# Recursive Lower and Dual Upper Bounds for Bermudan-style Options*

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**Abstract**

Although Bermudan options are routinely priced by simulation and least-squares methods using lower and dual upper bounds, these bounds are hardly optimized. We optimize recursive upper bounds (UB), which are more tractable than the original/nonrecursive ones, and derive two new results. (1) An UB based on (a martingale that depends on) stopping times is independent of the next-period exercise decision and hence cannot be optimized. So we optimize the recursive lower bound, and use its optimal recursive policy to evaluate the upper bound as well. (2) Less time-intensive UBs that are based on a continuation value function only need this function in the continuation region, where this continuation value is less nonlinear and easier to fit (than in the entire support). In the numerical exercise, the lower and upper bounds based on these two approaches are very tight, improving over state-of-the-art methods (including global least-squares and pathwise optimization).

Keywords: Finance; Bermudan options; optimal recursive lower/upper bounds; simulation and local least squares;

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# 1 Introduction

Pricing Bermudan options in high dimensions requires Monte Carlo (MC) methods, and two MC-based prices have been developed: lower bounds and dual upper bounds. Longstaff and Schwartz (2001) use a global least-squares MC approach to compute lower bounds; likewise, dual bounds are also based on global regression and simulation (Andersen and Broadie, 2004). Although these bounds converge, they are hardly optimized, which is important because simulation is time consuming, demanding a smart approach.[1]

In this paper, we fill this gap and optimize recursive lower and dual upper bounds, which are more tractable than the original (nonrecursive) ones. We show a recursive upper bound is independent of the next-period exercise decision and hence cannot be optimized. So we optimize the recursive lower bound, and use its optimal recursive policy to evaluate the upper bound as well.[2] We show these two bounds, which have a similar cost to the two bounds based on a recursive policy estimated by global least-squares, are very tight. Specifically, lower/upper bounds generated by simulation depend on an exercise policy, whereby the upper bound is derived from a martingale based on this policy. In addition, a less time-intesive yet more biased upper bound is generated from a martingale based on a continuation-value function. We study this latter upper bound separately, and show how to reduce its bias as well.

First, consider a given family of exercise policies/stopping times. Ibáñez and Velasco (2018) maximize a recursive Bermudan price/lower bound, $L$, with regard to this family at each exercise period. An open question is which exercise strategy minimizes a (dual) upper bound, $U$. We show the exercise strategy that maximizes a recursive lower bound also minimizes not the recursive upper bound itself, but rather the gap between them, $U - L$. We provide a recursive expression for the gap (Theorem 1), and show a recursive upper bound $U$ is independent of the next-period exercise policy (Proposition 2). Therefore, minimizing the gap, $U - L$, is equivalent to maximizing the Bermudan price, $L$, recursively.

[1] Tsitsiklis and Van Roy (2001), Clément et al. (2002), Stentoft (2004), Egloff (2005), Glasserman and (2004), and Zanger (2017), among others, study the convergence of least-squares. For instance, pricing a max-call barrier option, Desai et al. (2012) report that state-of-the-art methods (including global regression and pathwise optimization) yield a three-digit gap between lower and dual upper bounds, perhaps because this gap is costly to reduce by increasing the number of basis functions and simulated sample paths.

[2] Consider a Bermudan option that is exercisable from periods 1 to $T$. The exercise strategy at any time $t$, $1 \leq t \leq T-1$, is associated with an otherwise equivalent option but exercisable only from $t$ to $T$, where the strategy from $t+1$ to $T-1$ is given. That is, we consider stopping-times $\tau$ such that $\tau(1) = \tau(t) \geq t$ and $\tau(t+1)$ is given, going recursively from $t = T-1$ to $t = 1$. The same definition applies to upper bounds.

The following example illustrates the latter results: Consider a family of exercise strategies and a Bermudan option with three exercise dates, $t \in \{1, 2, 3\}$. The first-order conditions associated with maximizing the Bermudan price at $t = 0$ imply optimal exercise at $t \in \{1, 2\}$, *but* only for those paths that are alive for the exercise decision at $t = 2$ (Ibáñez and Velasco). Hence, if we consider all paths at $t = 2$, we can solve this problem recursively, which is more tractable yet close to the optimal one. Minimizing the upper bound, by contrast, depends only on the exercise decision at $t = 2$, not on $t = 1$ (Proposition 2).[3]

Second, consider a family of continuation-value functions, which lead to less time-intensive dual upper bounds based on a one-period subsimulation. We show (i) a recursive upper bound is independent of the next-period continuation-value function as well (Proposition 2); (ii) by factorizing the two martingales that are based on either stopping times or continuation values, the latter martingale includes a third error term, which ensures the process is actually a martingale though implies more biased upper bounds; and (iii) this third term, however, depends only on the option continuation value in the continuation/waiting region.[4]

The latter waiting-region constraint is important. Bermudan options become nonlinear near the exercise boundary but much less in the waiting region, and fitting a continuation value function only in this region is easier. This new upper bound (based on the waiting-region continuation value) is as accurate as an upper bound based on an exercise policy estimated by global regression (Anderson and Broadie), but in a fraction of the time. The former bound is especially accurate for at-/in-the-money options; in this case, this bound is computed mostly from sample paths that cross the exercise region (and do not depend on a continuation-value function).[5]

In the numerical exercise, we price up-and-out Bermudan max-options. We use Ibáñez and Velasco's local regression to derive the optimal recursive exercise policy. We compute the two bounds associated with this policy: The lower bound improves upon (Desai et al., 2012) global regression and pathwise optimization by more than 100 to 200 cents, and the upper bound yields a one-digit gap. This small gap implies the local-regression exercise policy is near optimal, and the two associated bounds are close to the true (yet unknown) price.

[3] Kaniel et al. (2008), Lemma 1, proved this result for a two-period Bermudan option, $t \in \{1, 2\}$; the upper bound does not depend on $t = 1$ exercise decision, implying a two-period Bermudan upper bound is unbiased.

[4] The martingale first two components are those of the standard factorization of the American/Bermudan option into an early-exercise-premium plus the European counterpart (Kim, 1991; Carr et al., 1992).

[5] Interestingly, this waiting-region constraint has the dual flavor of being the reciprocal constraint of using only in-the-money paths (hence, the exercise region), as suggested, in the least-squares primal method.

Although any upper bound is costly, the local policy is so good that reducing the number of subsimulation by 20, our upper bound increases only by a few cents. The decision on both tradeoffs—number of local least-squares iterations versus (slightly) less biased lower bounds—and number of nested simulation paths versus less biased upper bounds is left to the user.

Our upper bounds based on the local-regression exercise policy only change marginally with the number of subsimulation paths and are robust to all refinements, implying the upper bound is tighter (and closer to the true price) than the lower bound. With other methods that yield a nontrivial gap (e.g., global least-squares), this claim cannot be made. This result agrees with the two-period Bermudan upper bound, which is independent of the (one-period) exercise policy. A tighter upper bound implies that a mid point is lower biased.

Although a barrier option is an exotic security, the barrier makes this option very sensitive to suboptimal exercise. This sensitivity depends on the gap between the intrinsic value slope and the Bermudan option Delta at the exercise boundary. Consider a two-period Bermudan up-and-out call on a single stock, which depends on its European counterpart (in the first period) and easily illustrates the point: the call-payoff slope is equal to 1. A deep in-the-money European call Delta is also close to 1, but with an up-and-out barrier, the potentially large profits vanish and the price function flatters. Hence, 1 minus Delta is easily two digits larger for an up-and-out call than for a call if volatility is large (e.g., a max-call option).

The duality approach was developed by Rogers (2001) and Haugh and Kogan (2004) and extended by many others.[6] We tailor these results to our optimal recursive setting (Proposition 1; Theorem 1), which yields such tight lower/upper bounds. As a byproduct of the analysis of both bounds, we develop a new statistical test that enables us to evaluate whether an exercise strategy/stopping time is close to the optimal one. This test is formulated in terms of stopping times instead of a nonzero (upper minus lower bound) gap.

Section 2 reviews dual upper bounds; section 3 derives the main results of the paper and minimizes a recursive gap between lower and upper bounds; section 4 further studies upper bounds based on stopping times and continuation values; section 5 presents a numerical exercise; section 6 develops a test on optimal stopping times; and section 7 concludes. Appendix A includes proofs of the main results, and B provides the local-regression algorithm.

[6] Practitioners use dual upper bounds to appraise Bermudan swaptions prices (Andersen and Andreasen, 2004; Sventrup, 2005). Broadie and Detemple (1996) introduce lower and upper bounds for American options. For instance, Chen and Glasserman (2007) and Rogers (2010) study optimal dual bounds; Belomestny et al. (2013) use a multilevel approach; Glasserman (2004) studies primal/dual bounds based on regression methods.

## 2 Lower and Dual Upper Bounds: A Summary

Consider a Bermudan option that can be exercised at $t \in \{1, 2, ..., T\}$, where $t = 0$ is today. Consider a vector of $N$ stock prices $S_t$. Interest rates are stochastic, $R_t > 0$ is a bank account process, $R_0 = 1$, and $R_{j,t} = R_t / R_j$. If the interest rate $r$ is constant, $R_t = e^{rt\Delta t}$, $R_{t,t+1} = e^{r\Delta t}$, and $\Delta t$ is the time between $t$ and $t+1$. We assume a risk-neutral measure $Q$ exists.

From the Bellman principle, the continuation value $V^*(t, S_t)$ of a Bermudan option with intrinsic value $I \geq 0$ satisfies

$$V^*(t, S_t) = E_t^Q \left[ \frac{1}{R_{t,t+1}} \times \max \{ I_{t+1}, V^*(t+1, S_{t+1}) \} \right], \ t = 0, 1, ..., T-1, \tag{1}$$

and $V^*(T, S_T) = 0$. We refer to $V^*$ as the "first-best" Bermudan price. Although the results below can be derived in terms of a bank account (in which $R_t = 1$, $t = 0, 1, .., T$), we work in nominal terms for completeness; that way, all equations carry directly to the computer.

**Lower bounds** We rewrite equation (1) for a MC setting. Let $\mathcal{T}$ be the set of stopping-times, $\tau \in \{1, 2, ..., T\}$. For a given $\widetilde{\tau} \in \mathcal{T}$, a lower bound $V_0^{low}$ is defined as follows:

$$V_0^{low} := E_0^Q \left[ \frac{I_{\widetilde{\tau}}}{R_{\widetilde{\tau}}} \right] \leq \sup_{\tau \in \mathcal{T}} E_0^Q \left[ \frac{I_\tau}{R_\tau} \right] = E_0^Q \left[ \left. \frac{I_\tau}{R_\tau} \right|_{\tau = \tau^*} \right] := V_0^*, \tag{2}$$

where $V_0^* = V^*(0, S_0)$ is the Bermudan price and $\tau^*$ is the associated first-best stopping time.

**Dual upper bounds** A dual upper bound is also an estimator of the Bermudan price and allows us to build a mid-point and to assess a lower bound. A dual upper bound, however, depends on a martingale that is not specified (Rogers, 2001; Haugh and Kogan, 2004).

For a martingale $\frac{M_t}{R_t}$, $t \in \{0, 1, ..., T\}$, upper bounds $V_0^{up}$ are based on the following result:

$$V_0^{up} := M_0 + E_0^Q \left[ \max_{1 \leq t \leq T} \left\{ \frac{I_t - M_t}{R_t} \right\} \right] \geq M_0 + E_0^Q \left[ \left. \frac{I_\tau}{R_\tau} - \frac{M_\tau}{R_\tau} \right|_{\tau = \tau^*} \right] = V_0^*,$$

the last equality follows from the optional sampling theorem, and the inequality follows from

$$\max_{1 \leq t \leq T} \left\{ \frac{I_t - M_t}{R_t} \right\} \geq \left. \frac{I_\tau - M_\tau}{R_\tau} \right|_{\tau = \tau^*}.$$

$V_0^{up}$ does not depend on the initial value $M_0$ (see Appendix A). The upper bound is binding (i.e., $V_0^{up} = V_0^*$) for the process associated with the Bermudan price, $M^*$ (Rogers, 2001; Andersen and Broadie, 2004; or our Proposition 1). We define $M^*$ below (see section 2.1).

We now build a martingale by using stopping times following equation (2) (as in Andersen and Broadie, 2004). We build two martingales by using continuation values following equation

(1) in section 3.2. Let $\widetilde{\tau}(t) \in \{t, t+1, ..., T\}$, and hence $\widetilde{\tau}(t) \geq t$, be a stopping time indexed in $t$, for $t \in \{1, 2, ..., T\}$, and $\widetilde{\tau}(T) = T$. If $\widetilde{\tau}$ is not indexed in $t$, $\widetilde{\tau} = \widetilde{\tau}(1)$.

Let $\widehat{V}$ be the Bermudan price (continuation value) associated with $\widetilde{\tau}$; that is, $\widehat{V}_T = 0$ and

$$\widehat{V}_{t-1} = E_{t-1}^{Q}\left[\frac{1_{\{t<\widetilde{\tau}(t)\}}\widehat{V}_t + 1_{\{t=\widetilde{\tau}(t)\}}I_t}{R_{t-1,t}}\right], \; t = 1, 2, ..., T. \quad (3)$$

We define the lower bound as in equation (2), $V_0^{low} = \widehat{V}_0$ (given that necessarily $\widehat{V}_0 \leq V_0^*$).[7]

We then define the process $\widehat{M}$ from $\widetilde{\tau}$ as well; that is, $\widehat{M}_0 = \widehat{V}_0$ and

$$\widehat{M}_t = \widehat{M}_{t-1}R_{t-1,t} + \left(1_{\{t<\widetilde{\tau}(t)\}}\widehat{V}_t + 1_{\{t=\widetilde{\tau}(t)\}}I_t\right) - \widehat{V}_{t-1} \times R_{t-1,t}, \; t = 1, 2, ..., T. \quad (4)$$

$\widehat{M}_t/R_t$ is a martingale (i.e., $E_{t-1}^{Q}\left[\widehat{M}_t/R_{t-1,t}\right] = \widehat{M}_{t-1}$), which follows from $\widehat{V}$ definition.

The initial value of the process $\widehat{M}$ is not set to zero but $\widehat{M}_0 = \widehat{V}_0$, which implies, together with

$$V_0^{low} = \widehat{V}_0 = \widehat{M}_0 \leq V_0^* \leq V_0^{up},$$

that the following expectation is a proper gap:

$$E_0^{Q}\left[\max_{1\leq t\leq T}\left\{\frac{I_t - \widehat{M}_t}{R_t}\right\}\right] = V_0^{up} - \widehat{V}_0 \geq 0;$$

that is, the difference between the upper and the lower bound is nonnegative.

## 2.1 Factorizing the martingale

Importantly, the process $\widehat{M}$ is explicitly defined by $\widehat{M}_0 = \widehat{V}_0$ and

$$\widehat{M}_t = \sum_{j=1}^{t-1}\left(I_j - \widehat{V}_j\right) \times 1_{\{j=\widetilde{\tau}(j)\}} \times R_{j,t} + \left(1_{\{t<\widetilde{\tau}(t)\}}\widehat{V}_t + 1_{\{t=\widetilde{\tau}(t)\}}I_t\right), \quad (5)$$

which is equal to the sum of the early-exercise premium (reinvested in a bank account) plus the right to exercise at time $t$ (see Appendix A). For $t = T$, because $\widehat{V}_T = 0$, equation's (5) expectation implies the classical factorization of an American option into an early-exercise premium plus the equivalent European counterpart (if $\widetilde{\tau} = \tau^*$). This factorization is related to the Doob-decomposition theorem, in which the Bermudan-option price process is the Snell envelope (e.g., Carr et al., 1992).

[7] In our case, we estimate a second continuation-value function $\widetilde{V}_t$ by local least-squares (Ibáñez and Velasco, 2018), $t \in \{1, 2, ..., T-1\}$, from which the stopping time $\widetilde{\tau}(t)$ is recursively defined by

$$\widetilde{\tau}(t) = t \;\text{ if }\; I_t \geq \widetilde{V}_t; \;\; \widetilde{\tau}(t) = \widetilde{\tau}(t+1) \;\text{ otherwise},$$

and $\widetilde{\tau}(T) = T$.

From equation (5), it follows for $t \leq \widetilde{\tau}$ that

$$\widehat{M}_t = \widehat{V}_t, \text{ if } t < \widetilde{\tau}; \quad \text{and } \widehat{M}_t = I_t, \text{ if } t = \widetilde{\tau},$$

and therefore,

$$\max_{1 \leq t \leq T} \left\{ I_t - \widehat{M}_t \right\} \geq I_{\widetilde{\tau}} - \widehat{M}_{\widetilde{\tau}} = 0. \tag{6}$$

The martingale associated with the optimal stopping-time family, $\tau^*$, is denoted by $M^*/R$ and is defined in a similar way as in equation (5), where $M_0^* = V_0^*$. The next result complements the literature (Rogers, 2001) on dual upper bounds for the optimal $\tau^*$.

**Proposition 1.** (i) An upper bound based on the optimal stopping time $\tau^*$ is binding. And (ii), for any path,

$$\begin{aligned} \tau^* &= \inf \left( \arg \max_{1 \leq t \leq T} \left\{ I_t - M_t^* \right\} \right), \\ 0 &= \max_{1 \leq t \leq T} \left\{ I_t - M_t^* \right\}, \end{aligned}$$

in which the "inf" is taken in the case of multiple solutions.

Proof. See Appendix A. ■

*Remark.* Proposition 1 shows equation (6) inequality is binding and the maximum is equal to zero path by path for the optimal martingale $M^*$ (associated with $\tau^*$). It follows that the term $\max_{1 \leq t \leq T}\{I_t - \widehat{M}_t\}$, as well as the (sample) gap between the lower and the upper bound, will have little variance if the process $\widehat{M}$ is based on a good exercise policy $\widetilde{\tau}$ (and if, in addition, $\widehat{M}$ is estimated with little simulation error).

# 3 Recursive Bounds: An Optimal Recursive Gap

Because the original dual upper bound is not tractable, we study a recursive version. Ibáñez and Velasco (2018) maximize the Bermudan price with respect to a family of stopping times at each exercise period recursively; we refer to this price as a recursive Bermudan price, which is the objective function of the primal problem. The dual problem is to minimize the recursive upper bound and to determine whether the solution to these two (recursive) primal and dual problems are linked. If we consider a family of stopping times that are specified in a recursive way, we show a martingale based on the exercise strategy that maximizes the Bermudan price also minimizes not the upper bound itself, but rather the gap between the lower and the upper bound.

We derive a simple recursive expression for this gap (Theorem 1), which holds for martingales based on stopping times and continuation values, and from which we prove all results. An upper bound is independent of the next-period stopping time or continuation value (Proposition 2). Therefore, for a martingale based on stopping times, minimizing the gap is equivalent to maximizing the lower bound in a recursive way (Proposition 3). For a martingale based on a continuation value function, we show this function is only needed in the waiting region. In this continuation/waiting region, the continuation value is less nonlinear and easier to fit than in the entire support. We next define recursive lower/upper bounds, derive Theorem 1 and Proposition 2, and then minimize a recursive gap.

### 3.1 Recursive lower and dual upper bounds

We introduce three processes ($\widehat{Z}_t^{(st)}$, $\widehat{Z}_t^{(cv)}$, and $\widehat{Z}_t^{(stcv)}$), which are based on equations (1) and (2), but for simplicity, we denote any of them by $\widehat{Z}_t$. In this way, we do not need to distinguish between upper bounds based on stopping times ($st$), continuation values ($cv$), or both of them ($stcv$). Theorem 1 and Proposition 2 below hold for the three processes.

For $t \in \{1, 2, ..., T\}$, $Z_t$ takes any of these three forms:

$$\begin{aligned} \widehat{Z}_t^{(st)} &= 1_{\{t<\widetilde{\tau}(t)\}}\widehat{V}_t + 1_{\{t=\widetilde{\tau}(t)\}}I_t, \\ \widehat{Z}_t^{(cv)} &= \max\left\{\widetilde{V}_t, I_t\right\}, \\ \widehat{Z}_t^{(stcv)} &= 1_{\{t<\widetilde{\tau}(t)\}}\widetilde{V}_t + 1_{\{t=\widetilde{\tau}(t)\}}I_t, \end{aligned} \tag{7}$$

where $\widetilde{\tau}(t)$ and $\widetilde{V}_t$ are given (and $\widetilde{V}_T = \widehat{V}_T = 0$). $\widehat{Z}_t$ is akin to the value process of a Bermudan option, either the intrinsic value or the continuation value.

Similar to equation (3), we redefine the process $\widehat{V}$ associated with $\widehat{Z}$, satisfying $\widehat{V}_T = 0$ and

$$\widehat{V}_{t-1} = E_{t-1}^{Q}\left[\frac{\widehat{Z}_t}{R_{t-1,t}}\right], \; t = 1, 2, ..., T. \tag{8}$$

We also redefine the process $\widehat{M}_t$ in equation (4) in terms of $\widehat{Z}_t$. That is, $\widehat{M}_0 = \widehat{V}_0$ and

$$\widehat{M}_t = \widehat{M}_{t-1}R_{t-1,t} + \widehat{Z}_t - \widehat{V}_{t-1} \times R_{t-1,t}, \; 1 \le t \le T, \tag{9}$$

so that $\widehat{M}_t/R_t$ is also a martingale. In particular, $\widehat{M}_1 = \widehat{Z}_1$.

We define a new variable $GAP$ at time $s$ as follows:

$$GAP_s := \frac{\widehat{M}_s}{R_s} - \frac{\widehat{Z}_s}{R_s} + \max_{s \le t \le T}\left\{\frac{I_t - \widehat{M}_t}{R_t}\right\}, \; 1 \le s \le T. \tag{10}$$

where $\frac{\widehat{M}_s}{R_s} - \frac{\widehat{M}_t}{R_t}$, $s \le t$, are the Doob-martingale increments.[8] In particular, because $\widehat{M}_1 = \widehat{Z}_1$,

$$GAP_1 := \max_{1 \le t \le T} \left\{ \frac{I_t - \widehat{M}_t}{R_t} \right\},$$

and the upper bound is given by

$$V_0^{up} := \widehat{M}_0 + E_0^Q \left[ \max_{1 \le t \le T} \left\{ \frac{I_t - \widehat{M}_t}{R_t} \right\} \right] = \widehat{M}_0 + E_0^Q \left[ GAP_1 \right],$$

where $\widehat{M}_0 = \widehat{V}_0$.

The next result allows us to understand a recursive gap between lower and upper bounds (see Schoennemakers et al., 2013, for similar recursive statements).

**Theorem 1.** The process $GAP$ defined in equation (10) for $s = \{1, 2, ..., T\}$, with $GAP_{T+1} = 0$, satisfies that

$$GAP_s = \frac{-\widehat{Z}_s}{R_s} + \max \left\{ \frac{I_s}{R_s}, \frac{\widehat{V}_s}{R_s} + GAP_{s+1} \right\}. \tag{11}$$

Moreover, only the term $\widehat{Z}_s$, which is defined in three ways in equation (7), depends on the functions $\widetilde{\tau}(s)$ or $\widetilde{V}_s$.

Proof. See Appendix A. ■

*Remark.* From equation (10),

$$GAP_T = \frac{I_T - \widehat{Z}_T}{R_T}.$$

And from equation (11) (because $\widehat{V}_T = 0$, $GAP_{T+1} = 0$, and $I \ge 0$), it follows that

$$GAP_T = \left( -\widehat{Z}_T + I_T \right) \frac{1}{R_T}$$

as well. Then $GAP_T = 0$ if $\widehat{Z}_T = I_T$, that is, if the intrinsic value at maturity is known, which is always the case.

*Example.* Consider a Bermudan option with three exercise opportunities (i.e., $s = 1$ and $T = 3$),

$$\begin{aligned} GAP_1 + \frac{\widehat{Z}_1}{R_1} &= \max \left\{ \frac{I_1}{R_1}, \frac{\widehat{V}_1}{R_1} + GAP_2 \right\} \\ &= \max \left\{ \frac{I_1}{R_1}, \frac{\widehat{V}_1}{R_1} - \frac{\widehat{Z}_2}{R_2} + \max \left\{ \frac{I_2}{R_2}, \frac{\widehat{V}_2}{R_2} + \underbrace{GAP_3}_{=0} \right\} \right\}, \end{aligned}$$

[8] The process $\frac{\widehat{M}_s}{R_s} - \frac{\widehat{Z}_s}{R_s} - \frac{\widehat{M}_t}{R_t}$ is also a martingale. From equation (9), its initial value at $t = s-1$ is given by $-\left( \frac{\widehat{M}_s}{R_s} - \frac{\widehat{Z}_s}{R_s} - \frac{\widehat{M}_{s-1}}{R_{s-1}} \right) = \frac{\widehat{V}_{s-1}}{R_{s-1}}$, where $\widehat{V}_{s-1}$ is the price of a Bermudan option exercisable from $s$ to $T$.

which (from equations (7) and (8)) does not depend on the functions $\widetilde{\tau}(1)$ or $\widetilde{V}_1$, as claimed in Theorem 1.

For tractability, we analyze the upper bound recursively. We consider the following lower and upper bounds, which correspond to a Bermudan option that can only be exercised from $s$ to $T$, $1 \leq s \leq T-1$. That is,

$$V_{s,0}^{low} := E_0^Q \left[ \frac{\widehat{V}_{s-1}}{R_{s-1}} \right] \quad \text{and} \quad V_{s,0}^{up} := E_0^Q \left[ \frac{\widehat{V}_{s-1}}{R_{s-1}} \right] + E_0^Q \left[ GAP_s \right], \tag{12}$$

Consistent with our notation, $V_0^{low} = V_{1,0}^{low}$, $V_0^{up} = V_{1,0}^{up}$, and

$$V_{s,0}^{up} - V_{s,0}^{low} = E_0^Q \left[ GAP_s \right].$$

**Proposition 2.** $V_{s,0}^{up}$ does not depend on the variables $\widetilde{\tau}(s)$ or $\widetilde{V}_s$, which are the next-period exercise decision or continuation value, respectively.

Proof. It follows from Theorem 1 and

$$\begin{aligned} V_{s,0}^{up} \quad &: = E_0^Q \left[ \frac{\widehat{V}_{s-1}}{R_{s-1}} \right] + E_0^Q \left[ GAP_s \right] \\ &= E_0^Q \left[ \frac{\widehat{V}_{s-1}}{R_{s-1}} \right] + E_0^Q \left[ \frac{-\widehat{Z}_s}{R_s} + \max \left\{ \frac{I_s}{R_s}, \frac{\widehat{V}_s}{R_s} + GAP_{s+1} \right\} \right] \\ &= E_0^Q \left[ \max \left\{ \frac{I_s}{R_s}, \frac{\widehat{V}_s}{R_s} + GAP_{s+1} \right\} \right]. \quad \blacksquare \end{aligned}$$

Namely, $V_{s,0}^{up}$ does not depend on $\widetilde{\tau}(s)$ or $\widetilde{V}_s$ because the upper bound directly compares the intrinsic value and an estimated continuation value at time $s$ (i.e., $\max\{I_s, \widehat{V}_s\}$). In particular, a two-period Bermudan upper bound is always binding (Kaniel et al., 2008). For a two-period Bermudan, $T = 2$ (and $GAP_2 = 0$). Then

$$\begin{aligned} V_0^{up} &= V_{1,0}^{up} = \widehat{M}_0 + E_0^Q \left[ GAP_1 \right] \\ &= \widehat{V}_0 - \underbrace{E_0^Q \left[ \widehat{Z}_1 \times \frac{1}{R_1} \right]}_{=\widehat{V}_0} + E_0^Q \left[ \max \left\{ \frac{I_1}{R_1}, \frac{\widehat{V}_1}{R_1} + \underbrace{GAP_2}_{=0} \right\} \right] \\ &= E_0^Q \left[ \max \left\{ I_1, E_1^Q \left[ I_2 \times \frac{1}{R_{1,2}} \right] \right\} \times \frac{1}{R_1} \right] = V_0^*, \end{aligned}$$

where the last equality follows from $V_0^*$ definition (i.e., the maximum between exercise and the European option at $t = 1$).

Proposition 2 implies that we cannot minimize the upper bound $V_{s,0}^{up}$ but rather the gap, $E_0^Q[GAP_s]$, in a recursive way (where $\widetilde{\tau}(t)$ and $\widetilde{V}_t$ are given for $t > s$). We next analyze the three cases of $\widehat{Z}_t$ in this minimization.

*Remark.* Although $V_{s,0}^{low}$ defined in equation (12) denotes a lower bound, a negative bias is associated with $\widehat{Z}_t^{(st)}$ but not with $\widehat{Z}_t^{(cv)}$ and $\widehat{Z}_t^{(stcv)}$. The latter two processes depend on a continuation value $\widetilde{V}_t$, which may yield a positively biased $\widehat{V}_{t-1}$ (if $\widetilde{V}_t > V_t^*$). Yet we keep the lower-bound name, even if a bit misleading, because $\widetilde{V}$ is only used to compute an upper bound and because we define and analyze the three gaps in the same way.

### 3.2 An optimal recursive gap: A martingale based on stopping times, $\widetilde{\tau}(t)$

From equation (7), recall that $\widehat{Z}_t^{(st)} = 1_{\{t<\widetilde{\tau}(t)\}}\widehat{V}_t + 1_{\{t=\widetilde{\tau}(t)\}}I_t$. Define

$$\widetilde{\tau}^*(s) := \arg\max_{\widetilde{\tau}(s)\in\mathcal{T}|\widetilde{\tau}(s+1)} V_{s,0}^{low}, \tag{13}$$

where $V_{s,0}^{low}$ is given in equation (12). $\widetilde{\tau}^*(s)$ means optimal exercise at time $s$, conditional on $\widetilde{\tau}(s+1)$ and subject to a given set of stopping times $\mathcal{T}$ (in which now $\tau \in \{s, s+1, ..., T\}$). Namely, if $\widetilde{\tau}^*(s) > s$, $\widetilde{\tau}^*(s) = \widetilde{\tau}(s+1)$ where $\widetilde{\tau}(s+1)$ is computed in advance.

**Proposition 3.** Consider a Bermudan option that can only be exercised from $s$ to $T$, $1 \leq s \leq T-1$; that is, $s \leq \widetilde{\tau}(s) \in \mathcal{T}$ . Assume the stopping time $\widetilde{\tau}(s+1)$ is given. Then $\widetilde{\tau}^*(s)$, defined in equation (13), satisfies that

$$\widetilde{\tau}^*(s) = \arg\min_{\widetilde{\tau}(s)\in\mathcal{T}|\widetilde{\tau}(s+1)} E_0^Q[GAP_s].$$

Further, if $\widetilde{\tau}(s+1) = \tau^*(s+1)$ and $\tau^*(s) \in \mathcal{T}$, then $\widetilde{\tau}^*(s) = \tau^*(s)$ and $V_{s,0}^{low} = V_{s,0}^{up}$.

In particular, for $s = 1$ (where $R_0 = 1$, $\widehat{V}_0 = V_0^{low}$ and $\widehat{M}_0 = \widehat{V}_0$),

$$\widetilde{\tau}^*(1) := \arg\max_{\widetilde{\tau}(1)\in\mathcal{T}|\widetilde{\tau}(2)} V_0^{low} = \arg\min_{\widetilde{\tau}(1)\in\mathcal{T}|\widetilde{\tau}(2)} \left\{V_0^{up} - V_0^{low}\right\}.$$

Proof. See Appendix A. ■

Minimizing the gap $E_0^Q[GAP_s]$ is well defined and corresponds with optimally exercise at time $s$ conditional on $\widetilde{\tau}(s+1)$. The recursive problem of finding an optimal $\widetilde{\tau}^*(s) \in \mathcal{T}$ solving equation (13) given $\widetilde{\tau}(s+1)$ is tractable, being solved by a local-regression approach (Ibáñez and Velasco, 2018), which yields a continuation value function $\widetilde{V}_s$. Note $\widehat{V}$ and $\widetilde{V}$ are two different functions. $\widehat{V}$ is a lower bound (because $\widehat{V} \leq V^*$) associated with the exercise strategy $\widetilde{\tau}(s)$, which in turn is defined in terms of $\widetilde{V}$. By contrast, $\widetilde{V}$ can be under (over) valued, which leads to exercising too soon (too late).

**The lower/upper biases** From $E_0^Q[GAP_1] = \left(V_0^* - V_0^{low}\right) + \left(V_0^{up} - V_0^*\right)$,

$$\begin{aligned} 0 &\leq V_0^* - V_0^{low} \\ &= E_0^Q\left[\left(1_{\{1=\tau^*(1)\}} I_1 + 1_{\{1<\tau^*(1)\}} V_1^*\right) \times \frac{1}{R_1}\right] - E_0^Q\left[\left(1_{\{1=\widetilde{\tau}(1)\}} I_1 + 1_{\{1<\widetilde{\tau}(1)\}} \widehat{V}_1\right) \times \frac{1}{R_1}\right], \end{aligned}$$

which is the bias associated with the lower bound, and

$$\begin{aligned} 0 &\leq V_0^{up} - V_0^* \\ &= E_0^Q\left[\max\left\{I_1, \widehat{V}_1 + GAP_2 \times \left(\frac{1}{R_1}\right)^{-1}\right\} \times \frac{1}{R_1}\right] - E_0^Q\left[\max\left\{I_1, V_1^*\right\} \times \frac{1}{R_1}\right], \end{aligned}$$

which is the upper-bound bias (from $V_0^{up} = \widehat{M}_0 + E_0^Q[GAP_1]$ and equation (11) for $GAP_1$).

For instance, if $GAP_2 = 0$, because $\widehat{V}_1 \leq V_1^*$, then $\widehat{V}_1 = V_1^*$ and $V_0^{up} = V_0^*$ so that the upper bound is unbiased and independent of the $(t = 1)$ next-period exercise decision, as in Proposition 2. We just assume $\widehat{V}_1$ is computed without simulation error.

### 3.3 An optimal recursive gap: A martingale based on continuation values, $\widetilde{V}_t$

Recall that $\widehat{Z}_t^{(cv)} = \max\left\{\widetilde{V}_t, I_t\right\}$. From Theorem 1 (equation (11)),

$$E_0^Q[GAP_s] = -E_0^Q\left[\max\left\{I_s, \widetilde{V}_s\right\} \times \frac{1}{R_s}\right] + E_0^Q\left[\max\left\{\frac{I_s}{R_s}, \frac{\widehat{V}_s}{R_s} + GAP_{s+1}\right\}\right].$$

Here, minimizing the gap is not well defined, because the lower bound based on $\widetilde{V}$,

$$V_{s,0}^{low} := E_0^Q\left[\frac{\widehat{V}_{s-1}}{R_{s-1}}\right] = E_0^Q\left[\max\left\{I_s, \widetilde{V}_s\right\} \times \frac{1}{R_s}\right],$$

is not necessarily lower biased (where $\widehat{V}$ is defined as in equation (8) with $\widehat{Z}_t^{(cv)}$).

Let us impose the best case $E_0^Q[GAP_s] = 0$, and search for an $\widetilde{V}$ guaranteeing this. Then, if we assume $GAP_{s+1} = 0$,

$$E_0^Q\left[\max\left\{I_s, \widetilde{V}_s\right\} \times \frac{1}{R_s}\right] = E_0^Q\left[\max\left\{I_s, \widehat{V}_s\right\} \times \frac{1}{R_s}\right],$$

and the simple solution associated with the latter equation is that $\widetilde{V}_s = \widehat{V}_s$ subject to $\widehat{V}_s > I_s$.

This solution implies a (least-squares) fitting of the function $\widetilde{V}_s$ for a given sample of simulated paths $\widehat{V}_s$ subject to $\widehat{V}_s > I_s$.[9] $\widetilde{V}_s$ matches $\widehat{V}_s$ only in the waiting region, where $\widehat{V}_s > I_s$. This constraint is convenient because the option continuation value becomes nonlinear near the exercise boundary. This same result is derived in section 4.2 by factorizing the process $\widehat{M}$.

[9] In the case of stopping times, where $\widehat{Z}_t = 1_{\{t<\widetilde{\tau}(t)\}} \widehat{V}_t + 1_{\{t=\widetilde{\tau}(t)\}} I_t$, the last equality is given by

$$E_0^Q\left[\left(1_{\{s<\widetilde{\tau}(s)\}} \widehat{V}_s + 1_{\{s=\widetilde{\tau}(s)\}} I_s\right) \times \frac{1}{R_s}\right] = E_0^Q\left[\max\left\{I_s, \widehat{V}_s\right\} \times \frac{1}{R_s}\right],$$

and the same $\widehat{V}_s$ appears on both sides of the equality, where the only possible difference is derived from $\widetilde{\tau}(s)$.

**A martingale based on both stopping times and continuation values, $\widetilde{\tau}(t)$ and $\widetilde{V}_t$** Recall that $\widehat{Z}_t^{(cvst)} = 1_{\{t<\widetilde{\tau}(t)\}}\widetilde{V}_t + 1_{\{t=\widetilde{\tau}(t)\}}I_t$. The gap minimization is also not well defined. Assuming $E_0^Q[GAP_s] = 0$ and $GAP_{s+1} = 0$,

$$E_0^Q\left[\left(1_{\{s<\widetilde{\tau}(s)\}}\widetilde{V}_s + 1_{\{s=\widetilde{\tau}(s)\}}I_s\right) \times \frac{1}{R_s}\right] = E_0^Q\left[\max\left\{\frac{I_s}{R_s}, \frac{\widehat{V}_s}{R_s}\right\}\right].$$

We split this equality into two subproblems. We take $\widetilde{\tau}(s)$ from a local regression instead of the "max" function, and take $\widetilde{V}_s$ from a least-squares fitting, which also asks for setting $\widetilde{V}_s = \widehat{V}_s$ only in the waiting region (i.e., $\widehat{V}_s > I_s$ or $s < \widetilde{\tau}(s)$).

# 4 Upper bounds: Stopping Times versus Continuation Values

In this section, we compare the previous three upper bounds (based on stopping times, continuation values, and a combination of the two) by factorizing the three associated martingales. First, we show why an upper bound that is based on a good stopping time is less biased than the other two upper bounds, which are based on continuation values. In the latter case, the martingale has an additional third error term, which implies this larger bias (from Jensen inequality). Second, we show that fitting the continuation value in the waiting/continuation region is sufficient, a result that was also derived in the previous section. In the three cases, we include the simulation error in the martingale.

## 4.1 The stopping time, $\widetilde{\tau}(t)$

$\widehat{M}$ is based on $\widehat{Z}^{(st)}$, which depends on an stopping time $\widetilde{\tau}$ as in equation (4); that is, $\widehat{M}_0 = \widehat{V}_0$ and, for $t \in \{1, 2, ..., T\}$,

$$\widehat{M}_t = \widehat{M}_{t-1}R_{t-1,t} + \left(1_{\{t<\widetilde{\tau}(t)\}}\left(\widehat{V}_t + \widehat{\xi}_t\right) + 1_{\{t=\widetilde{\tau}(t)\}}I_t\right) - \left(\widehat{V}_{t-1} + \widehat{\xi}_{t-1}\right) \times R_{t-1,t}, \qquad (14)$$

where $\widehat{\xi}_t$ is a zero-mean error (i.e., $E[\widehat{\xi}_t] = 0$) and hence $\widehat{M}_t/R_t$ is a martingale.

$\widehat{V}$ is computed separately based on

$$\widehat{V}_{t-1} = E_{t-1}^Q\left[R_{t-1} \times \frac{I_{\widetilde{\tau}(t)}}{R_{\widetilde{\tau}(t)}}\right].$$

For every path, the process $\widehat{V}_t$ is computed for $t \in \{1, 2, ..., T-1\}$ by a new subsimulation (from $t$ to $\widetilde{\tau}(t+1)$). Namely, $\widehat{V}_t$, which is an expectation, is replaced by $\widehat{V}_t + \widehat{\xi}_t$ in equation (14). $\widehat{M}_t$ depends on both $\widehat{V}_t$ and $\widehat{V}_{t-1}$; $\widehat{\xi}_t$ introduces a second bias in the upper bound.

From equation (5), the martingale (i.e., $\widehat{M}/R$) is explicitly given by

$$\widehat{M}_t = \sum_{j=1}^{t-1} \left( I_j - \left( \widehat{V}_j + \widehat{\xi}_j \right) \right) \times 1_{\{j=\widetilde{\tau}(j)\}} \times R_{j,t} + \left( 1_{\{t<\widetilde{\tau}(t)\}} \left( \widehat{V}_t + \widehat{\xi}_t \right) + 1_{\{t=\widetilde{\tau}(t)\}} I_t \right). \tag{15}$$

### 4.1.1 Reducing the upper-bound computational cost

Consider a path "$\omega$" such that the stopping time satisfies that $t < \widetilde{\tau}(t) = t+1$ (and let $\widehat{\xi}_t = 0$). From $R_{j,t}/R_t = R_j^{-1}$,

$$\frac{\widehat{M}_t - I_t}{R_t} = \sum_{j=1}^{t-1} \left( I_j - \widehat{V}_j \right) \times 1_{\{j=\widetilde{\tau}(j)\}} \times \frac{1}{R_j} + \frac{\widehat{V}_t - I_t}{R_t}.$$

Likewise, noting the "$j = t$" term of the sum in $\widehat{M}_{t+1}$ is zero because $t < \widetilde{\tau}(t)$,

$$\frac{\widehat{M}_{t+1} - I_{t+1}}{R_{t+1}} = \sum_{j=1}^{t-1} \left( I_j - \widehat{V}_j \right) \times 1_{\{j=\widetilde{\tau}(j)\}} \times \frac{1}{R_j}.$$

Note that $t < \widetilde{\tau}(t)$ does not necessarily imply $\widehat{V}_t > I_t$; if it did, $\widetilde{\tau}(t) = \widetilde{\tau}(t+1)$ would be the optimal time–$t$ recursive exercise policy. Hence, we have no guarantee that

$$\frac{-(\widehat{M}_t - I_t)}{R_t} < \frac{-(\widehat{M}_{t+1} - I_{t+1})}{R_{t+1}},$$

which would imply computing $\widehat{V}_t$ is not necessary if $t < \widetilde{\tau}(t)$. Similarly, given $\tau(t) = t+1$, computing $\widehat{V}_{t-j}$ is also not necessary for any previous period $t - j$ $(j \geq 0)$ such that the path is in the continuation region, namely, $t - j < \widetilde{\tau}(t - j)$.

Hence, for any path $\omega$, it does not necessarily follow that

$$\max_{1 \leq t \leq T:\, t=\widetilde{\tau}(t)} \left\{ \frac{I_t - \widehat{M}_t}{R_t} \right\} = \max_{1 \leq t \leq T} \left\{ \frac{I_t - \widehat{M}_t}{R_t} \right\}.$$

Using the lhs, however, reduces the number of periods in which to launch a subsimulation (especially for at-the-money/out-of-the-money options, as paths start in the waiting region), but introduces a negative bias that lowers the upper bound. If $\widetilde{\tau}$ is a good exercise policy, this bias may be negligible, especially compared to the time saved in subsimulations.[10]

## 4.2 The continuation value, $\widetilde{V}_t$

Consider $\widehat{M}$ is based on $\widehat{Z}^{(cv)}$. Instead of using stopping times, we define a martingale based on a continuation value, $\widetilde{V}$; that is, $\widehat{M}_0 = \widehat{V}_0$, and for $t \in \{1, 2, ..., T\}$,

$$\widehat{M}_t = \widehat{M}_{t-1} R_{t-1,t} + \max\left\{ \widetilde{V}_t, I_t \right\} - \left( \widehat{V}_{t-1} + \widehat{\xi}_{t-1} \right) \times R_{t-1,t}, \tag{16}$$

[10] Broadie and Cao (2008) and Joshi (2007) introduce a similar idea to reduce the cost of dual upper bounds.

and

$$\widehat{V}_{t-1} = E^{Q}_{t-1}\left[R^{-1}_{t-1,t}\max\left\{\widetilde{V}_t, I_t\right\}\right].$$

Now, for every path, the process $\widehat{V}_t$ is computed for $t \in \{1, 2, ..., T-1\}$ by a one-period subsimulation (from $t$ to $t+1$), where $\widehat{\xi}$ is the one-period subsimulation error, $E[\widehat{\xi}_t] = 0$.[11]

Then (see Appendix A)

$$\widehat{M}_t = \sum_{j=1}^{t-1}\left\{I_j - \widetilde{V}_j\right\}^{+} \times R_{j,t} + \sum_{j=1}^{t-1}\left(\widetilde{V}_j - \left(\widehat{V}_j + \widehat{\xi}_j\right)\right) \times R_{j,t} + \max\left\{\widetilde{V}_t, I_t\right\}. \tag{17}$$

By comparing $\widehat{M}$ in equation (17) with the optimal $M^*$ in (15), the upper-bound bias depends mostly on the second sum, a martingale error-correcting term, in which the differences $\widetilde{V}_j - \widehat{V}_j$ can be large. However, $\widetilde{V}_j$ cancels if $\{I_j - \widetilde{V}_j\}^+ > 0$, from the first and second sums, implying the error $\widetilde{V}_j - \widehat{V}_j$ only matters when $I_j < \widetilde{V}_j$, which is the waiting region.

### 4.2.1 Computing the function $\widetilde{V}^{co}$ in the waiting region

We denote by $\widetilde{V}^{co}_s$ the continuation value in the waiting region. $\widetilde{V}^{co}_s$ is computed in two steps. First, we compute a stopping time, $\widetilde{\tau}(s) \geq s$. Second, we use least squares to fit $\widetilde{V}^{co}_s$ in the continuation region, where $\widetilde{\tau}(s) > s$. That is, $\widetilde{\tau}(s)$ is used to exercise a sample path (from $s+1$ on) and to determine if a path is in the waiting region (at time $s$).

Consider a family of continuation values $F$. $\widetilde{V}^{co}_T = 0$ and $\widetilde{V}^{co}_{t-1}$ is defined, for $t = \{T, T-1, ..., 2\}$, as follows:

$$\begin{aligned}
\widetilde{V}^{co}_{t-1} &= \arg\min_{f_{t-1}\in F}\sum_{\varpi}\frac{1}{R_{t-1}} \times \left(f_{t-1} - R^{-1}_{t-1,t} \times \widehat{U}_t\right)^2 \times 1_{\{\widetilde{\tau}(t-1)>t-1\}},\\
\widehat{U}_t &= R^{-1}_{t,\widetilde{\tau}(t)} \times I_{\widetilde{\tau}(t)},
\end{aligned}$$

and all variables depend on the sample paths $\omega$. At time $s = t-1$, we compute many samples of the discounted realized payoff, $R^{-1}_{t-1,t} \times \widehat{U}_t$, following the local-regression exercise strategy $\widetilde{\tau}(t)$. The payoffs in the waiting region (i.e., if $1_{\{\widetilde{\tau}(t-1)>t-1\}} = 1$) are approximated using a global regression by the family $F$. We proceed backward until $s = 1$. We then plug $\widetilde{V}^{co}$ in equation (16); that is, $\widetilde{V} = \widetilde{V}^{co}$.

Table 1's last two columns show a global-regression stopping time produces only slightly worse upper bounds than the local-regression stopping time. This finding implies estimating the continuation value in the waiting region (i.e., if $\widetilde{\tau}(t-1) > t-1$) is the key insight (more than the realized payoff $\widehat{U}_t$) for upper bounds based on a continuation value function.

[11] In some cases, the one-period expectation can be approximated analytically (Glasserman and Yu, 2004; Nadarajaha et al., 2017)

**A third alternative, $\widetilde{\tau}(t)$ and $\widetilde{V}_t$.** $\widehat{M}$ is based on $\widehat{Z}^{(stcv)}$. Extending equation (4), $\widehat{M}$ is defined from $\widetilde{\tau}$ but using a new $\widetilde{V}$; that is, $\widehat{M}_0 = \widehat{V}_0$, and for $t = \{1, 2, ..., T\}$,

$$\widehat{M}_t = \widehat{M}_{t-1} R_{t-1,t} + \left(1_{\{t<\widetilde{\tau}(t)\}} \widetilde{V}_t + 1_{\{t=\widetilde{\tau}(t)\}} I_t\right) - \left(\widehat{V}_{t-1} + \widehat{\xi}_{t-1}\right) \times R_{t-1,t},$$

and

$$\widehat{V}_{t-1} = E^Q_{t-1}\left[R^{-1}_{t-1,t} \times \left(1_{\{t<\widetilde{\tau}(t)\}} \widetilde{V}_t + 1_{\{t=\widetilde{\tau}(t)\}} I_t\right)\right].$$

We combine, in a one-period subsimulation, $\widetilde{\tau}$ from the local regression and $\widetilde{V}$ from least-squares (in the waiting region). This approach is intuitive if $\widetilde{\tau}$ is close to $\tau^*$.

Then

$$\begin{aligned}\widetilde{M}_t \quad = \quad & \sum_{j=1}^{t-1} \left(I_j - \widetilde{V}_j\right) \times 1_{\{t=\widetilde{\tau}(t)\}} \times R_{j,t} + \sum_{j=1}^{t-1} \left(\widetilde{V}_j - \left(\widehat{V}_j + \widehat{\xi}_j\right)\right) \times R_{j.t} \\ & + \left(1_{\{t<\widetilde{\tau}(t)\}} \widetilde{V}_t + 1_{\{t=\widetilde{\tau}(t)\}} I_t\right).\end{aligned} \quad (18)$$

Comparing the three factorizations, in (15), (17), and (18), based on $\widehat{Z}^{(st)}$, $\widehat{Z}^{(cv)}$, and $\widehat{Z}^{(stcv)}$, respectively, the first yields a martingale that is close to $M^*/R$ if $\widetilde{\tau}$ is close to $\tau^*$. The second and third martingales require only a one-period subsimulation, but include a second sum that depends on the error between the posed $\widetilde{V}$ and the sample realized $\widehat{V}$ (in the waiting region), implying a more biased upper bound. In an unreported numerical exercise, we find the latter martingale yields the most biased upper bound of the three.[12]

## 5 Numerical Exercise: Up-and-Out Bermudan Max-Options

We price an up-and-out max-call Bermudan option. This barrier feature makes call payoffs very sensitive to suboptimal exercise. We define $I_t = \{\max\{S_t\} - K\}^+$, $K$ is the strike price, and $B > K$ is the barrier. We introduce two auxiliary processes, $Y$ and $b$; $b_0 = 1$ and

$$Y_t = 1_{\{\max\{S_t\}<B\}} \text{ and } b_t = b_{t-1} \times Y_t, \; t = 1, 2, ..., T. \quad (19)$$

$b_t = 0$ indicates the up-and-out barrier ($B$) has been hit (i.e., $b_j = 0$, $j = t, t+1, ..., T$). The Bermudan payoff is given by $I_t \times b_t$.

[12] The third martingale works worse than the second martingale, because a low-biased $\widetilde{V}_t$ implies a process based on $\widehat{Z}^{(cv)} = \max\left\{\widetilde{V}_t, I_t\right\}$ is closer to the optimal $Z^*_t = \max\{V^*_t, I_t\}$ than the one based on $\widehat{Z}^{(stcv)} = (1_{\{t<\widetilde{\tau}(t)\}} \widetilde{V}_t + 1_{\{t=\widetilde{\tau}(t)\}} I_t)$. That is, for any stopping time $\widetilde{\tau}(t)$ (including $\widetilde{\tau}(t) = \tau^*(t)$),

$$\text{if } \widetilde{V}_t \le V^*_t, \; \max\{V^*_t, I_t\} \ge \max\left\{\widetilde{V}_t, I_t\right\} \ge 1_{\{t<\widetilde{\tau}(t)\}} \widetilde{V}_t + 1_{\{t=\widetilde{\tau}(t)\}} I_t.$$

We use Ibáñez and Velasco's local regression to derive the optimal recursive policy, and compute the associated lower/dual upper bounds. We follow the MC exercise in Table 1 in Desai et al. (DFM). This table has nine examples, corresponding to three numbers of stocks ($N = \{4, 8, 16\}$) and three initial stock prices ($S_0 = \{90, 100, 110\}$). The strike price $K = 100$ is common across all scenarios. The up-and-out barrier is $B = 170$. To derive the $\tau$ associated with the local regression, we also use 200,000 paths but exclude those points that are out-of-the-money or hit the barrier (i.e., if $\max\{S_t\} \leq K$ or if $\max\{S_t\} \geq B$). The local-regression algorithm is in Appendix B, where we provide further details.

We use the same linear basis of $N + 2$ variables as DFM, which includes a constant, every component of the price vector $S = \left(S^{(1)}, S^{(2)}, ..., S^{(N)}\right)$, and $\{\max\{S_t\} - K\}^+$;[13] that is,

$$\left(1, S_t, \{\max\{S_t\} - K\}^+\right) \times 1_{\{K<\max\{S_t\}<B\}}.$$

The regression's dependent variable is also multiplied by $1_{\{K<\max\{S_t\}<B\}}$. To compute $V_0^{low}$, we use 2 million paths. We report the mean and standard error over 10 independent trials.

## 5.1 Lower/upper bounds based on stopping times

In our Table 1, we provide the lower bound $V_0^{low}$ produced by two regression methods: the global method and the local method (first to third iterations). From the exercise strategies associated with global regression and the local third iteration, we also generate upper bounds. The upper-bound mean and standard error are also based on 10 independent trials.

***** to include Table 1 *****

As our Table 1 shows, the local approach improves upon the global (and DFM) lower bounds. The local lower bounds improves upon the global lower bounds by 100 to 280 cents (upon DFM lower bounds by 85 to 160 cents). The lowest gap in DFM Table 1, which includes two lower and three upper bounds, varies from 105 cents ($N = 16$ and $S_0 = 110$) to 200 cents ($N = 4$ and $S_0 = \{90, 100\}$). The third iteration of the local regression reduces this gap to less than 10 cents. In the nine examples, the first iteration yields the most significant improvement; for four assets, the price rises only by a few cents after the third iteration; for eight and 16 assets, the lower bound converges in one iteration.[14]

[13] Because we consider in-the-money paths, the regressors $\left(1, \{\max\{S\} - K\}^+\right)$ and $(1, \max\{S\})$ are alike.

[14] The largest gap corresponds to four stocks; for 16 stocks, the gap is less than 6 cents. For 16 stocks, if the option is in-the-money, the chance of hitting the barrier the next period and all being lost is high. Hence,

**Standard errors** The standard errors (s.e.) of the local-regression method are between 0.2 to 0.4 cents. These s.e. correspond to the average of 10 trials, where the price of each trial is computed with two million paths. Because the s.e. of a single price is approximately 0.3 cents, the similar (0.2 to 0.4 cents) s.e. of the average of 10 prices, which use 20 million paths, is mostly due to computing the 10 different continuation values in the first simulation step. The error of these estimated continuation values, which are based on 200,000 paths, does not have a significant impact on the lower bound (i.e., $\sqrt{10} \times 0.3$, 1 cent approximately).

**Improvement and robustness of the lower bound** In Table 2, we increase the number of paths that are used in the local regression to improve Table 1 numbers. We consider $N = 4$ stocks (the hardest problem) and compute the average of 10 independent trials. Improving these lower bounds is difficult. We just reduce the s.e. and get smoother prices through all iterations, which is intuitive in a least-squares setting.

In Table 3, we show the lower bound's robustness to the local regression kernel. More (less) than 1% percent of the paths that are used in the Table 1 kernel imply lower (slightly larger but erratic) prices. This 1% is our optimal kernel choice, yet a kernel with a constant bandwidth of 1.2 (from try and error) also works. A bandwidth larger (lower) than 1.2 implies lower (larger but erratic) prices; for eight assets, this constant is 2.0.

Tables 2 and 3 show the robustness of both upper bounds to the estimation of the continuation value by local least squares (i.e., number of simulation paths and kernel). Because the upper bound based on stopping times is robust to (and does not improve with) the number of iterations, this upper bound is tighter (closer to the true price) than the lower bound.

***** to include Table 2 and 3 *****

## 5.2 Upper bounds

In the case of (a martingale based on) stopping times, the upper bound is defined as the lower bound plus the gap. Because we have computed a lower bound with a large number of paths in Table 1, we just compute (and report) the gap in Table 4. The gap has much less variability, requiring a lower number of paths, for example, 3,000 (against the 2 million for the lower

the optimal exercise policy seems to be exercising as soon as the largest stock price is sufficiently in-the-money. We also include the two-asset Bermudan, where the price is approximated by a binomial method and linear extrapolation (and differs from the upper bound by 1 cent). A sixth iteration of the local LSM increases the lower-bound to 31.05, which cannot be further improved by additional iterations.

bound) for a similar standard error. In the case of continuation values, we directly compute the upper bound as explained in section 4.2.

First, the upper bound based on the third iteration of the local exercise strategy produces very good prices. Table 4 shows this upper bound deteriorates little with the number of paths in the sub-simulation. We can reduce the upper-bound cost without losing accuracy; for example, in reducing the number of subsimulation paths from 10,000 to 500 (to 100), the gap rises by only 3 (15) cents.

***** to include Table 4 *****

Second, from Table 1, an upper bound based on continuation values is more biased. Yet fitting a continuation value in the waiting region ($V^{co}$) yields upper bounds that, especially if the option is at-/in-the-money, are as accurate as those based on stopping times and global regression but in a fraction of the time. The difference between splitting the waiting region by using a global versus a local approach is less important; see Table 1, last two columns.[15]

**Computational cost of optimal recursive lower/upper bounds** We make two remarks. First, in the case of the lower bound, optimizing the kernel means optimizing the bandwith. Although the latter can be optimized in each exercise period, a constant bandwidth (which is obtained from try and error) is robust; it produces prices a couple of cents below the best price. This kernel optimization is a fixed or learning cost, which is made once.

A kernel regression is a weighted-global regression, which has a cost similar to a global regression. Therefore, the variable cost of a local approach over the global regression is given by the number of iterations minus one. For example, for $N = \{8, 16\}$, the lower bound converges in one iteration; hence, the extra cost of the local approach is given by the time invested in finding the constant bandwidth. In addition, the local approach is independent of moneyness, which is convenient in the case of portfolios of derivatives.

Second, in the case of the upper bound based on stopping times, our upper bound is so good that we can reduce the number of subsimulation paths (e.g., from 10,000 to 500 paths) without losing accuracy. This is a notable saving for the upper bound. So the combined cost of the lower/upper bounds based on a local approach (i.e., a constant bandwidth, moneyness free, a couple of iterations, and less subsimulation paths) compares well with the cost of the

[15] In addition (not reported here for brevity), for four assets, we can reduce this upper bound by another 10 bps if we use a quadratic (instead of a linear) function $V^{co}$, and by another few points if $V^{co}$ is estimated from paths simulated exactly from $S_0$ (to compute $\widetilde{V}$ for the $\widetilde{\tau}$ stopping-time, it is convenient to simulate paths not from $S_0$ and $t = 0$ but from in-the-money values and $t < 0$).

two bounds based on global regression, yet the local approach produces very tight bounds.

## 6 Testing Whether a Stopping Time Is Optimal

Consider a martingale $\frac{M}{R}$ defined in terms of stopping times/exercise strategies, $\widetilde{\tau}$. The gap $V_0^{up} - \widehat{V}_0$ enables us to test if $\widetilde{\tau}$ is optimal. $V_0^{up} - \widehat{V}_0 = 0$ if and only if $\widetilde{\tau} = \tau^*$ (if we forget the simulation error) as shown by the literature. In this section, we introduce a second test about the optimality of $\widetilde{\tau}$, where $\widetilde{\tau} = \widetilde{\tau}\,(1)$.

For any stopping time $\widetilde{\tau}$, see equations (5) and (6), $\widehat{M_{\widetilde{\tau}}} = I_{\widetilde{\tau}} \times b_{\widetilde{\tau}}$. It follows that

$$0 \leq \max_{1 \leq t \leq T} \left\{ \frac{I_t - \widehat{M}_t}{R_t} \right\} = GAP_1,$$

In the case of $\tau^*$, the inequality is binding (see Proposition 1).

Define $t^* = t^*(\omega)$ for any path $\omega$,

$$t^*(\omega) := \inf \left( \arg \max_{1 \leq t \leq T} \left\{ \frac{I_t - \widehat{M}_t}{R_t} \right\} \right).$$

From $\widehat{M_{\widetilde{\tau}}} = I_{\widetilde{\tau}}$, it follows $\widehat{M}_{t^*} = I_{t^*}$ if $t^* = \widetilde{\tau}$. Then,

$$\begin{aligned} V_0^* &\leq V_0^{up} = \widehat{V}_0 + E_0^Q \left[ \left. \frac{I_t - \widehat{M}_t}{R_t} \right|_{t=t^*} \right] \\ &= \widehat{V}_0 + E_0^Q \left[ \left. \frac{I_t - \widehat{M}_t}{R_t} \right|_{t=t^*} \times 1_{\{t^* \neq \widetilde{\tau}\}} \right]. \end{aligned} \tag{20}$$

The latter equation implies that if $\widetilde{\tau}$ is not the optimal stopping-time (and $V_0^* - \widehat{V}_0 > 0$),

$$E_0^Q \left[ \left. \frac{I_t - \widehat{M}_t}{R_t} \right|_{t=t^*} \times 1_{\{t^* \neq \widetilde{\tau}\}} \right] = V_0^{up} - \widehat{V}_0 \geq V_0^* - \widehat{V}_0 > 0.$$

The gap, $V_0^{up} - \widehat{V}_0$, depends only on those paths in which $\widetilde{\tau}$ is different from $t^*$; $t^* \neq \widetilde{\tau}$. For the optimal stopping time $\tau^*$, this set has zero probability because $t^* = \tau^*$ (Proposition 1).

**Proposition 4.** Assume that $\left. \frac{I_t - \widehat{M}_t}{R_t} \right|_{t=t^*} < \infty$ (a.s.) and $E_0^Q \left[ 1_{\{\tau^* \neq \widetilde{\tau}(1)\}} \right] > 0$ implies $\widehat{V}_0 < V_0^*$. Then $E_0^Q \left[ 1_{\{\tau^* \neq \widetilde{\tau}(1)\}} \right] > 0$ (i.e., $\widetilde{\tau}$ is not the optimal stopping time) if and only if

$$E_0^Q \left[ 1_{\{t^* \neq \widetilde{\tau}(1)\}} \right] > 0. \tag{21}$$

Proof. See Appendix A.

That is, $\widetilde{\tau} \neq \tau^*$ with positive probability if and only if $\widetilde{\tau} \neq t^*$ with positive probability. Although $\tau^*$ is unknown, the inequality (21) is feasible because $t^*$ can be simulated. Next, because of the following bound,

$$0 \leq E_0^Q \left[ 1_{\{t^* \neq \widetilde{\tau}(1)\}} \right] \leq 1,$$

an optimality test based on "counting" (i.e., $t^*$ and $\widetilde{\tau}$) can be more intuitive than one based on computing lower and dual upper bounds,

$$0 \leq V_0^{up} - \widehat{V}_0 = E_0^Q \left[ \left. \frac{I_t - \widehat{M}_t}{R_t} \right|_{t=t^*} \times 1_{\{t^* \neq \widetilde{\tau}(1)\}} \right],$$

where the right-hand-side equality is also a novel result. Finally, alternative tests can also be proposed using time deviations, because $E_0^Q \left[ 1_{\{t^* \neq \widetilde{\tau}(1)\}} \right] > 0$ implies $E_0^Q \left[ |t^* - \widetilde{\tau}(1)| \right] > 0$.

## 7 Conclusions

In this paper, we show the exercise strategy that maximizes the Bermudan price/lower bound (Ibáñez and Velasco, 2018) also minimizes the gap between the lower and the dual upper bound. We assume both bounds are specified recursively, and show the upper bound is independent of the next-period policy. Lower/upper bounds based on this optimal recursive exercise policy are very tight, as we show for barrier Bermudan max-options. Upper bounds are tighter but more time intensive than lower bounds. In addition, a better upper bound based on continuation values, not as accurate but more efficient than based on stopping times, requires reestimating the continuation value *only* in the waiting region.

Specifically, the difference between the payoff slope and the option Delta at the exercise boundary gives the sensitivity to suboptimal exercise for Bermudan options. The up-and-out barrier feature means the option Delta is well below 1, implying optimal exercise matters for these securities. For more examples of lower/upper bounds in equity models with stochastic volatility and interest rates, see Ibáñez and Velasco (2016); for term-structure applications, Joshi and Tang (2014); for energy real options, Nadarajaha et al. (2017); and see Kogan and Mitra (2013) and Bender et al. (2017) for extensions to other economic problems.

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# 8 Appendix A: Proofs

**Proof of Equation (5).** From the same equation (5),

$$\begin{aligned}
\widehat{M}_t - \widehat{M}_{t-1}R_{t-1,t} &= \left(I_{t-1} - \widehat{V}_{t-1}\right) \times 1_{\{t-1=\widetilde{\tau}(t-1)\}} \times R_{t-1,t} + \left(1_{\{t<\widetilde{\tau}(t)\}}\widehat{V}_t + 1_{\{t=\widetilde{\tau}(t)\}}I_t\right) \\
&\quad - \left(1_{\{t-1<\widetilde{\tau}(t-1)\}}\widehat{V}_{t-1} + 1_{\{t-1=\widetilde{\tau}(t-1)\}}I_{t-1}\right) \times R_{t-1,t} \\
&= \left(1_{\{t<\widetilde{\tau}(t)\}}\widehat{V}_t + 1_{\{t=\widetilde{\tau}(t)\}}I_t\right) - \widehat{V}_{t-1} \times R_{t-1,t},
\end{aligned}$$

which is equation (4). ■

**Proof of Proposition 1.** $M^*$ is explicitly defined as $\widehat{M}$ in equation (5) for $\widetilde{\tau} = \tau^*$. The definition of $\tau^*$ (i.e., $\tau^*(t) = t$, if $I_t \geq V_t^*$; $\tau^*(t) > t$, otherwise) implies

$$(I_t - V_t^*) \times 1_{\{t=\tau^*(t)\}} = \{I_t - V_t^*\}^+ \geq 0.$$

From equation (5) for $\widetilde{\tau} = \tau^*$ and from the last equation, it follows that

$$M_t^* > I_t, \text{ if } t \neq \tau^*; \text{ and } M_t^* = I_t, \text{ if } t = \tau^*,$$

where $\tau^*$ means $\tau^*(1)$, and therefore,

$$\begin{aligned} & \max_{1\leq t\leq T} \{I_t - M_t^*\} = I_{\tau^*} - M_{\tau^*}^* = 0, \text{ and} \\ V_0^{up} := \quad & M_0^* + E_0^Q \left[ \max_{1\leq t\leq T} \left\{ \frac{I_t - M_t^*}{R_t} \right\} \right] = V_0^* + E_0^Q [0] = V_0^*. \quad \blacksquare \end{aligned}$$

$V_{1,0}^{up}$ **does not depend on the martingale initial value** $\widehat{M}_0$. Consider a different initial value $\widehat{x}_0 \neq \widehat{V}_0$. From

$$\begin{aligned} V_{s,0}^{up} \quad : \quad & = \widehat{V}_0 + E_0^Q \left[ \max_{1\leq t\leq T} \left\{ \frac{I_t - \widehat{M}_t}{R_t} \right\} \right] \\ & = \widehat{x}_0 + E_0^Q \left[ \max_{1\leq t\leq T} \left\{ \frac{I_t - \left( \widehat{M}_{s,t} - \left( \widehat{V}_0 - \widehat{x}_0 \right) \times R_t \right)}{R_t} \right\} \right], \end{aligned}$$

and a more general martingale $\widehat{m}_t / R_t$ is given by $\widehat{m}_0 = \widehat{x}_0$ and $\widehat{m}_t = \widehat{M}_t - \left( \widehat{V}_0 - \widehat{x}_0 \right) \times R_t$ for $t \geq 1$. And $\widehat{m}_t = \widehat{M}_t$ if $\widehat{x}_0 = \widehat{V}_0$. $\blacksquare$

**Proof of Theorem 1.** From equation (10),

$$GAP_s = \frac{-\widehat{Z}_s}{R_s} + \max \left\{ \frac{I_s}{R_s}, \frac{\widehat{M}_s}{R_s} + \max_{s+1\leq t\leq T} \left\{ \frac{I_t - \widehat{M}_t}{R_t} \right\} \right\} \tag{22}$$

$$= \frac{-\widehat{Z}_s}{R_s} + \max \left\{ \frac{I_s}{R_s}, \frac{\widehat{V}_s}{R_s} + \frac{\widehat{M}_s R_{s,s+1} + \widehat{Z}_{s+1} - \widehat{V}_s R_{s,s+1}}{R_{s+1}} - \frac{\widehat{Z}_{s+1}}{R_{s+1}} + \max_{s+1\leq t\leq T} \left\{ \frac{I_t - \widehat{M}_t}{R_t} \right\} \right\}$$

$$= \frac{-\widehat{Z}_s}{R_s} + \max \left\{ \frac{I_s}{R_s}, \frac{\widehat{V}_s}{R_s} + \underbrace{\frac{\widehat{M}_{s+1}}{R_{s+1}} - \frac{\widehat{Z}_{s+1}}{R_{s+1}} + \max_{s+1\leq t\leq T} \left\{ \frac{I_t - \widehat{M}_t}{R_t} \right\}}_{=GAP_{s+1}} \right\},$$

the last equality follows from the definition of $\widehat{M}$ in equation (9).

Then it is easy to show by induction that $GAP_s + \frac{\widehat{Z}_s}{R_s}$ does not depend on $\widetilde{\tau}(s)$ or $\widetilde{V}_s$ because $GAP_{s+1}$ does not either. $\blacksquare$

**Proof of Propostion 3.** $V_{s,0}^{up}$ does not depend on $\widetilde{\tau}(s)$, and hence, from equation (12),

$$\begin{aligned}\arg\min_{\widetilde{\tau}(s)\in\mathcal{T}|\widetilde{\tau}(s+1)} E_0^Q\left[GAP_s\right] &= \arg\min_{\widetilde{\tau}(s)\in\mathcal{T}|\widetilde{\tau}(s+1)} \left\{V_{s,0}^{up} - V_{s,0}^{low}\right\} = \arg\min_{\widetilde{\tau}(s)\in\mathcal{T}|\widetilde{\tau}(s+1)} \left\{-V_{s,0}^{low}\right\} \\ &= \arg\max_{\widetilde{\tau}(s)\in\mathcal{T}|\widetilde{\tau}(s+1)} V_{s,0}^{low} = \widetilde{\tau}^*(s). \quad \blacksquare\end{aligned}$$

**Proof of equation (17).** We assume $\widehat{\xi} = 0$ for simplicity. From the same equation (17),

$$\begin{aligned}\widehat{M}_t - \widehat{M}_{t-1}R_{t-1,t} &= \left(\left\{I_{t-1} - \widetilde{V}_{t-1}\right\}^+ + \widetilde{V}_{t-1} - \widehat{V}_{t-1}\right) \times R_{t-1,t} + \max\left\{\widetilde{V}_t, I_t\right\} \\ &\quad - \max\left\{\widetilde{V}_{t-1}, I_{t-1}\right\} \times R_{t-1,t} \\ &= \max\left\{\widetilde{V}_t, I_t\right\} - \widehat{V}_{t-1} \times R_{t-1,t},\end{aligned}$$

which is equation (16), because $\left\{I_{t-1} - \widetilde{V}_{t-1}\right\}^+ + \widetilde{V}_{t-1} = \max\left\{\widetilde{V}_{t-1}, I_{t-1}\right\}$. $\blacksquare$

**Proof of Proposition 4.** We assume $E_0^Q\left[1_{\{\tau^*\neq\widetilde{\tau}\}}\right] > 0$ implies $V_0^* > \widehat{V}_0$.

If $V_0^* > \widehat{V}_0$,

$$E_0^Q\left[\left.\frac{I_t - \widehat{M}_t}{R_t}\right|_{t=t^*} \times 1_{\{t^*\neq\widetilde{\tau}\}}\right] = V_0^{up} - \widehat{V}_0 \geq V_0^* - \widehat{V}_0 > 0,$$

where $\left.\frac{I_t - \widehat{M}_t}{R_t}\right|_{t=t^*} \geq 0$ and $1_{\{t^*\neq\widetilde{\tau}\}} \geq 0$. Next, if

$$E_0^Q\left[\left.\frac{I_t - \widehat{M}_t}{R_t}\right|_{t=t^*} \times 1_{\{t^*\neq\widetilde{\tau}\}}\right] > 0,$$

then $E_0^Q\left[1_{\{t^*\neq\widetilde{\tau}\}}\right] > 0$ if we assume $\left.\frac{I_t - \widehat{M}_t}{R_t}\right|_{t=t^*} < \infty$ (a.s.).

Finally, because $t^* = \tau^*$ path by path (see Proposition 1), $E_0^Q\left[1_{\{t^*\neq\widetilde{\tau}\}}\right] > 0$ directly implies $E_0^Q\left[1_{\{\tau^*\neq\widetilde{\tau}\}}\right] > 0$. $\blacksquare$

# 9 Appendix B: The Local Least-Squares Algorithm

We follow Ibáñez and Velasco (2018) and adapt the local algorithm to the up-and-out barrier. Let $ni \geq 1$ be the number of iterations. We specify the final period $t = T$, and recursively solve the continuation value for $T-1$, $T-2$, until $t = 1$.

The intrinsic value is given by

$$I_t = \left\{\max\{S_t\} - K\right\}^+, \ t = \{1, 2, ..., T\} \ , \tag{23}$$

Consider a set of simulated paths, $\varpi \in \Omega$.

**The local LSM algorithm:**

**0.** Set $t = T$. Define $y_{T+1} = 0$ and $\widetilde{V}_T^* = 0$.

**1.** UPDATING PATHS, $\varpi \in \Omega$

$$y_t = 1_{\{\max\{S_t\}<B\}} \times \begin{cases} I_t, & \text{if } I_t \geq \widetilde{V}_t^* \\ e^{-r\Delta t} \times y_{t+1}, & \text{otherwise.} \end{cases}$$

With the indicator $1_{\{\max\{S_t\}<B\}}$, we cancel (the value is zero) a path that hits the barrier. Set $t = t - 1$.

**2.** The new CONTINUATION VALUE

Set $n = 1$ and $\widetilde{V}_t^0 = \widetilde{V}_{t+1}^*$. If $t = T - 1$, set $\widetilde{V}_{T-1}^0 = \widetilde{V}_{T-1}^{LSM}$ from a global least-squares method.

**2.1.** LOCALIZING THE EXERCISE BOUNDARY

Provide a kernel $\mathcal{K}$. Then

$$\begin{aligned} \widetilde{V}_t^n &= \arg\min_{f_t \in F} \sum_{\varpi \in \Omega} \left( f_t\left(x_t\right) - \frac{y_{t+1}}{e^{r\Delta t}} \right)^2 \times \mathcal{K}\left( \widetilde{V}_t^{n-1}\left(x_t\right) - I_t, \sigma \right) \times 1_{\{K<\max\{S_t\}<B\}}, \\ \widetilde{V}_t^n &\leftarrow \frac{\widetilde{V}_t^n + \widetilde{V}_t^{n-1}}{2}. \end{aligned} \tag{24}$$

Set $n = n + 1$. Go back to step **2.1** until $n = ni$.

Set $\widetilde{V}_t^* = \widetilde{V}_t^{ni}$. Go back to step **1** until $t = 1$.

**End of the local LSM algorithm**

At $t = 1$, we have estimated all continuation values: $\widetilde{V}_{T-1}^*, \widetilde{V}_{T-2}^*, ..., \widetilde{V}_1^*$. ■

Finally, like Desai, Farias, and Moallemi (2012), we use the same $N + 2$ regressors,

$$x_t = \left(1, S_t, \{\max\{S_t\} - K\}^+\right),$$

and the same linear function, namely, $f_t\left(x_t\right) = b_t' \times x_t$, where $b_t$ is the $1 \times (N + 2)$ vector of coefficients.

## 9.1 The local-regression implementation

The distinctive feature of the local-regression method is estimating the continuation value near the exercise boundary. Note four points regarding this algorithm.

First, for a local regression, having paths close to the (unknown) exercise boundary is necessary; otherwise, we have no information to rely on. We start to simulate paths three months before the initial period $t = 0$, so rich price dispersion is present at the first exercise dates. For $N = \{2, 4, 8\}$, we simulate paths from an in-the-money point (i.e., 120 for all assets). If the boundary is well above $K = 100$, and we simulate paths from 90 or 100 (and from $t = 0$), few paths overshoot the boundary at the first exercise dates. For $N = 16$ assets, we simulate from 100 because many paths will eventually hit the barrier. The simulated paths are the same for the global-regression method. These changes improve the robustness of the local method (our global-regression prices are in line with those reported in DFM).

Second, by using the continuation value estimated in the previous period to define the kernel, one local regression produces very good prices. We iterate this local regression a couple of times to further increase this price a few cents. The local exercise strategy does not depend on moneyness; that is, neither the initial simulation point (120 for $N = \{2, 4, 8\}$ and 100 for $N = 16$) nor the estimated continuation values depend on the initial stock price $S_0$.

Third, the up-and-out barrier implies the Bermudan price is not monotonic near the exercise boundary. To avoid potential cycles and guarantee smooth price convergence, we define the new continuation value as one half the local regressions of the present and previous periods (the last two iterations, in the case of more than one iteration).

Finally, the optimal kernel uses approximately 1%–5% of the 200,000 simulated points that are closer to the exercise boundary. A larger (lower) number of paths implies more biased (more erratic) prices; see Table 3 below. For 8 and 16 stocks, many of those 200,000 paths eventually hit the barrier near expiry, which implies many fewer available points for the local regression, requiring a less localized kernel of 5%.

## 10 Tables

Table 1: Lower and upper bounds

| | | Lower bound, $V_0^{low}$ | | | | Upper bound, $V_0^{up}$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| $S_0$ | binomial price | LSM | local LSM iterations | | | $\tau$–based | | $V^{co}$–based | |
| | $V_0^*$ | | 1st | 2nd | 3rd | 3rd | LSM | 3rd | LSM |
| | | | $n = 2$ assets (kernel= 0.5%) | | | | | | |
| 100 | 31.074 | 28.799 (.006) | 30.869 (.008) | 30.988 (.006) | 31.016 (.006) | 31.083 (.001) | 31.278 (.028) | 31.347 (.006) | 31.331 (.007) |
| | $[V_{0,\text{DFM}}^{low}, V_{0,\text{DFM}}^{up}]$ | | $n = 4$ assets (kernel= 1%) | | | | | | |
| 90 | $[33.011, 34.989]$ | 32.706 (.008) | 34.612 (.004) | 34.656 (.004) | 34.667 (.004) | 34.749 (.005) | 34.934 (.015) | 34.976 (.013) | 34.962 (.010) |
| 100 | $[41.541, 43.587]$ | 40.328 (.008) | 43.117 (.003) | 43.138 (.004) | 43.161 (.004) | 43.251 (.004) | 43.630 (.024) | 43.558 (.011) | 43.557 (.010) |
| 110 | $[48.169, 49.909]$ | 47.197 (.007) | 49.398 (.004) | 49.429 (.004) | 49.430 (.004) | 49.482 (.004) | 49.998 (.028) | 49.780 (.007) | 49.851 (.007) |
| | | | $n = 8$ assets (kernel= 5%) | | | | | | |
| 90 | $[44.113, 45.847]$ | 43.321 (.006) | 45.460 (.004) | 45.460 (.004) | 45.460 (.004) | 45.580 (.003) | 46.743 (.015) | 45.847 (.007) | 45.830 (.008) |
| 100 | $[50.252, 51.814]$ | 49.523 (.007) | 51.357 (.003) | 51.360 (.003) | 51.360 (.003) | 51.433 (.003) | 51.646 (.015) | 51.668 (.003) | 51.667 (.005) |
| 110 | $[53.488, 54.890]$ | 52.319 (.006) | 54.525 (.002) | 54.527 (.002) | 54.527 (.002) | 54.564 (.001) | 54.898 (.018) | 54.697 (.004) | 54.744 (.003) |
| | | | $n = 16$ assets (kernel= 5%) | | | | | | |
| 90 | $[50.885, 52.316]$ | 49.779 (.005) | 51.916 (.003) | 51.923 (.002) | 51.925 (.002) | 51.981 (.002) | 52.252 (.015) | 52.158 (.005) | 52.184 (.006) |
| 100 | $[53.638, 54.883]$ | 52.574 (.002) | 54.601 (.002) | 54.603 (.002) | 54.603 (.002) | 54.633 (.002) | 53.806 (.017) | 54.718 (.002) | 54.800 (.003) |
| 110 | $[55.146, 56.201]$ | 54.968 (.005) | 55.994 (.003) | 55.995 (.003) | 55.995 (.003) | 56.025 (.002) | 56.200 (.018) | 56.070 (.002) | 56.125 (.003) |

**Table 1.** Prices of Bermudan up-and-out max-call options for $n = \{4, 8, 16\}$ uncorrelated stocks in a Black-Scholes setting ($r = 0.05$, $\delta = 0$, and $\sigma = 0.20$). $K = 100$ is the strike price, $B = 170$ is the barrier, $T = 3$ is maturity, and 54 exercise opportunities. The first column is the stock price, and the second is the best lower- and upper-bound, $[V_{0,\text{DFM}}^{low}, V_{0,\text{DFM}}^{up}]$, reported by DFM (Desai, Farias, and Mollemi, 2012). The third to sixth and seventh to tenth are the lower and upper bounds, respectively. The third column is the global regression/LSM method (Longstaff-Schwartz), and the fourth to sixth columns are the first three iterations of the local LSM method (Ibáñez and Velasco, 2018). The seventh and eighth are upper-bounds based on a stopping time $\widetilde{\tau}$, which are associated with the LSM (as Andersen-Broadie) and local LSM third-iteration exercise strategies, respectively. The last two columns are upper bounds

based on a continuation value $V^{co}$, which is reestimated in the continuation region, which is associated with the LSM and third-iteration local LSM exercise strategies, respectively. For the LSM and local LSM methods, we use 200,000 paths to recursively compute the continuation values and then 2 million paths to compute the Bermudan price. We report the mean and standard error (over 10 independent trials). For the gap of the upper bound based on $\widetilde{\tau}$, we use 3,000 external paths and 10,000 subsimulation paths. For the upper bound based on $V^{co}$, we use 10,000 external paths and 500 subsimulation paths. We also report the two-asset case, in which the true price is derived from the binomial method and linear extrapolation (to correct the erratic binomial prices).

Table 2: Increasing the number of paths in the regression ($S_0 = 100$, $n = 4$, kernel= 1%)

| | Lower bound, $V_0^{low}$ | | | | | | | Upper bound, $V_0^{up}$ | |
|---|---|---|---|---|---|---|---|---|---|
| paths | LSM | local LSM iterations | | | | | | $\tau$–based | $V^{co}$–based |
| $m$ | | 1st | 2nd | 3rd | 4th | 5th | 10th | 3rd | 3rd |
| $5*10^4$ | 40.327 (.010) | 43.106 (.004) | 43.128 (.005) | 43.147 (.004) | 43.153 (.004) | 43.160 (.004) | 43.178 (.004) | 43.251 (.004) | 43.558 (.011) |
| $10^5$ | 40.329 (.008) | 43.112 (.005) | 43.135 (.004) | 43.154 (.005) | 43.161 (.005) | 43.167 (.005) | 43.183 (.004) | 43.250 (.005) | 43.558 (.011) |
| $2*10^5$ | 40.328 (.008) | 43.117 (.003) | 43.138 (.004) | 43.161 (.004) | 43.168 (.004) | 43.175 (.004) | 43.191 (.004) | 43.251 (.004) | 43.558 (.011) |
| $4*10^5$ | 40.325 (.006) | 43.119 (.003) | 43.143 (.003) | 43.164 (.002) | 43.172 (.002) | 43.179 (.002) | 43.193 (.002) | 43.250 (.002) | 43.559 (.011) |
| $10^6$ | 40.328 (.002) | 43.118 (.002) | 43.143 (.002) | 43.163 (.002) | 43.170 (.002) | 43.177 (.002) | 43.193 (.002) | 43.250 (.003) | 43.558 (.011) |
| $2*10^6$ | 40.322 (.003) | 43.120 (.003) | 43.144 (.002) | 43.165 (.002) | 43.172 (.002) | 43.179 (.002) | 43.194 (.002) | 43.250 (.003) | 43.559 (.011) |

**Table 2.** Prices of Bermudan up-and-out max-call options for $n = 4$ uncorrelated stocks in a Black-Scholes setting ($r = 0.05$, $\delta = 0$, and $\sigma = 0.20$ and kernel= 1%). $S_0 = 100$ is the initial value, $K = 100$ is the strike price, $B = 170$ is the barrier, $T = 3$ is maturity, and 54 exercise opportunities. The first column "$m$" is the number of paths in the backward regressions to estimate the continuation values. The second column is the LSM method and third to eighth columns are the first five and the tenth iteration of the local LSM method (Ibáñez and Velasco, 2018) lower-bounds. The ninth and tenth are the upper bounds. We report the mean and standard error over 10 trials. Lower and upper bounds are as in Table 1.

Table 3: Different kernels ($S_0 = 100$, $n = 4, m = 200,000$)

| | Lower bound, $V_0^{up}$ | | | | | | | Upper bound, $V_0^{up}$ | |
|---|---|---|---|---|---|---|---|---|---|
| kernel | LSM | local LSM iterations | | | | | | $\tau$–based | $V^{co}$–based |
| $p$ | | 1st | 2nd | 3rd | 4th | 5th | 10th | 3rd | 3rd |
| 0.5% | 40.328 (.008) | 43.119 (.003) | 43.151 (.005) | 43.175 (.004) | 43.186 (.004) | 43.194 (.004) | 43.208 (.003) | 43.250 (.003) | 43.559 (.011) |
| 1% | 40.328 (.008) | 43.117 (.003) | 43.138 (.004) | 43.161 (.004) | 43.168 (.004) | 43.175 (.004) | 43.191 (.004) | 43.251 (.004) | 43.558 (.011) |
| 3% | 40.328 (.008) | 43.116 (.002) | 43.138 (.003) | 43.159 (.002) | 43.166 (.002) | 43.173 (.002) | 43.189 (.003) | 43.249 (.004) | 43.558 (.011) |
| 5% | 40.328 (.008) | 43.073 (.005) | 43.072 (.005) | 43.081 (.005) | 43.086 (.005) | 43.089 (.005) | 43.101 (.005) | 43.262 (.006) | 43.558 (.011) |

**Table 3.** Prices of Bermudan up-and-out max-call options for $n = 4$ uncorrelated stocks in a Black-Scholes setting ($r = 0.05$, $\delta = 0$, and $\sigma = 0.20$). $S_0 = 100$ is the initial value, $K = 100$ is the strike price, $B = 170$ is the barrier, $T = 3$ is maturity, and 54 exercise opportunities. The first column is the kernel used in the local regressions, where $p$ is the approximated number of points used from a total of $m = 200,000$. The second column is the LSM method and the third to eighth columns are the first five and the tenth iterations of the local LSM method (Ibáñez and Velasco, 2018) lower-bounds. The ninth and tenth are the upper bounds. We report the mean and standard error over 10 trials. Lower and upper bounds are as in Table 1.

Table 4: Lower and upper bounds gaps $\tau$–based ($S_0 = 100$, $n = 4, m = 200,000$)

| | Lower bound, $V_0^{low}$ | | Gap $\tau$–based | | | | | | Upper bound, $V_0^{up}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | local LSM | 100 sub-paths | | 500 sub-paths | | 10,000 sub-paths | | 10,000 sub-paths | |
| $S_0$ | LSM | 3rd iter | LSM | 3rd iter | LSM | 3rd iter | LSM | 3rd iter | LSM | 3rd iter |
| 90 | 32.706 (.008) | 34.647 (.005) | 2.405 (.015) | 0.306 (.007) | 2.257 (.015) | 0.133 (.006) | 2.228 (.015) | 0.082 (.005) | 34.934 (.015) | 34.752 (.005) |
| 100 | 40.328 (.008) | 43.159 (.002) | 3.483 (.025) | 0.284 (.004) | 3.329 (.026) | 0.120 (.004) | 3.302 (.024) | 0.090 (.005) | 43.630 (.024) | 43.249 (.004) |
| 110 | 46.197 (.007) | 49.429 (.003) | 3.965 (.029) | 0.234 (.004) | 3.833 (.027) | 0.082 (.003) | 3.801 (.028) | 0.052 (.004) | 49.998 (.028) | 49.480 (.003) |

**Table 4.** Gaps of lower and $\tau$–based upper bounds for up-and-out Bermudan max-call options for $n = 4$ uncorrelated stocks in a Black-Scholes setting ($r = 0.05$, $\delta = 0$, and $\sigma = 0.20$). $S_0$ is the initial value, $K = 100$ is the strike price, $B = 170$ is the barrier, $T = 3$ is maturity, and 54 exercise opportunities. We directly compute the gap for different numbers of subsimulation paths (sub-paths): 100, 500, and 10,000. The upper bound is defined as the lower bound plus the gap (i.e., $V_0^{up}$=$V_0^{low}$+Gap). We report the mean and standard error over 10 trials. Table 1 represents the case using 10,000 subsimulation paths.