DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2013

Website: www.aimSciences.org

pp. **375–384**

FINITE-DIMENSIONAL BEHAVIOR IN A THERMOSYPHON WITH A VISCOELASTIC FLUID

A. Jiménez-Casas

Grupo de Dinámica No Lineal
Universidad Pontificia Comillas de Madrid
C/Alberto Agulilera 23, 28015 Madrid, Spain

Mario Castro

Grupo Interdisciplinar de Sistemas Complejos (GISC)
and Grupo de Dinmica No Lineal (DNL)
Escuela Tcnica Superior de Ingeniera (ICAI)
Universidad Pontificia Comillas, E28015, Madrid, Spain

Justine Yassapan

Grupo de Dinmica No Lineal (DNL)
Departamento de Matemtica Aplicada y Computacin
Escuela Tcnica Superior de Ingeniera (ICAI)
Universidad Pontificia Comillas, E28015, Madrid, Spain

Abstract. We analyse the motion of a viscoelastic fluid in the interior of a closed loop thermosyphon under the effects of natural convection. We consider a viscoelastic fluid described by the Maxwell constitutive equation. This fluid presents elastic-like behavior and memory effects. We study the asymptotic properties of the fluid inside the thermosyphon and derive the exact equations of motion in the inertial manifold that characterize the asymptotic behavior. Our work is a generalization of some previous results on standard (Newtonian) fluids.

1. **Introduction.** Chaos in fluids subject to temperature gradients has been a subject of intense work for its applications in the field of engineering and atmospheric sciences. A thermosyphon is a device composed of a closed loop *pipe* containing a fluid whose motion is driven by the effect of several actions such as gravity and natural convection [7, 12, 11]. The flow inside the loop is driven by an energetic balance between thermal energy and mechanical energy. The interest on this system comes from engineering and as a *toy* model of natural convection (for instance, to understand the origin of chaos in atmospheric systems).

As viscoelasticity is, in general, strongly dependent on the material composition and working regime, here we will approach this problem by studying the most essential feature of viscoelastic fluids: memory effects. To this aim we restrict ourselves to the study of the so-called Maxwell model [8]. In this model, both Newton's law of viscosity and Hooke's law of elasticity are generalized and complemented through

2010 *Mathematics Subject Classification.* 35M33, 35B42, 76A10.

Key words and phrases. Thermosyphon, viscoelastic fluid, heat flux, asymptotic behaviour.

Partially supported by Projects FIS2009-12964-C05-03, and MTM2009-07540, GR58/08 Grupo 920894 BSCH-UCM, Spain.

an evolution equation for the stress tensor, σ. The stress tensor comes into play in the equation for the conservation of momentum:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla p+\nabla\cdot\sigma \tag{1}$$

For a Maxwellian fluid, the stress tensor takes the form:

$$\frac{\mu}{E}\frac{\partial\sigma}{\partial t}+\sigma=\mu\dot{\gamma}, \tag{2}$$

where μ is the fluid viscosity, E the Young's modulus and $\dot{\gamma}$ the shear strain rate (or rate at which the fluid deforms). Under stationary flow, the equation (2) reduces to Newton's law, and consequently, the equation (1) reduces to the celebrated Navier-Stokes equation. On the contrary, for short times, when *impulsive* behavior from rest can be expected, Equation (2) reduces to Hooke's law of elasticity.

The derivation of the thermosyphon equations of motion is similar to that in [7, 12, 11]. The simplest way to incorporate Equation (2) into Equation (1) is by differentiating equation (1) with respect to time and replacing the resulting time derivative of σ with Equation (2). This way to incorporate the constitutive equation allows to reduce the number of unknowns (we remove σ from the system of equations) at the cost of increasing the order of the time derivatives to second order. The resulting second order equation is then averaged along the loop section (as in Ref.[7]). Finally, after adimensionalizing the variables (to reduce the number of free parameters) we arrive at the ODE/PDE system

$$\left\{\begin{array}{rcll} \varepsilon\dfrac{d^2v}{dt^2}+\dfrac{dv}{dt}+G(v)v & = & \displaystyle\oint Tf, & v(0)=v_0,\, \frac{dv}{dt}(0)=w_0 \\ \dfrac{\partial T}{\partial t}+v\dfrac{\partial T}{\partial x} & = & h(x)+\nu\frac{\partial^2 T}{\partial x^2}, & T(0,x)=T_0(x) \end{array}\right. \tag{3}$$

where $v(t)$ is the velocity, $T(t,x)$ is the distribution of the temperature of the viscoelastic fluid in the loop, ν is the temperature diffusion coefficient, $G(v)$ is the friction law at the inner wall of the loop, the function f is the geometry of the loop and the distribution of gravitational forces, $h(x)$ is the general heat flux and ε is the viscoelastic parameter, which is the dimensionless version of the viscoelastic time, $t_V=\mu/E$. Roughly speaking, it gives the time scale in which the transition from elastic to fluid-like occurs in the fluid. We consider G and h are given continuous functions, such that $G(v)\geq G_0>0$. Finally, for physical consistency, it is important to note that all functions considered must be 1-periodic with respect to the spatial variable.

Our contribution in this paper is: first, to prove the existence of an attractor and an inertial manifold and next, using this, to obtain an explicit reduction to low-dimensional systems of the behavior of viscoelastic fluids; extending the results in [9] for this kind of fluid, in order to get the similar results like [13] when we consider a given heat flux instead of Newton's linear cooling law.

2. **Well posedness and asymptotic bounds.**

2.1. **Existence and uniqueness of solutions.**

First, we note if we integrate the equation for the temperature along the loop, $\oint T(t)=\oint T_0+t\oint h$. Therefore, $\oint T(t)$ is unbounded, as $t\mapsto\infty$, unless $\oint h=0$.

However, taking $\theta = T - \oint T$ and $h^* = h - \oint h$ reduces to the case $\oint T(t) = \oint T_0 = \oint h = 0$, since θ would satisfy

$$\frac{\partial \theta}{\partial t} + v\frac{\partial \theta}{\partial x} = h(x) + \nu \frac{\partial^2 \theta}{\partial x^2}$$

and $\oint Tf = \oint \theta f$, since $\oint f = 0$. Therefore, hereafter we consider the system (3) where all functions have zero average. Also, the operator $\nu A = -\nu \frac{\partial^2}{\partial x^2}$, together with periodic boundary conditions, is an unbounded, self-adjoint operator with compact resolvent in $L^2_{per}(0,1)$, that is positive when restricted to the space of zero average functions $\dot{L}^2_{per}(0,1)$. Hence, the equation for the temperature T in (3) is of parabolic type for $\nu > 0$.

Hereafter we denote by $w = \frac{dv}{dt}$ and we write the system (3) as the following evolution system for the acceleration, velocity and temperature:

$$\left\{ \begin{array}{rcll} \dfrac{dw}{dt} + \dfrac{1}{\varepsilon} w & = & -\frac{1}{\varepsilon} G(v)v + \frac{1}{\varepsilon} \displaystyle\oint Tf, & w(0) = w_0 \\ \frac{dv}{dt} & = & w, & v(0) = v_0 \\ \dfrac{\partial T}{\partial t} + v\dfrac{\partial T}{\partial x} - \nu \dfrac{\partial^2 T}{\partial x^2} & = & h(x), & T(0,x) = T_0(x) \end{array} \right. \tag{4}$$

this is:

$$\frac{d}{dt}\begin{pmatrix} w \\ v \\ T \end{pmatrix} + \begin{pmatrix} \frac{1}{\varepsilon} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\nu \frac{\partial^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} w \\ v \\ T \end{pmatrix} = \begin{pmatrix} F_1(w,v,T) \\ F_2(w,v,T) \\ F_3(w,v,T) \end{pmatrix} \tag{5}$$

with

$$F_1(w,v,T) = -\frac{1}{\varepsilon} G(v)v + \frac{1}{\varepsilon}\oint Tf, F_2(w,v,T) = w \text{ and } F_3(w,v,T) = -v\frac{\partial T}{\partial x} + h(x). \tag{6}$$

The operator $B = \begin{pmatrix} \frac{1}{\varepsilon} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\nu \frac{\partial^2}{\partial x^2} \end{pmatrix}$ is a sectorial operator in $Y = \mathbb{R}^2 \times \dot{L}^2_{per}(0,1)$ with domain $D(B) = \mathbb{R}^2 \times \dot{H}^2_{per}(0,1)$ and has compact resolvent, where

$$\dot{L}^2_{per}(0,1) = \{u \in L^2_{loc}(\mathbb{R}), u(x+1) = u(x) a.e., \oint u = 0\} \text{ and}$$

$$\dot{H}^m_{per}(0,1) = H^m_{loc}(\mathbb{R}) \cap \dot{L}^2_{per}(0,1).$$

Thus, using the result and techniques about sectorial operator of [4] we obtain Theorem 1.

Theorem 1. *We suppose that $H(r) = rG(r)$ is locally Lipschitz, $f, h \in \dot{L}^2_{per}(0,1)$. Then, given $(w_0, v_0, T_0) \in Y = \mathbb{R}^2 \times \dot{L}^2_{per}(0,1)$, there exists a unique solution of (3) satisfying $(w,v,T) \in C([0,\infty), \mathbb{R}^2 \times \dot{L}^2_{per}(0,1)) \cap C(0,\infty), \mathbb{R}^2 \times \dot{H}^2_{per}(0,1))$, $(\dot{w}, w, \frac{\partial T}{\partial t}) \in C(0,\infty), \mathbb{R}^2 \times \dot{H}^{2-\delta}_{per}(0,1))$, where $w = \dot{v} = \frac{dv}{dt}$ and $\dot{w} = \frac{d^2 v}{dt^2}$ for every $\delta > 0$. In particular, (3) defines a nonlinear semigroup, $S(t)$ in $Y = \mathbb{R}^2 \times \dot{L}^2_{per}(0,1)$, with $S(t)(w_0, v_0, T_0) = (w(t), v(t), T(t))$.*

Proof. **Step 1.** First, we prove the local existence and regularity. This follows easily from the variation of constants formula and [4]. In order to prove this we write the system as (5), and we have:

$$U_t + BU = F(U), \text{ with } U = \begin{pmatrix} w \\ v \\ T \end{pmatrix}, B = \begin{pmatrix} \frac{1}{\varepsilon} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\nu\frac{\partial^2}{\partial x^2} \end{pmatrix} \text{ and } F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

where the operator B is also sectorial operator in $\mathcal{Y} = I\!R^2 \times \dot{H}^{-1}_{per}(0,1)$ with domain $D(B) = \mathbb{R}^2 \times \dot{H}^1_{per}(0,1)$ and has compact resolvent. Note that in this context the operator $A = -\frac{\partial^2}{\partial x^2}$ must be understood in the variational sense, i.e., for every $T, \varphi \in \dot{H}^1_{per}(0,1)$, $< A(T), \varphi >= \oint \frac{\partial T}{\partial x}\frac{\partial \varphi}{\partial x}$ and $\dot{L}^2_{per}(0,1)$ coincides with the fractional space of exponent $\frac{1}{2}$ [4]. Hereafter we denote by $\|.\|$ the norm on the space $\dot{L}^2_{per}(0,1)$.

Now, under the above notations, using that $H(v) = G(v)v$ is locally Lipschitz together with $f, h \in \dot{L}^2_{per}(0,1)$, we obtain that the nonlinearity (6) $F : Y = \mathbb{R}^2 \times \dot{L}^2_{per}(0,1) \mapsto Y^{-\frac{1}{2}} = \mathbb{R}^2 \times \dot{H}^{-1}_{per}(0,1)$ is well defined and is Lipschitz and bounded on bounded sets.

Therefore, using the techniques of variations of constants formula [4], we get the unique local solution $(w, v, T) \in C([0,\tau], Y = \mathbb{R}^2 \times \dot{L}^2_{per}(0,1))$ of (4) which are given by

$$w(t) = w_0 e^{-\frac{1}{\varepsilon}t} - \frac{1}{\varepsilon}\int_0^t e^{-\frac{1}{\varepsilon}(t-r)} H(r)dr + \frac{1}{\varepsilon}\int_0^t e^{-\frac{1}{\varepsilon}(t-r)}(\oint T(r)f)dr \tag{7}$$

with $H(r) = G(v(r))v(r)$.

$$v(t) = v_0 + \int_0^t w(r)dr \tag{8}$$

$$T(t,x) = e^{-\nu A t}T_0(x) + \int_0^t e^{-\nu A(t-r)}h(x)dr - \int_0^t e^{-\nu A(t-r)}v(r)\frac{\partial T(r,x)}{\partial x}dr \tag{9}$$

where $(w, v, T) \in C([0,\tau], Y)$ and using again the results of [4](smoothing effect of the equations together with bootstrapping method), we get the above regularity of solutions.

Step 2. Now, we prove the solutions of (4) are defined for every time $t \geq 0$. To prove the global existence, we must show that solutions are bounded in the $Y = \mathbb{R}^2 \times \dot{L}^2_{per}(0,1)$ norm on finite time intervals.

First, to obtain the norm of T is bounded in finite time, we note that multiplying the equations for the temperature by T in $\dot{L}^2_{per}$ and integrating by parts, we have that:

$$\frac{1}{2}\frac{d}{dt}\|T\|^2 + \nu\|\frac{\partial T}{\partial x}\|^2 = \oint hT$$

since $\oint T\frac{\partial T}{\partial x} = 0$. Using Cauchy-Schwarz and the Young inequality and then the Poincaré inequality for functions with zero average, since $\oint T = 0$, we obtain $\frac{1}{2}\frac{d}{dt}\|T\|^2 + \nu\pi^2\|T\|^2 \leq C_\delta\|h\|^2 + \delta\|T\|^2$ for every $\delta > 0$ with $C_\delta = \frac{1}{4\delta}$, since π^2 is the first nonzero eigenvalue of A in $\dot{L}^2_{per}(0,1)$. Thus, taking $\delta = \frac{\nu\pi^2}{2}$, $C_\delta = \frac{1}{2\nu\pi^2}$ we obtain

$$\frac{d}{dt}\|T\|^2 + \nu\pi^2\|T\|^2 \leq \frac{\|h\|^2}{\nu\pi^2}. \tag{10}$$

Now, by integrating we get $\|T\|$ is bounded for finite time, and so are $|v(t)|$ and $|w(t)|$, hence we have a global solution and nonlinear semigroup in $Y = \mathbb{R}^2 \times \dot{L}^2_{per}(0,1)$. $\square$

3. **Asymptotic bounds on the solutions.** In this section we use the results and techniques from [13] to prove the existence of the global attractor for the semigroup defined by (3) in the space $Y = \mathbb{R}^2 \times \dot{L}^2_{per}(0, 1)$.

In order to obtain asymptotic bounds on the solutions as $t \to \infty$, we consider the friction function G as in [13] i.e., satisfying the hypotheses from the previous section and we also assume that there exits a constant $g_0 \geq 0$ such that:

$$\limsup_{t\to\infty} \frac{|G'(t)|}{G(t)} = 0 \text{ and } \limsup_{t\to\infty} \frac{|tG'(t)|}{G(t)} \leq g_0. \tag{11}$$

Now, using the l'Hopital's lemma proved in [9] we have the following Lemma proved in [13].

Lemma 1. *If we assume $G(r)$ and $H(r) = rG(r)$ satisfy the hypothesis from Theorem 1 together with (11), then:*

$$\limsup_{t\to\infty} \frac{\left|H(t) - \frac{1}{\varepsilon}\int_0^t e^{-\frac{1}{\varepsilon}(t-r)} H(r)dr\right|}{G(t)} \leq H_0 \tag{12}$$

with $H_0 = (1 + g_0)\varepsilon$ a positive constant such that $H_0 \to 0$ if $\varepsilon \to 0$.

Remark 1. We note that the conditions (11) are satisfied for all friction functions G considered in the previous works, i.e., the thermosyphon models where G is constant or linear or quadratic law. Moreover, its conditions (11) are also true for $G(s) \approx A|s|^n$, as $s \to \infty$.

Finally, in order to obtain the asymptotic bounds on the solutions we obtain the asymptotic bounds for the temperature in this diffusion case and we will use the following result from [13] [Theorem 2.3 Part I] to get the asymptotic bounds for the velocity and the acceleration functions.

Lemma 2. *Under the above notations and hypothesis from Theorem 1, if we assume also that G satisfies (12) for some constant $H_0 \geq 0$ and*

$$\varepsilon\frac{d^2v}{dt^2} + \frac{dv}{dt} + G(v)v = \oint Tf, v(0) = v_0, \frac{dv}{dt}(0) = w_0, \text{ then}$$

$$i) \limsup_{t\to\infty} |v(t)| \leq \frac{1}{G_0} \limsup_{t\mapsto\infty} \left|\oint T(t,\cdot)f(\cdot)\right| + H_0 \tag{13}$$

In particular: If $\limsup_{t\mapsto\infty} \|T\| \in \mathbb{R}$ then

$$\limsup_{t\to\infty} |v(t)| \leq \frac{1}{G_0}\|f\| \limsup_{t\mapsto\infty} \|T\| + H_0 \in \mathbb{R}. \tag{14}$$

ii)If $\limsup_{t\mapsto\infty} \|T\| \in \mathbb{R}$ and we denote now by $G_0^* = \limsup_{t\to\infty} G(v(t))$, with $w(t) = \frac{dv}{dt}$, then

$$\limsup_{t\to\infty} |w(t)| \leq G_0^* H_0 + \left(1 + \frac{G_0^*}{G_0}\right) I \text{ with } I = \limsup_{t\mapsto\infty} \left|\oint T(t,\cdot)f(\cdot)\right| \text{ and} \tag{15}$$

$$\limsup_{t\to\infty} |w(t)| \leq G_0^* H_0 + \left(1 + \frac{G_0^*}{G_0}\right) \|f\| \limsup_{t\mapsto\infty} \|T\| \in \mathbb{R}. \tag{16}$$

Proof. First we obtain $\frac{dv}{ds} + G(s)v = w(0)e^{-\frac{1}{\varepsilon}s} + \frac{1}{\varepsilon}\int_0^s (\oint T(r)\cdot f)e^{-\frac{1}{\varepsilon}(s-r)}dr + I(s)$, with $I(s) = H(s) - \frac{1}{\varepsilon}\int_0^s e^{-\frac{1}{\varepsilon}(s-r)} H(r)$, and then from Lemma 1 together with l'Hopital's lemma we conclude (see [13] [Theorem 2.3 Part I]). □

Proposition 1. *If $f, h \in \dot{L}^2_{per}(0,1)$ and $H(r) = rG(r)$ is locally Lipschitz with $G(v) \geq G_0 > 0$ and satisfies (12) for some constant $H_0 \geq 0$. Then for any solution of (3) in the space $Y = \mathbb{R}^2 \times \dot{L}^2_{per}(0,1)$ we have: i)*

$$\limsup_{t\to\infty} \|T(t)\| \leq \frac{\|h\|}{\nu\pi^2} \text{ and } \limsup_{t\to\infty} |v(t)| \leq \frac{\|f\|\|h\|}{\nu\pi^2 G_0} + H_0 \tag{17}$$

ii) If we denote now by $G_0^* = \limsup_{t\to\infty} G(v(t))$ we get

$$\limsup_{t\to\infty} |w(t)| \leq G_0^* H_0 + G_2 \frac{\|h\|\|f\|}{\nu\pi^2} \text{ with } G_2 = \left(1 + \frac{G_0^*}{G_0}\right). \tag{18}$$

Therefore, (3) has a global compact and connected attractor, A, in $Y = \mathbb{R}^2 \times \dot{L}^2_{per}(0,1)$.

Proof. i) From (10) we get

$$\|T\|^2 \leq \frac{\|h\|^2}{\nu^2\pi^4} + \left(\|T_0\|^2 - \frac{\|h\|^2}{\nu^2\pi^4}\right)_+ e^{-\pi^2\nu t} \tag{19}$$

and thus we obtain the asymptotic bounded of $\|T(t)\|$. Next, from Lemma 2 we get (17) and (18). Since the sectorial operator B, defined in the above section 2.1.1., has compact resolvent; the existence of global compact and connected attractor A, follows from [[3], Theorem 4.2.2 and 3.4.8]. ☐

4. **Finite-dimensional behavior.** Now we take a close look at the dynamics of (3) by considering the Fourier expansions of each function and observing the dynamics of each Fourier mode.

We assume that $h, f, T_0 \in \dot{L}^2_{per}(0,1)$ are given by the following Fourier expansions

$$h(x) = \sum_{k\in\mathbb{Z}^*} b_k e^{2\pi kix}, f(x) = \sum_{k\in\mathbb{Z}^*} c_k e^{2\pi kix}, T_0(x) = \sum_{k\in\mathbb{Z}^*} a_{k0} e^{2\pi kix} \tag{20}$$

with $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. Finally assume that $T(t,x) \in \dot{L}^2_{per}(0,1)$ is given by

$$T(t,x) = \sum_{k\in\mathbb{Z}^*} a_k(t) e^{2\pi kix}. \tag{21}$$

Then, the coefficients $a_k(t)$ in (21), verify the equations:

$$\dot{a_k}(t) + (2\pi kvi + 4\nu\pi^2 k^2) a_k(t) = b_k, \quad a_k(0) = a_{k0}, \quad k \in \mathbb{Z}^* \tag{22}$$

Therefore, (3) is equivalent to the infinite system of ODEs consisting of (22) coupled with

$$\varepsilon \frac{d^2 v}{dt} + \frac{dv}{dt} + G(v)v = \sum_{k\in\mathbb{Z}^*} a_k(t)\bar{c}_k.$$

The two equations reflect two of the main features of (3): the coupling between modes enter only through the velocity, while diffusion acts as a linear damping term. In what follows, we will exploit this explicit equation for the temperature modes to analyze the asymptotic behavior of the system and to obtain the explicit low-dimensional models.

A similar explicit construction was given by Bloch and Titi in [1] for a nonlinear beam equation where the nonlinearity occurs only through the appearance of the L^2 norm of the unknown. A related construction was given by Stuart in [10] for a nonlocal reaction-diffusion equation.

We note that the system (3) is equivalent to the system (4) for the acceleration, velocity and temperature and this is equivalent now to the following infinite system of ODEs (23)

$$\begin{cases} \dfrac{dw}{dt}+\dfrac{1}{\varepsilon}w=-\dfrac{1}{\varepsilon}G(v)v+\dfrac{1}{\varepsilon}\displaystyle\sum_{k\in\mathbb{Z}^*}a_k(t)\bar{c}_k, & =w(0)=w_0\\ \frac{dv}{dt}=w, & v(0)=v_0\\ \dot{a_k}(t)+(2\pi kvi+4\nu\pi^2k^2)a_k(t)=b_k, & a_k(0)=a_{k0}, \quad k\in\mathbb{Z}^*. \end{cases} \tag{23}$$

Next, we obtain the boundedness of these coefficients that improve the boundedness of temperature of the previous section and in particular, allow as to prove the existence of the inertial manifold for the system (3).

4.1. **Inertial manifold.**

Proposition 2. *For every solution of the system (3), (w,v,T), and for every $k\in\mathbb{Z}^*$ we have*

$$\limsup_{t\to\infty}|a_k(t)|\le\frac{|b_k|}{4\nu\pi^2k^2},\ \textit{in particular}\ \limsup_{t\to\infty}\|T(t,.)\|\le\frac{1}{4\nu\pi^2}\|h\| \tag{24}$$

$$\limsup_{t\to\infty}|v(t)|\le\frac{I_0}{G_0}+H_0,\ \textit{with}\ I_0=\sum_{k\in\mathbb{Z}^*}\frac{|b_k||c_k|}{4\nu\pi^2k^2} \tag{25}$$

and G_0 positive constant such that $G(v)\ge G_0$.

$$iii)\ \limsup_{t\to\infty}|w(t)|\le G_0^*H_0+(1+\frac{G_0^*}{G_0})I_0,\ \textit{with}\ G_0^*=\limsup_{t\to\infty}G(v(t)). \tag{26}$$

Proof. From (22), we have that

$$a_k(t)=a_{k0}e^{-4\nu\pi^2k^2t}e^{-2\pi ki\int_0^t v}+e^{-4\nu\pi^2k^2t}b_k\int_0^t e^{4\nu\pi^2k^2s}e^{-2\pi ki\int_s^t v}ds$$

and taking into account that $|e^{-2\pi ki\int_0^t v}|=|e^{-2\pi ki\int_s^t v}|=1$ we obtain:

$$|a_k(t)|\le|a_{k0}|e^{-4\nu\pi^2k^2t}+\frac{|b_k|}{4\nu\pi^2k^2}(1-e^{-4\nu\pi^2k^2t}) \tag{27}$$

and we get $\limsup_{t\to\infty}|a_k(t)|\le\frac{|b_k|}{4\nu\pi^2k^2}$. Using Lemma 2 together with $\oint Tf=\displaystyle\sum_{k\in\mathbb{Z}^*}a_k(t)\bar{c}_k$, the rest is obvious. $\square$

Corollary 1. *i) If $|a_{k0}|\le\frac{|b_k|}{4\nu\pi^2k^2}$ then $|a_k(t)|\le\frac{|b_k|}{4\nu\pi^2k^2}$ for every $t\ge0$.*

ii) If A is the global attractor in the space $Y=\mathbb{R}^2\times\dot{L}^2_{per}(0,1)$, then for every $(w_0,v_0,T_0)\in A$, with $T_0(x)=\sum_{k\in\mathbb{Z}^*}a_ke^{2\pi kix}$ we get,

$$|a_k|\le\frac{|b_k|}{4\nu\pi^2k^2},\quad k\in\mathbb{Z}^*. \tag{28}$$

In particular, if $h\in\dot{H}^m_{per}$ with $m\ge1$, the global attractor $A\hookrightarrow\mathbb{R}^2\times\dot{H}^{m+2}_{per}$ and is compact in this space.

Proof. i) From (27) we have $|a_k(t)|\le\frac{|b_k|}{4\nu\pi^2k^2}+(|a_{k0}|-\frac{|b_k|}{4\nu\pi^2k^2})_+e^{-4\nu\pi^2k^2t}$ thus, if $|a_{k0}|\le\frac{|b_k|}{4\nu\pi^2k^2}$ then $|a_k(t)|\le\frac{|b_k|}{4\nu\pi^2k^2}$ for every $t\ge0$ and $k\in\mathbb{Z}^*$.

ii) To prove the last statement we take into account that, from i), if $h(x) = \sum_{k\in\mathbb{Z}^*} b_k e^{2\pi kix} \in \dot{H}^m_{per}$, then $\sum_{k\in\mathbb{Z}^*} k^{2m}|b_k|^2 < \infty$, and therefore $T_0 \in C = \{R(x) = \sum_{k\in\mathbb{Z}^*} r_k e^{2\pi kix} \in \dot{H}^{m+2}_{per},\ k^2|r_k| \leq \frac{1}{4\pi^2\nu}|b_k|\}$. □

In the next result we will prove that there exists an inertial manifold M for the semigroup $S(t)$ in the phase space $Y = \mathbb{R}^2 \times \dot{H}^m_{per}$, $m \geq 1$ according to [2], i.e., a submanifold of Y such that i) $S(t)M \subset M$ for every $t \geq 0$, ii) there exists $M > 0$ verifying that for every bounded set $B \subset Y$, there exists $C(B) \geq 0$ such that $dist(S(t), M \leq C(B)e^{-Mt}, t \geq 0$ see, for example, [2]. Assume then that $h \in \dot{H}^m_{per}$ with

$$h(x) = \sum_{k\in K} b_k e^{2\pi kix}$$

with $b_k \neq 0$ for every $k \in K \subset \mathbb{Z}^*$ with $0 \notin K$, since $\oint h = 0$. We denote by V_m the closure of the subspace of $\dot{H}^m_{per}$ generated by $\{e^{2\pi kix}, k \in K\}$.

Theorem 2. *Assume that $h \in \dot{H}^m_{per}$ and $f \in \dot{L}^2_{per}$. Then the set $M = \mathbb{R}^2 \times V_m$ is an inertial manifold for the flow of $S(t)(w_0, v_0, T_0) = (w(t), v(t), T(t))$ in the space $Y = \mathbb{R}^2 \times \dot{H}^m_{per}$. Moreover if K is a finite set, then the dimension of M is $|K| + 2$, where $|K|$ is the number of elements in K.*

Proof. **Step I.** First, we show that M is invariant. Note that if $k \notin K$, then $b_k = 0$, and therefore if $a_{k0} = 0$, from (27), we get that $a_k(t) = 0$ for every t, i.e., $T(t,x) = \sum_{k\in K} a_k(t)e^{2\pi kix}$. Therefore, if $(w_0, v_0, T_0) \in M$, then $(w(t), v(t), T(t)) \in M$ for every t, i.e., is invariant.

Step II. From previous assertions, $\oint T(t) \cdot f = \sum_{k\in K} a_k(t) \cdot \bar{c}_k$ and the flow on M is given by

$$\begin{gathered}
\dot{w} + \frac{1}{\varepsilon}w + \frac{1}{\varepsilon}G(v)v = \frac{1}{\varepsilon}\sum_{k\in K} a_k(t) \cdot \bar{c}_k \\
\dot{v} = w \\
\dot{a_k}(t) + (2\pi kvi + 4c\pi^2k^2)a_k(t) = b_k, \quad k \in K \qquad (29) \\
a_k = 0, k \notin K.
\end{gathered}$$

Now, we consider the following decomposition in $\dot{H}^m_{per}$, $T = T^1 + T^2$, where T^1 is the projection of T on V_m and T^2 is the projection of T on the subspace generated by $\{e^{2\pi kix}, k \in \mathbb{Z}^* \setminus K\}$ i.e. $T^1 = \sum_{k\in K} a_k e^{2\pi kix}$ and $T^2 = \sum_{k\in\mathbb{Z}^*\setminus K} a_k e^{2\pi kix} = T - T^1$.

Then, given $(w_0, v_0, T_0) \in Y$ we decompose $T_0 = T_0^1 + T_0^2$, and $T(t) = T^1(t) + T^2(t)$ and we consider $(w(t), v(t), T^1(t)) \in M$ and then

$$(w(t), v(t), T(t)) - (w(t), v(t), T^1(t)) = (0, 0, T^2(t)).$$

From (27) taking into account that $b_k = 0$ for $k \in \mathbb{Z}^* \setminus K$, we have that $|a_k(t)| \leq |a_{k0}|e^{-\nu\pi^2k^2t}$ and this together with $\nu\pi^2k^2t \geq \nu\pi^2t$ for every $k \in \mathbb{Z}^*$ implies that $\|T^2(t)\|_{\dot{H}^m_{per}} \leq \|T_0^2\|_{\dot{H}^m_{per}} e^{-\nu\pi^2t}$ i.e., $T^2(t) \to 0$ in $\dot{H}^m_{per}$ if $t \to \infty$.

Therefore, we have that $\|T^2(t)\|_{\dot{H}^m_{per}} \to 0$ as $t \to \infty$ with exponential decay rate $e^{-\nu\pi^2t}$. Thus M attracts $(w(t), v(t), T(t))$ with exponential rate $e^{-\nu\pi^2t}$. □

4.2. **The reduced explicit system.** Under the hypotheses and notations of Theorem 2, we suppose moreover that

$$f(x) = \sum_{k\in J} c_k e^{2\pi kix},$$

with $c_k \neq 0$ for every $k \in J \subset \mathbb{Z}$. Note that since all functions involved are real, one has $\bar{a}_k = a_{-k}$, $\bar{b}_k = b_{-k}$ and $\bar{c}_k = c_{-k}$. Then, on the inertial manifold $\oint T(t) \cdot f = \sum_{k\in K} a_k(t)\bar{c_k} = \sum_{k\in K\cap J} a_k(t).c_{-k}$. So, the evolution of the velocity v, and the acceleration w depends only on the coefficients of T which belong to the set $K\cap J$. Note that in (29) the set of equations for a_k with $k \in K\cap J$, together with the equation for v and w, are a subsystem of coupled equations. After solving this, we must solve the equations for $k \notin K\cap J$ which are linear autonomous equations.

We note that $0 \notin K\cap J$ and since $K = -K$ and $J = -J$ then the set $K\cap J$ has an even number of elements, that we denote by $2n_0$.

Corollary 2. *Under the notations and hypotheses of the Theorem 2, we suppose that the set $K\cap J$ is finite and then $|K\cap J| = 2n_0$. Then the asymptotic behavior of the system (3), is described by a system of $N = 2n_0 + 2$ coupled equations in $\mathbb{R}^N$, which determine $(w, v, a_k), k \in K\cap J$, and a family of $|K \setminus (K\cap J)|$ linear non-autonomous equations.*

In particular, if $K\cap J = \emptyset$, and $G(v) = G_0$ then for every $(w_0, v_0, T_0) \in \mathbb{R}^2 \times \dot{L}^2_{per}(0,1)$ we have that the associated solution verifies that $v(t) \to 0$, $w(t) \to 0$ and $T(t) \to \theta_\infty$ in $\dot{L}^2_{per}(0,1)$, i.e., the global attractor is given by $A = \{(0,0,\theta_\infty)\}$, where $\theta_\infty(x)$ is the unique solution in $\dot{H}^2_{per}(0,1)$ of the equation $-\nu\frac{\partial^2\theta_\infty}{\partial x^2} = h(x)$.

Proof. Note that on the inertial manifold $\oint T\cdot f = \sum_{k\in K} a_k(t)\bar{c_k} = \sum_{k\in K\cap J} a_k(t).c_{-k}$. Thus, the dynamics of the system depends only on the coefficients in $K\cap J$. Moreover the equations for a_{-k} are conjugated of the equations for a_k and therefore we have that $\sum_{k\in K\cap J} a_k(t)c_{-k} = 2Re\left(\sum_{k\in(K\cap J)_+} a_k(t)c_{-k}\right)$. From this, and taking real and imaginary parts of $a_k, (a_1^k, a_2^k), k \in (K\cap J)_+$ in (23) where $n_0 = |(K\cap J)_+|$, we conclude.

If $K\cap J = \emptyset$, and $G(v) = G_0$ then on the inertial manifold we get a homogeneous linear equation for the velocity with positive coefficients, and by this $\limsup_{t\to\infty}|v(t)| = 0$, and therefore the equation for w on the inertial manifold is $\frac{dw}{dt} + \frac{1}{\varepsilon}w = -\frac{1}{\varepsilon}G_0 v = \delta(t)$. Next, using $\delta(t) \to 0$ we get $w(t) \to 0$ as $t \to \infty$.

Moreover from the equation for the temperature in (3) we have that the function $\theta = T - \theta_\infty$ satisfies the equation: $\frac{\partial\theta}{\partial t} + v\frac{\partial\theta}{\partial x} = -v\frac{\partial\theta_\infty}{\partial x} + \nu\frac{\partial^2\theta}{\partial x^2}$.

We can multiply by θ in $\dot{L}^2_{per}$ and taking into account that $\oint \frac{\partial\theta}{\partial x}\theta = \frac{1}{2}\oint\frac{\partial(\theta^2)}{\partial x} = 0$ since θ is periodic, we obtain $\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \nu\|\frac{\partial\theta}{\partial x}\|^2 = -v\oint\frac{\partial(\theta_\infty)}{\partial x}\theta$, and using Cauchy-Schwarz and the Young inequality with $\delta = \frac{\nu\pi^2}{2}, C_\delta = \frac{1}{4\delta}$ and then the Poincaré inequality, since $\oint\theta = 0$, we have that $\frac{d}{dt}\|\theta\|^2 + \nu\pi^2\|\theta\|^2 \le |v|^2\frac{1}{2\nu\pi^2}\|\frac{\partial\theta_\infty}{\partial x}\|^2$. Next, from Gronwall's lemma we get $\lim_{t\to\infty}\|\theta\|^2 \le \lim_{t\to\infty}|v|^2\frac{1}{2\nu^2\pi^4}\|\frac{\partial\theta_\infty}{\partial x}\|^2$ and using $v(t) \to 0$ we prove that $\theta(t) \to 0$ in $\dot{L}^2_{per}$. ∎

5. **Conclusion.** Taking real and imaginary parts of coefficients $a_k(t)$ (temperature), b_k (heat flux at the wall of the loop) and c_k (geometry of circuit)

$$a_k(t) = a_1^k(t) + ia_2^k(t), b_k = b_1^k + ib_2^k \text{ and } c_k = c_1^k + ic_2^k,$$

the asymptotic behavior of the system (3) is given by a reduced explicit system in $\mathbb{R}^N$, where $N = 2n_0 + 2$ $(w(t), v(t), a_1^k(t), a_2^k(t), k \in (K \cap J)_+)$ and $n_0 = |(K \cap J)_+|$.

Observe that from the analysis above, it is possible to design the geometry of circuit and/or the external heating, by properly choosing the functions f and/or the heat flux, h, so that the resulting system has an arbitrary number of equations of the form $N = 2n_0 + 2$.

Note that it may be the case that K and J are infinite sets, but their intersection is finite. Also, for a circular circuit we have $f(x) \sim asin(x) + bcos(x)$, i.e.; $J = \{\pm 1\}$ and then $K \cap J$ is either $\{\pm 1\}$ or the empty set. Also, if in the original variables for (3), h is constant we get $K \cap J = \emptyset$ for any choice of f.

This conclusion about the finite-dimensional asymptotic behavior is similar to the previous models like [5, 6, 9] between others. Next, using these results the physical and mathematical implications of the resulting system of ODEs which describes the dynamics at the inertial manifold has been analyzed numerically in [14].

REFERENCES

[1] A. M. Bloch and E. S. Titi, *On the dynamics of rotating elastic beams*, in New Trends in System Theory, Progr. Systems Control Theory 7, Birkhäuser Boston, Boston, MA, 128–135, (1991).

[2] C. Foias, G. R. Sell and R. Temam, *Inertial Manifolds for Nonlinear Evolution Equations*, J. Diff. Equ., **73** (1988), 309–353.

[3] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, AMS, Providence, RI, (1988).

[4] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lectures Notes in Mathematics, **840** (1982), Springer-Verlag, Berlin, New York.

[5] A. Jiménez-Casas and A. M. L. Ovejero, *Numerical analysis of a closed-loop thermosyphon including the Soret effect*, Appl. Math. Comput., **124** (2001), 289–318.

[6] A. Jiménez-Casas and A. Rodríguez-Bernal, *Finite-dimensional asymptotic behavior in a thermosyphon including the Soret effect*, Math. Meth. in the Appl. Sci., **22**, (1999), 117–137.

[7] B. Keller, *Periodic oscillations in a model of thermal convection*, J. Fluid Mech., **26** (1966), 599–606.

[8] F. Morrison, *Understanding rheology*, Oxford University Press, 2001, USA.

[9] A. Rodríguez-Bernal and E. S. Van Vleck, *Diffusion Induced Chaos in a Closed Loop Thermosyphon*, SIAM J. Appl. Math., **58** (1998), 1072–1093(electronic).

[10] A. M. Stuart, "Perturbation Theory of Infinite-Dimensional Dynamical Systems," in Theory and Numerics of Ordinary and Partial differential Equations, M. Ainsworth, J. Levesley, W.A. Light and M. Marletta, eds. (1994), Oxford University Press, Oxford, UK.

[11] J. J. L. Velázquez, *On the dynamics of a closed thermosyphon*, SIAM J. Appl. Math. **54** (1994), 1561–1593.

[12] P. Welander, *On the oscillatory instability of a differentially heated fluid loop*, J. Fluid Mech., **29** (1967), 17–30.

[13] J. Yasappan, A. Jiménez-Casas and Mario Castro, "Asymptotic Behavior of a Viscoelastic Fluid in a Closed Loop Thermosyphon: Physical Derivation, Asymptotic Analysis and Numerical Experiments," Abstr. Appl. Anal., (2013).

[14] J. Yasappan, A. Jiménez-Casas and Mario Castro, *Chaotic behavior of the closed loop thermosyphon model with memory effects*, submitted, (2012).

Received for publication July 2012.

E-mail address: `ajimenez@upcomillas.es`
E-mail address: `marioc@upcomillas.es`
E-mail address: `justemmasj@gmail.com`