On Fuzzy Arithmetic, Possibility Theory and Theory of Evidence

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Abstract

This paper explores the existing relationship between Possibility Theory and Theory of Evidence, when they are both applied to fuzzy arithmetic. Possibility Theory arithmetic is based on the extension principle (projection of the joint possibility distribution), while in Theory of Evidence, the consonant bodies of evidence obtained from each operand are combined into a new joint body of evidence, which can in general be non consonant. Identical behaviour is found when the joint possibility distribution is calculated using the *min* operator, while Possibility Theory gives more specific results when others T-norms are used. This has been considered by some authors as a Theory of Evidence drawback (Dubois & Prade 1989). This paper shows that Theory of Evidence may be a more realistic uncertainty model when input data are obtained from random experiments with imprecise outcomes.

1 INTRODUCTION

There is a straight forward relationship between Possibility Theory and Theory of Evidence, when consonant bodies of evidence are involved. In this case possibility and plausibility measures coincide. Interpreting basic assignments as density functions, where the random variables are the focal elements, simulations can be performed from given possibility distributions.

This paper shows how the sum of two fuzzy numbers A and B can be calculated, applying both the extension principle and the theory of evidence, and compares the results.

Section 2 reviews some basic definitions. Section 3 shows how the joint possibility/plausibility distribution can be obtained. Section 4 compares the possibility/plausibility distribution of the union of two points. Section 5 compares the results of summing two fuzzy numbers using both approaches, and finally some conclusions are presented in Section 6.

2 DEFINITIONS REVIEW

Plausibility and Belief measures are fuzzy measures defined by (see (Klir 1988) (Shafer 1987)):

$$Pl : P(X) \to [0,1]$$

$$Bel : P(X) \to [0,1]$$

such that:

$$Pl(A_1 \cap A_2 \cap \ldots A_n) \le \sum_i Pl(A_i) \ - \ \sum_{i<j} Pl(A_i \cup A_j) \ + \ \ldots$$
$$+ \ (-1)^{n+1} Pl(A_1 \cup A_2 \cup \ldots A_n)$$

$$Bel(A_1 \cup A_2 \cup \ldots A_n) \ge \sum_i Bel(A_i) \ - \ \sum_{i<j} Pl(A_i \cap A_j) \ + \ \ldots$$
$$+ \ (-1)^{n+1} Bel(A_1 \cap A_2 \cap \ldots A_n)$$

where $P(X)$ is the power set of crisp subsets of X.

Plausibility/Belief measures can also be defined, given a body of evidence *(F,m)*, as:

$$Pl(B) = \sum_{A_i \cap B \neq 0} m(A_i)$$

$$Bel(B) = \sum_{A_i \subseteq B} m(A_i)$$

where A_i are the focal elements and *m* is the basic probability assignment (Alvarez 1994).

When the body of evidence is consonant, that is, its focal elements are nested, then the plausibility (resp belief) measure is called possibility (resp necessity) measure, and the following properties hold:

$$Pl(A \cup B) = max[Pl(A), Pl(B)] \ \to \pi(A \cup B) = max[\pi(A), \pi(B)]$$

$$Bel(A \cap B) = min[Bel(A), Bel(B)] \ \to N(A \cap B) = min[N(A), N(B)]$$

Taking into account the body of evidence, a fuzzy set can be given a probabilistic interpretation. The basic assignment is view as a probability density function whose random variable is the set of focal elements. The possibility/plausibility distribution function is defined from the plausibility measure definition by:

$$\mu(x) = Pl(\{x\}) = \sum_{A_i \cap \{x\} \neq \varnothing} m(A_i) = \sum_{x \in A_i} m(A_i)$$

$\mu(x)$ can then be interpreted as a probability distribution function of the focal elements, and Montercarlo method can be used to obtain a realisation of the random variable, that is, to obtain a set A_i.

A possibility distribution can be also be represented in terms of its alpha-cuts (Dubois & Prade 1989) (Dubois & Prade 1986 a):

$$\left\{ F_\alpha \mid \alpha \in (0,1] \right\} \text{ where } F_\alpha = \left\{ w \mid \mu_F(w) \geq \alpha \right\}$$

$$\mu_F(w) = \sup\left\{ \alpha \in (0,1] \mid w \in F_\alpha \right\}$$

In the following, only possibility and plausibility measures will be considered.

3 JOINT POSSIBILITY / PLAUSIBILITY DISTRIBUTION

3.1 JOINT DISTRIBUTION IN POSSIBILITY THEORY

Having two fuzzy numbers A and B with possibility distributions $\pi_A(a)$ and $\pi_B(b)$ defined over U_A and U_B, the joint possibility distribution π_{AxB} can be obtained combining them with a T-norm:

$$\pi_{AxB}\left(\left\{(a_1, b_1)\right\}\right) = T(\pi_A(a_1), \pi_B(b_1))$$

where $a_i \in U_A$ and $b_i \in U_B$.

Figure 1 shows A and B possibility distributions and the joint distribution π_{AxB} for the minimum, product or Luckasievicz T-norms. As it can be seen, the min T-norm gives the least specific result.

When both numerical variables a_i and b_i correspond the same physical variable, a possibility distribution can be obtained cutting the previous surfaces with $a_i = b_i$, as shown in Figure 2 (Zadeh 1977). It is supposed that the sources are completely reliable, as conjunctive consensus has been used (Dubois & Prade 1988).

3.2 JOINT DISTRIBUTION IN THEORY OF EVIDENCE

If we consider the consonant body of evidence of each fuzzy set (F_A, m_A), (F_B, m_B) as the random variables density functions, the relationship between both random variables can be used to calculate the joint basic assignment m_{AxB}. The plausibility measure is then given by:

$$Pl((a_1, b_1)) = \sum_{C_i \cap \{(a_1, b_1)\} \neq \varnothing} m_{AxB}(C_i)$$

where C_i are the focal elements of the joint body of evidence, defined in $A_i \times B_i$, or a subset, depending on the relationship between the random variables.
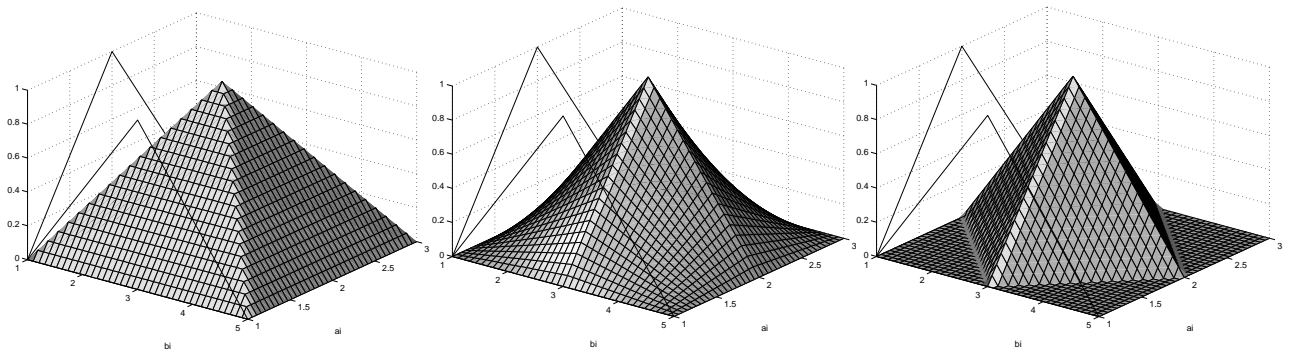

Figure 1: Joint possibility distribution, with different T-norms:
a) minimum, b) product, c) Lukasiewicz
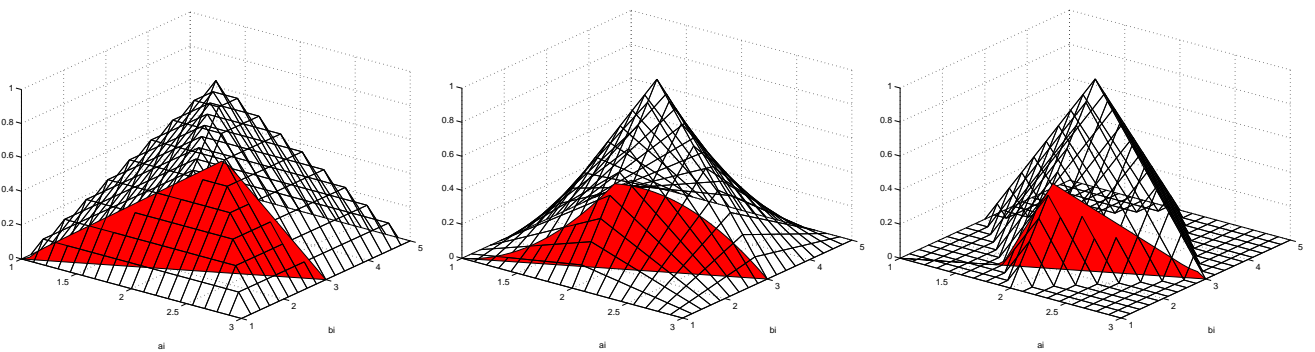

Figure 2: A and B possibility distribution when $a_i = b_i$, with different T-norms:
a) minimum, b) product, c) Lukasiewicz

If the joint body of evidence is consonant, this plausibility measure is also a possibility measure.

In the following, three kinds of relationships between the random variables will be analysed: $\alpha_A = \alpha_B$, independence, and $\alpha_A = 1 - \alpha_B$. As explained in (Tan 1993), when probability is concentrated and uniformly distributed on the main diagonal of the joint domain, α_A and α_B are in perfect positive correlation. It can also be interpreted as concordance between A and B sources of knowledge (for example, the same instrument has been used to measure A and B intervals).

When the whole probability is concentrated and uniformly distributed on the anti-diagonal, α_A and α_B are in perfect negative correlation. It can be interpreted as a discrepancy between both sources of knowledge, or between precision in measurements. Independence remains with its usual interpretation.

3.2.1 $\alpha_A = \alpha_B$ relationship

In this case the joint basic assignment is only defined in the line shown in Figure 3. A and B basic assignments are displayed in X and Y axes, and the joint basic probability assignment in the corresponding subset of $A_i \times B_i$.

Each point of the A basic assignment represents an interval that is a (F_A, m_A) body of evidence focal element. Every interval is represented by its lower limit, and thus the domain of the basic assignment is the domain of the intervals lower limits. This graphical representation is similar to (Tan 1993), where focal elements are represented by their corresponding alpha value. But lower limit representation allows to show plausibility measures in the basic assignment graph.
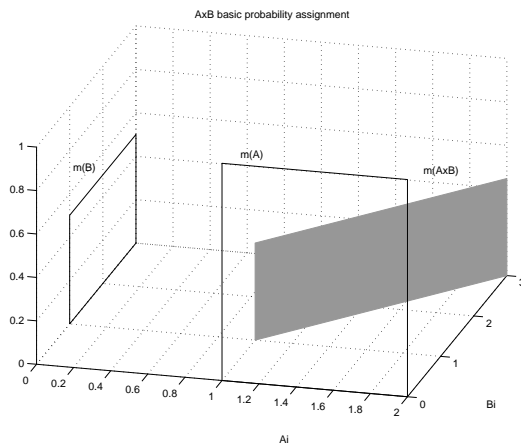

Figure 3: Joint basic assignment when $\alpha_A = \alpha_B$

Every pair (a_i, b_j) located in the domain of the joint basic assignment m_{AxB} represents a focal element of the joint body of evidence (F_{AxB}, m_{AxB}). Figure 4 shows that in this case the joint focal elements are nested, and thus plausibility measures will be possibility measures.

To calculate the possibility of a_1 and b_1 (see Figure 4), every joint focal element containing the pair (a_1, b_1) has to be considered. That is,

$$\pi((a_1, b_1)) = \sum_{k=i}^{n} m_{AxB}(C_k)$$

or, expressed for continuous variables:

$$\pi((a_1, b_1)) = \int_{C_k = C_i}^{C_n} m_{AxB}(C_k) \cdot dC_k$$
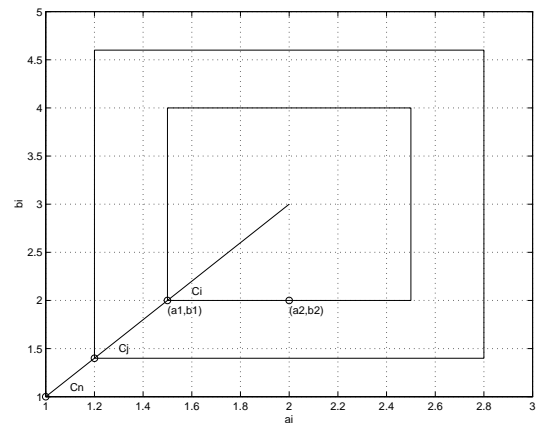
which is equal to the area indicated in Figure 5.

Figure 4: Joint body of evidence focal elements when $\alpha_A = \alpha_B$

Since this point belongs to the line corresponding to $\alpha_A = \alpha_B$, it is $\pi((a_1, b_1)) = \pi_A(a_1) = \pi_B(b_1)$.

To calculate the possibility of the pair (a_2, b_2) shown in Figure 4, it should be noted that the joint focal elements containing this point are the same joint focal elements that contain the previous one (a_1, b_1). That is,

$$\pi((a_2, b_2)) = \pi((a_1, b_1)) = \pi_B(b_2) = min(\pi_A(a_2), \pi_B(b_2))$$
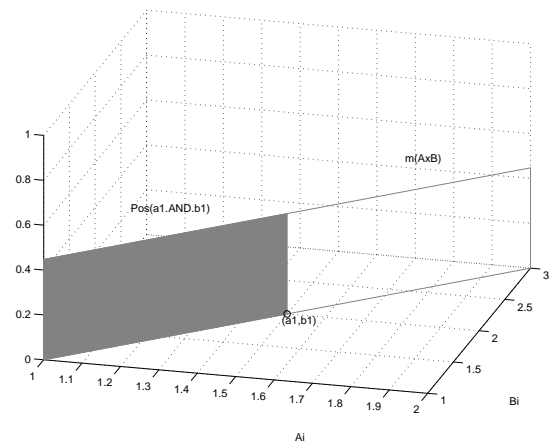

Figure 5: Possibility of (a_1, b_1) when $\alpha_A = \alpha_B$

Calculating the possibility of every pair (a_i, b_j), the joint possibility distribution is the one obtained in possibility theory when the *min* T-norm is used.

Numerical simulation can be applied to obtain the same result R using both approaches, when A and B refer to the same physical variable. (Figure 2.a). To perform the simulation, random variables are obtained by Montecarlo method: a value of alpha is generated as a uniform distribution between 0 and 1. With this value, alpha-cuts of A and B are obtained, which are realisations of the random variables. The intersection of both intervals is calculated, and the result is a focal element of (F_R, m_R). These focal elements are nested and the possibility measure associated to the resulting body of evidence can be obtained by the formula

$$\pi(r_1) = \sum_{k=i}^{n} m_R(R_k)$$

3.2.2 Independence relationship

If A_i and B_i random variables are independent, the joint basic assignment domain is the whole cartesian product $A_i \times B_i$, where the basic assignment is uniformly distributed (see Figure 6).
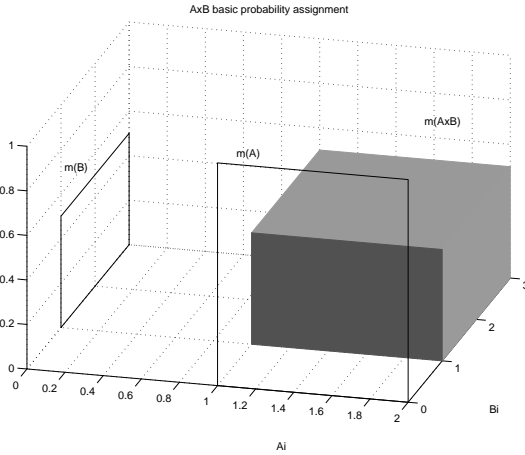

Figure 6: Joint basic assignment when A_i and B_i are independent

Again, every point in the domain represents a joint focal element, build from a focal element of (F_A, m_A) and a focal element of (F_B, m_B) (represented both by their lower limits). Two of these joint focal elements are shown in Figure 7. As the domain is the whole cartesian product, the focal elements are not nested, and the plausibility measures associated to the joint body of evidence are not possibility measures.
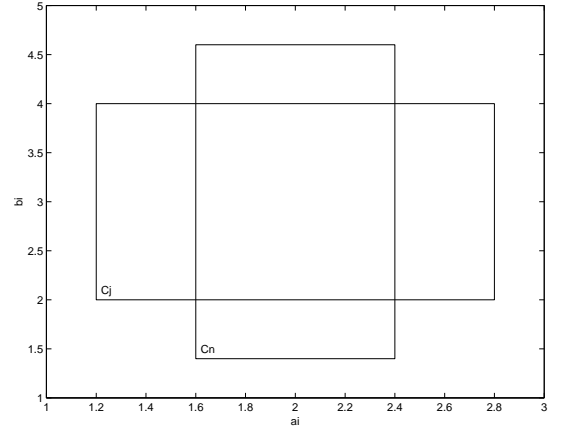

Figure 7: Joint body of evidence focal elements when A_i and B_i are independent

Plausibility of (a_1 and b_1) is calculated considering every joint focal element that contains the point (a_1, b_1). Every focal element, represented by a point located in the volume base in Figure 8, contains (a_1, b_1). Applying the plausibility formula for continuous variable, the plausibility is the volume shown in Figure 8.

$$Pl((a_1, b_1)) = \int_{C_i \cap (a_1, b_1) \neq \varnothing} m_{AxB}(C_i) \cdot dC_i =$$

$$= \int_{A_i = A_j}^{A_n} \int_{B_i = B_k}^{B_n} m_A(A_i) \cdot m_B(B_i) \cdot dA_i \cdot dB_i =$$

$$= \int_{A_i = A_j}^{A_n} m_A(A_i) \cdot dA_i \cdot \int_{B_i = B_k}^{B_n} m_B(B_i) \cdot dB_i = \pi_A(a_1) \cdot \pi_B(b_1)$$

Furthermore, (a_1, b_1) plausibility is equal to the product of possibilities $\pi_A(a_1)$, $\pi_B(b_1)$, and thus the plausibility distribution obtained from the theory of evidence is the same as the possibility distribution obtained from the possibility theory.

As it will be explained later, the difference is that the underlying body of evidence in possibility theory is consonant, while in the theory of evidence it is not.
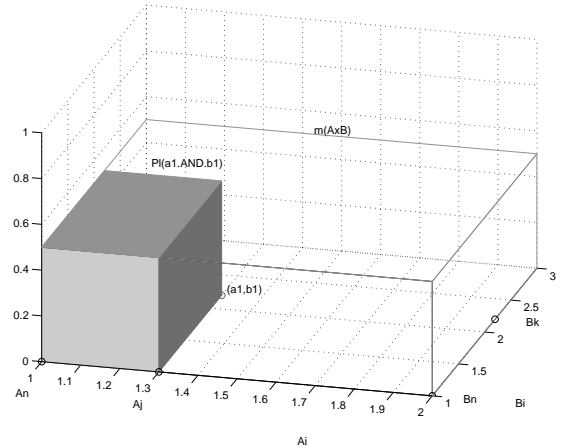

Figure 8: Plausibility of (a_1, b_1) when A_i and B_i are independent

The same conclusion was reached using numerical simulation, when A and B refer to the same variable. Numerical simulation with independent random variables gives the same result as the possibility theory approach, when the product t-norm is used.

Numerical simulation has been performed as before, but two different values of alpha are obtained independently, one for A an the other one for B.

3.2.3 $\alpha_A = 1 - \alpha_B$ relationship

When the relationship between A_i and B_i is somehow contradictory, $\alpha_A = 1 - \alpha_B$, the joint focal elements are only defined in the line shown in figure 9, where the joint basic assignment is uniformly distributed (Tan 1993).
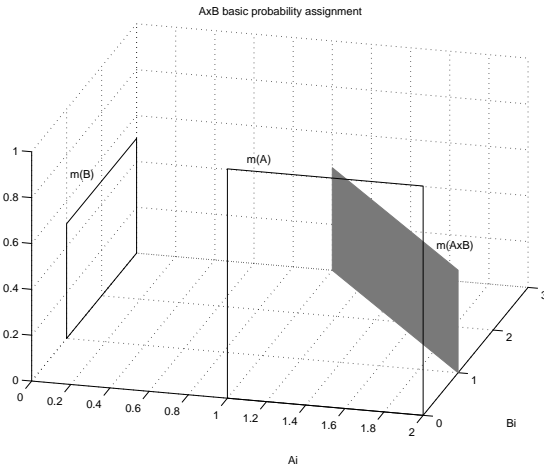

Figure 9: Joint basic assignment when $\alpha_A = 1 - \alpha_B$

In this case, as shown in Figure 10, focal elements are again not nested, giving a non consonant body of evidence, and thus, plausibility measures are not possibility measures.
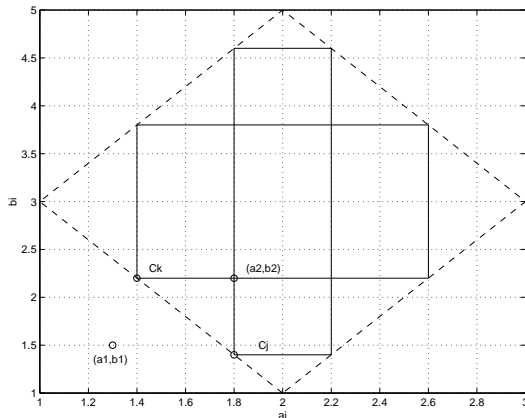

Figure 10: Joint body of evidence focal elements when $\alpha_A = 1 - \alpha_B$

Let us consider the point (a_1, b_1) in Figure 10. Since no joint focal element contains that point, its plausibility is null. On the other hand, (a_2, b_2) plausibility is not null.

Every joint focal element from C_k to C_j has to be considered, because it contains the point (a_2, b_2) giving:

$$Pl((a_2, b_2)) = \int_{C_i = C_k}^{C_j} m_{AxB}(C_i) \cdot dC_i$$

which can be interpreted as the area shown in Figure 11.
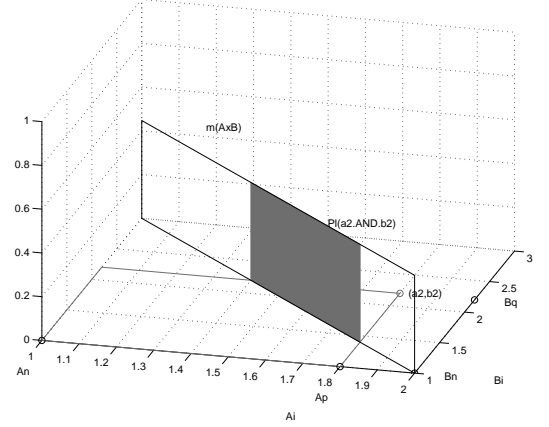

Figure 11: Plausibility of (a_2, b_2) when $\alpha_A = 1 - \alpha_B$

Plausibility measure can be expressed in terms of the initial bodies of evidence:

$$Pl((a_2, b_2)) = \int_{C_i = C_k}^{C_j} m_{AxB}(C_i) \cdot dC_i =$$

$$= \int_{C_i = C_n}^{C_j} m_{AxB}(C_i) \cdot dC_i - \int_{C_i = C_n}^{C_k} m_{AxB}(C_i) \cdot dC_i =$$

$$= \int_{A_i = A_p}^{A_n} m_A(A_i) \cdot dA_i - \int_{B_i = B_1}^{B_q} m_B(B_i) \cdot dB_i =$$

$$= \int_{A_i = A_p}^{A_n} m_A(A_i) \cdot dA_i - \left(1 - \int_{B_i = B_q}^{B_n} m_B(B_i) \cdot dB_i\right) =$$

$$= \pi_A(a_2) - 1 + \pi_B(b_2)$$

And in general, for any pair (a_i, b_j), it is:

$$Pl((a_i, b_j)) = max\left(0, \pi_A(a_i) + \pi_B(b_j) - 1\right)$$

that is, Lukasiewicz T-norm. The joint plausibility distribution in this case is the same as the possibility distribution obtained by possibility theory, the difference being again the underlying body of evidence.

Numerical simulation gives the same result. A value of alpha is obtained (uniformly distributed between 0 and 1) and the other one is calculated according to $\alpha_A = 1 - \alpha_B$.

4 PLAUSIBILITY OF $(a_1, b_1) \cup (a_2, b_2)$

4.1 POSSIBILITY THEORY

The possibility of a set is the maximum possibility of every point belonging to it, that is:

$$\pi\big((a_1, b_1) \cup (a_2, b_2)\big) = max\big(\pi(a_1, b_1), \pi(a_2, b_2)\big)$$

where the joint possibility is obtained with a T-norm.

4.2 THEORY OF EVIDENCE

Given a body of evidence (F_{AxB}, m_{AxB}), the plausibility measure of a two points set is given by:

$$Pl\big((a_1, b_1) \cup (a_2, b_2)\big) = \sum_{\substack{C_i \cap (a_1, b_1) \neq \varnothing \\ or \\ C_i \cap (a_2, b_2) \neq \varnothing}} m_{AxB}(C_i)$$

that is, every joint focal element containing at least one of the two points must be considered.

Figure 12.a shows the joint focal elements that contain (a_1, b_1) or (a_2, b_2), when the relationship between random variables is $\alpha_A = \alpha_B$. In this case, the plausibility of the union is equal to the plausibility of (a_2, b_2), which is the maximum plausibility of both points (*max* operator is obtained when focal elements are nested, (Klir 1988)). The result is the same as in possibility theory because the underlying body of evidence is also the same.

Figure 12.b shows the joint focal elements when the random variables are independent. All the joint focal elements located in the marked area contain at least one of the two points. Expressing the formula for continuous variables, the plausibility is the integral of the uniform distribution m_{AxB} over the indicated area (that is, the volume whose base is the indicated area). This volume is in general greater or equal than the volumes obtained for $Pl\big((a_1, b_1)\big)$, or $Pl\big((a_2, b_2)\big)$. Then it is:

$$Pl\big((a_1, b_1) \cup (a_2, b_2)\big) \geq max\big(Pl(a_1, b_1), Pl(a_2, b_2)\big)$$

This discrepancy between both approaches is due to the fact that the body of evidence obtained from the theory of evidence is not consonant, while possibility theory always considers, among all the different bodies of evidence with the same possibility/plausibility distribution, the underlying consonant body of evidence. The use of *max* operator in the extension principle means that the considered body of evidence is the consonant one.

Figure 12.c shows the joint focal elements that contain at least one of the points, when the relationship between the random variables is $\alpha_A = 1 - \alpha_B$. Again, expressing the plausibility for continuous variables, its value is given by the area whose base is marked, and in general it is greater or equal than the individual plausibility measures. As in the previous case, the body of evidence considered by the theory of evidence is not consonant, while the body of evidence underlying in possibility theory calculus is the consonant one.

5 SUM OF A AND B

In this section the sum of two fuzzy numbers *A* and *B* will be discussed using the previous results. Given *A* and *B* their sum *R=A+B* will be obtained computing the possibility/plausibility of the union of points located in the line defined by $a_j + b_k = r_i$, where $r_i \in U_R$.

5.1 POSSIBILITY THEORY

Applying the extension principle, the possibility of each r_i is obtained as the maximum of the possibilities of the pairs (a_j, b_k) verifying $a_j + b_k = r_i$, which defines a section in the joint possibility distribution. Figure 13 shows the section obtained for a particular r_i.
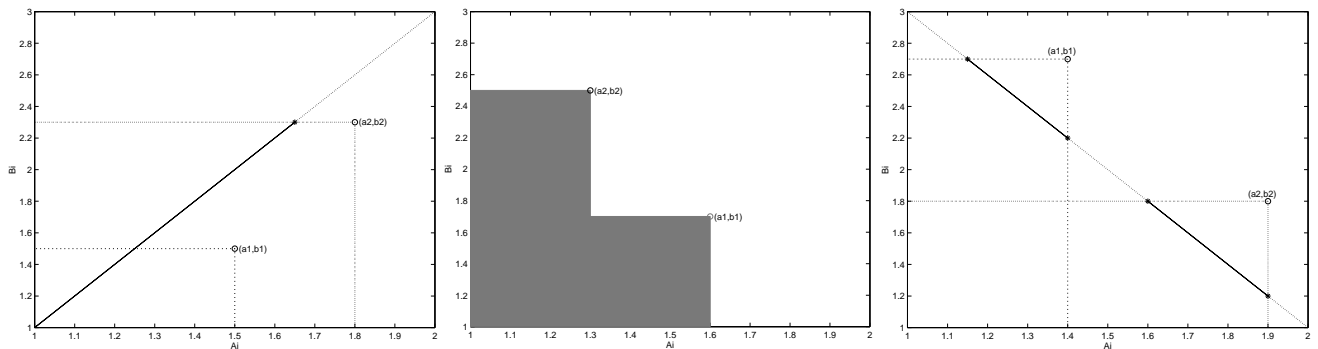

Figure 12: Joint focal elements containing (a_1, b_1) or (a_2, b_2).
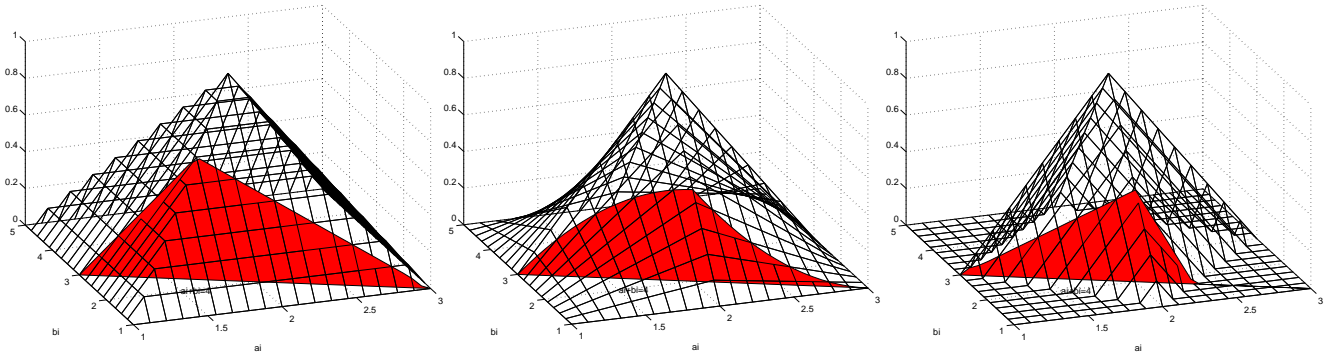a) $\alpha_A = \alpha_B$, b) independence, c) $\alpha_A = 1 - \alpha_B$

Figure 13: $a_j + b_k = constant$ cut with different T-norms:
a) minimum, b) product, c) Lukasiewicz

5.2 THEORY OF EVIDENCE

Given a line defined by $a_j + b_k = constant = r_i$, the plausibility of r_i is calculated summing the basic probability assignment of every joint focal element containing any of the points of the line.

Figure 14.a shows these focal elements when the random variables are related by $\alpha_A = \alpha_B$. For example the possibility of $r_i = 3$ is given by the possibility of the point where the diagonal and the line $a_j + b_k = 3$ intersect.

Figure 14.b shows the joint focal elements where the random variables are independent. The plausibility measure is the volume whose base is the marked area.

When the random variables are related by $\alpha_A = 1 - \alpha_B$, see figure 14.c, there are no joint focal elements located in $\alpha_A = 1 - \alpha_B$ line containing any point defined by $a_j + b_k = 3$. This means that the plausibility of 3, in this case, is zero. The figure also shows the joint focal elements that must be taken into account to calculate the plausibility of 3.5.

Again it can be checked that the plausibility of the union is greater or equal to the maximum of all of them:

$$Pl(r_i) \geq \max_{a_j + b_k = r_i} \left(Pl(a_j, b_k) \right)$$

The values of R under these three assumptions were also computed using numerical simulations. Again Montecarlo method was used to obtain the focal elements of A and B that had to be added. Apart from the case where $\alpha_A = \alpha_B$ that gives the same result using both approaches, in general theory of evidence leads to less specific distributions than possibility theory.
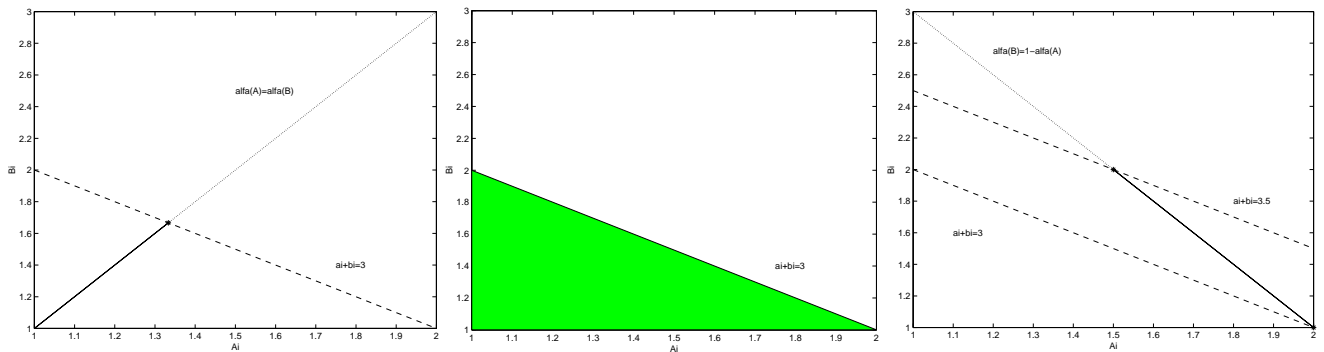

Figure 14: Joint focal elements containing any point of $a_j + b_k = constant$:
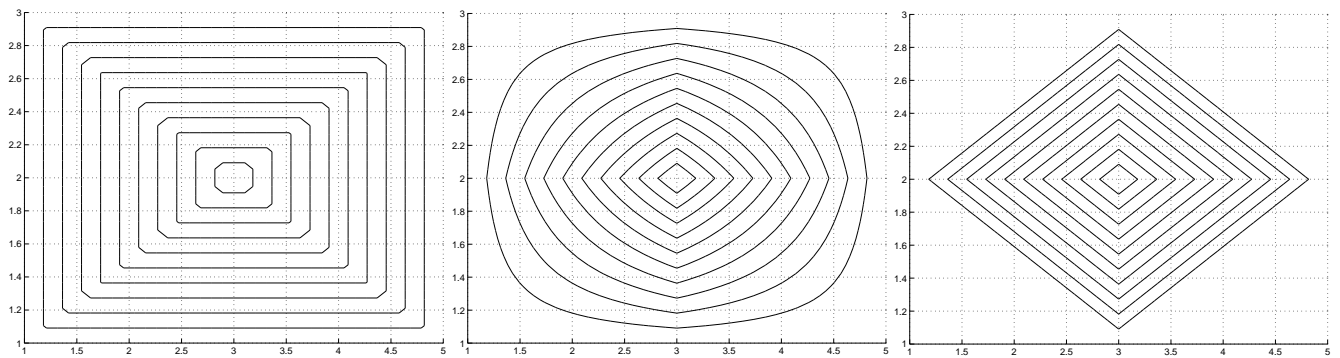a) $\alpha_A = \alpha_B$, b) independence, c) $\alpha_A = 1 - \alpha_B$

Figure 15: Joint focal elements with different T-norms:
a) minimum, b) product, c) Lukasiewicz

6 CONCLUSIONS

This paper analyses two different approaches to aggregate possibility distributions and to operate fuzzy numbers, using possibility theory and fuzzy arithmetic in one hand, and theory of evidence on the other hand.

Given the focal elements of each of the operands (interpreted as random variables), the conjunctive joint possibility/plausibility distributions obtained from both methods are identical when:

- the random variables are related by $\alpha_A = \alpha_B$ and the *and* operator used in possibility theory is the *min* t-norm.

- the random variables are independent and the *and* operator used in possibility theory is the product T-norm.

- the random variables are related by $\alpha_A = 1 - \alpha_B$ and the *and* operator used in possibility theory is the Lukasiewicz t-norm.

Given a subset of the conjunctive joint possibility distribution, its possibility is calculated using the *max* operator, and thus implicitly assuming a consonant underlying body of evidence. However in theory of evidence, the plausibility distribution must be obtained from an explicitly calculated body of evidence, which is in general not consonant, leading to different and less specific results.

If we suppose (see (Dubois & Prade 1986 b)) that fuzzy numbers *A* and *B* are obtained from random experiments with imprecise outcomes, that is, each measure is an interval where no distinctions can be made, the theory of evidence seems a more realistic model. Additionally this means that Montecarlo simulation can be used to perform the computations.

On the contrary the nested underlying focal elements obtained from the joint possibility distributions (using product and Lukasiewicz t-norms) can not be interpreted the same way (see figure 15), since they can not be obtained combining *A* and *B* focal elements. As seen

previously, the combination of *A* and *B* focal elements only produce rectangles parallel to the axes.

Theory of evidence approach leads to less specific results than possibility theory, although it could be considered a more realistic uncertainty model under the above assumptions, not just a possibility theory approximation (see (Dubois & Prade 1989)).

References

E. Alvarez, E. Castillo (1996). "Uncertainty Measures in Expert Systems". Microcomputers in Civil Engineering 9, 359-366

D. Dubois, H. Prade (1986 a). "A Set –Theoretic View of Believe Functions". International Journal of General Systems 12, 193-226.

D. Dubois, H. Prade (1986 b). "Fuzzy Sets and Statistical Data". European Journal of Operational Research 25, 345-356.

D. Dubois, H. Prade (1988). "Representation and Combination of Uncertainty with Belief Functions and Possibility Measures". Computational Intelligence 4, 244-264.

D. Dubois, H. Prade (1989). "Fuzzy Sets, Probability and Measurement". European Journal of Operational Research 40, 135-154

G.J. Klir, T.A. Folger (1988). "Fuzzy Sets, Uncertainty, and Information". Prentice-Hall International Editions

G. Shafer (1987). "Belief Functions and Possibility Measures". Analysis fuzzy information, vol. 1, 51-84

S.K. Tan, P.Z. Wang, X.Z. Zhang (1993). "Fuzzy Inference Relation based on the Theory of Falling Shadows". Fuzzy Sets and Systems 53, 179-188

L.A. Zadeh (1977). "Fuzzy Sets as a basis for a Theory of Possibility". Fuzzy Sets and Systems 1, 3-28