper, we will explore the relation between the product and composition of formal Laurent series and the product of (not mono-infinite matrices but) bi-infinite matrices. These matrices are objects of the type

$$[a_{ij}]_{i,j\in\mathbb{Z}} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & a_{-1,-1} & a_{-1,0} & a_{-1,1} & \cdots \\ \cdots & a_{0,-1} & \boxed{a_{0,0}} & a_{0,1} & \cdots \\ \cdots & a_{1,-1} & a_{1,0} & a_{1,1} & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad a_{ij} \in \mathbb{K}.$$

On the other hand, a bi-infinite column vector is an object of the type $\boldsymbol{v} = [\dots, v_{-1}, \boxed{v_0}, v_1, \dots]^T$, $v_i \in \mathbb{K}$.

Note that the multiplication of two of these bi-infinite matrices $[a_{ij}]_{i,j\in\mathbb{Z}}$, $[b_{ij}]_{i,j\in\mathbb{Z}}$ (sim. for the multiplication of one bi-infinite matrix and a bi-infinite column vector) is not always well defined. For this paper, we consider that the product is well defined if, for every $i,j \in \mathbb{Z}$, only a finite number of terms in the summation $\sum_{k\in\mathbb{Z}} a_{ik}b_{kj}$ are nonzero (we do not accept the case in which $\sum_{k\in\mathbb{Z}} a_{ik}b_{kj}$ is convergent, even for those choices of the field $\mathbb{K}$ for which this concept makes sense).

In [9] the reader may find a group of bi-infinite matrices isomorphic to the Riordan group. In the following section, we will briefly discuss some properties of the multiplication of bi-infinite matrices and the multiplication of a bi-infinite matrix with a bi-infinite column vector.

Outline of this paper

- In Section 2 we will provide some comments and introduce notation concerning bi-infinite matrices to improve the readability of the rest of the paper.
- In Section 3 we will study the matricial approach to the multiplication of elements in $\mathbb{L}$. To do so, we will assign to each element $\alpha \in \mathbb{L}$ a bi-infinite matrix $\boldsymbol{A}_\alpha$ in such a way that, when the product is defined,

$$\boldsymbol{A}_\alpha \boldsymbol{A}_\beta = \boldsymbol{A}_{\alpha\cdot\beta}.$$

This leads to a first generalization of the 1FTRM (Theorem 10) and to the description of two groups of bi-infinite matrices:

$$\{\boldsymbol{A}_\alpha : \alpha \in \mathbb{K}((x)) \setminus \{0\}\}, \qquad \left\{\boldsymbol{A}_\alpha : \alpha \in \mathbb{K}\left(\left(\frac{1}{x}\right)\right) \setminus \{0\}\right\},$$

generalizing the Toeplitz subgroup of the Riordan group (Theorem 11).
- In Section 4 we will study the matricial approach to the composition of elements in $\mathbb{L}$. To do so, we will assign to each element $\omega \in (\mathbb{K}((x)) \cup \mathbb{K}((\frac{1}{x})))$ a bi-infinite matrix $\boldsymbol{B}_\omega$ in such a way that, when the product is defined,

$$\boldsymbol{B}_\omega \boldsymbol{B}_\chi = \boldsymbol{B}_{\chi \circ \omega}.$$

This leads to a second generalization of the 1FTRM (Theorem 13) and to the study of the multiplication and inversion of the matrices $\boldsymbol{B}_\omega$ (Theorem 15) in order to obtain a generalization of the classical Lagrange subgroup.
- In Section 5, we discuss the possibility of generalizing the concept of Riordan matrix (associated to a pair of formal power series) to the one of bi-infinite Riordan matrix (associated to a pair of formal Laurent series). To do so, we need to study when the multiplication between elements $\boldsymbol{A}_\alpha$ and $\boldsymbol{B}_\omega$ is well defined.
- Finally, in Section 6, we present an application of the results from previous sections. The (classical) Riordan group has its origin in Combinatorics, and the generalized Riordan matrices introduced here can also be used to derive combinatorial identities.

2. Some comments on the multiplication of bi-infinite matrices

Let us start with the following:

Remark. It is possible to define the product of a bi-infinite matrix $\boldsymbol{M} = [m_{ij}]_{i,j\in\mathbb{Z}}$ by an infinite column vector $\boldsymbol{v} = [\dots, v_{-1}, v_0, v_1, \dots]^T$, to be the bi-infinite column vector $[\dots, w_{-1}, w_0, w_1, \dots]^T$ given by

$$w_i = \sum_{j=-\infty}^{\infty} m_{ij} v_j, \qquad \forall i \in \mathbb{Z}, \tag{4}$$

if and only if, for every $i \in \mathbb{Z}$, the number of nonzero products $m_{ij}v_j$ is finite. In particular, this happens if only a finite number of the elements v_j are nonzero.

In a similar fashion, for two such bi-infinite matrices $\boldsymbol{M} = [m_{ij}]_{i,j\in\mathbb{Z}}$, $\boldsymbol{N} = [n_{ij}]_{i,j\in\mathbb{Z}}$, the product $\boldsymbol{MN} = [p_{ij}]_{i,j\in\mathbb{Z}}$, given by

$$p_{ij} = \sum_{k=-\infty}^{\infty} m_{ik} n_{kj}, \qquad \forall i, j \in \mathbb{Z}, \tag{5}$$

is well defined if, for every $i, j \in \mathbb{Z}$, the number of nonzero products $m_{ik}n_{kj}$ is finite.

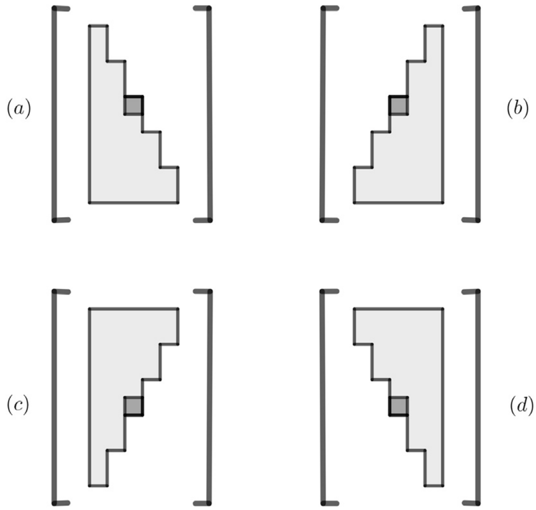


Fig. 1. Types of *echelon forms* of bi-infinite matrices.

We have the following elementary result, that we will need later, but whose proof we will omit:

Lemma 4.

(a) *For bi-infinite matrices with coefficients in* $\mathbb{K}$ *we have, in some sense, an* Associative Law*: if we have three bi-infinite matrices* $\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{P}$ *such that the products* $\boldsymbol{MN}$*,* $\boldsymbol{NP}$*,* $\boldsymbol{M}(\boldsymbol{NP})$*,* $(\boldsymbol{MN})\boldsymbol{P}$ *are well defined, then* $\boldsymbol{M}(\boldsymbol{NP}) = (\boldsymbol{MN})\boldsymbol{P}$*.*

(b) *In the following,* $\boldsymbol{I}$ *will denote the bi-infinite matrix* $[\delta_{ij}]_{i,j\in\mathbb{Z}}$*, where* δ_{ij} *stands for the Kronecker delta. For any bi-infinite matrix* $\boldsymbol{M}$*, we have* $\boldsymbol{MI} = \boldsymbol{IM} = \boldsymbol{M}$*.*

We define **lower** (resp. **upper**) column vectors $[\ldots, v_{-1}, v_0, v_1, \ldots]^T$ to be those satisfying that there exits some $k \in \mathbb{Z}$ such that, for $i > k$ (resp. $i < k$), $v_i = 0$.

In this paper we use some bi-infinite matrices whose shape resembles that of the *reduced echelon form*. We have four cases of interest, all of them appear in Fig. 1.

We will denote by $\mathcal{L}_+$, $\mathcal{L}_-$, $\mathcal{U}_-$ and $\mathcal{U}_+$ to the sets of those bi-infinite matrices corresponding to the pictures (a), (b), (c) and (d), respectively, in Fig. 1. Formally, a bi-infinite matrix $\boldsymbol{M} = [m_{ij}]_{i,j\in\mathbb{Z}}$ belongs to $\mathcal{L}_+$ (similar for $\mathcal{L}_-$, $\mathcal{U}_-$ and $\mathcal{U}_+$) if and only if there exists a bi-infinite vector of strictly increasing integer numbers $[\ldots, v_{-1}, v_0, v_1, \ldots]$ such that

$$m_{ij} \begin{cases} = 0 & \text{if } i < v_j \\ \neq 0 & \text{if } i = v_j \end{cases} \qquad \forall j \in \mathbb{Z}.$$

We have the following:

Lemma 5. *Let* $\boldsymbol{M} = [m_{ij}]_{i,j\in\mathbb{Z}}$ *and* $\boldsymbol{v} = [\ldots, v_{-1}, v_0, v_1, \ldots]^T$.

(a) *If* $\boldsymbol{M} \in \mathcal{L}_+$ *and* $\boldsymbol{v}$ *is a lower vector, then* $\boldsymbol{Mv}$ *is well defined.*
(b) *If* $\boldsymbol{M} \in \mathcal{L}_-$ *and* $\boldsymbol{v}$ *is an upper vector, then* $\boldsymbol{Mv}$ *is well defined.*
(c) *If* $\boldsymbol{M} \in \mathcal{U}_-$ *and* $\boldsymbol{v}$ *is a lower vector, then* $\boldsymbol{Mv}$ *is well defined.*
(d) *If* $\boldsymbol{M} \in \mathcal{U}_+$ *and* $\boldsymbol{v}$ *an upper vector, then* $\boldsymbol{Mv}$ *is well defined.*

Proof. In any of the cases above, for every j, the number of nonzero products $m_{ij}v_j$ in the sum in Equation (4) is finite. □

Let us define the matrix $\boldsymbol{J} = [\xi_{ij}]_{i,j\in\mathbb{Z}}$ given, for every $i, j \in \mathbb{Z}$, by

$$\xi_{ij} = \begin{cases} 1 & \text{if } i + j = 0 \\ 0 & \text{in other case} \end{cases} \tag{6}$$

that is, $\boldsymbol{J}$ is the bi-infinite matrix whose entries are 0, except for those in the secondary diagonal. This matrix will simplify our proofs from this moment on. We have the following:

Lemma 6.

(a) $\boldsymbol{J}$ *is an* ***involution****, that is,* $\boldsymbol{J}^2 = \boldsymbol{I}$.
(b) *For every bi-infinite matrix* $\boldsymbol{C} = [c_{ij}]_{i,j\in\mathbb{Z}}$*, the product* $\boldsymbol{JC} = [d_{ij}]_{i,j\in\mathbb{Z}}$ *is well defined and the result is a bi-infinite matrix obtained from* $\boldsymbol{C}$ *by a* reflection across the 0-row*, that is,* $\forall i, j \in \mathbb{Z}$, $d_{ij} = c_{-i,j}$.
(c) *For every bi-infinite matrix* $\boldsymbol{C} = [c_{ij}]_{i,j\in\mathbb{Z}}$*, the product* $\boldsymbol{CJ} = [d_{ij}]_{i,j\in\mathbb{Z}}$ *is well defined and the result is a bi-infinite matrix obtained from* $\boldsymbol{C}$ *by a* reflection across the 0-column*, that is,* $\forall i, j \in \mathbb{Z}$, $d_{ij} = c_{i,-j}$.

Proof. The three statements are a consequence of the multiplication formula for bi-infinite matrices given in Equation (5). □

As a consequence of the previous lemma, we have the following:

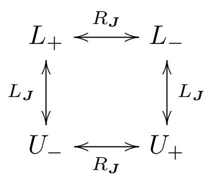
Corollary 7.

(a) *Every element in* $\mathcal{L}_-$ *can be written as* $\boldsymbol{MJ}$*, for some* $\boldsymbol{M} \in \mathcal{L}_+$*, that is, the following sets are equal:* $\mathcal{L}_- = \mathcal{L}_+\boldsymbol{J}$ *and* $\mathcal{L}_-\boldsymbol{J} = \mathcal{L}_+$.
(b) *Every element in* $\mathcal{U}_+$ *can be written as* $\boldsymbol{JMJ}$*, for some* $\boldsymbol{M} \in \mathcal{L}_+$*, that is, the following sets are equal:* $\mathcal{U}_+ = \boldsymbol{J}\mathcal{L}_+\boldsymbol{J}$ *and* $\mathcal{L}_+ = \boldsymbol{J}\mathcal{U}_+\boldsymbol{J}$.
(c) *Every element in* $\mathcal{U}_-$ *can be written as* $\boldsymbol{JM}$*, for some* $\boldsymbol{M} \in \mathcal{L}_+$*, that is, the following sets are equal:* $\mathcal{U}_- = \boldsymbol{J}\mathcal{L}_+$ *and* $\mathcal{L}_+ = \boldsymbol{J}\mathcal{U}_-$.

Table 1
With *not defined* we mean *not defined in general*, noting that the intersection of the sets $\mathcal{L}_+$, $\mathcal{L}_-$, $\mathcal{U}_-$ and $\mathcal{U}_+$ may be non-empty.

	$\boldsymbol{N} \in \mathcal{L}_+$	$\boldsymbol{N} \in \mathcal{L}_-$	$\boldsymbol{N} \in \mathcal{U}_+$	$\boldsymbol{N} \in \mathcal{U}_-$
$\boldsymbol{M} \in \mathcal{L}_+$	$(\boldsymbol{MN}) \in \mathcal{L}_+$	$(\boldsymbol{MN}) \in \mathcal{L}_-$	not defined	not defined
$\boldsymbol{M} \in \mathcal{L}_-$	not defined	not defined	$(\boldsymbol{MN}) \in \mathcal{L}_-$	$(\boldsymbol{MN}) \in \mathcal{L}_+$
$\boldsymbol{M} \in \mathcal{U}_+$	not defined	not defined	$(\boldsymbol{MN}) \in \mathcal{U}_+$	$(\boldsymbol{MN}) \in \mathcal{U}_-$
$\boldsymbol{M} \in \mathcal{U}_-$	$(\boldsymbol{MN}) \in \mathcal{U}_-$	$(\boldsymbol{MN}) \in \mathcal{U}_+$	not defined	not defined

This information can also be depicted in the following commutative diagram:



where $L_{\boldsymbol{J}}$ *and* $R_{\boldsymbol{J}}$ *denote the left and right multiplications by the matrix* $\boldsymbol{J}$*, respectively.*

Let us conclude this section with the following:

Lemma 8. *Let* $\boldsymbol{M} = [m_{ij}]_{i,j\in\mathbb{Z}}$ *and* $\boldsymbol{N} = [n_{ij}]_{i,j\in\mathbb{Z}}$ *be two bi-infinite matrices. The product* $\boldsymbol{MN}$ *follows the rules appearing in Table 1.*

Proof. Let us start by finding the cases in which the product $\boldsymbol{MN}$ is well defined. To do so, we use Lemma 5.

Second, we need to check that, in the case $\boldsymbol{M} = [m_{ij}]_{i,j\in\mathbb{Z}}, \boldsymbol{N} = [n_{ij}]_{i,j\in\mathbb{Z}} \in \mathcal{L}_+$, $\boldsymbol{MN}$ belongs to $\mathcal{L}_+$. We will omit details, since the proof is quite similar to the one showing that the product of lower triangular matrices is lower triangular in the classical context.

Finally, to complete the information appearing in Table 1, we can use Corollary 7. Let us prove, as a matter of example, that

$$\boldsymbol{M} \in \mathcal{L}_+, \boldsymbol{N} \in \mathcal{L}_- \implies \boldsymbol{MN} \in \mathcal{L}_-$$

and we leave the rest of the cases to the reader. If $\boldsymbol{N} \in \mathcal{L}_-$, then $\boldsymbol{N} = \widetilde{\boldsymbol{N}}\boldsymbol{J}$, for some $\widetilde{\boldsymbol{N}} \in \mathcal{L}_+$. So $\boldsymbol{MN} = (\boldsymbol{M}\widetilde{\boldsymbol{N}})\boldsymbol{J}$, where $(\boldsymbol{M}\widetilde{\boldsymbol{N}}) \in \mathcal{L}_+$ and so $(\boldsymbol{M}\widetilde{\boldsymbol{N}})\boldsymbol{J} \in \mathcal{L}_-$. □

3. First partial generalization of the 1FTRM and the generalized Toeplitz group

Definition 9. For every element $\alpha = \sum_{k=-\infty}^{\infty} \alpha_k x^k \in \mathbb{L}$, we define the bi-infinite matrix

$$\boldsymbol{A}_\alpha = [a_{ij}]_{i,j\in\mathbb{Z}} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & \alpha_0 & \alpha_{-1} & \alpha_{-2} & \alpha_{-3} & \alpha_{-4} & \dots \\ \dots & \alpha_1 & \alpha_0 & \alpha_{-1} & \alpha_{-2} & \alpha_{-3} & \dots \\ \dots & \alpha_2 & \alpha_1 & \boxed{\alpha_0} & \alpha_{-1} & \alpha_{-2} & \dots \\ \dots & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \alpha_{-1} & \dots \\ \dots & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \text{ where } a_{ij} = \alpha_{i-j}. \tag{7}$$

We say that this bi-infinite matrix is a **Toeplitz matrix**.

Note that, if $\alpha \in \mathbb{K}((x))$ (resp. $\alpha \in \mathbb{K}((\frac{1}{x}))$), then $\boldsymbol{A}_\alpha \in \mathcal{L}_+$ (resp. $\boldsymbol{A}_\alpha \in \mathcal{U}_+$). As a consequence, we have the following:

Theorem 10 *(First Partial Generalization of the 1FTRM). Let*

$$\alpha = \sum_{k=-\infty}^{\infty} \alpha_k x^k \in \mathbb{L}, \qquad \beta = \sum_{k=-\infty}^{\infty} \beta_k x^k \in \mathbb{L}.$$
$$\boldsymbol{v} = [\dots, \beta_{-1}, \beta_0, \beta_1, \dots]^T.$$

(a) *If $\alpha, \beta \in \mathbb{K}((x))$, then $\boldsymbol{A}_\alpha \boldsymbol{v}$ is well defined.*
(b) *If $\alpha, \beta \in \mathbb{K}((\frac{1}{x}))$, then $\boldsymbol{A}_\alpha \boldsymbol{v}$ is well defined.*

The product is not necessarily well defined in any other case. In any of the previous cases, let $\boldsymbol{w} = [\dots, w_{-1}, w_0, w_1, \dots]^T$ be the resulting vector and define the power series $\gamma = \sum_{k=-\infty}^{\infty} w_k x^k$. Then $\gamma = \alpha \cdot \beta$.

Proof. Statements (a) and (b) of the present theorem follow from the discussion preceding it and from Lemma 5.

Once this has been stated, the second part of the current proof derives from the comparison of the formula for the multiplication of two formal power series (see Equation (1)) and the formula for the multiplication of a bi-infinite matrix by a bi-infinite column vector (see Equation (4)) for the particular case of the matrix described in Equation (7). □

As a direct consequence of the previous, we have the following

Theorem 11. *The set of matrices $\boldsymbol{A}_\alpha$ such that $\alpha \in \mathbb{K}((x))\backslash\{0\}$ (resp. $\alpha \in \mathbb{K}((\frac{1}{x}))\backslash\{0\}$) is a group that will be called the* ***generalized lower (resp. upper) Toeplitz group****, isomorphic to the group $(\mathbb{K}((x)),\cdot)$. (resp. $(\mathbb{K}((\frac{1}{x})),\cdot)$). For every $\alpha,\beta \in \mathbb{K}((x))$ (resp. in $\mathbb{K}((\frac{1}{x}))$),*

- $\boldsymbol{A}_\alpha \boldsymbol{A}_\beta = \boldsymbol{A}_{\alpha\cdot\beta}$.
- *The identity element is* $\boldsymbol{A}_1 = \boldsymbol{I}$.
- $\boldsymbol{A}_\alpha^{-1} = \boldsymbol{A}_{1/\alpha}$.

Proof. To prove the first statement, for $j \in \mathbb{Z}$, let us denote by $\boldsymbol{v}_j$ to the column vector corresponding to the j-th column in $\boldsymbol{A}_\beta$. This column vector contains the coefficients of $x^j\beta$. The j-th column in the product $\boldsymbol{A}_\alpha \boldsymbol{A}_\beta$ is $\boldsymbol{A}_\alpha \boldsymbol{v}_j$. According to Theorem 10, this column contains the coefficients of $x^j\alpha\beta$. This completes the proof of this statement, and the rest of them are a consequence of this it. □

Some further questions concerning the existence of multiplicative inverses of elements in $\mathbb{L}$ were already studied in [2,5], but the information provided in the previous result will be sufficient for the rest of the paper.

4. Second partial generalization of the 1FTRM and the generalized Lagrange group

Some questions concerning the existence of the composition $\chi \circ \omega$, for $\chi, \omega \in \mathbb{L}$ can be found in [2]. In any case, we will follow our own approach for this problem.

Note that $\mathbb{K}((x))$ is not closed under composition.

Example. For instance,

$$\chi = 1 + x + x^2 + x^3 + \ldots \in \mathbb{K}((x)),\ \omega = \frac{1}{x} \in \mathbb{K}((x)),$$
$$\chi \circ \omega = 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \ldots \notin \mathbb{K}((x)).$$

With respect to this, we also have a problem of interpretation.

Example (continuation). Let us consider the previous example in the case $\mathbb{K} = \mathbb{C}$. In this setting, some series can be interpreted as expansions of complex functions of one complex variable. We can identify

$$\chi = \frac{1}{1-x}$$

and then

$$\chi \circ \omega = \frac{1}{1-\frac{1}{x}} = -\frac{x}{1-x} = -x - x^2 - x^3 - \ldots$$

Therefore

$$1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \dots, \qquad -x - x^2 - x^3 - \dots$$

are *different as formal Laurent series* but *equal in some sense*. This is because they correspond to two different expansions of the same complex function of one complex variable. For us, these two formal Laurent series will be different and any other analytic interpretation will be left apart in this paper.

Definition 12. For every element $\omega \in \mathbb{K}((x)) \cup \mathbb{K}((\frac{1}{x}))$, we define the matrix $\boldsymbol{B}_\omega$ to be the bi-infinite matrix $\boldsymbol{B}_\omega = [b_{ij}]_{-\infty<i,j<\infty}$ such that, for every $j \in \mathbb{Z}$, $\omega^j = \sum_{i=-\infty}^{\infty} b_{ij} x^i$.

The condition $\omega \in \mathbb{K}((x)) \cup \mathbb{K}(\frac{1}{x}))$ ensures that the powers

$$\dots, \omega^{-1}, \omega^0, \omega^1, \omega^2, \omega^3, \dots$$

are well defined.

As a first example of such a matrix, note that $\boldsymbol{J} = \boldsymbol{B}_{1/x}$ (the definition appears in Equation (6)). On the other hand, we have that:

- If $\omega \in \mathbb{K}((x))$ and its order is positive, then $\boldsymbol{B}_\omega \in \mathcal{L}_+$.
- If $\omega \in \mathbb{K}((x))$ and its order is negative, then $\boldsymbol{B}_\omega \in \mathcal{L}_-$.
- If $\omega \in \mathbb{K}((\frac{1}{x}))$ and its order is positive, then $\boldsymbol{B}_\omega \in \mathcal{U}_+$.
- If $\omega \in \mathbb{K}((\frac{1}{x}))$ and its order is negative, then $\boldsymbol{B}_\omega \in \mathcal{U}_-$.

We have the following:

Theorem 13 *(Second Partial Generalization of the 1FTRM). Let*

$$\chi = \sum_{k=-\infty}^{\infty} \chi_k x^k \in \mathbb{L}(\mathbb{K}), \qquad \boldsymbol{v} = [\dots, \chi_{-1}, \chi_0, \chi_1, \dots]^T.$$

(a) *If $\omega \in \mathbb{K}((x))$ and has positive order and $\chi \in \mathbb{K}((x))$, then $\boldsymbol{B}_\omega \boldsymbol{v}$ is well defined.*
(b) *If $\omega \in \mathbb{K}((\frac{1}{x}))$ and has negative order and $\chi \in \mathbb{K}((x))$, then $\boldsymbol{B}_\omega \boldsymbol{v}$ is well defined.*
(c) *If $\omega \in \mathbb{K}((x))$ and has negative order and $\chi \in \mathbb{K}((\frac{1}{x}))$, then $\boldsymbol{B}_\omega \boldsymbol{v}$ is well defined.*
(d) *If $\omega \in \mathbb{K}((\frac{1}{x}))$ and has positive order and $\chi \in \mathbb{K}((\frac{1}{x}))$, then $\boldsymbol{B}_\omega \boldsymbol{v}$ is well defined.*
(e) *If $\omega \in \mathbb{K}((x)) \cup \mathbb{K}\left(\left(\frac{1}{x}\right)\right)$ and $\chi \in \mathbb{K}((x)) \cap \mathbb{K}\left(\left(\frac{1}{x}\right)\right)$, for some $n \in \mathbb{Z}$, then $\boldsymbol{B}_\omega \boldsymbol{v}$ is well defined.*

In any of the previous cases, the resulting vector $\boldsymbol{w} = [\dots, w_{-1}, w_0, w_1, \dots]^T$ corresponds to the formal Laurent series ψ

$$\psi = \sum_{k=-\infty}^{\infty} w_k x^k = \chi \circ \omega$$

where this composition of elements in $\mathbb{L}$ is also well defined.

Proof. We can prove Statements (a), (b), (c), (d) and (e) using the discussion before this theorem and Lemma 5.

To prove the final statement, let $\boldsymbol{B}_\omega = [b_{ij}]_{i,j\in\mathbb{Z}}$ and let us denote by $\boldsymbol{u}_j$ to the column vector $[\ldots, b_{-1,j}, b_{0j}, b_{1j}, \ldots]^T$. The column vector

$$\boldsymbol{w} = \ldots + \chi_{-1}\boldsymbol{u}_{-1} + \chi_0\boldsymbol{u}_0 + \chi_1\boldsymbol{u}_1 + \ldots$$

that we have ensured that is well defined (at each level there are only finitely many nonzero entries), corresponds to the formal Laurent series

$$\chi \circ \omega = \sum_{k=-\infty}^{\infty} \chi_k \omega^k. \quad \square$$

Lemma 6 has an interesting translation to this context. We have the following:

Corollary 14. *For every $\omega \in \mathbb{K}((x)) \cup \mathbb{K}((\frac{1}{x}))$, $\boldsymbol{J}\boldsymbol{B}_\omega = \boldsymbol{B}_{\omega\circ\frac{1}{x}}$, $\boldsymbol{B}_\omega\boldsymbol{J} = \boldsymbol{B}_{1/\omega}$.*

Proof. This result is a direct consequence of Theorem 13, particularly, Statement (e). $\square$

Theorem 15.

(a) *When defined, the product $\boldsymbol{B}_\omega\boldsymbol{B}_\chi$ equals $\boldsymbol{B}_{\chi\circ\omega}$.*

(b) *In view of the previous lemma, to decide if $\boldsymbol{B}_\omega\boldsymbol{B}_\chi$ is well defined, it suffices to know that,*
- *For every bi-infinite matrix $\boldsymbol{IC} = \boldsymbol{CI} = \boldsymbol{C}$.*
- *$\boldsymbol{JJ} = \boldsymbol{I}$.*
- *For every $\boldsymbol{B}_\omega, \boldsymbol{B}_\chi \in \mathcal{L}_+$, the following products are well defined:*

$$\boldsymbol{B}_\omega\boldsymbol{B}_\chi, \qquad \boldsymbol{J}\boldsymbol{B}_\omega\boldsymbol{B}_\chi, \qquad \boldsymbol{B}_\omega\boldsymbol{B}_\chi\boldsymbol{J}, \qquad \boldsymbol{J}\boldsymbol{B}_\omega\boldsymbol{B}_\chi\boldsymbol{J}$$

- *For every $\boldsymbol{B}_\omega, \boldsymbol{B}_\chi \in \mathcal{L}_+$, in general, the product $\boldsymbol{B}_\omega\boldsymbol{J}\boldsymbol{B}_\chi$ is not well defined.*

Equivalently, we can use the comments after Definition 12 (to determine, in each case, if the corresponding matrix is in $\mathcal{L}_+$, $\mathcal{L}_-$, $\mathcal{U}_+$ or $\mathcal{U}_-$) along with Lemma 8.

(c) *Let $\omega \in \mathbb{K}((x)) \cup \mathbb{K}((\frac{1}{x}))$ not of order 0. There exists some $\chi \in \mathbb{K}((x)) \cup \mathbb{K}((\frac{1}{x}))$ such that $\boldsymbol{B}_\chi$ is the inverse of $\boldsymbol{B}_\omega$ if and only if the order of ω is ± 1. This χ is a compositional inverse of ω in $\mathbb{L}$, belongs to the same set $\mathbb{K}((x)), \mathbb{K}((\frac{1}{x}))$ than ω and has the same order.*

(d) *The set of matrices $\boldsymbol{B}_\omega \in \mathcal{L}_+$ and the set of matrices $\boldsymbol{B}_\omega \in \mathcal{U}_+$, endowed with the multiplication, are monoids.*
(e) *The set of matrices $\mathcal{B}_\omega \in \mathcal{L}_+$ such that ω has order 1 and the set of matrices $\mathcal{B}_\omega \in \mathcal{U}_+$ such that ω has order 1, endowed with the multiplication, are groups that will be called* ***generalized lower/upper Lagrange group****, respectively. The function given by*

$$\boldsymbol{B}_\omega \mapsto \boldsymbol{J}\boldsymbol{B}_\omega\boldsymbol{J},$$

is an isomorphism (from the first one to the second one and from the second one to the first one) and an involution (the iterated composition of this isomorphism is the identity).

Proof. Let us start with the proof of Statement (a). We denote by $\boldsymbol{v}_j$ to the j-th column of the matrix $\boldsymbol{B}_\chi$. Then $\boldsymbol{B}_\omega\boldsymbol{v}$, according to Definition 12 and to Theorem 13, contains the coefficients of $(\chi\circ\omega)^j$. So the product $\boldsymbol{B}_\omega\boldsymbol{B}_\chi$ is the matrix whose j-column contains the coefficients of $(\chi\circ\omega)^j$ and thus, by Definition 12, equals $\boldsymbol{B}_{\chi\circ\omega}$.

The first part of Statement (b) appears in Lemma 4, while the second part is in Lemma 6. Finally, the third and fourth parts are a consequence of Theorem 13.

For Statement (c), let us prove the necessity of the condition of ω having order ± 1.

Let us start with the case $\boldsymbol{B}_\omega \in \mathcal{L}_+$. Let us see that ω necessarily is of order 1. Consider the bi-infinite linear system:

$$\boldsymbol{B}_\omega \begin{bmatrix} \vdots \\ \chi_{-2} \\ \chi_{-1} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}. \tag{8}$$

Suppose that the order of ω is $n > 1$ (we are in the Case (a) of Fig. 1). Let us consider, in turn, the following linear system (obtained deleting some rows)

$$\begin{bmatrix} \ddots & & & & \\ \cdots & b_{-n,-1} & & & \\ \cdots & b_{0,-1} & 1 & & \\ \cdots & b_{n,-1} & 0 & b_{n,1} & \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \chi_{-1} \\ \chi_0 \\ \chi_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}. \tag{9}$$

Note that if the system showed in Equation (8) has a solution, then the system showed in Equation (9) also does. Note that, if $n = 1$, the column vector is non-trivial.

If $\chi \in \mathbb{K}((x)) \cup \mathbb{K}((\frac{1}{x}))$ and has order k, this system reduces to one of the following types (depending on $\chi \in \mathbb{K}((x))$ or $\chi \in \mathbb{K}((\frac{1}{x}))$):

$$\begin{bmatrix} b_{kk} & & \\ b_{k+n,k} & b_{k+n,k+1} & \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \chi_k \\ \chi_{k+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}, \qquad \begin{bmatrix} \ddots & & \\ \dots & b_{k-n,k-1} & \\ \dots & b_{k,k-1} & b_{kk} \end{bmatrix} \begin{bmatrix} \vdots \\ \chi_{k-1} \\ \chi_k \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ 0 \end{bmatrix}.$$

Note that the matrices of coefficients in the previous equation are not bi-infinite and that the elements in the main diagonal are different from 0.

In both cases we can conclude that the only solution is $\chi = 0$ (performing some kind of *Gaussian elimination* that, although requires to replace a row by a sum of infinitely many rows, it is well defined), which is a contradiction, since $\boldsymbol{B}_\omega \boldsymbol{B}_0 \neq \boldsymbol{I}$.

For the cases $\boldsymbol{B}_\omega \in \mathcal{L}_-$, $\boldsymbol{B}_\omega \in \mathcal{U}_-$, $\boldsymbol{B}_\omega \in \mathcal{U}_+$, we have that

$$\boldsymbol{B}_\omega = \boldsymbol{B}_{\widetilde{\omega}} \boldsymbol{J}, \qquad \boldsymbol{B}_\omega = \boldsymbol{J} \boldsymbol{B}_{\widetilde{\omega}}, \qquad \boldsymbol{B}_\omega = \boldsymbol{J} \boldsymbol{B}_{\widetilde{\omega}} \boldsymbol{J}, \tag{10}$$

respectively, for some $\boldsymbol{B}_{\widetilde{\omega}} \in \mathcal{L}_+$.

Let us see that this condition is sufficient for having an inverse. Suppose that $\boldsymbol{B}_\omega \in \mathcal{L}_+$ and ω has order 1. In this case, the classical theory for the group $(\mathcal{F}_1, \circ)$ ensures that ω has a compositional inverse $\omega^{[-1]}$ of order 1 and so $\boldsymbol{B}_{\omega^{[-1]}}$ is the inverse of $\boldsymbol{B}_\omega$. Again, using Corollary 7, we can study the rest of the cases.

For the matrices in Equation (10), if ω has order n, then $\widetilde{\omega}$ has order $\pm n$. Moreover, the matrices $\boldsymbol{B}_\omega$ in Equation (10) have an inverse if and only if so does $\boldsymbol{B}_{\widetilde{\omega}}$. This proves the necessity of the condition of ω having order ± 1.

Statement (d) is a consequence of Lemma 4.

The first part of Statement (e) is a consequence of Statement (c). The second part is a consequence of Statement (b) and Corollary 7. □

5. Generalized Riordan group

Definition 16. Let α, ω be two elements, both in $\mathbb{K}((x))$ or both in $\mathbb{K}((\frac{1}{x}))$. We define the **bi-infinite Riordan matrix** $\boldsymbol{R}_{\alpha,\omega}$ to be the bi-infinite matrix $\boldsymbol{R}_{\alpha,\omega} = [r_{ij}]_{-\infty<i,j<\infty}$ such that, for every $j \in \mathbb{Z}$, $\alpha \cdot \omega^j = \sum_{i=-\infty}^{\infty} r_{ij} x^i$.

The condition at the beginning ensures that the powers $\dots, \frac{\alpha}{\omega^2}, \frac{\alpha}{\omega}, \alpha, \alpha\omega, \alpha\omega^2, \dots$ are well defined. Note that $\boldsymbol{R}_{\alpha,x} = \boldsymbol{A}_\alpha$ and $\boldsymbol{R}_{1,\omega} = \boldsymbol{B}_\omega$.

Let us remark that the case $(\alpha, \omega) \in \mathcal{F}_0 \times \mathcal{F}_1$ was already introduced in [9]. In this case, the corresponding bi-infinite matrix $\boldsymbol{R}_{\alpha,\omega}$ is lower triangular and of the type

$$\left[\begin{array}{ccc|c} \ddots & & & \\ \dots & d_{-2,-2} & & \\ \dots & d_{-1,-2} & d_{-1,-1} & \\ \hline \dots & d_{0,-2} & d_{0,-1} & \\ \dots & d_{1,-2} & d_{1,-1} & R(\alpha,\omega) \\ & \vdots & \vdots & \end{array}\right]$$

where $R(\alpha, \omega)$ is a classical (mono-infinite) Riordan matrix. Additional comments concerning this particular case will appear at the end of this section, after discussing the algebraic properties of the matrices $\boldsymbol{R}_{\alpha,\omega}$.

Lemma 17. *Every bi-infinite Riordan matrix $\boldsymbol{R}_{\alpha,\omega}$ satisfies $\boldsymbol{R}_{\alpha,\omega} = \boldsymbol{A}_\alpha \boldsymbol{B}_\omega$. Furthermore, $\boldsymbol{R}_{\alpha,\omega}$ lies in any of the sets $\mathcal{L}_+$, $\mathcal{L}_-$, $\mathcal{U}_+$, $\mathcal{U}_-$ if and only if $\boldsymbol{B}_\omega$ does.*

Proof. The first part is a direct consequence of Theorem 10 and the second part ir a direct consequence of Lemma 8. □

Theorem 18 *(Final Generalization of the 1FTRM). Let*

$$\chi = \sum_{k=-\infty}^{\infty} \chi_k x^k \in \mathbb{L}(\mathbb{K}), \qquad \boldsymbol{v} = [\ldots, \chi_{-1}, \chi_0, \chi_1, \ldots]^T.$$

(a) *If $\omega \in \mathbb{K}((x))$ and has positive order and $\chi \in \mathbb{K}((x))$, then $\boldsymbol{R}_{\alpha,\omega}\boldsymbol{v}$ is well defined.*
(b) *If $\omega \in \mathbb{K}((\frac{1}{x}))$ and has negative order and $\chi \in \mathbb{K}((x))$, then $\boldsymbol{R}_{\alpha,\omega}\boldsymbol{v}$ is well defined.*
(c) *If $\omega \in \mathbb{K}((x))$ and has negative order and $\chi \in \mathbb{K}((\frac{1}{x}))$, then $\boldsymbol{R}_{\alpha,\omega}\boldsymbol{v}$ is well defined.*
(d) *If $\omega \in \mathbb{K}((\frac{1}{x}))$ and has positive order and $\chi \in \mathbb{K}((\frac{1}{x}))$, then $\boldsymbol{R}_{\alpha,\omega}\boldsymbol{v}$ is well defined.*
(e) *If $\omega \in \mathbb{K}((x)) \cup \mathbb{K}\left(\left(\frac{1}{x}\right)\right)$ and $\chi \in \mathbb{K}((x)) \cap \mathbb{K}\left(\left(\frac{1}{x}\right)\right)$, for some $n \in \mathbb{Z}$, then $\boldsymbol{R}_{\alpha,\omega}\boldsymbol{v}$ is well defined.*

In any of the previous cases, the resulting vector $\boldsymbol{w} = [\ldots, w_{-1}, w_0, w_1, \ldots]^T$ corresponds to the formal Laurent series ψ

$$\psi = \sum_{k=-\infty}^{\infty} w_k x^k = \alpha \cdot (\chi \circ \omega).$$

Proof. First, in view of Lemma 17, we write $\boldsymbol{R}_{\alpha,\omega} = \boldsymbol{A}_\alpha \boldsymbol{B}_\omega$. Then, we use Theorem 13 to give meaning to $\boldsymbol{B}_\omega \boldsymbol{v}$. Finally, we use Theorem 10 to give meaning to $\boldsymbol{A}_\alpha(\boldsymbol{B}_\alpha \boldsymbol{v})$. □

The set of all classical (mono-infinite) Riordan matrices is a group. But, in general, the multiplication is not closed in the set of bi-infinite Riordan matrices $\boldsymbol{R}_{\alpha,\omega}$. Consequently, this set is not a group. Also, not every bi-infinite Riordan matrix $\boldsymbol{R}_{\alpha,\omega}$ is invertible. Some important cases are considered in the following result.

Theorem 19.

(a) *A bi-infinite Riordan matrix $\boldsymbol{R}_{\alpha,\omega}$ is invertible if and only if $\alpha \neq 0$ and $\boldsymbol{B}_\omega$ is invertible, that is, the order of ω is ± 1. In this case,*

$$(\boldsymbol{R}_{\alpha,\omega})^{-1} = \boldsymbol{R}_{1/(\alpha \circ \omega^{[-1]}), \omega^{[-1]}}.$$

(b) *The multiplication of two bi-infinite Riordan matrices* $\boldsymbol{R}_{\alpha,\omega}\boldsymbol{R}_{\beta,\chi}$ *is well defined if and only if* $\boldsymbol{B}_\omega\boldsymbol{B}_\chi$ *is well defined. In this case,* $\boldsymbol{R}_{\alpha,\omega}\boldsymbol{R}_{\beta,\chi} = \boldsymbol{R}_{\alpha\cdot(\beta\circ\omega),\chi\circ\omega}$.

(c) *The sets of matrices* $\boldsymbol{R}_{\alpha,\omega} \in \mathcal{L}_+$ *and the set of matrices* $\boldsymbol{B}_\omega \in \mathcal{U}_+$ *endowed with the multiplication, are monoids.*

(d) *The sets of matrices* $\boldsymbol{R}_{\alpha,\omega} \in \mathcal{L}_+$ *such that* $\alpha \neq 0$ *and* ω *has order 1 and the set of matrices* $\boldsymbol{R}_{\alpha,\omega} \in \mathcal{U}_+$ *such that* $\alpha \neq 0$ *and* ω *has order 1, endowed with the multiplication, are groups, that will be called the* ***generalized lower/upper Riordan group****, respectively. The function given by*

$$\boldsymbol{R}_{\alpha,\omega} \mapsto \boldsymbol{J}\boldsymbol{R}_{\alpha,\omega}\boldsymbol{J},$$

is an isomorphism (from the first one to the second one and from the second one to the first one) and an involution (the iterated composition of this isomorphism is the identity).

Proof. First, Statement (a) is a consequence of the decomposition in Lemma 17 and of the existence of inverses appearing in Theorems 11 and 15.

Second, the first part of Statement (b) is a consequence of Lemma 17 and Lemma 8. To prove the second part, note that, according to Theorem 13, the product $\boldsymbol{B}_\omega\boldsymbol{A}_\alpha$ is well defined and $\boldsymbol{B}_\omega\boldsymbol{A}_\alpha = \boldsymbol{A}_{\alpha\circ\omega}\boldsymbol{B}_\omega$.

So,

$$\boldsymbol{R}_{\alpha,\omega}\boldsymbol{R}_{\beta,\chi} = \boldsymbol{A}_\alpha\boldsymbol{B}_\omega\boldsymbol{A}_\beta\boldsymbol{B}_\chi = \boldsymbol{A}_\alpha\boldsymbol{A}_{\beta\circ\omega}\boldsymbol{B}_\omega\boldsymbol{B}_\chi.$$

Third, Statements (c) and (d) are a combination of Theorems 11 and 15. □

What it is called *generalized lower Riordan group* is actually the set of Riordan matrices $\{\boldsymbol{R}_{\alpha,\omega} : \alpha \in \mathcal{F}_0, \omega \in \mathcal{F}_1\}$, already studied in [9] and isomorphic to the classical Riordan group.

The generalized lower (resp. upper) Toeplitz group and the generalized lower (resp. upper) Lagrange group are subgroups of the generalized lower (resp. upper) Riordan group. Furthermore, the first one is a normal subgroup (we omit details).

6. Application: palindromic polynomials

When the study of the Riordan group began in the paper [16], the goal was not to analyze an algebraic abstract object, but to investigate an interesting structure due to its applications in Combinatorics. In this section, we aim to demonstrate that, like the classical Riordan group, the generalized Toeplitz, Lagrange, and Riordan groups presented here also provide a framework that facilitates the proof of various combinatorial identities, including some not available for the classical Riordan group. Furthermore, these identities can be studied within the context of the *Concrete Mathematics* (in the sense of the term used by R. L. Graham, E. D. Knuth and O. Patashnik in [6]). As an example, we present a really short proof of a well-known identity from the theory of finite simplicial complexes.

For the sake of brevity, we refer the reader to [10] for the concepts of *finite simplicial complex* and *f-polynomial*. None of these concepts are crucial. We only need to know that the *extended f-polynomial* of a simplicial complex is a polynomial $f(x) = f_{-1} + f_0 x + \ldots + f_d x^{d+1}$, being d the dimension of the simplicial complex. In the aforementioned paper [10], the relation of this kind of problems regarding f-vectors and Riordan matrices was explored. The reader may find more information about problems concerning f-vectors in [7,18] and the references therein.

For some families of finite simplicial complexes (like partitionable simplicial complexes), it makes sense to study, not only the extended f-polynomial but also the **h-polynomial**, defined as

$$h(x) = h_0 + h_1 x + \ldots + h_{d+1} x^{d+1} = (1-x)^{d+1} f\left(\frac{x}{1-x}\right). \tag{11}$$

(Section 8.3 in [18]), since, in this case, the coefficients $h_0, \ldots, h_{d+1}$ have combinatorial meaning.

We also need to recall that, for many important families of finite simplicial complexes (such as simplicial polytopes), the h-polynomial is **palindromic**, that is,

$$h_k = h_{d+1-k}, \qquad \forall k = 0, \ldots, d+1. \tag{12}$$

These conditions are known as the **Dehn-Sommerville Equations** (Theorem 8.21 in [18]).

Our approach allows us to find very quickly another version of interest of the Dehn-Sommerville equations stated only in terms of the f-vector that can be found in Section 9.2 in [7].

Theorem 20. *The Dehn Sommerville equations hold if and only if the following equations hold:*

$$\sum_{j=k}^{d} (-1)^j \binom{j+1}{k+1} f_j = (-1)^d f_k, \qquad k = -1, \ldots, d. \tag{13}$$

Proof. Let $\boldsymbol{f}, \boldsymbol{h}$ be the bi-infinite vectors containing the coefficients of the f-polynomial and the h-polynomial, respectively. Following the notation shown in this paper, Equation (11) is equivalent to $\boldsymbol{R}_{x^{d+1}, \frac{1}{x}} \boldsymbol{h} = \boldsymbol{h}$ and the Dehn-Sommerville Equations (12) are just $\boldsymbol{h} = \boldsymbol{R}_{(1-x)^{d+1}, \frac{x}{1-x}} \boldsymbol{f}$.

Starting with

$$\boldsymbol{R}_{x^{d+1}, \frac{1}{x}} \boldsymbol{h} = \boldsymbol{h},$$

and substituting $\boldsymbol{h} = \boldsymbol{R}_{(1-x)^{d+1}, \frac{x}{1-x}} \boldsymbol{f}$, we get

$$\boldsymbol{R}_{x^{d+1}, \frac{1}{x}} \boldsymbol{R}_{(1-x)^{d+1}, \frac{x}{1-x}} \boldsymbol{f} = \boldsymbol{R}_{(1-x)^{d+1}, \frac{x}{1-x}} \boldsymbol{f}.$$

Now we use the multiplication rule for the product:

$$\boldsymbol{R}_{(x-1)^{d+1}, \frac{1}{x-1}} \boldsymbol{f} = \boldsymbol{R}_{(1-x)^{d+1}, \frac{x}{1-x}} \boldsymbol{f}.$$

Left multiplying both sides of the equation by the inverse of $\boldsymbol{R}_{(1-x)^{d+1}, \frac{x}{1-x}}$, we obtain

$$\boldsymbol{R}_{(1+x)^{d+1}, \frac{x}{1+x}} \boldsymbol{R}_{(x-1)^{d+1}, \frac{1}{x-1}} \boldsymbol{f} = \boldsymbol{f}.$$

Using one more time the multiplication rule, we reach:

$$\boldsymbol{R}_{(-1)^{d+1}, -(1+x)} \boldsymbol{f} = \boldsymbol{f}$$

The expression in Equation (13) follows from expanding the previous matrix:

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \boxed{\pm 1} & \mp 1 & \pm 1 & \mp 1 & \cdots \\ \cdots & & \mp 1 & \pm 2 & \mp 2 & \cdots \\ \cdots & & & \pm 1 & \mp 3 & \cdots \\ \cdots & & & & \mp 1 & \cdots \\ & & & & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ 0 \\ \boxed{f_{-1}} \\ f_0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ \boxed{f_{-1}} \\ f_0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix}. \quad \square$$

7. Some perspectives

As the reader may have imagined, for the (classical) Riordan group, we can also find a 2FTRM in the literature, concerning the existence of a power series whose coefficients form the so called *A-sequence*. This has been deliberately omitted in the paper for the sake of brevity, but these ideas should be extended to the generalized Riordan group.

At the beginning of Section 4 we have included an example, for the case $\mathbb{K} = \mathbb{C}$, for which two different formal Laurent series correspond to different expansions (at 0 and at ∞) of the same meromorphic function. Finding relations between one expansion and the other is a classical problem in Complex Analysis. We have intentionally abandoned the study of convergence in this setting, but maybe the framework provided here may help to approach these kinds of problem.

Also, for the (classical) Riordan group, the study of eigenproblems has received some recent attention (see, for example, [4,11–13] and the references therein). In the setting of generalized Riordan matrices that we have just described, these problems seem to be more challenging, but equally interesting (see, for example, the previous proof).

In relation to these eigenproblems, several papers in the literature address involutions in the (classical) Riordan group (see, for example, [3,11,12] and the references therein). We have mentioned here that the matrix $\boldsymbol{J}$ is an involution. It could be interesting to study, in general, the problem of finding and describing all the involutions in this new generalized context.

At the end of Section 5, we have already pointed out that the generalized lower Riordan group appeared already in [9]. In that paper, some concepts with combinatorial meaning, such as complementary and dual matrices, are discussed. These questions would also be interesting in this new framework.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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