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# VAN DOOREN'S INDEX SUM THEOREM AND RATIONAL MATRICES WITH PRESCRIBED STRUCTURAL DATA*

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*Dedicated to Paul Van Dooren on becoming Emeritus Professor*

**Abstract.** The structural data of any rational matrix $R(\lambda)$, i.e., the structural indices of its poles and zeros together with the minimal indices of its left and right nullspaces, is known to satisfy a simple condition involving certain sums of these indices. This fundamental constraint was first proved by Van Dooren in 1978; here we refer to this result as the *rational index sum theorem*. An analogous result for polynomial matrices has been independently discovered (and rediscovered) several times in the past three decades. In this paper we clarify the connection between these two seemingly different index sum theorems, describe a little bit of the history of their development, and discuss their curious apparent unawareness of each other. Finally, we use the connection between these results to solve a fundamental inverse problem for rational matrices—for which lists $\mathcal{L}$ of prescribed structural data does there exist some rational matrix $R(\lambda)$ that realizes exactly the list $\mathcal{L}$? We show that Van Dooren's condition is the *only* constraint on rational realizability; that is, a list $\mathcal{L}$ is the structural data of some rational matrix $R(\lambda)$ if and only if $\mathcal{L}$ satisfies the rational index sum condition.

**Key words.** eigenvalues, index sum theorem, structural indices, rational matrices, poles, zeros, invariant orders, minimal indices, polynomial matrices

**AMS subject classifications.** 15A18, 15A29, 65F15, 65F18, 93B10, 93B55, 93B60

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## 1. Introduction.

Rational matrices, i.e., matrices whose entries are scalar rational functions in one variable, lie at the heart of control theory and linear systems theory. As a consequence, essentially any book on these disciplines will contain a wealth of information on rational matrices; see, for instance, the seminal references [23, 37], or the recent [6]. In the numerical linear algebra community, increased attention has recently been directed at rational matrices, due to their relationship with the numerical solution of nonlinear eigenvalue problems (NEP) arising in modern applications. Some of these NEPs arise immediately as rational eigenvalue problems (REPs) expressed in terms of rational matrices. Even more importantly, there are many NEPs that are not rational but can be reliably approximated by REPs; these rational approximations can then be linearized and solved by standard methods for linear eigenvalue problems, such as the QZ algorithm in the case of dense medium size

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problems, or various Krylov methods for large-scale problems. Some recent references on this subject are [1, 2, 3, 14, 19, 20, 33, 39, 44].

This renewed interest in rational matrices is motivating a rethinking of a number of classical results and concepts, looking for simplifications, revisions, and extensions that might make them more amenable to numerical treatment and applications, as well as to the development of new lines of research on rational matrices themselves. It is instructive to contrast the study of rational matrices with the parallel investigation of polynomial matrices, a particular subclass of rational matrices. Polynomial matrices have been the subject of intense research activity over the last 15 years, despite already being a fundamental topic in matrix analysis and numerical linear algebra that is well covered in classic references [17, 18, 23, 46]. Some representative sources for various aspects of recent research on polynomial matrices, among a great many others, are [5, 7, 8, 9, 10, 13, 15, 21, 22, 26, 27, 28, 29, 35, 40, 41, 42, 44, 51]. This small sample should make clear that modern research on rational matrices is in some respects well behind modern research on polynomial matrices; indeed, many of the problems considered and solved for polynomials remain open for rational matrices.

In this context, we solve a problem on rational matrices corresponding to a very recently settled problem for polynomial matrices [10]; informally stated, this might be called the general inverse "eigenstructure" problem for rational matrices. Although this informal statement immediately establishes a connection with REPs, we emphasize that the problem considered in this paper is much more than just an inverse "eigenstructure" problem. Our goal is to realize not just a prescribed list of eigenvalues but a complete list of structural data, comprising finite and infinite zeros and poles together with their structural indices, as well as the minimal indices of left and right rational null spaces. More precisely, then, *the inverse problem solved in this work is to find a necessary and sufficient condition for the existence of a rational matrix when a complete list of "structural data" is prescribed.* This necessary and sufficient condition is that the prescribed structural data satisfy a fundamental relation that we baptize as the *rational index sum theorem*, or *Van Dooren's index sum theorem*, since it was proved for the first time by Paul Van Dooren in 1978 and published in [49] (more information on the history of this result will be provided later). The condition in Van Dooren's index sum theorem is extremely easy to check, since it simply says that *for any rational matrix the total number of its poles* (*counting orders*) *is equal to the total number of its zeros* (*counting orders*) *plus the sum of all its minimal indices.*

We expect the inverse problem solved in this paper to have numerical applications, since the corresponding result for polynomial matrices has already found one such application, in particular, the development of stratification hierarchies of polynomial matrices in terms of their complete eigenstructures [13, 22]. These stratification hierarchies determine what are the possible eigenstructures of all the polynomial matrices in a neighborhood of a given one. Such results, combined with backward stable algorithms for computing eigenstructures of polynomial matrices [15], allow us to determine numerically the defective eigenstructures compatible under roundoff errors with a given polynomial matrix. The polynomial inverse problem solved in [10] has recently found another interesting application [12] in the description of sets of polynomial matrices with bounded rank and degree. We expect that the results in this paper can also be applied to solve related problems for rational matrices.

An important byproduct of this paper is to bring the attention of the numerical linear algebra community to Van Dooren's index sum theorem for rational matrices, a result that has remained essentially unknown in this community. It is also expected that this rational index sum theorem will find other relevant applications in addition to the inverse problem solved in this work, since its polynomial matrix counterpart [36, 9] has already been applied to solve a number of interesting problems. For example, it has been used to show that many structured classes of even degree polynomial matrices that arise in applications contain polynomials that cannot be "strongly linearized" via a pencil with the same structure [9, section 7]. It is important to note that the polynomial and rational index sum theorems can be seen to be easy corollaries of each other, as we discuss in section 3 of this paper.

The rest of the paper is organized as follows. Section 2 reviews some basic notions. Van Dooren's index sum theorem is revisited in section 3 from two perspectives: first, a new proof of this result valid in arbitrary fields is presented, then its history as well as its relation with the so-called polynomial index sum theorem are briefly discussed. Section 4 includes the most important original result of this paper, i.e., the solution of the inverse problem for rational matrices when a complete list of structural data is prescribed. An alternative formulation of the inverse problem is studied in section 5. Finally, some conclusions and lines of future work are discussed in section 6.

## 2. Basic concepts, auxiliary results, and notation.

The results summarized in this section on rational matrices, as well as many others, can be found in the classic references [23, Chapter 6] for real and complex rational functions and [37] for rational functions with coefficients in arbitrary fields. Another interesting reference is [48, Chapters 1 and 3], which only considers real rational functions. We also strongly recommend the recent reference [4], which works in the general setting of matrices over principal ideal domains and the corresponding fields of fractions.

In this paper an arbitrary field $\mathbb{F}$ is considered. In those results where $\mathbb{F}$ is required to contain infinitely many elements, this property is explicitly stated. The algebraic closure of $\mathbb{F}$ is denoted by $\overline{\mathbb{F}}$, $\mathbb{F}[\lambda]$ stands for the ring of polynomials in the variable $\lambda$ with coefficients in $\mathbb{F}$, and $\mathbb{F}(\lambda)$ stands for the field of rational functions in the variable $\lambda$ with coefficients in $\mathbb{F}$. A polynomial matrix is a matrix whose entries are elements of $\mathbb{F}[\lambda]$, and a rational matrix is a matrix whose entries are elements of $\mathbb{F}(\lambda)$. The set of $m \times n$ constant matrices is denoted by $\mathbb{F}^{m\times n}$, the set of $m \times n$ polynomial matrices by $\mathbb{F}[\lambda]^{m\times n}$, and the set of $m \times n$ rational matrices by $\mathbb{F}(\lambda)^{m\times n}$. Row or column polynomial (resp., rational) vectors are just $m \times n$ polynomial (resp., rational) matrices with $m = 1$ or with $n = 1$. For any pair of scalar polynomials $p(\lambda), q(\lambda) \in \mathbb{F}[\lambda]$, the expression $p(\lambda) \,|\, q(\lambda)$ means that $p(\lambda)$ divides $q(\lambda)$. Given two matrices $A$ and $B$, $A \oplus B$ denotes their direct sum, i.e., $A \oplus B = \operatorname{diag}(A, B)$. Throughout the paper, the unspecified entries of a matrix are zero.

The degree of a polynomial matrix $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ is the largest degree of its entries and is denoted by $\deg(P)$. If $\deg(P) = d$, then $P(\lambda)$ can be written as

$$P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0 \quad \text{with } P_0, P_1, \ldots, P_d \in \mathbb{F}^{m\times n} \text{ and } P_d \neq 0. \tag{2.1}$$

For any $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ of degree $d$ as in (2.1), we define its reversal polynomial as

$$(\operatorname{rev} P)(\lambda) := \lambda^d P\left(\frac{1}{\lambda}\right) = P_0\lambda^d + \cdots + P_{d-1}\lambda + P_d. \tag{2.2}$$

It is well known that any rational function has infinitely many representations as a ratio of polynomials but can be uniquely simplified to reduced form.

Definition 2.1 (Reduced form). *Any nonzero $r(\lambda) \in \mathbb{F}(\lambda)$ can be uniquely expressed in* reduced form $r(\lambda) = \frac{\alpha\cdot u(\lambda)}{\ell(\lambda)}$, *where the polynomials $u(\lambda)$ and $\ell(\lambda)$ are coprime and monic, and $\alpha \in \mathbb{F}$. The associated expression $\widetilde{r}(\lambda) = \frac{u(\lambda)}{\ell(\lambda)}$ is the* normalized reduced form *of $r(\lambda)$.*

Also, any rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{m\times n}$ can be uniquely expressed as

$$R(\lambda) = P(\lambda) + R_{sp}(\lambda), \tag{2.3}$$

where $P(\lambda)$ is a polynomial matrix and $R_{sp}(\lambda)$ is a strictly proper rational matrix, i.e., a rational matrix such that for each of its nonzero entries the degree of the denominator is strictly larger than the degree of its numerator. $P(\lambda)$ is called the polynomial part of $R(\lambda)$ and $R_{sp}(\lambda)$ the strictly proper part of $R(\lambda)$.

The key tool for working with rational matrices is the Smith–McMillan form, introduced by McMillan [31, 32] via the Smith form of polynomial matrices [17, Chapter VI]. The Smith–McMillan form is the canonical form of a rational matrix under multiplication by *unimodular* matrices, i.e., square polynomial matrices with nonzero constant determinant. We state this result here with the notation we will use throughout the rest of the paper (see also [37, Chapter 3]).

THEOREM 2.2 (Smith–McMillan form). *For any rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{m\times n}$ there exist unimodular matrices $U(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ and $V(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ and a nonnegative integer $r \le \min\{m, n\}$ such that*

$$U(\lambda)R(\lambda)V(\lambda) = \left[\begin{array}{ccc|c} d_1(\lambda) & & & \\ & \ddots & & 0_{r\times(n-r)} \\ & & d_r(\lambda) & \\ \hline & 0_{(m-r)\times r} & & 0_{(m-r)\times(n-r)} \end{array}\right] =: D(\lambda), \tag{2.4}$$

*where for $i = 1, \ldots, r$ the diagonal entries $d_i(\lambda) = \frac{\varepsilon_i(\lambda)}{\psi_i(\lambda)} \in \mathbb{F}(\lambda)$ are in normalized reduced form, and for $j = 1, \ldots, r-1$ we have $\varepsilon_j(\lambda) \,|\, \varepsilon_{j+1}(\lambda)$ and $\psi_{j+1}(\lambda) \,|\, \psi_j(\lambda)$. Moreover, the rational diagonal matrix $D(\lambda)$ is unique.*

The diagonal matrix $D(\lambda)$ in (2.4) is called the Smith–McMillan form of $R(\lambda)$, and $d_1(\lambda), \ldots, d_r(\lambda)$ are the invariant rational functions of $R(\lambda)$. The integer $r$ is the rank of $R(\lambda)$ when viewed as a matrix over the field $\mathbb{F}(\lambda)$ and is denoted by $r = \operatorname{rank}(R)$. Those polynomials $\varepsilon_1(\lambda), \ldots, \varepsilon_r(\lambda)$ and $\psi_1(\lambda), \ldots, \psi_r(\lambda)$ in (2.4) that are different from 1 are called the nontrivial numerators and denominators, respectively, of the Smith–McMillan form of $R(\lambda)$. Note that $\psi_1(\lambda)$ is the monic least common multiple of the denominators of the entries of $R(\lambda)$, when they are expressed in reduced form.

Suppose $p(\lambda) \in \mathbb{F}[\lambda]$ is a nonzero polynomial, and let $\pi(\lambda)$ be any nonconstant monic irreducible polynomial with coefficients in $\mathbb{F}$. Then there is a unique nonnegative integer $k$ and a unique polynomial $q(\lambda) \in \mathbb{F}[\lambda]$ with $q(\lambda), \pi(\lambda)$ coprime such that

$$p(\lambda) = [\pi(\lambda)]^k q(\lambda). \tag{2.5}$$

The integer $k$ is called the *structural index of* $p(\lambda)$ *at* $\pi(\lambda)$ and is denoted $S(p, \pi)$. It follows immediately that for any two nonzero polynomials $p_1(\lambda), p_2(\lambda) \in \mathbb{F}[\lambda]$,

$$S(p_1p_2, \pi) = S(p_1, \pi) + S(p_2, \pi)\,. \tag{2.6}$$

A natural extension of this concept to rational functions follows from the next lemma.

LEMMA 2.3. *Let $\pi(\lambda)$ be a nonconstant monic irreducible polynomial over the field $\mathbb{F}$, and let $r(\lambda)$ be any nonzero rational function in $\mathbb{F}(\lambda)$. Then there is a unique integer $k$ (possibly zero or negative) and a rational function $s(\lambda)$ in reduced form $s(\lambda) = \alpha \cdot u(\lambda)/\,\ell(\lambda)$ where $\pi(\lambda)$ is coprime to $u(\lambda)$ and to $\ell(\lambda)$ such that*

$$r(\lambda) \;=\; [\pi(\lambda)]^k\, s(\lambda)\,.$$

As before, we call the integer $k$ the *structural index of* $r(\lambda)$ *at* $\pi(\lambda)$ and denote it by $S(r,\pi)$. The structural index $S(r,\pi)$ can also be easily calculated from *any* representation $n(\lambda)/d(\lambda)$ of $r(\lambda)$ as a fraction of polynomials by

$$S(r,\pi) \;=\; S(n,\pi) - S(d,\pi)\,. \tag{2.7}$$

The following properties are now easy consequences of (2.6) and (2.7).

LEMMA 2.4. *Consider any nonconstant monic* $\mathbb{F}$*-irreducible polynomial* $\pi(\lambda)$ *and any nonzero rational functions* $r_1(\lambda), r_2(\lambda) \in \mathbb{F}(\lambda)$. *Then*

$$S(r_1 r_2,\pi) \;=\; S(r_1,\pi) + S(r_2,\pi)\,. \tag{2.8}$$

*Also, for any nonzero rational function* $f(\lambda) \in \mathbb{F}(\lambda)$,

$$S(r_1,\pi) = S(r_2,\pi) \;\Rightarrow\; S(fr_1,\pi) = S(fr_2,\pi)\,, \tag{2.9}$$

$$S(r_1,\pi) < S(r_2,\pi) \;\Rightarrow\; S(fr_1,\pi) < S(fr_2,\pi)\,. \tag{2.10}$$

There is a further natural extension to rational matrices. Let $R(\lambda)$ be a rational matrix over $\mathbb{F}$, with its Smith–McMillan form given by (2.4). Let $d_i(\lambda) = \varepsilon_i(\lambda)/\psi_i(\lambda)$ for $1 \le i \le r$, and define the *structural index sequence of* $R(\lambda)$ *at* $\pi(\lambda)$ to be

$$S(R,\pi) := \big(S(d_1,\pi), S(d_2,\pi), \ldots, S(d_r,\pi)\big). \tag{2.11}$$

Then by Lemma 2.3, $S(R,\pi)$ is the sequence of integers $(h_1, h_2, \ldots, h_r)$ such that

$$\frac{\varepsilon_i(\lambda)}{\psi_i(\lambda)} = \pi(\lambda)^{h_i}\,\frac{\tilde{\varepsilon}_i(\lambda)}{\tilde{\psi}_i(\lambda)}\,, \tag{2.12}$$

where the triples of polynomials $\big(\tilde{\varepsilon}_i(\lambda), \tilde{\psi}_i(\lambda), \pi(\lambda)\big)$ are pairwise coprime for $i = 1, \ldots, r$. The sequence of integer exponents $(h_1, h_2, \ldots, h_r)$ is unique and satisfies the nondecreasing condition $h_1 \le h_2 \le \cdots \le h_r$ by the divisibility properties of the numerators and denominators of the Smith–McMillan form and (2.7); indeed, these divisibility properties are *equivalent* to the structural index sequence at every irreducible $\pi(\lambda)$ being nondecreasing. Note that in [4, p. 204], $h_1, h_2, \ldots, h_r$ are called the "invariant orders" at $\pi(\lambda)$ of $R(\lambda)$. In this paper we use "structural indices" instead, since it is used in the classical reference [23, p. 447] (see also [45, p. 2.4]).

Note that $S(R,\pi)$ contains nonzero terms if and only if $\pi(\lambda)$ in (2.12) divides either $\varepsilon_r(\lambda)$ or $\psi_1(\lambda)$ (or both); otherwise, $(h_1, h_2, \ldots, h_r) = (0, 0, \ldots, 0)$. Including sequences with all structural indices equal to zero in the definition allows us flexibility and the ability to state certain results in a concise way.

Given the Smith–McMillan form (2.4) of $R(\lambda) \in \mathbb{F}(\lambda)^{m\times n}$ over a field $\mathbb{F}$, the roots of $\varepsilon_1(\lambda), \ldots, \varepsilon_r(\lambda)$ in the algebraic closure $\overline{\mathbb{F}}$ are the *finite zeros* of $R(\lambda)$. Analogously, the roots of $\psi_1(\lambda), \ldots, \psi_r(\lambda)$ in $\overline{\mathbb{F}}$ are the *finite poles* of $R(\lambda)$. The *finite eigenvalues* of $R(\lambda)$ are those finite zeros that are not poles. Observe that the uniqueness of $D(\lambda)$ in (2.4) implies that the Smith–McMillan form of $R(\lambda)$ does not change under field extensions. Thus, $D(\lambda)$ is also the Smith–McMillan form of $R(\lambda)$ considered as a matrix in $\overline{\mathbb{F}}(\lambda)^{m\times n}$. This makes it possible to consistently define the structural index sequence of $R(\lambda)$ at any $\lambda_0 \in \overline{\mathbb{F}}$, denoted $S(R,\lambda_0)$ for simplicity, by identifying $S(R,\lambda_0)$ with $S(R,\pi(\lambda))$ for $\pi(\lambda) = \lambda - \lambda_0$. With this notation, observe that

(1) $\lambda_0$ is a finite zero of $R(\lambda)$ if and only if the last term of $S(R,\lambda_0)$ is positive;
(2) $\lambda_0$ is a finite pole of $R(\lambda)$ if and only if the first term of $S(R,\lambda_0)$ is negative;

(3) $\lambda_0$ is neither a finite zero nor a finite pole of $R(\lambda)$ if and only if all the terms of $S(R, \lambda_0)$ are zero;
(4) $\lambda_0$ is a finite eigenvalue of $R(\lambda)$ if and only if $S(R, \lambda_0)$ contains only nonnegative terms, the last of which is positive.

The following example illustrates these ideas.

*Example* 2.5. Given the rational matrix

$$R(\lambda) = \operatorname{diag}\left(\frac{\lambda}{\lambda-1}, \frac{1}{\lambda-1}, (\lambda-1)^2\right) \oplus \begin{bmatrix} 1 & \lambda^2 & 0 \\ 0 & 1 & \lambda^7 \end{bmatrix} \in \mathbb{F}(\lambda)^{5\times 6},$$

it is easy to check that the Smith–McMillan form of $R(\lambda)$ is

$$D(\lambda) = \left[\ \operatorname{diag}\left(\frac{1}{\lambda-1}, \frac{1}{\lambda-1}, 1, 1, \lambda(\lambda-1)^2\right) \ \middle|\ 0\ \right].$$

The only nonzero structural index sequences at finite $\lambda_0 \in \overline{\mathbb{F}}$ for this $R(\lambda)$ are $S(R, 1) = (-1, -1, 0, 0, 2)$ and $S(R, 0) = (0, 0, 0, 0, 1)$; note that the sequence length is 5 since $\operatorname{rank}(R) = 5$. Thus, 1 is simultaneously a pole and a zero of $R(\lambda)$, while 0 is a zero but not a pole. Therefore, 0 is the only finite eigenvalue of $R(\lambda)$.

Notice that the Smith–McMillan form of a rational matrix $R(\lambda)$ over $\mathbb{F}$ can be uniquely reconstructed from the nontrivial structural index sequences of $R(\lambda)$ at nonconstant monic irreducible polynomials. Suppose $\pi_1(\lambda), \pi_2(\lambda), \ldots, \pi_k(\lambda)$ are the only nonconstant monic irreducible polynomials in $\mathbb{F}[\lambda]$ such that $S(R, \pi_i)$ is nontrivial, and let $S(R, \pi_i) = (h_1^{(i)}, h_2^{(i)}, \ldots, h_r^{(i)})$ with $h_1^{(i)} \le h_2^{(i)} \le \cdots \le h_r^{(i)}$. Then the invariant rational functions $d_1(\lambda), d_2(\lambda), \ldots, d_r(\lambda)$ of $R(\lambda)$ are given by

$$d_j(\lambda) = \prod_{i=1}^{k} [\pi_i(\lambda)]^{h_j^{(i)}} = \frac{\varepsilon_j(\lambda)}{\psi_j(\lambda)}. \tag{2.13}$$

The following simple result on Smith–McMillan forms and structural index sequences is fundamental in proving the main results in this paper.

Lemma 2.6. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{m\times n}$ be a rational matrix with rank $r$ and Smith–McMillan form $D(\lambda)$. Also let $f(\lambda) \in \mathbb{F}(\lambda)$ be any nonzero scalar rational function and $\pi(\lambda) \in \mathbb{F}[\lambda]$ any nonconstant monic irreducible polynomial. Then*

$$S\big(f(\lambda)R(\lambda), \pi(\lambda)\big) = S\big(f(\lambda)D(\lambda), \pi(\lambda)\big) = S\big(R(\lambda), \pi(\lambda)\big) + (s, s, \ldots, s), \tag{2.14}$$

*where $s = S(f(\lambda), \pi(\lambda))$. Furthermore, the Smith–McMillan form of $f(\lambda)R(\lambda) \in \mathbb{F}(\lambda)^{m\times n}$ can be obtained from the diagonal rational matrix $f(\lambda)D(\lambda)$ simply by replacing each nonzero entry of $f(\lambda)D(\lambda)$ by its normalized reduced form.*

*Proof.* It is immediate that $f(\lambda)R(\lambda)$ is unimodularly equivalent to $f(\lambda)D(\lambda)$. Thus $f(\lambda)R(\lambda)$ and $f(\lambda)D(\lambda)$ have the same Smith–McMillan form and hence identical structural index sequences $S\big(f(\lambda)R(\lambda), \pi(\lambda)\big)$ and $S\big(f(\lambda)D(\lambda), \pi(\lambda)\big)$. But by Lemma 2.4, the structural indices of the diagonal entries of $f(\lambda)D(\lambda)$ form a nondecreasing sequence for any irreducible $\pi(\lambda)$. Thus, aside from expressing these diagonal entries in normalized reduced form, the matrix $f(\lambda)D(\lambda)$ is already essentially in Smith–McMillan form, so that

$$S\big(f(\lambda)D(\lambda), \pi(\lambda)\big) = \big(\ldots, S\big(f(\lambda)d_i(\lambda), \pi(\lambda)\big), \ldots\big) = S\big(R(\lambda), \pi(\lambda)\big) + (s, s, \ldots, s),$$

by (2.8) and (2.11). ∎

*Remark* 2.7. For brevity, in situations such as those in Lemma 2.6, we will informally say that $f(\lambda)D(\lambda)$ is the Smith–McMillan form of $f(\lambda)R(\lambda)$.

So far, only finite poles and zeros of rational matrices have been defined. Next, we define the structure at $\infty$ as it was originally done by McMillan [31, 32] (see also [23, p. 450]). In Definition 2.8, bear in mind that $0 \in \mathbb{F}$.

DEFINITION 2.8. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{m\times n}$ be a rational matrix. The structural index sequence of $R(\lambda)$ at infinity, denoted $S(R,\infty)$, is defined to be identical with the structural index sequence of $R(1/\lambda)$ at 0 or, equivalently, as the structural index sequence of $R(1/\lambda)$ at $\pi(\lambda) = \lambda$, i.e.,*

$$S(R,\infty) := S\big(R(1/\lambda),0\big) := S\big(R(1/\lambda),\lambda\big)\,.$$

According to this definition, then, $R(\lambda)$ has a pole (resp., a zero) at $\infty$ if $R(1/\lambda)$ has a pole (resp., a zero) at 0. The following simple result about rational functions will be very useful when calculating the structural indices at $\infty$ for a rational matrix.

LEMMA 2.9. *Suppose $r(\lambda) = \frac{n(\lambda)}{d(\lambda)} \in \mathbb{F}(\lambda)$ is a nonzero scalar rational function, where $n(\lambda), d(\lambda) \in \mathbb{F}[\lambda]$ are scalar polynomials. Then $r(\frac{1}{\lambda})$ can be expressed in the form*

$$r\left(\frac{1}{\lambda}\right) = \frac{f(\lambda)}{g(\lambda)}\,\lambda^{\deg(d)-\deg(n)},$$

*where $f(\lambda), g(\lambda) \in \mathbb{F}[\lambda]$ are each coprime to $\lambda$, so that $S(r,\infty) = \deg(d) - \deg(n)$.*

*Proof.* $r(1/\lambda) = \frac{n(1/\lambda)}{d(1/\lambda)} = \frac{\lambda^{\deg(d)}\lambda^{\deg(n)}n(1/\lambda)}{\lambda^{\deg(n)}\lambda^{\deg(d)}d(1/\lambda)} = \lambda^{\deg(d)-\deg(n)}\,\frac{\operatorname{rev} n(\lambda)}{\operatorname{rev} d(\lambda)}$ . ∎

The following example illustrates the structure at infinity of a rational matrix.

*Example* 2.10. Consider again the matrix $R(\lambda)$ in Example 2.5 and note that the degree of its polynomial part is 7. In addition, it is easy to check that $R(1/\lambda)$ has as Smith–McMillan form the matrix

$$\left[\ \operatorname{diag}\!\left(\frac{1}{\lambda^7(\lambda-1)},\frac{1}{\lambda^2(\lambda-1)},\frac{1}{\lambda^2},1,\lambda(\lambda-1)^2\right)\ \middle|\ 0\ \right].$$

So, the sequence of structural indices at infinity of $R(\lambda)$ is $S(R,\infty) = (-7,-2,-2,0,1)$ $(= S(R(1/\lambda),0))$. Therefore, $R(\lambda)$ has a pole and also a zero at infinity.

The next definitions will play key roles in Van Dooren's index sum theorem.

DEFINITION 2.11 (Total numbers of poles and zeros). *Let $R(\lambda) \in \mathbb{F}(\lambda)^{m\times n}$ be a rational matrix. Then the* total number of poles *of $R(\lambda)$, denoted $\delta_p(R)$, is minus the sum of the negative structural indices at all the poles (finite or infinite) of $R(\lambda)$; equivalently, the summation of negative structural indices may be taken over all $\lambda_0 \in \overline{\mathbb{F}} \cup \{\infty\}$. Similarly, the* total number of zeros *of $R(\lambda)$, denoted $\delta_z(R)$, is the sum of the positive structural indices at all the zeros (finite or infinite) of $R(\lambda)$ or, equivalently, the sum of positive structural indices over all $\lambda_0 \in \overline{\mathbb{F}} \cup \{\infty\}$.*

In [4, 23, 45], the positive entries of $S(R,\lambda_0)$ for any $\lambda_0 \in \overline{\mathbb{F}} \cup \{\infty\}$ are called the *orders of the zero at* $\lambda_0$, while the negative entries of $S(R,\lambda_0)$ with their signs changed are called the *orders of the pole at* $\lambda_0 \in \overline{\mathbb{F}} \cup \{\infty\}$. In this terminology, $\delta_z(R)$ (resp., $\delta_p(R)$) is simply the sum of the orders of all zeros (resp., poles) in $\overline{\mathbb{F}} \cup \{\infty\}$.

*Remark* 2.12. The descriptions of $\delta_p(R)$ and $\delta_z(R)$ given so far all require passing to the algebraic closure $\overline{\mathbb{F}}$. This can be avoided by directly using the invariant rational functions in the Smith–McMillan form over $\mathbb{F}$ given by (2.4). If $S(R,\infty) = (q_1, \ldots, q_r)$, then it is easy to see that

$$\delta_p(R) = \sum_{i=1}^{r} \deg(\psi_i) - \sum_{q_i<0} q_i \quad \text{and} \quad \delta_z(R) = \sum_{i=1}^{r} \deg(\varepsilon_i) + \sum_{q_i>0} q_i \,. \tag{2.15}$$

*Example* 2.13. The only nonzero structural index sequences of the matrix $R(\lambda)$ in Example 2.5 are $S(R,1) = (-1,-1,0,0,2)$, $S(R,0) = (0,0,0,0,1)$, and $S(R,\infty) = (-7,-2,-2,0,1)$. Therefore, $\delta_p(R) = 13$ and $\delta_z(R) = 4$. Using the Smith–McMillan form of $R(\lambda)$ in Example 2.5, it is easy to check that (2.15) yields the same result.

The final concept we need to complete our survey of the structural data of a rational matrix is that of minimal indices. Their definition is analogous to the corresponding one for polynomial matrices and is recalled here. Note that a rational matrix $R(\lambda)$ is said to be regular if $R(\lambda)$ is square and $\det R(\lambda)$ is not identically zero; otherwise, $R(\lambda)$ is said to be singular. Any singular $R(\lambda) \in \mathbb{F}(\lambda)^{m\times n}$ has nontrivial left and/or right *rational* null spaces (here rational means *over the field* $\mathbb{F}(\lambda)$),

$$\begin{aligned} \mathcal{N}_\ell(R) &:= \{y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \quad \text{such that} \quad y(\lambda)^T R(\lambda) = 0\}, \\ \mathcal{N}_r(R) &:= \{x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \quad \text{such that} \quad R(\lambda)x(\lambda) = 0\}\,, \end{aligned}$$

which are just particular examples of rational subspaces as described in [16]. Any rational subspace $\mathcal{V} \subseteq \mathbb{F}(\lambda)^n$ has bases formed entirely of polynomial vectors, which are called polynomial bases of $\mathcal{V}$. The order of any polynomial basis of $\mathcal{V}$ is defined as the sum of the degrees of its vectors [16, Definition 2]. Among all of the polynomial bases of $\mathcal{V}$, those with least order are called minimal bases of $\mathcal{V}$ [16, Definition 3]. Although there are infinitely many minimal bases of $\mathcal{V}$, the ordered list of degrees of the polynomial vectors in any minimal basis of $\mathcal{V}$ is always the same [16, Remark 4, p. 497] and is called the list of minimal indices of $\mathcal{V}$. The left and right minimal indices and bases of a rational matrix $R(\lambda)$ are defined as those of $\mathcal{N}_\ell(R)$ and $\mathcal{N}_r(R)$, respectively. The next example illustrates minimal bases and minimal indices.

*Example* 2.14. Consider again the $5\times 6$ rational matrix $R(\lambda)$ from Example 2.5. Note that $\operatorname{rank}(R) = 5$, so $\dim \mathcal{N}_\ell(R) = 0$ and $\dim \mathcal{N}_r(R) = 1$. This means that $R(\lambda)$ has no left minimal indices and exactly one right minimal index. It can easily be checked that $\{v(\lambda) := [0,0,0,\lambda^9,-\lambda^7,1]^T\}$ is a polynomial basis of $\mathcal{N}_r(R)$; that $\{v(\lambda)\}$ is a minimal basis follows from all polynomial vectors in $\mathcal{N}_r(R)$ being scalar polynomial multiples of $v(\lambda)$. So, the unique right minimal index of $R(\lambda)$ is 9.

By contrast with Example 2.14, determining directly from the definition whether a general polynomial basis is minimal may be very hard. Interested readers can find useful criteria for minimality in [16] and a recent one in [47, section 3].

The concepts previously introduced give rise to the following definition.

DEFINITION 2.15. *Given a rational matrix* $R(\lambda) \in \mathbb{F}(\lambda)^{m\times n}$ *with rank* $r$*, the* complete structural data *of* $R(\lambda)$ *consists of the following four components:*

(i) "Finite structure": *the invariant rational functions* $\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}$ *defining the Smith–McMillan form of* $R(\lambda)$*;*

(ii) "Infinite structure": *the structural index sequence* $S(R,\infty)$*;*

(iii) "Left singular structure": *the left minimal indices* $\eta_1, \ldots, \eta_{m-r}$ *of* $R(\lambda)$*; and*

(iv) "Right singular structure": *the right minimal indices* $\alpha_1, \ldots, \alpha_{n-r}$ *of* $R(\lambda)$.

It is worth emphasizing some constraints on the complete structural data of $R(\lambda)$: first, the number of invariant rational functions and the number of structural indices at infinity are both equal to $\operatorname{rank}(R)$, and second, the numbers of left and right minimal indices, the size of $R(\lambda)$, and $\operatorname{rank}(R)$ are related via the rank-nullity theorem.

*Remark* 2.16. There are several alternative, but equivalent, ways to specify the finite structure of a rational matrix $R(\lambda)$ as presented in Definition 2.15(i). Staying inside the field $\mathbb{F}$, one could list all the nonconstant monic irreducible polynomials $\pi(\lambda) \in \mathbb{F}[\lambda]$ such that $S(R,\pi)$ is nonzero, together with their structural index sequences. As discussed in (2.13), this information is sufficient to uniquely reconstruct the invariant rational functions. At the cost of passing to the algebraic closure $\overline{\mathbb{F}}$, one could instead list all the finite poles and zeros of $R(\lambda)$, together with their structural index sequences. This description may be more natural in the important case $\mathbb{F} = \mathbb{C}$.

**2.1. Polynomial matrices: Structure at infinity.** Polynomial matrices can be viewed as rational matrices with the denominators of all entries equal to one. The Smith–McMillan form of any polynomial matrix is identical to its Smith form [17, Chapter VI], so the invariant rational functions are the same as the invariant polynomials. Therefore, polynomial matrices do not have any finite poles, and all their finite zeros are finite eigenvalues. The structural index sequences at the finite zeros are nonnegative and are exactly the same as what in the literature on polynomial matrices [9, 18, 30] are called the *partial multiplicity sequences* at the finite eigenvalues.

Consider any $m \times n$ polynomial matrix $P(\lambda)$ of degree $d$, over the field $\mathbb{F}$. Using Definition 2.8, we can directly compute the smallest structural index at infinity of $P(\lambda)$, viewing $P(\lambda)$ as a rational matrix. Write $P(\lambda) = [p_{ij}(\lambda)]_{m\times n}$ with

$$d_{ij} = \deg p_{ij}(\lambda) \quad \text{and} \quad d = \deg P(\lambda) = \max_{ij} d_{ij}. \tag{2.16}$$

By the definition of rev, we have $p_{ij}\left(\frac{1}{\lambda}\right) = \frac{\operatorname{rev} p_{ij}(\lambda)}{\lambda^{d_{ij}}}$. It follows that

$$P\left(\frac{1}{\lambda}\right) = \left[p_{ij}\left(\frac{1}{\lambda}\right)\right] = \left[\frac{\operatorname{rev} p_{ij}(\lambda)}{\lambda^{d_{ij}}}\right] = \frac{1}{\lambda^d}\left[\lambda^{d-d_{ij}}\operatorname{rev} p_{ij}(\lambda)\right] =: \frac{1}{\lambda^d}\, Q(\lambda),$$

where $Q(\lambda) = \operatorname{rev} P(\lambda)$ is a polynomial matrix. Note that at any $(i,j)$ where $d_{ij} = d$, the entry $Q_{ij}(\lambda)$ will be coprime to $\lambda$, since the reversal of any scalar polynomial is coprime to $\lambda$. Consequently the first invariant polynomial of $Q(\lambda)$ will be coprime to $\lambda$, since it is the gcd of the entries of $Q(\lambda)$. This means that $S(Q,0) = (0,*,\ldots,*)$, where each $*$ is nonnegative. Hence, by Lemma 2.6, we have

$$S(P,\infty) = S\big(P(\tfrac{1}{\lambda}),0\big) = S(Q,0) + (-d,\ldots,-d) = (-d,*,\ldots,*), \tag{2.17}$$

where each $*$ is greater than or equal to $-d$. Thus *any polynomial matrix of degree* $d > 0$ *has a pole at infinity of order* $d$ (and perhaps other orders as well). This simple calculation provides an example of a more general result for rational matrices found in [4, section 5] and [48, Chapter 3], which we state here without proof.

PROPOSITION 2.17. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{m\times n}$ be a rational matrix with each nonzero entry expressed as $r_{ij}(\lambda) = n_{ij}(\lambda)/d_{ij}(\lambda)$, where $n_{ij}(\lambda), d_{ij}(\lambda) \in \mathbb{F}[\lambda]$. Then the smallest structural index at infinity of $R(\lambda)$ is*

$$\omega := \min_{r_{ij}(\lambda)\neq 0} \big(\deg(d_{ij}) - \deg(n_{ij})\big). \tag{2.18}$$

*Let $P(\lambda)$ be the polynomial part of $R(\lambda)$ as in* (2.3). *If $P(\lambda) \neq 0$, then $\omega = -\deg(P)$, while if $P(\lambda) = 0$, then $\omega > 0$.*

Proposition 2.17 is illustrated in Example 2.10.

As we have seen so far in this section, it is possible to coherently define the structure at infinity of a polynomial matrix $P(\lambda)$ by viewing it as a rational matrix, and then using the structural indices at infinity of this (special) rational matrix. However, this has not been the typical practice in the literature on polynomial matrices [9, 11, 18, 29, 30], or even for matrix pencils [17, 38, 43]. Instead, the standard way to define the structure at infinity of polynomial matrices has been via the reversal polynomial (2.2), as in the following definition. Another classical way to define the structure at infinity is through the use of homogeneous polynomial formulations [11, 17, 38, 43]; note that this is equivalent to the definition via the reversal polynomial [50].

DEFINITION 2.18. *Let $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ be a polynomial matrix. Then infinity is an eigenvalue of $P(\lambda)$ if* 0 *is an eigenvalue of the polynomial matrix* $\mathrm{rev}P(\lambda)$. *Let $M(\mathrm{rev}P, 0) := S(\mathrm{rev}P, 0)$ be the partial multiplicity sequence of* $\mathrm{rev}P(\lambda)$ *at* 0. *The partial multiplicity sequence of $P(\lambda)$ at $\infty$, denoted $M(P,\infty)$, is defined as*

$$M(P,\infty) := \; M(\mathrm{rev}P, 0)\,.$$

We reemphasize that for polynomial matrices, structural index sequences and partial multiplicity sequences are identical for all *finite* $\lambda_0$. They *only* differ at infinity; however, there is a simple relationship between the two, as shown in the next proposition.

PROPOSITION 2.19. *Let $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ be a polynomial matrix of degree $d$. Then*

$$M(P,\infty) \;=\; S(P,\infty) \,+\, (d, d, \ldots, d)\,.$$

*Proof.* Since $S(P,\infty) = S(P(1/\lambda), 0)$ and $M(P,\infty) = S(\lambda^d P(1/\lambda), 0)$, the result follows from Lemma 2.6 with $R(\lambda) = P(1/\lambda)$, $\pi(\lambda) = \lambda$, and $f(\lambda) = \lambda^d$. ∎

*Remark* 2.20. An immediate corollary of (2.17) and Proposition 2.19 is that the smallest partial multiplicity at infinity of any matrix polynomial is always zero. This fact also follows easily from Definition 2.18 and the definition of reversal polynomial in (2.2), which implies that $\mathrm{rev}\, P(0) = P_d \neq 0$. Now if $M(P,\infty) = (t_1, \ldots, t_r)$, then the Smith form of $\mathrm{rev}P$ can be written as $\mathrm{diag}(\lambda^{t_1} p_1(\lambda), \ldots, \lambda^{t_r} p_r(\lambda)) \oplus 0$. So $t_1 > 0$ would imply $\mathrm{rev}P(0) = 0$, contradicting the nonzeroness of $P_d$; hence $t_1 = 0$.

*Remark* 2.21. By definition, a polynomial matrix $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$ of degree $d$ has an eigenvalue at $\infty$ if $M(P,\infty)$ contains nonzero terms, i.e., if $t_r > 0$. In light of the Smith form for $\mathrm{rev}P$ in Remark 2.20, having $t_r > 0$ is equivalent to $\mathrm{rank}\, P_d = \mathrm{rank}\big(\mathrm{rev}P(0)\big) < r = \mathrm{rank}(\mathrm{rev}P) = \mathrm{rank}(P)$. That is, $P(\lambda)$ has an eigenvalue at $\infty$ if and only if the rank of $P_d$ is smaller than the rank of the polynomial matrix. This rank deficiency is related to the need to impose differentiability conditions on the right-hand side of the system of differential-algebraic equations $P(\frac{d}{dt})\, u = f$ to guarantee the existence of solutions [18, Chapter 8].

**2.2. Polynomial matrices: Index sum theorem and inverse problem.** A fundamental result on polynomial matrices is the polynomial index sum theorem, proved over the real field in [36, 34] and extended to arbitrary fields in [9, Theorem 6.5].

THEOREM 2.22 (Polynomial Index Sum Theorem). *Let $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ be a polynomial matrix of degree $d$ and rank $r$, with invariant polynomials $p_1(\lambda), \ldots, p_r(\lambda)$,*

*$M(P,\infty) = (t_1, \ldots, t_r)$, left minimal indices $\eta_1, \ldots, \eta_{m-r}$, and right minimal indices $\alpha_1, \ldots, \alpha_{n-r}$. Then*

$$\sum_{j=1}^{r} \deg(p_j) + \sum_{j=1}^{r} t_j + \sum_{j=1}^{m-r} \eta_j + \sum_{j=1}^{n-r} \alpha_j \;=\; d\,r. \tag{2.19}$$

A second fundamental result, proved in [10, Theorem 3.3], solves the most general form of inverse problem for polynomial matrices with prescribed complete structural data.

Theorem 2.23 (Fundamental Realization Theorem for Polynomial Matrices). *Let $\mathbb{F}$ be an infinite field, let $m$, $n$, $d$, and $r \leq \min\{m, n\}$ be given positive integers, let $p_1(\lambda)|\cdots|p_r(\lambda)$ be a divisibility chain of arbitrary monic polynomials in $\mathbb{F}[\lambda]$, and let $t_1 \leq \cdots \leq t_r$, $\eta_1 \leq \cdots \leq \eta_{m-r}$, and $\alpha_1 \leq \cdots \leq \alpha_{n-r}$ be given lists of nonnegative integers. Then there exists an $m \times n$ polynomial matrix $P(\lambda)$ with coefficients in $\mathbb{F}$, with rank $r$ and degree $d$, with invariant polynomials $p_1(\lambda), \ldots, p_r(\lambda)$ and partial multiplicities at infinity $t_1, \ldots, t_r$, and with left and right minimal indices respectively equal to $\eta_1, \ldots, \eta_{m-r}$ and $\alpha_1, \ldots \alpha_{n-r}$, if and only if $t_1 = 0$ and (2.19) holds.*

We emphasize that although the proof of Theorem 2.23 given in [10] is constructive, it is also involved. As an unfortunate side effect, the constructed polynomial matrix $P(\lambda)$ does not transparently display any of the prescribed structural data.

*Remark* 2.24. The proof given in [10, Theorem 3.3] for Theorem 2.23 uses the assumption that $\mathbb{F}$ is an infinite field *only* to guarantee the existence for any prescribed polynomials $p_1(\lambda)|\cdots|p_r(\lambda)$ of a constant $\beta \in \mathbb{F}$ such that $p_r(\beta) \neq 0$. Such a $\beta$ allows the general inverse problem to be reduced via a Möbius transformation [10, Lemma 3.4 and p. 319] to an inverse problem where there are *no* prescribed eigenvalues at infinity. Thus the assumption that $\mathbb{F}$ is infinite can be replaced by the weaker assumption that there exists some $\beta \in \mathbb{F}$ such that $p_r(\beta) \neq 0$. Note that for finite fields there always exist some choices of polynomials $p_1(\lambda)|\cdots|p_r(\lambda)$ that do not satisfy this assumption.

*Remark* 2.25. The assumption in Theorem 2.23 that $m, n, r$, and $d$ are positive integers was made in [10] to avoid consideration of the trivial cases of empty matrices (when $m = 0$ or $n = 0$), zero matrices (when $r = 0$), and constant matrices (when $d = 0$). However, it is not hard to see that Theorem 2.23 still holds even if $r = 0$ or $d = 0$, making the right-hand side of (2.19) zero, and thus forcing all summands on the left-hand side to be zero. The simple proof of this fact is omitted for brevity.

**3. Van Dooren's index sum theorem revisited.** This section has two parts. First, a new proof of Van Dooren's index sum theorem is provided, which in contrast to previous proofs is valid over arbitrary fields. The second part, in subsection 3.2, has a historical nature and is not essential for understanding the rest of the paper.

**3.1. A new proof of the rational index sum theorem.** The first proof of Van Dooren's index sum theorem can be found in [45, Proposition 5.10] and [49, Theorem 3]; a different proof can be found in [23, Theorem 6.5-11]. These proofs are briefly discussed in subsection 3.2; for now we only emphasize that both assume that $\mathbb{F} = \mathbb{C}$. In this section, we offer a proof based on Theorem 2.22 that is valid in arbitrary fields. In order to see that our argument is not "circular", the main steps of the proof of Theorem 2.22 provided in [9, Theorem 6.5] are now summarized:

(1) the relation between the structural data of any polynomial matrix and those of its first Frobenius companion form $C_1(\lambda)$ is established in [9, Theorem 5.3];
(2) Theorem 2.22 is proved for pencils in [9, Lemma 6.3];

(3) the results in steps (1) and (2) are combined in [9, Theorem 6.5] to prove Theorem 2.22 by counting the rank of $C_1(\lambda)$ in two different ways.

Next, we state and prove Van Dooren's index sum theorem over arbitrary fields.

THEOREM 3.1 (rational index sum theorem). *Let $R(\lambda)$ be a rational matrix over an arbitrary field $\mathbb{F}$. Let $\delta_p(R)$ and $\delta_z(R)$ be the total number of poles and zeros, respectively, of $R(\lambda)$, and let $\mu(R)$ be the sum of the left and right minimal indices of $R(\lambda)$. Then*

$$\delta_p(R) \;=\; \delta_z(R) + \mu(R)\,. \tag{3.1}$$

*Proof.* Let us assume that $R(\lambda)$ has the Smith–McMillan form $D(\lambda)$ given in (2.4), that $S(R,\infty) = (q_1,\dots,q_r)$, and recall that $\psi_1(\lambda)$ is the monic least common multiple of the denominators of the entries of $R(\lambda)$. The proof follows easily from applying the polynomial index sum theorem, i.e., Theorem 2.22, to the polynomial matrix $P(\lambda) := \psi_1(\lambda)R(\lambda)$ and from the relation between the structural data of $R(\lambda)$ and $P(\lambda)$. Note first that the minimal indices of $P(\lambda)$ and $R(\lambda)$ are identical, since $\psi_1(\lambda)$ is just a nonzero scalar in the field $\mathbb{F}(\lambda)$. Lemma 2.6 and (2.4) imply that the invariant polynomials of $P(\lambda)$ are $\psi_1(\lambda)\frac{\varepsilon_j(\lambda)}{\psi_j(\lambda)}$ for $j = 1,\dots,r$, which have degrees $\deg(\psi_1) + \deg(\varepsilon_j) - \deg(\psi_j)$. From Lemmas 2.6 and 2.9, we conclude that

$$S(P,\infty) \;=\; S\left(\psi_1\left(\tfrac{1}{\lambda}\right)\, R\left(\tfrac{1}{\lambda}\right), 0\right) \;=\; \big(q_1 - \deg(\psi_1)\,,\, \dots\,,\, q_r - \deg(\psi_1)\big)\,.$$

Letting $d := \deg(P)$, Proposition 2.19 then yields

$$M(P,\infty) \;=\; \big(\, q_1 - \deg(\psi_1) + d\,,\, \dots\,,\, q_r - \deg(\psi_1) + d\,\big)\,. \tag{3.2}$$

The combination of all this information with (2.19), applied to $P(\lambda)$, and with (2.15) yields (3.1). ◻

*Remark* 3.2. The proof of Theorem 3.1 proceeds in the same spirit as the original proof of the Smith–McMillan form given in [31, 32] (see also [23, p. 443]); both proofs first reduce the rational problem to a "polynomial problem", then leverage known results about polynomial matrices before converting back to rational matrices. As discussed in subsection 3.2, other proofs of Van Dooren's index sum theorem available in the literature follow different paths, which by contrast might informally be termed "intrinsically rational". It is interesting to observe that there also exist "intrinsically rational" proofs of the Smith–McMillan form [48, p. 10].

*Example* 3.3. The results in Examples 2.13 and 2.14 allow us to check immediately that the matrix $R(\lambda)$ in Example 2.5 satisfies (3.1).

*Remark* 3.4. It is worth mentioning here the following way to formulate the rational index sum theorem. Rearrange (3.1) to the form $\delta_z(R) - \delta_p(R) + \mu(R) = 0$, and recall that $\delta_p(R)$ from Definition 2.11 is *minus* the sum of all the negative structural indices. Then this form of the rational index sum theorem simply says that the *sum of all the indices* (*minimal and structural, positive and negative, finite and infinite, over all* $\lambda_0 \in \overline{\mathbb{F}} \cup \{\infty\}$) *is zero for any rational matrix.* Observe that, with the notation in (2.15), this formulation is equivalent to

$$\sum_{j=1}^{r} \deg(\varepsilon_j) - \sum_{j=1}^{r} \deg(\psi_j) + \sum_{j=1}^{r} q_j + \mu(R) \;=\; 0. \tag{3.3}$$

**3.2. History of Van Dooren's index sum theorem and its relation with the polynomial index sum theorem.** The rational and polynomial index sum theorems have a rather curious history. In the first place, they seem to have completely ignored each other until only recently, when the polynomial index sum theorem was finally recognized to be a corollary of the rational index sum theorem in [10, Remark 3.2].[1] This "mutual ignorance" is probably related to two facts:

(a) each index sum theorem uses a different definition of structure at infinity, which may have created a certain amount of confusion;

(b) the statements of these results appear on their face to be very different from one another—Theorem 2.22 explicitly displays the rank and degree of the polynomial matrix, while in Theorem 3.1 there is no explicit reference either to the rank or to any degree associated with the rational matrix.

Connected to these facts, we also emphasize that the original proofs of these two index sum theorems have completely different flavors and use rather different techniques.

Second, both index sum theorems seem to have remained unnoticed by many researchers in the linear algebra community, which is surprising since they establish basic relationships between the structural data of rational and polynomial matrices.

As far as we know, the first published index sum theorem is the rational index sum theorem in [49, Theorem 3], a paper published in 1979 but submitted in 1978. The authors of [49] write the following footnote (on p. 241) concerning the rational index sum theorem: "First obtained, in a slightly different way, by Van Dooren in earlier unpublished research"; this is the reason we have referred to this theorem as "Van Dooren's index sum theorem". The same result appears as Proposition 5.10 in the thesis [45], with the same proof as in [49]. The proof of Van Dooren's index sum theorem presented in [45] is far from trivial and relies heavily on the theory of realizations of rational matrices in terms of polynomial matrices.

The rational index sum theorem can also be found in the classic reference [23, Theorem 6.5-11], with a proof very different from that in [45, 49]. The proof in [23] uses valuations of rational matrices, defined as in [16] via the valuations of the minors of the considered rational matrix. Thus the proof in [23] has very much a "determinantal" flavor. The rational index sum theorem restricted to real rational matrices with full column rank is also proved via valuations in [48, p. 137].

The first statement that we know of the polynomial index sum theorem is given in [36, Theorem 3] for real polynomial matrices. The proof in [36] is essentially the same as that outlined in the first paragraph of subsection 3.1. Surprisingly, no connections with the original rational index sum theorem in [45, 49] are mentioned at all in [36]. It is worth noting that the polynomial index sum theorem is used in [36, 34] mainly as a tool supporting the primary goal of the authors, the development of a numerically reliable algorithm for column reduction of polynomial matrices. It is perhaps this auxiliary role that allowed the polynomial index sum theorem to go unrecognized as a fundamental result for so long in the linear algebra community and to remain essentially forgotten until its importance was highlighted in [9], where, in addition, it was extended to arbitrary fields and given its current name. Nevertheless, note that the polynomial index sum theorem has appeared in some scattered references

[1]This nice connection was pointed out to the authors of [10] by an anonymous referee. During the ALAMA meeting held in Alicante, Spain, May–June 2018, this referee identified himself as Ion Zaballa. The authors of [10] can now publicly acknowledge Ion Zaballa for his constructive and generous report on the first submitted version of [10].

such as [24, Proposition 1], but always as a nameless auxiliary result, and without establishing any connection with Van Dooren's index sum theorem in [45, 49].

Finally, we note that on p. 3093 of the long survey paper [25] by Kublanovskaya, one can see the rational and the polynomial index sum theorems stated one right after the other(!),[2] without proofs, and again without establishing (or even mentioning) any connection between them. As we have seen, the connection had to wait until [10, Remark 3.2]. For completeness we end this section with the following theorem, which the alert reader has undoubtedly already anticipated.

THEOREM 3.5. *The polynomial and rational index sum theorems are equivalent.*

*Proof.* The proof of Theorem 3.1 shows that the polynomial implies the rational index sum theorem. The reverse implication follows immediately from the fact that if $P(\lambda)$ is a polynomial matrix with $\operatorname{rank}(P) = r$, $\deg(P) = d$, and $M(P,\infty) = (t_1,\dots,t_r)$, then $S(P,\infty) = (t_1 - d,\dots,t_r - d)$, by Proposition 2.19. Then, (3.1) applied to $P(\lambda)$ implies Theorem 2.22. ☐

**4. Rational matrices with prescribed complete structural data.** This section presents in Theorem 4.1 the most important original result of this paper, which solves the basic form of the general inverse problem for structural data of rational matrices. In section 5, another formulation of this problem is studied.

THEOREM 4.1. *Let $\mathbb{F}$ be an infinite field, let $m$, $n$, and $r \leq \min\{m,n\}$ be given positive integers, and let $\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)},\dots,\frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}$ be $r$ rational functions in normalized reduced form, such that the monic polynomials in their numerators and denominators form divisibility chains $\varepsilon_1(\lambda)\mid\cdots\mid\varepsilon_r(\lambda)$ and $\psi_r(\lambda)\mid\cdots\mid\psi_1(\lambda)$. Also let $q_1\leq\cdots\leq q_r$ be arbitrary integers (i.e., positive, negative, or zero) and $\eta_1\leq\cdots\leq\eta_{m-r}$ and $\alpha_1\leq\cdots\leq\alpha_{n-r}$ be two lists of nonnegative integers. Then there exists a rational matrix $R(\lambda)\in\mathbb{F}(\lambda)^{m\times n}$ of rank $r$, with invariant rational functions $\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)},\dots,\frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}$ and $S(R,\infty)=(q_1,\dots,q_r)$, and with left and right minimal indices equal to $\eta_1,\dots,\eta_{m-r}$ and $\alpha_1,\dots,\alpha_{n-r}$, respectively, if and only if*

$$\sum_{j=1}^{r}\deg(\psi_j)+\sum_{q_j<0}(-q_j) \;=\; \sum_{j=1}^{r}\deg(\varepsilon_j)+\sum_{q_j>0}q_j+\sum_{j=1}^{m-r}\eta_j+\sum_{j=1}^{n-r}\alpha_j\,. \tag{4.1}$$

Before proving Theorem 4.1, we emphasize that, taking into account (2.15) and (3.1), the necessary and sufficient condition (4.1) can be stated in plain words as "the prescribed complete structural data satisfy the condition in Van Dooren's index sum theorem". Note also that (4.1) is written to correspond exactly with (3.1).

*Proof of Theorem* 4.1. The fact that the existence of $R(\lambda)$ with the specified properties implies (4.1) is just Theorem 3.1. The proof that (4.1) implies the existence of $R(\lambda)$ with the prescribed complete structural data consists of three simple steps: (a) the prescribed "rational" structural data are transformed into "polynomial" data; (b) a polynomial matrix $P(\lambda)$ whose structural data are these "polynomial" data is provided by Theorem 2.23; and (c) the desired rational matrix is proved to be $R(\lambda) := (1/\psi_1(\lambda))\,P(\lambda)$.

[2]The references given in [25] for the index sum theorems are imprecise. With a considerable degree of interpretation it can be inferred that [25] attributes the rational index sum theorem to Van Dooren in [45] and the polynomial index sum theorem to V. B. Khazanov in his Ph.D. thesis, written in Russian in 1983, which we have not seen.

The "polynomial" data in step (a) are

(i) the divisibility chain $\psi_1(\lambda)\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)} \,\Big|\, \cdots \,\Big|\, \psi_1(\lambda)\frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}$ of scalar monic polynomials with coefficients in $\mathbb{F}$,

(ii) the list of $r$ nonnegative integers $0 \leq q_2 - q_1 \leq \cdots \leq q_r - q_1$, whose first term is zero,

(iii) the two lists of nonnegative integers $\eta_1 \leq \cdots \leq \eta_{m-r}$ and $\alpha_1 \leq \cdots \leq \alpha_{n-r}$.

Observe that (4.1) (or equivalently (3.3)) implies that these "polynomial" data satisfy

$$\sum_{j=1}^{r} \deg\left(\psi_1(\lambda)\frac{\varepsilon_j(\lambda)}{\psi_j(\lambda)}\right) + \sum_{j=1}^{r}(q_j - q_1) + \sum_{j=1}^{m-r}\eta_j + \sum_{j=1}^{n-r}\alpha_j = r\left[\deg(\psi_1) - q_1\right]. \tag{4.2}$$

Since all summands in the left-hand side of (4.2) are nonnegative, $\deg(\psi_1) - q_1 \geq 0$. Thus, (4.2) shows that the "polynomial" data in (i), (ii), and (iii) satisfy the conditions of Theorem 2.23 (see also Remark 2.25). Therefore, there exists a polynomial matrix $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ with rank $r$, degree $d = \deg(\psi_1) - q_1$, invariant polynomials given in (i), partial multiplicities at infinity given in (ii), and with left and right minimal indices given in (iii).

Finally, we prove that the rational matrix

$$R(\lambda) := \frac{1}{\psi_1(\lambda)}\, P(\lambda) \in \mathbb{F}(\lambda)^{m\times n} \tag{4.3}$$

has the complete structural data prescribed in the statement. The minimal indices of $R(\lambda)$ are obviously equal to those of $P(\lambda)$. Lemma 2.6 implies that the invariant rational functions of $R(\lambda)$ are $\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}$. Proposition 2.19 implies that

$$S(P,\infty) = M(P,\infty) - (d,\ldots,d) = (q_1 - \deg(\psi_1), q_2 - \deg(\psi_1), \ldots, q_r - \deg(\psi_1))$$

and Lemmas 2.6 and 2.9 imply that $S(R,\infty) = S(P,\infty) + (\deg(\psi_1),\ldots,\deg(\psi_1))$. This yields $S(R,\infty) = (q_1, q_2, \ldots, q_r)$ and completes the proof. ∎

The proof of Theorem 4.1 is constructive, since it relies on applying Theorem 2.23 to construct the polynomial matrix $P(\lambda)$ in (4.3). However, from the comments in the paragraph just after Theorem 2.23, we deduce that the constructed rational matrix $R(\lambda)$ does not transparently reveal any of the prescribed structural data.

*Remark* 4.2. Remark 2.24 and the proof of Theorem 4.1 imply that we can state a version of Theorem 4.1 valid for *any* field, but at the cost of adding the assumption "further suppose that there exists $\beta \in \mathbb{F}$ such that $\psi_1(\beta)\varepsilon_r(\beta)/\psi_r(\beta) \neq 0$".

**5. Rational matrices with prescribed nontrivial structural data.** Theorem 4.1 allows some of the prescribed invariant rational functions to be equal to 1, as well as some of the prescribed structural indices at infinity to be zero, i.e., to be trivial. In this context the word "trivial" refers to data that do not carry any information on the orders of the poles and/or zeros of the rational matrix whose existence is guaranteed by Theorem 4.1. The purpose of this section is to provide (in Theorem 5.2) necessary and sufficient conditions for the existence of a rational matrix when only the nontrivial structural data are prescribed. We emphasize that Theorem 5.2 is a direct corollary of Theorem 4.1. In order to state Theorem 5.2 in a concise way, we first make the following definitions.

Definition 5.1. *Let $\mathbb{F}$ be an arbitrary field. A list $\mathcal{L}_{\text{fin}}$ of nontrivial finite structural data is a list of the form*

$$\mathcal{L}_{\text{fin}} := \Big\{ \pi_1(\lambda)^{s_{11}}, \pi_1(\lambda)^{s_{21}}, \dots, \pi_1(\lambda)^{s_{g_1 1}}, \\ \pi_2(\lambda)^{s_{12}}, \pi_2(\lambda)^{s_{22}}, \dots, \pi_2(\lambda)^{s_{g_2 2}}, \\ \vdots \\ \pi_t(\lambda)^{s_{1t}}, \pi_t(\lambda)^{s_{2t}}, \dots, \pi_t(\lambda)^{s_{g_t t}} \Big\},$$

*where $\pi_1(\lambda), \dots, \pi_t(\lambda)$ are distinct nonconstant monic irreducible polynomials in $\mathbb{F}[\lambda]$ and, for each $j = 1, \dots, t$, $s_{1j} \le \cdots \le s_{g_j j}$ is a sequence of nonzero integers* (*that may be negative or positive*)*. Moreover, for any rational matrix $R(\lambda)$ with entries in $\mathbb{F}(\lambda)$, we say that $\mathcal{L}_{\text{fin}}$ is the list of nontrivial finite structural data of $R(\lambda)$ if the nonzero structural indices of $R(\lambda)$ at $\pi_j(\lambda)$ are exactly $s_{1j} \le \cdots \le s_{g_j j}$ for $j = 1, \dots, t$, while the structural indices of $R(\lambda)$ at $\pi(\lambda)$ are all equal to zero for any $\pi(\lambda) \in \mathbb{F}[\lambda]$ such that $\pi(\lambda) \ne \pi_j(\lambda)$ for $j = 1, \dots, t$.*

*The sum of the "signed degrees" of the rational functions in $\mathcal{L}_{\text{fin}}$ and the length of the longest chain of $\mathcal{L}_{\text{fin}}$ associated with the same irreducible polynomial are denoted, respectively, by*

$$\delta(\mathcal{L}_{\text{fin}}) := \sum_{j=1}^{t} \sum_{i=1}^{g_j} s_{ij} \, \deg \pi_j(\lambda) \quad \text{and} \quad g(\mathcal{L}_{\text{fin}}) := \max_{1 \le j \le t} g_j.$$

Recall that if $\mathbb{F} = \mathbb{C}$, every $\pi_j(\lambda)$ in Definition 5.1 is of the form $\pi_j(\lambda) = (\lambda - \lambda_j)$ with $\lambda_j \in \mathbb{C}$, while if $\mathbb{F} = \mathbb{R}$, either $\pi_j(\lambda) = (\lambda - \lambda_j)$ with $\lambda_j \in \mathbb{R}$ or $\pi_j(\lambda) = \lambda^2 + a_j \lambda + b_j$ with $a_j, b_j \in \mathbb{R}$ and with two complex conjugate nonreal roots.

Theorem 5.2. *Let $\mathbb{F}$ be an infinite field, let $\mathcal{L}_{\text{fin}}$ be a list of nontrivial finite structural data as in Definition* 5.1*, let $c_1 \le \cdots \le c_{g_\infty}$ be a list of nonzero integers, and let $\eta_1 \le \cdots \le \eta_q$ and $\alpha_1 \le \cdots \le \alpha_p$ be two lists of nonnegative integers. Then there exists a rational matrix $R(\lambda)$ of rank $r$ with entries in $\mathbb{F}(\lambda)$, with list of nontrivial finite structural data equal to $\mathcal{L}_{\text{fin}}$ and nonzero structural indices at infinity equal to $c_1 \le \cdots \le c_{g_\infty}$, and with left and right minimal indices equal to $\eta_1, \dots, \eta_q$ and $\alpha_1, \dots, \alpha_p$, respectively, if and only if the following two conditions hold:*

(a) $r \ge \max\{g(\mathcal{L}_{\text{fin}}), g_\infty\}$ *, and*

(b) $0 = \delta(\mathcal{L}_{\text{fin}}) + \sum_{j=1}^{g_\infty} c_j + \sum_{j=1}^{q} \eta_j + \sum_{j=1}^{p} \alpha_j$.

*In particular, if* (b) *holds, then for any choice of $r$ satisfying* (a) *there exists a rational matrix $R(\lambda)$ of rank $r$ with the prescribed structural data; such an $R(\lambda)$ will have size $(q + r) \times (p + r)$. If $r$ does not satisfy* (a)*, then there does not exist any rational matrix with rank $r$ and the prescribed structural data.*

*Proof.* First, we prove that the existence of $R(\lambda)$ with rank $r$ and with the prescribed structural data implies that (a) and (b) hold. According to the definition in (2.11) and Definition 2.8, $r$ is the length of the sequence of all the structural indices at any $\pi_j(\lambda) \in \mathbb{F}(\lambda)$ of $R(\lambda)$, as well as the length of the sequence of structural indices at infinity of $R(\lambda)$. Therefore, $r$ is larger than or equal to the number of nonzero structural indices at $\pi_j(\lambda)$ or at infinity of $R(\lambda)$, which is condition (a). Using (2.13) and (3.3), we see that (b) is just the condition in the rational index sum theorem and thus holds for $R(\lambda)$.

Next, we prove that conditions (a) and (b) imply the existence of $R(\lambda)$ with the prescribed structural data and rank $r$. Let $r$ be any integer satisfying (a). To each of the $t$ sequences of nonzero integers $s_{1j} \leq \cdots \leq s_{g_j j}$ from $\mathcal{L}_{\text{fin}}$, $j = 1, \ldots, t$, append $r - g_j$ zeroes to form $t$ new integer sequences $\widetilde{s}_{1j} \leq \cdots \leq \widetilde{s}_{rj}$, each of length $r$. From these $t$ sequences, define the following rational functions in normalized reduced form:

$$\frac{\varepsilon_i(\lambda)}{\psi_i(\lambda)} := \pi_1(\lambda)^{\widetilde{s}_{i1}} \cdots \pi_t(\lambda)^{\widetilde{s}_{it}} \quad \text{for } i = 1, \ldots, r\,. \tag{5.1}$$

Note that the polynomials $\psi_i(\lambda) := \prod_{\widetilde{s}_{ij}<0} \pi_j(\lambda)^{-\widetilde{s}_{ij}}$ and $\varepsilon_i(\lambda) := \prod_{\widetilde{s}_{ij}>0} \pi_j(\lambda)^{\widetilde{s}_{ij}}$ clearly satisfy $\varepsilon_1(\lambda) \mid \cdots \mid \varepsilon_r(\lambda)$ and $\psi_r(\lambda) \mid \cdots \mid \psi_1(\lambda)$. Analogously, append $r - g_\infty$ zeroes to the sequence $c_1 \leq \cdots \leq c_{g_\infty}$ of nonzero integers to get an integer sequence $q_1 \leq \cdots \leq q_r$ of length $r$. With these definitions and $m := q + r$, $n := p + r$, condition (b) is equivalent to (4.1) (or to (3.3)). Theorem 4.1 can now be applied to prove the existence of $R(\lambda) \in \mathbb{F}(\lambda)^{(q+r)\times(p+r)}$ with the prescribed structural data and rank $r$. $\square$

*Remark* 5.3. Remark 4.2 and the proof of Theorem 5.2 imply that we can state a version of Theorem 5.2 valid in any field, but at the cost of adding the assumption "further suppose that there exists $\omega \in \mathbb{F}$ such that $\pi_j(\omega) \neq 0$ for $j = 1, \ldots, t$, where $\pi_j(\lambda)$ are the nonconstant monic irreducible polynomials in the definition of $\mathcal{L}_{\text{fin}}$".

**6. Conclusions and future work.** We have proved that there exists a rational matrix with prescribed complete structural data if and only if such data satisfies the very easily checked necessary condition in Van Dooren's rational index sum theorem of 1978. In addition, this rational index sum theorem has itself been revisited from two points of view: we have extended it to arbitrary fields and have discussed some of its history and relationship with the polynomial index sum theorem. These two rational matrix results are based on, and significantly extend, previous results valid only for polynomial matrices that can be found in [9, 10]. These previous polynomial matrix results have already been applied to the solution of a number of problems, some of them related to numerical algorithms. Consequently, we anticipate that the results in this paper will have similar applications in the context of rational eigenvalue problems.

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