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Combinatorial bounds on paired and multiple domination in triangulations

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Abstract

In this article we study some variants of domination in triangulation graphs. We establish tight combinatorial bounds for paired domination and 2-domination in maximal outerplanar graph, and for double domination and connected 2-domination in triangulations.

Keywords: Domination, Maximal Outerplanar Graph, Triangulation.

1 Introduction

Given a graph $G = (V, E)$ a *dominating set* is a set $D \subseteq V$ such that every vertex not in D is adjacent to a vertex in D. The *domination number* $\gamma(G)$ is the number of vertices in a smallest dominating set for G. In graph theory,

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dominating set problems have received much attention in numerous articles and books. The fundamental reference is the book of Haynes, Hedetniemi and Slater [6], where many variants of domination that take into account the connectivity of the subgraph generated by the dominant set or the multiplicity of the domination are analyzed. In recent years, the problem of domination in outerplanar graphs has received special attention (e.g., [1,2,10]) A graph is *outerplanar* if it has a crossing-free embedding in the plane such that all vertices lie on the boundary of its outer face (the unbounded face). An outerplanar graph is *maximal* if it is not possible to add an edge such that the resulting graph is still outerplanar. A maximal outerplanar graph (abbreviated MOP) embedded in the plane corresponds to a triangulation of a polygon. The works mentioned above continued the work started by Matheson and Tarjan [8], where the authors proved that the domination number of a triangulated disc of order $n \geq 3$ is at most $\frac{n}{3}$. A *triangulated disc*, or triangulation graph, is a plane graph such that all its faces, except the outer face, are triangles. In this article we establish tight combinatorial bounds for the following domination variants: paired, 2-domination and total 2-domination for maximal outerplanar graphs, and double and connected 2-domination for triangulation graphs.

Paired domination was introduced by Haynes and Slater [7], k-domination by Fink and Jacobson [3], total k-domination by Hansberg [4] and double domination by Harary and Haynes [5].

In the next section we describe the terminology that will be used throughout this paper and the relation between the different parameters of connectivity and domination in a triangulation. In sections 3 and 4 we present the results obtained for paired, 2-domination and total 2-domination, for maximal outerplanar graphs. In section 5 we present the results for triangulation graphs.

2 Parameters of domination, connectivity and multiplicity

Let $G = (V, E)$ be a graph of order n and S a dominating set of G. By imposing different connectivity or multiplicity conditions, we have different kinds of domination. In this work we will study the following ones related with connectivity: connected domination ($G[S]$ is connected), paired domination ($G[S]$ has a perfect matching) and total domination ($G[S]$ has not isolated vertices). Regarding the multiplicity we remember that $S \subset V$ is a k-dominating set of G if for every $u \in V - S$ we have $|N(u) \cap S| \geq k$. In this work, G is a triangulation graph, that can have vertices of degree two, so we will only study the case $k = 2$. A set D of V is a double dominating set if it is 2-dominating set and $|N(v) \cap D| \geq 1$ for every $v \in D$. Now we are going to define the notation that we use in a general way. Let H designate any property about domination of G. The minimum number of elements in a dominant set verifying the property H in a graph G is called domination number under H of G and it is denoted by $h(G)$. That is,

$$h(G) = min\{|S| : S \text{ is a domination set of G that satifies } H\}$$

In the combinatorial study of any kind of domination, say dominations satisfying property H, our goal is to determine bounds for

$$h(n) = max\{h(G) : G \text{ is a triangulation (or MOP) with } n \text{ vertices}\}$$

In the following sections we are going to establish tight bounds for each of the previously described domination variants for triangulation graphs (some of them only for maximal outerplanar graphs). When we are referring to paired, 2-domination, total 2-domination, double domination and connected 2-domination, $h(n)$ is replaced by $\gamma_{pr}(n)$, $\gamma_2(n)$, $\gamma_2^t(n)$, $\gamma_{\times 2}(n)$ and $\gamma_2^c(n)$ respectively.

3 Paired domination in MOP's

Theorem 3.1 *If G is a maximal outerplanar graph of order n then $\gamma_{pr}(G) \leq 2\lfloor \frac{n}{4} \rfloor$. This bound is tight, it exists a mop T of order n such that* $\gamma_{pr}(T) = 2\lfloor \frac{n}{4} \rfloor$

Proof. We use a result proved by O'Rourke [9] about guarding polygons with mobile guard, that can patrol throughout an edge or a diagonal. *"Every triangulation graph of a polygon of n vertices can be guarded by $\lfloor \frac{n}{4} \rfloor$ mobile guards"*. Guarding means that each triangle has a vertex which is incident with one guard. We prove that the mobile guards set is an independent edge set. Replacing each mobile guard by its extreme vertices, it gives a paired dominating set of $2\lfloor \frac{n}{4} \rfloor$ vertices. The Figure 1 shows a mop T with $\gamma_{pr}(T) = 2\lfloor \frac{n}{4} \rfloor$.

□

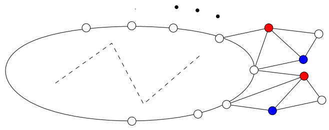

Fig. 1. A maximal outerplanar graph with $\gamma_{pr}(T) = 2\lfloor \frac{n}{4} \rfloor$

4 2-domination and total 2-domination in MOP's

In this section we study 2-domination and total 2-domination in maximal outerplanar graphs. There are numerous works about multiple domination, but none of them study this parameter in maximal outerplanar graphs.

The combinatorial bound about 2-domination is established in the following theorem

Theorem 4.1 *If G is a maximal outerplanar graph of order n then $\gamma_2(G) \leq \lceil \frac{n}{2} \rceil$. This bound is tight.*

Proof. For the upper bound it is sufficient to observe that every maximal outerplanar graph is a hamiltonian graph.

The Figure 2 shows that there are maximal outerplanar graphs in which any 2-dominating set has at least $\lceil \frac{n}{2} \rceil$ vertices.

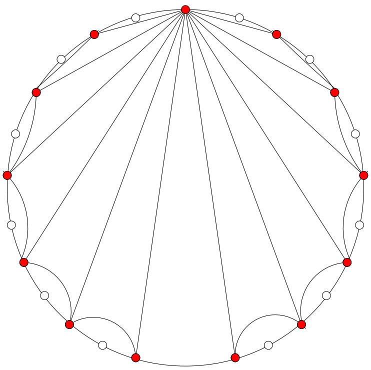

Fig. 2. A maximal outerplanar graph G with $\gamma_2(G) \geq \lceil \frac{n}{2} \rceil$

All the vertices of degree 2 need two neighbors to be 2-dominated. Therefore the red vertices are a 2-dominant minimum set of cardinal number $\lceil \frac{n}{2} \rceil$. □

Theorem 4.2 *If T is a maximal outerplanar graph with $n > 3$ vertices then $\gamma_2^t(T) \leq \lfloor \frac{3n}{4} \rfloor$. Furthermore, this bound is tight.*

Proof. The bound is tight as shown by the graph in Figure 1. To prove the upper bound, we apply a lemma of Tokunaga [10] about 4-coloring of mop's.

5 Double domination and connected 2-domination in triangulation graphs

In this section we determine combinatorial bounds for double and connected 2-domination on triangulation graphs. The bounds are equal to those obtained for maximal outerplanar graphs. Since the results are the same for both types of domination, we present them together.

Theorem 5.1 *If T is a triangulation graph with $n > 3$ vertices then $\gamma_{\times 2}(T) \leq \gamma_2^c(T) \leq \lfloor \frac{2n}{3} \rfloor$. Furthermore, these bounds are tight.*

Proof. The bound is tight for both types of domination as shown by the graphs in Figure 3. Therefore $\gamma_{\times 2}(n) = \gamma_2^c(n) = \lfloor \frac{2n}{3} \rfloor$

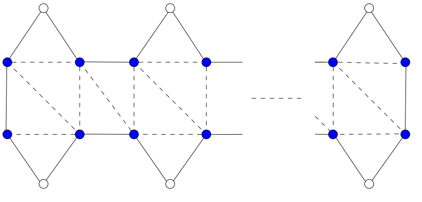

Fig. 3. A maximal outerplanar graph G such that $\gamma_{\times 2}(G) \geq \lfloor \frac{2n}{3} \rfloor$ and $\gamma_2^c(G) \geq \lfloor \frac{2n}{3} \rfloor$

To prove the inequalities of the theorem we follow the ideas of Matheson and Tarjan [8], we prove that any n-vertex triangulation graph T admits a 3-coloring (not necessarily a proper coloring) that verifies:

(1) The 3-coloring of the outer face vertices is proper. (2) There is no monocolor triangle. (3) The vertices of any two colors form a double dominating (resp. connected 2-dominating) set of G

The proof is done by induction on n. For $n = 4$, there are two triangulation graphs and both admit a 3-coloration that satisfies (1), (2) and (3).

Assume that the result holds for $4 \leq m < n$. We distinguish two cases, T can have a separating edge or not.

Case 1) T has a separating edge uv

This edge splits G into two triangulations, T_1 and T_2. By the induction hypothesis there exist 3-colorings in T_1 and T_2. By exchanging colors we achieve that the colors of u and v are the same in T_1 and T_2. Thus we obtain a 3-coloring that meets the required conditions.

Case 2) T has no separating edge

In this case we delete a vertex u of the outer face of T, obtaining a triangulation T^* with $n-1$ vertices. By the induction hypothesis T^* admits a 3-coloration with the required conditions. We consider the vertices v and v', adjacent to u in the outer face of T. We distinguish two subcases, depending on wether the colors of v and v' in T^* are equal or not.

If color(v) $\neq$ color(v'), we assign the third color to vertex u. Thus we obtain a 3-coloring of T.

If color(v) = color(v') and there are only two colors in the neighbors of u in T^*, we assign the third color to vertex u. Otherwise we assign any color to vertex u.

In any subcase, if T^* meets the required conditions then T also meets them. □

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