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Influence of the spatial heterogeneities in the existence of positive solutions of logistic BVPs with sublinear mixed boundary conditions

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It is my pleasure to contribute with this paper in this special issue on the occasion of the 60th birthday of Professor J. López-Gómez. I am very grateful with him for getting involved me in a research field which passionates me, and I am indebted to him for all the scientific knowledge he has conveyed me. Clearly, Science is a great part of him and He is pivotal for Science. With all my affection and gratitude.

Abstract. *In this paper we analyze the influence of the spatial heterogeneities in the existence of positive solutions of Logistic problems with heterogeneous sublinear boundary conditions. We will show that the relative positions of the vanishing sets of the potentials in front of the nonlinearities, in the PDE and on the boundary conditions, play a crucial role as for the amplitude of the range of values of the bifurcation parameter for which the problems possess positive solutions. We will compare the cases of the logistic problem with linear and nonlinear boundary conditions. Also, we will show the global bifurcation diagram of positive solutions of the logistic problem with heterogeneous nonlinear boundary conditions, considering the amplitude of the nonlinearity in the boundary conditions as bifurcation-continuation parameter.*

Keywords: principal eigenvalues, positive solutions, nonlinear mixed boundary conditions, spatial heterogeneities, logistic problems.
MS Classification 2010: 35J65, 35J25, 35B09, 35B35, 35B40.

1. Introduction and Main Result

In this paper we consider the logistic elliptic boundary value problem with sublinear mixed boundary conditions and spatial heterogeneities given by

$$\begin{cases} -\Delta u = \lambda u - a(x)u^p & \text{in } \Omega\,, \quad p>1\,, \\ u=0 & \text{on } \Gamma_0\,, \\ \partial_\nu u = -b(x)u^q & \text{on } \Gamma_1\,, \quad q>1\,, \end{cases} \tag{1}$$

where:

i) Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, with boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are two disjoint components of the boundary,

ii) $-\Delta$ stands for the minus Laplacian operator in $\mathbb{R}^N$ and $\lambda \in \mathbb{R}$ is the bifurcation parameter,

iii) The potential $a \in \mathcal{C}(\bar{\Omega})$ with $a > 0$ measures the spatial heterogeneities in Ω and satisfies that

$$\Omega_0 := int\,\{x \in \Omega : a(x) = 0\} \neq 0\,, \qquad \Omega_0 \in \mathcal{C}^2$$

and

$$a \text{ is bounded away from zero in any compact subset of } \Omega \setminus \bar{\Omega}_0\,. \tag{2}$$

In some case, when it is pointed out, we will assume that

$$a \text{ is bounded away from zero in any compact subset of } (\Omega\setminus\bar{\Omega}_0)\cup\Gamma_1 \tag{3}$$

instead of (2)

iv) The potential $b \in \mathcal{C}(\Gamma_1)$ with $b > 0$ measures the spatial heterogeneities on Γ_1.

v) $\partial_\nu u(x)$ stands for the outward normal derivative of u at each $x \in \Gamma_1$.

By a positive solution of (1) for the value λ of the parameter we mean a strong positive solution, that is, any positive function $u \in W^2_r(\Omega)$ for some $r > N$ which satisfies (1) a.e. in Ω for such a value λ of the parameter.

This kind of elliptic problems has been widely analyzed under linear boundary conditions in some previous works (cf. [4, 6, 13, 14, 15, 18]) and under nonlinear boundary conditions (cf. [7, 9, 11, 16, 21, 22]).

The main goal of this work is to analyze the existence of positive solutions of (1) and to ascertain the global bifurcation diagram of positive solutions of it, depending on the nodal behavior of the spatial heterogeneities a and b, in the domain and on the boundary conditions, respectively. Namely, as for the nodal behavior of the potential a we will distinguish the cases

$$\Gamma_1 \subset \partial\Omega_0\,, \qquad dist(\partial\Omega_0 \cap \Omega, \Gamma_1) > 0 \tag{4}$$

and

$$\bar{\Omega}_0 \subset \Omega \cup \Gamma_0\,, \tag{5}$$

and as for the profile of the potential b we will distinguish the case when

$$b(x) \geq \underline{b} > 0 \qquad \text{for all } x \in \Gamma_1 \tag{6}$$

and the case when

$$b(x) = 0 \quad \forall\, x \in \Gamma_1^0 \qquad \text{and} \qquad b(x) > 0 \quad \forall\, x \in \Gamma_1^+\,, \tag{7}$$

being Γ_1^0 and Γ_1^+ two disjoint connected pieces of Γ_1, closed and open, respectively as $N-1$ dimensional manifolds, such that $\Gamma_1 = \Gamma_1^0 \cup \Gamma_1^+$. Hereafter, assuming that Γ_1^0 and Γ_1^+ satisfy the previous assumptions, we will denote

$$\mathcal{C}^+(\Gamma_1^+) := \left\{ V \in \mathcal{C}(\Gamma_1) \,:\, V(x) = 0 \;\forall\, x \in \Gamma_1^0 \text{ and } V(x) > 0 \quad \forall\, x \in \Gamma_1^+ \right\}. \tag{8}$$

In Figures 1 and 2 we show two possible configurations of the subdomain Ω_0 with respect to Γ_1, satisfying (4) in Figure 1 and satisfying (5) in Figure 2. In

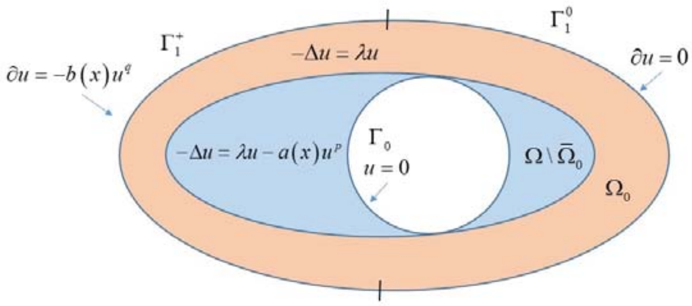

Figure 1: $\Gamma_1 \subset \partial\Omega_0, \quad b \in \mathcal{C}^+(\Gamma_1^+)$.

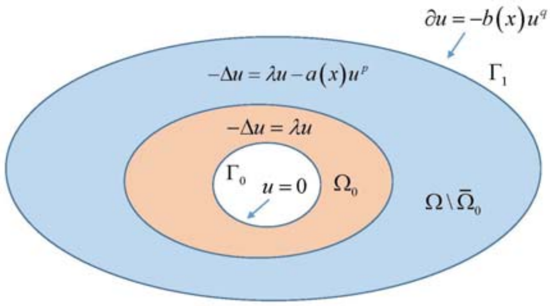

Figure 2: $\bar{\Omega}_0 \subset \Omega \cup \Gamma_0, \quad b \in \mathcal{C}(\Gamma_1)$.

[7, 9] it was analyzed, among other results, the existence of positive solutions of (1), in the particular case when $\bar{\Omega}_0 \subset \Omega$ and the potential a is bounded away from zero in any compact subset of $(\Omega \setminus \bar{\Omega}_0) \cup \Gamma_1$. In [10] it was analyzed

the particular special case when $\Omega = \Omega_0$ and (6) holds. The results obtained in this work extend the previous ones obtained in [7, 9], to cover the case when either (4) or (5) hold, and either (6) or (7) hold. Moreover, it should be noted that since we assume that (2) holds instead of (3) (as it was assumed in [7, 9]), now when $\operatorname{dist}(\bar{\Omega}_0, \Gamma_1) > 0$ we let that a vanishes on Γ_1 or in some subregion of Γ_1 and therefore, in this case a is not bounded away from zero in a neighborhood of Γ_1. The extensions carried out in this work are not straight with respect to the previous results, mainly when (4) and (7) hold, because to obtain them it is necessary to apply a great variety of very sharp results about principal eigenvalues. To obtain the new results under conditions (4) and (7) it is necessary to work with a family of singular boundary eigenvalue problems which possess Dirichlet and Neumann boundary conditions on the component Γ_1 of $\partial\Omega$ in a non-separated way. In this way, the results about principal eigenvalues recently obtained in [5] play a crucial role to develop our analysis.

Hereafter we denote by $\sigma_0^*[b, \Omega_0]$ the principal eigenvalue defined

$$\sigma_0^*[b, \Omega_0] := \begin{cases} \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}] & \text{if (5) and (6) hold,} \\ \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}] & \text{if (5) and (3) hold,} \\ \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}] & \text{if (4) and (6) hold,} \\ \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*(\Gamma_1^0)] & \text{if (4) and (7) hold,} \end{cases} \tag{9}$$

where $\mathcal{D}$ stands for the Dirichlet boundary operator, $\mathfrak{B}^*(\Gamma_1^0)$ denotes the boundary operator defined

$$\mathfrak{B}^*\left(\Gamma_1^0\right)\varphi = \begin{cases} \varphi & \text{on } \Gamma_0\,, \\ \partial_\nu \varphi & \text{on } \Gamma_1^0\,, \\ \varphi & \text{on } \Gamma_1^+\,, \end{cases} \qquad \Gamma_1^+ = \Gamma_1 \setminus \Gamma_1^0 \tag{10}$$

(cf. (7)) and where $\sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]$ and $\sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*(\Gamma_1^0)]$ stand for the principal eigenvalues of the problems $(-\Delta, \Omega_0, \mathcal{D})$ and $\left(-\Delta, \Omega_0, \mathfrak{B}^*(\Gamma_1^0)\right)$, respectively. It must be pointed out that, as show (9) and (41), when (4) and (7) hold, then

$$\sigma_0^*[b, \Omega_0] = \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*(\Gamma_1^0)] < \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]\,.$$

In (9) we can observe the dependence of $\sigma_0^*[b, \Omega_0]$ with respect to the potential b, since $\Gamma_1^0 = b^{-1}(0)$, and with respect to the relative position of the vanishing set Ω_0 of the potential a with respect to Γ_1. When (4) and (7) hold, the dependence of $\sigma_0^*(b, \Omega_0)$ with respect to b, is not with respect to the size of b but with respect to the amplitude of the piece Γ_1^0 where b vanishes. That is, $\sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*\left(\Gamma_1^0\right)]$ is decreasing with respect to the amplitude of Γ_1^0 and however, if $b_i \in \mathcal{C}^+(\Gamma_1^+)$, $i = 1, 2$, then we have that $\sigma_0^*[b_1, \Omega_0] = \sigma_0^*[b_2, \Omega_0] = \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*\left(\Gamma_1^0\right)]$, independently of the size of them.

Let consider the logistic boundary value problem with linear mixed boundary conditions given by

$$\begin{cases} -\Delta u = \lambda u - a(x)u^p & \text{in } \Omega\,, \\ u = 0 & \text{on } \Gamma_0\,, \\ \partial_\nu u = 0 & \text{on } \Gamma_1\,. \end{cases} \tag{11}$$

In the sequel we denote by $\Lambda_{NL}(\Omega_0, b)$ and $\Lambda_L(\Omega_0)$ the ranges of values of λ for which (1) and (11) possess positive solutions, respectively. Also we denote

$$\sigma_1 := \sigma_1^\Omega[-\Delta, \mathfrak{B}(0)]\,, \qquad \sigma_0^* := \sigma_0^*[b, \Omega_0]\,,$$

and we will say that a positive function u is strongly positive in Ω, and we will denote it by $u \gg 0$, if

$$u(x) > 0 \quad \forall\, x \in \Omega \cup \Gamma_1 \quad \text{and} \quad \partial_\nu u(x) < 0 \quad \forall\, x \in \Gamma_0 \quad \text{with} \quad u(x) = 0\,.$$

In the sequel we denote

$$W^2(\Omega) := \bigcap_{p>1} W_p^2(\Omega)\,, \qquad W^2_{\mathfrak{B}(V)} := \left\{ u \in W^2(\Omega) \,:\, \mathfrak{B}(V)u = 0 \right\}\,,$$

$$\mathcal{C}^\infty_{\Gamma_0 \cup \Gamma_1^+}(\Omega) := \left\{ \phi \in \mathcal{C}^\infty(\Omega) \,:\, supp\, \phi \subset \bar{\Omega} \setminus (\Gamma_0 \cup \Gamma_1^+) \right\}$$

and by $H^1_{\Gamma_0 \cup \Gamma_1^+}(\Omega)$ the clousure in $H^1(\Omega)$ of the set of functions $\mathcal{C}^\infty_{\Gamma_0 \cup \Gamma_1^+}(\Omega)$.

The following is the main result of this work. It gives the structure of the global bifurcation diagram of positive solutions of (1) and it compares $\Lambda_L(\Omega_0)$ with $\Lambda_{NL}(\Omega_0, b)$ depending on the nodal behavior and profiles of the potentials a and b.

THEOREM 1.1. *Under any pair of assumptions of* (9)*, the following assertions are true:*

i) (1) *possesses a positive solution if, and only if*

$$\sigma_1 < \lambda < \sigma_0^*\,. \tag{12}$$

Moreover, the positive solution if it exists, it is unique and strongly positive in Ω. We will denote it by u_λ. Moreover,

$$u_\lambda \in W^2(\Omega) \subset \mathcal{C}^{1+\alpha}(\bar{\Omega})\,, \quad \forall \alpha \in (0,1)\,. \tag{13}$$

ii) The following hold:

a) If (5) *and* (6) *hold, then*

$$\Lambda_L(\Omega_0) = \Lambda_{NL}(\Omega_0, b) = \Big(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]\Big)\,. \tag{14}$$

b) If (4) *and* (6) *hold, then*

$$\Lambda_L(\Omega_0) = \Big(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(0,\Omega_0)]\Big)\,, \tag{15}$$

$$\Lambda_{NL}(\Omega_0, b) = \Big(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]\Big) \tag{16}$$

and therefore,

$$\Lambda_L(\Omega_0) \subset \Lambda_{NL}(\Omega_0, b)\,. \tag{17}$$

c) If (4) *and* (7) *hold, then*

$$\Lambda_L(\Omega_0) = \Big(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(0,\Omega_0)]\Big)\,, \tag{18}$$

$$\Lambda_{NL}(\Omega_0, b) = \Big(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*\left(\Gamma_1^0\right)]\Big) \tag{19}$$

and therefore,

$$\Lambda_L(\Omega_0) \subset \Lambda_{NL}(\Omega_0, b)\,. \tag{20}$$

iii) Each positive solution u_λ of (1) *is linearly asymptotically stable, i.e., the principal eigenvalue of the linearization of* (1) *around (λ, u_λ) is positive. Moreover, the function*

$$\dot{u}_\lambda := \frac{d\,u_\lambda}{d\lambda} \gg 0 \qquad \text{in } \Omega \tag{21}$$

and in particular, for each $x \in \Omega \cup \Gamma_1$ the map $(\sigma_1, \sigma_0^*) \to (0,\infty)$ defined

$$\lambda \to u_\lambda(x)$$

is strictly increasing.

iv) There exists uniform $L^\infty(\Omega)$ bounds for the positive solutions of (1) *in any compact interval I of values of λ with $I \subset [\sigma_1, \sigma_0^*)$.*

v) The positive solutions of (1) *belong to a differentiable continuum $\mathfrak{C}^+(\sigma_1)$ of positive solutions. It emanates supercritically from the trivial branch $(\lambda, u) = (\lambda, 0)$ at the unique bifurcation value to positive solutions of* (1) *$\lambda = \sigma_1$, bifurcates from infinity at the unique bifurcation value to positive solutions from infinity $\lambda = \sigma_0^*$ and it is increasing in $\|\cdot\|_{L^\infty(\Omega)}$ with respect to the λ-parameter. In particular,*

$$\mathcal{P}_\lambda\left(\mathfrak{C}^+(\sigma_1)\right) = [\sigma_1, \sigma_0^*)\,, \tag{22}$$

and

$$\lim_{\lambda \downarrow \sigma_1} \|u_\lambda\|_{L_\infty(\Omega)} = 0, \qquad \lim_{\lambda \uparrow \sigma_0^*} \|u_\lambda\|_{L_\infty(\Omega)} = \infty\,,$$

where $\mathcal{P}_\lambda\left(\mathfrak{C}^+(\sigma_1)\right)$ denotes the λ-projection of the continuum $\mathfrak{C}^+(\sigma_1)$ over the λ-axis.

vi) Let b_1, $b_2 \in \mathcal{C}(\Gamma_1)$ *be such that*

$$b_2 > b_1 > 0\,, \tag{23}$$

let λ *be satisfying*

$$\lambda \in \Lambda_{NL}(\Omega_0, b_1) \cap \Lambda_{NL}(\Omega_0, b_2) \tag{24}$$

and let u_i, $i = 1, 2$ *denote the unique positive solution of* (1) *for* $b = b_i$, $i = 1, 2$. *Then,*

$$u_1 - u_2 \gg 0 \qquad \textit{in } \Omega\,, \tag{25}$$

that is,

$$u_1(x) > u_2(x) \quad \forall\, x \in \Omega \cup \Gamma_1 \qquad \textit{and} \qquad \partial_\nu u_1(x) < \partial_\nu u_2(x) \quad \forall\, x \in \Gamma_0.$$

vii) Let $b \in \mathcal{C}(\Gamma_1)$ *be such that* $b > 0$, *let* $\lambda \in \Lambda_L(\Omega_0)$ *be and let* $\tilde{u}$, u_0 *be the unique positive solution of* (1) *and* (11), *respectively, for such a value* λ *of the parameter. Then,*

$$u_0 > \tilde{u} \qquad \textit{in } \Omega\,. \tag{26}$$

viii) Assume that (4) *holds and let* b_1, $b_2 \in \mathcal{C}^+(\Gamma_1^+)$ *be satisfying* (23). *Let* $\lambda \in \Big(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*(\Gamma_1^0)]\Big)$ *be and let* u_i, $i = 1, 2$ *denote the unique positive solution of* (1) *for* $b = b_i$, $i = 1, 2$. *Then,* (25) *holds.*

ix) Assume that (4) *or* (5) *holds and let* b_1, $b_2 \in \mathcal{C}(\Gamma_1)$ *be bounded away from zero satisfying* (23). *Let* $\lambda \in \Big(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]\Big)$ *be and let* u_i, $i = 1, 2$ *denote the unique positive solution of* (1) *for* $b = b_i$, $i = 1, 2$. *Then,* (25) *holds.*

Taking into account the results of Theorem 1.1, Figure 3 shows the global bifurcation diagrams of positive solutions of (1) (red dashed curve) and (11) (blue curve), constituted by the global continuum $\mathcal{C}^+(\sigma_1)$ of positive solutions emanating from $\lambda = \sigma_1$, where $\Lambda_L(\Omega_0) = (\sigma_1, \sigma_0)$ and $\Lambda_{NL}(\Omega_0, b) = (\sigma_1, \sigma_0^*)$, with $\sigma_0 \leq \sigma_0^*$. Also, Figure 4 shows the global bifurcation diagrams of positive solutions of (1) for b_1, $b_2 \in \mathcal{C}^+(\Gamma_1^+)$ satisfying (23). The blue curve stands for the continuum $\mathcal{C}^+(\sigma_1)$ of (1) for $b = b_1$ and the the red dashed curve the global continuum $\mathcal{C}^+(\sigma_1)$ of (1) for $b = b_2$.

Following similar arguments to the given in the previous works [6, 7, 9, 12, 14], the results obtained in this paper may be generalized to ascertain the global bifurcation diagram of positive solutions of the following nonlinear elliptic weighted boundary value problem

$$\begin{cases} -\Delta u = \lambda W(x) u - a(x) f(x, u) u & \text{in } \Omega\,, \\ u = 0 & \text{on } \Gamma_0\,, \\ \partial_\nu u + V(x) u = -b(x) g(x, u) u & \text{on } \Gamma_1\,, \end{cases}$$

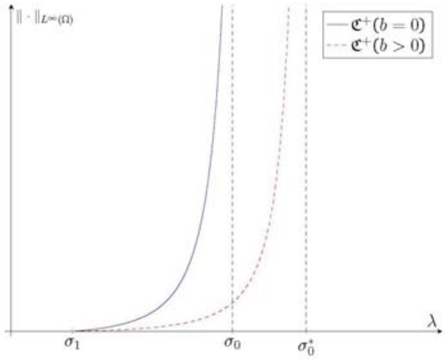

Figure 3: $\mathcal{C}^+(\sigma_1)$ for $b=0$ and $b>0$.

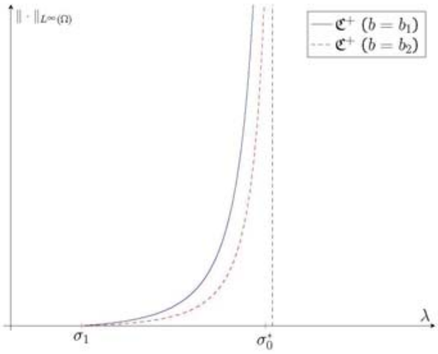

Figure 4: $\mathcal{C}^+(\sigma_1)$ for $b_1,\, b_2 \in \mathcal{C}^+(\Gamma_1^+),\ b_2 > b_1$.

where:

- $W \in L^\infty(\Omega)$ and $V \in \mathcal{C}(\Gamma_1)$ possess arbitrary sign in each point of Ω and Γ_1, respectively.
- The function $f \in \mathcal{C}^1(\bar{\Omega} \times [0,\infty); \mathbb{R})$ satisfies the following assumptions:
$$f(x,0) = 0\,, \qquad \frac{\partial f}{\partial u}(x,u) > 0 \quad (x,u) \in \Omega \times (0,\infty)$$
and
$$\lim_{u\uparrow\infty} f(x,u) = +\infty \text{ uniformly in } \bar{\Omega}\,.$$
- The function $g \in \mathcal{C}^1(\Gamma_1 \times [0,\infty); \mathbb{R})$ satisfies the following assumptions:
$$g(x,0) = 0\,, \qquad \frac{\partial g}{\partial u}(x,u) > 0 \quad (x,u) \in \Gamma_1 \times (0,\infty)$$

and

$$\lim_{u\uparrow\infty} g(x,u) = +\infty \text{ uniformly on } \Gamma_1 .$$

- The piece of the boundary Γ_0 possesses finitely many components satisfying

$$\Gamma_0 = \bigcup_{k=1}^{l} \Gamma_0^k \bigcup_{k=l+1}^{m} \Gamma_0^k ,$$

with

$$\Gamma_0^k \cap \partial\Omega_0 = \emptyset \quad k = 1,\dots,l , \qquad \Gamma_0^k \cap \partial\Omega_0 \neq \emptyset \quad k = l+1,\dots,m .$$

- The piece of the boundary Γ_1 possesses finitely many components, some of them where the potential b in the nonlinear boundary condition is bounded away from zero, and the rest where b vanishes in some subregions of them.

Also, following the arguments and taking into account the results given in [13], the results of this paper may be generalized to cover the very general case when the vanishing set Ω_0 of the potential a is not a nice subdomain of Ω with $\Omega_0 \in \mathcal{C}^2$, but a very general set with no special restriction on its structure.

The main technical tools used to develop our analysis are bifurcation and monotonicity techniques.

The distribution of the rest of this paper is the following. Section 2 contains, without proofs, all the previous results about principal eigenvalues coming from [5, 12, 17, 20] that we will need to prove the main result. Section 3 contains the proof of Theorem 1.1. Finally, Section 4 includes without proof, the main result coming from [11] about the global structure of the diagram of positive solutions of (1) for a fixed λ in a suitable interval, considering the amplitude of the potential b on the boundary conditions as bifurcation-continuation parameter.

2. Preliminaries results about principal eigenvalues

In this section we collect the main results about principal eigenvalues coming from [5, 12, 17, 20] that are going to be used throughout the rest of this paper.

Hereafter, for each $k \in L^\infty(\Omega)$, $\mathcal{L}_k$ stands for the linear second order differential operator

$$\mathcal{L}_k : -\Delta + k(x) ,$$

$\mathcal{D}$ stands for the Dirichlet boundary operator and for each $V \in \mathcal{C}(\Gamma_1)$, $\mathfrak{B}(V)$ denotes the boundary operator defined

$$\mathfrak{B}(V)\varphi = \begin{cases} \varphi & \text{on } \Gamma_0 , \\ \partial_\nu \varphi + V\varphi & \text{on } \Gamma_1 , \end{cases}$$

where $\partial_\nu \varphi$ stands for the outward normal derivative on Γ_1. It is known that for each $r > 1$

$$\mathfrak{B}(V) \in \mathcal{L}\left(W_r^2(\Omega), W_r^{2-\frac{1}{r}}(\Gamma_0) \times W_r^{1-\frac{1}{r}}(\Gamma_1)\right)$$

(cf. [2]). Also, given any proper subdomain Ω_0 of Ω of class $\mathcal{C}^2$ with

$$dist(\Gamma_1, \partial\Omega_0 \cap \Omega) > 0\,, \tag{27}$$

we denote $\mathfrak{B}(V, \Omega_0)$ the boundary operator built from $\mathfrak{B}(V)$ through by

$$\mathfrak{B}(V,\Omega_0)\varphi := \begin{cases} \varphi & \text{on } \partial\Omega_0 \cap \Omega\,, \\ \mathfrak{B}(V)\varphi & \text{on } \partial\Omega_0 \cap \partial\Omega\,. \end{cases}$$

It should be pointed out that when $\bar{\Omega}_0 \subset \Omega \cup \Gamma_0$, then $\mathfrak{B}(V, \Omega_0) = \mathcal{D}$.

By *principal eigenvalue* of an eigenvalue problem we mean any eigenvalue of it which possesses a one-signed eigenfunction, and in particular a positive eigenfunction.

It follows from [2, Theorem 12.1] that the eigenvalue problem

$$\begin{cases} \mathcal{L}_k\varphi = \sigma\varphi & \text{in } \Omega\,, \\ \mathfrak{B}(V)\varphi = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{28}$$

possesses a unique principal eigenvalue, denoted in the sequel by $\sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V)]$, which is simple and the least eigenvalue of (28). Moreover, the positive eigenfunction φ^* associated to it, unique up multiplicative constant, is strongly positive in Ω, that is,

$$\varphi^*(x) > 0 \quad \forall\, x \in \Omega \cup \Gamma_1 \quad \text{and} \quad \partial_\nu\varphi^*(x) < 0 \quad \forall\, x \in \Gamma_0\,, \tag{29}$$

and in addition

$$\varphi^* \in W^2_{\mathfrak{B}(V)}(\Omega) \subset \mathcal{C}^{1+\alpha}(\bar{\Omega}) \quad \text{for all } \alpha \in (0,1)\,. \tag{30}$$

The following result collects all the monotonicity properties of $\sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V)]$ coming from [17, Proposition 3.2], [12, Propositions 3.1, 3.2, 3.3 and 3.5] and [20, Chapter 8] that we will use to develop our analysis.

Proposition 2.1. *The following monotonicity properties hold:*

i) Let k_1, $k_2 \in L^\infty(\Omega)$ and $V \in \mathcal{C}(\Gamma_1)$ be such that $k_1 < k_2$. Then

$$\sigma_1^\Omega[\mathcal{L}_{k_1}, \mathfrak{B}(V)] < \sigma_1^\Omega[\mathcal{L}_{k_2}, \mathfrak{B}(V)]\,. \tag{31}$$

ii) Let $V_1, V_2 \in \mathcal{C}(\Gamma_1)$ *and* $k \in L^\infty(\Omega)$ *be such that* $V_1 < V_2$. *Then*

$$\sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V_1)] < \sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V_2)]\,. \tag{32}$$

iii) For any $V \in \mathcal{C}(\Gamma_1)$ *and* $k \in L^\infty(\Omega)$ *the following holds*

$$\sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V)] < \sigma_1^\Omega[\mathcal{L}_k, \mathcal{D}]\,. \tag{33}$$

iv) Let Ω_0 *be a proper subdomain of* Ω *of class* $\mathcal{C}^2$ *satisfying* (27). *Then, for any* $k \in L^\infty(\Omega)$

$$\sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V)] < \sigma_1^{\Omega_0}[\mathcal{L}_k, \mathfrak{B}(V, \Omega_0)]\,. \tag{34}$$

Let Ω_0 be a subdomain of Ω of class $\mathcal{C}^2$ with boundary $\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1$ such that $\Gamma_0^0 \cap \Gamma_1 = \emptyset$, where $\Gamma_0^0 = \partial\Omega_0 \cap \Omega$, and Ω_n, $n \geq 1$, a sequence of bounded domains of $\mathbb{R}^N$ with boundary $\partial\Omega_n = \Gamma_0^n \cup \Gamma_1$ of class $\mathcal{C}^2$ such that $\Gamma_0^n \cap \Gamma_1 = \emptyset$, $n \geq 1$, where $\Gamma_0^n = \partial\Omega_n \cap \Omega$. It is said that Ω_n converges to Ω_0 from the exterior if for each $n \geq 1$

$$\Omega_0 \subset \Omega_{n+1} \subset \Omega_n\,, \qquad \bigcap_{n\geq 1} \bar{\Omega}_n = \bar{\Omega}_0\,. \tag{35}$$

The following result collects all the asymptotic behaviors of $\sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V)]$, coming from [17, Theorems 4.2 and 5.1], [12, Theorems 7.1, 8.2, 9.1 and 10.1] and [20, Chapter 8], that we will need later.

Proposition 2.2. *Let* $k \in L^\infty(\Omega)$ *be. Then the following hold:*

i) Let $B_1 := \{x \in \mathbb{R}^N : |x| < 1\}$ *be, where* $|\cdot|$ *stands for the Lebesgue measure of* $\mathbb{R}^N$, *then*

$$\liminf_{|\Omega|\downarrow 0} \sigma_1^\Omega[\mathcal{L}_k, \mathcal{D}] \geq |B_1|^{\frac{2}{N}} \sigma_1^{B_1}[-\Delta, \mathcal{D}]|\Omega|^{-\frac{2}{N}}\,. \tag{36}$$

ii) For any sequence $V_n \in \mathcal{C}(\Gamma_1)$, $n \geq 1$ *satisfying*

$$\lim_{n\uparrow\infty} \min_{x\in\Gamma_1} V_n(x) = \infty$$

yields

$$\lim_{n\to\infty} \sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V_n)] = \sigma_1^\Omega[\mathcal{L}_k, \mathcal{D}]\,. \tag{37}$$

iii) Let Ω_0 *be a subdomain of* Ω *with boundary* $\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1$ *such that* $\Gamma_0^0 \cap \Gamma_1 = \emptyset$ *where* $\Gamma_0^0 = \partial\Omega_0 \cap \Omega$, *and let* Ω_n, $n \geq 1$ *be any sequence of* *bounded domains of* $\mathbb{R}^N$ *of class* $\mathcal{C}^2$ *converging to* Ω_0 *from the exterior in the sense of* (35). *Then*

$$\lim_{n\to\infty} \sigma_1^{\Omega_n}[\mathcal{L}_k, \mathfrak{B}_n(V)] = \sigma_1^{\Omega_0}[\mathcal{L}_k, \mathfrak{B}(V)]\,, \tag{38}$$

where $\mathfrak{B}_n(V)$ *denotes the boundary operator defined*

$$\mathfrak{B}_n(V)u := \begin{cases} u & \text{on } \Gamma_0^n\,, \\ \partial_\nu u + Vu & \text{on } \Gamma_1\,, \end{cases} \qquad \Gamma_0^n := \partial\Omega_n \cap \Omega\,.$$

iv) Let $V_n \in \mathcal{C}(\Gamma_1)$, $n \geq 1$, *be an arbitrary sequence satisfying*

$$\lim_{n\to\infty} \|V_n - V\|_{L^\infty(\Gamma_1)} = 0\,,$$

with $V \in \mathcal{C}(\Gamma_1)$. *Then,*

$$\lim_{n\to\infty} \sigma_1^{\Omega}[\mathcal{L}_k, \mathfrak{B}(V_n)] = \sigma_1^{\Omega}[\mathcal{L}_k, \mathfrak{B}(V)]\,. \tag{39}$$

Now, let consider the boundary operator $\mathfrak{B}^*(\Gamma_1^0)$ defined in (10), where Γ_1^0 and Γ_1^+ are two disjoint connected pieces of Γ_1, closed and open, respectively as $N-1$ dimensional manifolds, such that $\Gamma_1 = \Gamma_1^0 \cup \Gamma_1^+$

The following result collects all the properties about the principal eigenvalue of the problem $\left(\mathcal{L}_k, \Omega, \mathfrak{B}^*\left(\Gamma_1^0\right)\right)$, coming from [5, Th. 1.1, Prop. 3.2, Cor. 3.4 and 3.5], that we will use in the sequel.

Proposition 2.3. *Let* $k \in L^\infty(\Omega)$ *be and let consider the eigenvalue problem*

$$\begin{cases} \mathcal{L}_k\varphi = \sigma\varphi & \text{in } \Omega\,, \\ \mathfrak{B}^*(\Gamma_1^0)\varphi = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{40}$$

where $\partial\Omega = \Gamma_0 \cup \Gamma_1$ *and* $\Gamma_1 = \Gamma_1^0 \cup \Gamma_1^+$, *being* Γ_1^0 *and* Γ_1^+ *two disjoint connected pieces of* Γ_1, *closed and open, respectively as* $N-1$ *dimensional manifolds. Then,* (40) *possesses a unique principal eigenvalue, denoted in the sequel by* $\sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}^*(\Gamma_1^0)]$, *which is simple and the smallest eigenvalue of all other eigenvalues of* (40). *Moreover, any eigenfunction of* (40) *associated to the principal eigenvalue is one-signed in* Ω *and if we denote by* $\varphi_1 \in H^1_{\Gamma_0\cup\Gamma_1^+}(\Omega)$ *the positive eigenfunction associated to it, unique up multiplicative constant, yields*

$$\varphi_1(x) > 0 \quad \text{a.e. in} \quad \Omega\,.$$

In addition:

i) If $\psi^{\mathcal{D}}$ denotes the principal eigenfunction associated to $\sigma_1^\Omega[\mathcal{L}_k,\mathcal{D}]$ normalized so that $\|\psi^{\mathcal{D}}\|_{L^\infty(\Omega)}=1$, then

$$\sigma_1^\Omega[\mathcal{L}_k,\mathfrak{B}^*(\Gamma_1^0)] = \sigma_1^\Omega[\mathcal{L}_k,\mathcal{D}] + \frac{\int_{\Gamma_1^0}\partial_\nu\psi^{\mathcal{D}}\varphi_1}{\int_\Omega \psi^{\mathcal{D}}\varphi_1} < \sigma_1^\Omega[\mathcal{L}_k,\mathcal{D}]\,. \tag{41}$$

ii) For each $V\in\mathcal{C}^+(\Gamma_1^+)$ the following hold

$$\sigma_1^\Omega[\mathcal{L}_k,\mathfrak{B}(0)] < \sigma_1^\Omega[\mathcal{L}_k,\mathfrak{B}(V)] < \sigma_1^\Omega[\mathcal{L}_k,\mathfrak{B}^*(\Gamma_1^0)] < \sigma_1^\Omega[\mathcal{L}_k,\mathcal{D}]\,. \tag{42}$$

iii) The following characterization holds

$$\sigma_1^\Omega\left[\mathcal{L}_k,\mathfrak{B}^*(\Gamma_1^0)\right] = \sup_{V\in\mathcal{C}^+(\Gamma_1^+)} \sigma_1^\Omega[\mathcal{L}_k,\mathfrak{B}(V)]\,. \tag{43}$$

3. Proof of Theorem 1.1

In this section we prove some previous results that we need to prove Theorem 1.1 and finally we prove it.

Proposition 3.1. *Let u_λ be a positive solution of (1) for the value λ of the parameter. Then,*

$$\lambda = \sigma_1^\Omega\left[-\Delta + a(x)u_\lambda^{p-1},\mathfrak{B}\left(b(x)u_\lambda^{q-1}\right)\right]\,, \tag{44}$$

$$\lambda > \sigma_1^\Omega[-\Delta,\mathfrak{B}(0)] \tag{45}$$

and

$$u_\lambda \gg 0 \text{ in } \Omega\,,\quad u_\lambda\in W^2(\Omega)\subset\mathcal{C}^{1+\alpha}(\bar\Omega)\quad \forall\,\alpha\in(0,1)\,. \tag{46}$$

Moreover:

i) If either (5) is satisfied or (4) and (6) are satisfied, then

$$\lambda < \sigma_1^{\Omega_0}[-\Delta,\mathcal{D}]\,. \tag{47}$$

ii) If (4) and (7) are satisfied, then

$$\lambda < \sigma_1^{\Omega_0}[-\Delta,\mathfrak{B}^*(\Gamma_1^0)]\,. \tag{48}$$

Proof. Let u_λ be a positive solution of (1) for the value λ of the parameter. Then, $u_\lambda\in W_r^2(\Omega)$ for some $r>N$ and since $a\in\mathcal{C}(\bar\Omega)$ and $b\in\mathcal{C}(\Gamma_1)$, the following hold

$$\begin{cases}\left(-\Delta + a(x)u_\lambda^{p-1}\right)u_\lambda = \lambda u_\lambda & \text{in } \Omega\,,\\ \mathfrak{B}\left(b(x)u_\lambda^{q-1}\right)u_\lambda = 0 & \text{on } \partial\Omega\,,\end{cases} \tag{49}$$

where

$$a(x)u_\lambda^{p-1} \in \mathcal{C}(\bar{\Omega}), \qquad b(x)u_\lambda^{q-1} \in \mathcal{C}(\Gamma_1).$$

Then, (49) fits into the framework of (28) and u_λ is a positive eigenfunction of (49) associated to the eigenvalue λ. Thus, (44) and (46) follow owing to the existence and uniqueness of the principal eigenvalue of (49), joint with the strongly positivity and regularity of its principal eigenfunction (cf.(28), (29) and (30)).

Now, since $a > 0$ and $b > 0$, owing to (31) and (32) it follow from (44) that

$$\lambda = \sigma_1^\Omega \left[-\Delta + a(x)u_\lambda^{p-1}, \mathfrak{B}\left(b(x)u_\lambda^{q-1}\right)\right] > \sigma_1^\Omega[-\Delta, \mathfrak{B}(0)],$$

which proves (45). Also, owing to the monotonicity of the principal eigenvalue with respect to the domain (cf.(34)) it follows from (44) that

$$\lambda = \sigma_1^\Omega \left[-\Delta + a(x)u_\lambda^{p-1}, \mathfrak{B}\left(b(x)u_\lambda^{q-1}\right)\right] < \sigma_1^{\Omega_0} \left[-\Delta, \mathfrak{B}\left(b(x)u_\lambda^{q-1}, \Omega_0\right)\right]. \tag{50}$$

We now prove (47). Indeed, let assume that (5) holds. Then,

$$\mathfrak{B}\left(b(x)u_\lambda^{q-1}, \Omega_0\right) = \mathcal{D} \tag{51}$$

and hence, (50) and (51) imply (47) under condition (5). In the same way, let assume now that (4) and (6) are satisfied. Then, since $b(x)u_\lambda^{q-1} \in \mathcal{C}(\Gamma_1)$, (47) follows from (50) owing to (33).

Finally we now prove (48). Indeed, let assume that (4) and (7) are satisfied. Then, since $u_\lambda(x) > 0$ for all $x \in \Gamma_1$ and $b \in \mathcal{C}^+(\Gamma_1^+)$ (cf. (8)), we have that $b(x)u_\lambda^{q-1} \in \mathcal{C}^+(\Gamma_1^+)$ and hence, (48) follows from (50) owing to (42).

This completes the proof. □

Proposition 3.2. *For each*

$$\lambda > \sigma_1^\Omega[-\Delta, \mathfrak{B}(0)] \tag{52}$$

(1) *possesses a positive strict subsolution arbitrarily small and strongly positive in* Ω.

Proof. Let λ be satisfying (52). Owing to (32) and (39) we have that for each $\varepsilon > 0$

$$\sigma_1^\Omega[-\Delta, \mathfrak{B}(0)] < \sigma_1^\Omega[-\Delta, \mathfrak{B}(\varepsilon)],$$

and

$$\lim_{\varepsilon \downarrow 0} \sigma_1^\Omega[-\Delta, \mathfrak{B}(\varepsilon)] = \sigma_1^\Omega[-\Delta, \mathfrak{B}(0)],$$

and therefore since (52) holds, there exists $\varepsilon_1 > 0$ such that for each $\varepsilon \in (0, \varepsilon_1]$ the following hold

$$\sigma_1^\Omega[-\Delta, \mathfrak{B}(0)] < \sigma_1^\Omega[-\Delta, \mathfrak{B}(\varepsilon)] < \lambda\,. \tag{53}$$

Fix $\varepsilon \in (0, \varepsilon_1]$ satisfying (53) and let us denote $\sigma_1^\varepsilon := \sigma_1^\Omega[-\Delta, \mathfrak{B}(\varepsilon)]$ and φ_ε the principal eigenfunction associated to σ_1^ε normalized so that

$$\|\varphi_\varepsilon\|_{L^\infty(\Omega)} = 1\,. \tag{54}$$

Now, let us consider the function

$$\underline{u} = \alpha\varphi_\varepsilon\,, \tag{55}$$

where $\alpha > 0$ is an small constant to determine later.

Since φ_ε is strongly positive in Ω, to complete the proof it remains to prove that there exists $\tilde{\alpha} > 0$ small enough such that for each $\alpha \in (0, \tilde{\alpha})$ the function (55) provides us with a positive strict subsolution of (1). Indeed, pick up $\tilde{\alpha}$ satisfying

$$0 < \tilde{\alpha} < \min\left\{ \left(\frac{\lambda - \sigma_1^\varepsilon}{\|a\|_{L^\infty(\Omega)}} \right)^{\frac{1}{p-1}}, \left(\frac{\varepsilon}{\|b\|_{L^\infty(\Gamma_1)}} \right)^{\frac{1}{q-1}} \right\}.$$

Then, taking into account (53) and (54), we find that for each $\alpha \in (0, \tilde{\alpha}]$ the following estimate is satisfied in Ω

$$\begin{array}{rcl} -\Delta\underline{u} - \lambda\underline{u} + a(x)\underline{u}^p & = & \alpha\varphi_\varepsilon\left(\sigma_1^\varepsilon - \lambda + a(x)\alpha^{p-1}\varphi_\varepsilon^{p-1}\right) \\ & < & \alpha\varphi_\varepsilon\left(\sigma_1^\varepsilon - \lambda + \|a\|_{L^\infty(\Omega)}\tilde{\alpha}^{p-1}\right) < 0\,. \end{array} \tag{56}$$

Also, by construction the following estimate is satisfied on Γ_1

$$\begin{array}{rcl} \partial_\nu\underline{u} + b(x)\underline{u}^q & = & \alpha\varphi_\varepsilon\left(-\varepsilon + b(x)\alpha^{q-1}\varphi_\varepsilon^{q-1}\right) \\ & < & \alpha\varphi_\varepsilon\left(-\varepsilon + \|b\|_{L^\infty(\Gamma_1)}\tilde{\alpha}^{q-1}\right) < 0\,. \end{array} \tag{57}$$

Finally the following holds on Γ_0

$$\underline{u} = \alpha\varphi_\varepsilon = 0 \qquad \text{on } \Gamma_0\,. \tag{58}$$

Therefore, (56)-(58) prove that $\underline{u}$ provides us with a positive strict subsolution of (1) for each $\alpha \in (0, \tilde{\alpha}]$, which by construction is strongly positive in Ω.

This completes the proof. □

PROPOSITION 3.3. *Assume that either*

i) (4) *and* (7), *or*

ii) (4) *and* (6)*, or*

iii) (5) *and* (6)*, or*

iv) (5) *and* (3)

hold. Then, for each

$$\lambda < \sigma_0^*[b, \Omega_0]\,, \tag{59}$$

(1) *possesses a positive strict supersolution, arbitrarily large and strongly positive in* Ω.

Proof. Taking into account the definition of $\sigma_0^*[b,\Omega_0]$ (cf. (9)), we have that under condition *i*), (59) becomes

$$\lambda < \sigma_1^{\Omega_0}\left[-\Delta, \mathfrak{B}^*(\Gamma_1^0)\right] \tag{60}$$

and under conditions *ii*), *iii*) or *iv*), (59) becomes

$$\lambda < \sigma_1^{\Omega_0}\left[-\Delta, \mathcal{D}\right]\,. \tag{61}$$

We now prove the result under conditon *i*).

Let us denote $\sigma_0^* := \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*\left(\Gamma_1^0\right)]$ and let λ be satisfying (60).

Necessarily, either

$$\partial\Omega_0 \cap \Gamma_0 = \emptyset \tag{62}$$

or

$$\partial\Omega_0 \cap \Gamma_0 \neq \emptyset\,. \tag{63}$$

Assume (62) holds. Since (7) is satisfied we have that $b \in \mathcal{C}^+(\Gamma_1^+)$ and owing to (42) and (43) it follows from (60) that there exists $V \in \mathcal{C}^+(\Gamma_1^+)$ such that

$$\lambda < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V)] < \sigma_0^*\,. \tag{64}$$

Set for each interval $I \subset (0,\infty)$

$$\Gamma_1^I = \left\{x \in \Gamma_1^+ \,:\, \mathrm{dist}_{\Gamma_1}(x, \Gamma_1^0) \in I\right\}\,,$$

where $\mathrm{dist}_{\Gamma_1}(\cdot, \Gamma_1^0)$ stands for the $N-1$ dimensional minimal distance along Γ_1. Now, for each $\varepsilon > 0$ sufficiently small, let us take a continuous perturbation $V_\varepsilon \in \mathcal{C}^+(\Gamma_1^+ \setminus \Gamma_1^{(0,\varepsilon]})$ of V satisfying

$$V_\varepsilon(x) \leq V(x) \qquad \text{for all } x \in \Gamma_1$$

and

$$\lim_{\varepsilon\to 0} \|V - V_\varepsilon\|_{L^\infty(\Gamma_1)} = 0\,. \tag{65}$$

By construction we have that

$$V_\varepsilon(x) = 0 \quad \forall x \in \Gamma_1^0 \cup \Gamma_1^{(0,\varepsilon]} \qquad \text{and} \qquad V_\varepsilon(x) > 0 \quad \forall x \in \Gamma_1^+ \setminus \Gamma_1^{(0,\varepsilon]} \tag{66}$$

and

$$V_\varepsilon < V \qquad \text{on } \Gamma_1\,. \tag{67}$$

Owing to (67) and (32) we find that

$$\sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V_\varepsilon)] < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V)] \tag{68}$$

and owing to (65), it follows from (39) that

$$\lim_{\varepsilon \downarrow 0} \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V_\varepsilon)] = \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V)]\,. \tag{69}$$

Then, (64), (68) and (69) imply the existence of $\varepsilon_1 > 0$ such that for each $\varepsilon \in (0, \varepsilon_1]$ the following hold

$$\lambda < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V_\varepsilon)] < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V)] < \sigma_0^*\,. \tag{70}$$

Fix $\varepsilon \in (0, \varepsilon_1]$ satisfying (70). Also, since $b \in \mathcal{C}^+(\Gamma_1^+)$, there exists a constant $\beta_\varepsilon > 0$ such that

$$b(x) \geq \beta_\varepsilon > 0 \quad \forall x \in \Gamma_1^+ \setminus \Gamma_1^{(0,\varepsilon)}\,. \tag{71}$$

On the other hand, for each $\delta > 0$ sufficiently small, let consider the δ-neighborhoods

$$\Omega_\delta := (\Omega_0 + B_\delta) \cap \Omega\,, \qquad \mathcal{N}_\delta := (\Gamma_0 + B_\delta) \cap \Omega\,, \tag{72}$$

where $B_\delta \subset \mathbb{R}^N$ denotes the ball of radius δ centered at the origin, and set

$$\Gamma_\delta := \partial\Omega_\delta \cap \Omega\,.$$

Then, $\partial\Omega_\delta = \Gamma_\delta \cup \Gamma_1$. Since $\Gamma_0 \cap \Gamma_1 = \emptyset$ and (62) holds, there exists $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$

$$\bar{\Omega}_\delta \cap \bar{\mathcal{N}}_\delta = \emptyset\,. \tag{73}$$

By construction we have that Ω_0 is a proper subdomain of Ω_δ and Ω_δ converges to Ω_0 from the exterior in the sense of (35). Then, it follows from (34) and (38) that

$$\sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(V_\varepsilon)] < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V_\varepsilon)]\,, \quad 0 < \delta < \delta_0 \tag{74}$$

and

$$\lim_{\delta \downarrow 0} \sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(V_\varepsilon)] = \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V_\varepsilon)] \tag{75}$$

and therefore, (70), (74) and (75) imply the existence of $\delta_1 \in (0, \delta_0)$ such that

$$\lambda < \sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(V_\varepsilon)] < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V_\varepsilon)] < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V)] < \sigma_0^* \tag{76}$$

for each $\delta \in (0, \delta_1)$. Let us denote in the sequel $\sigma_1^{\delta,\varepsilon} := \sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(V_\varepsilon)]$. Also, since $\lim_{\delta\downarrow 0} |\mathcal{N}_\delta| = 0$, it follows from (36) the existence of $\delta_2 \in (0, \delta_1)$ such that for each $\delta \in (0, \delta_2)$

$$\sigma_1^{\mathcal{N}_\delta}[-\Delta, \mathcal{D}] > \sigma_0^* \,. \tag{77}$$

Now, fix $\delta \in (0,\delta_2)$ satisfying (76) and (77) and let $\varphi_\delta^\varepsilon$ and η_δ denote the principal eigenfunctions associated with the principal eigenvalues $\sigma_1^{\delta,\varepsilon}$ and $\sigma_1^{\mathcal{N}_\delta}[-\Delta, \mathcal{D}]$, respectively, normalized so that $\|\varphi_\delta^\varepsilon\|_{L^\infty(\Omega_\delta)} = 1$ and $\|\eta_\delta\|_{L^\infty(\mathcal{N}_\delta)} = 1$.

Then, let consider now the positive function

$$\bar{u} := K\Phi \,,$$

where $K > 0$ is a sufficiently large constant to be determined later and $\Phi : \bar{\Omega} \to [0, \infty)$ is defined by

$$\Phi := \begin{cases} \varphi_\delta^\varepsilon & \text{in } \bar{\Omega}_{\frac{\delta}{2}} \,, \\ \eta_\delta & \text{in } \bar{\mathcal{N}}_{\frac{\delta}{2}} \,, \\ \xi_\delta^\varepsilon & \text{in } \bar{\Omega} \setminus \left(\bar{\Omega}_{\frac{\delta}{2}} \cup \bar{\mathcal{N}}_{\frac{\delta}{2}} \right) \,, \end{cases}$$

where ξ_δ^ε is any regular positive extension of $\varphi_\delta^\varepsilon$ and η_δ from $\bar{\Omega}_{\frac{\delta}{2}} \cup \bar{\mathcal{N}}_{\frac{\delta}{2}}$ to $\bar{\Omega}$ which is bounded away from zero in $\bar{\Omega} \setminus \left(\bar{\Omega}_{\frac{\delta}{2}} \cup \bar{\mathcal{N}}_{\frac{\delta}{2}} \right)$. The existence of ξ_δ^ε is guaranteed since the functions

$$\varphi_\delta^\varepsilon|_{\Gamma_{\frac{\delta}{2}}} \,, \qquad \eta_\delta|_{\partial\mathcal{N}_{\frac{\delta}{2}} \cap \Omega}$$

are bounded away from zero. Let $\mu_\delta > 0$ be such that

$$\xi_\delta^\varepsilon(x) \geq \mu_\delta > 0 \qquad \forall\, x \in \bar{\Omega} \setminus \left(\bar{\Omega}_{\frac{\delta}{2}} \cup \bar{\mathcal{N}}_{\frac{\delta}{2}} \right) \,. \tag{78}$$

Also, since a is bounded away from zero in any compact subset of $\Omega \setminus \bar{\Omega}_0$, there exists $\underline{a}_\delta > 0$ such that

$$a(x) \geq \underline{a}_\delta > 0 \qquad \forall\, x \in \bar{\Omega} \setminus \left(\bar{\Omega}_{\frac{\delta}{2}} \cup \bar{\mathcal{N}}_{\frac{\delta}{2}} \right) \,. \tag{79}$$

To complete the proof it remains to show that there exists $\kappa > 0$ sufficiently large such that $\bar{u} = K\Phi$ provides us with a positive strict supersolution of (1) for each $K \geq \kappa$. Indeed, since $a > 0$ it follows from (76) that in $\Omega_{\frac{\delta}{2}}$ the following estimate is satisfied for each $K > 0$

$$\begin{aligned} -\Delta\bar{u} - \lambda\bar{u} + a(x)\bar{u}^p &= K\varphi_\delta^\varepsilon \left(\sigma_1^{\delta,\varepsilon} - \lambda + a(x)K^{p-1}(\varphi_\delta^\varepsilon)^{p-1} \right) \\ &\geq K\varphi_\delta^\varepsilon \left(\sigma_1^{\delta,\varepsilon} - \lambda \right) > 0 \,. \end{aligned} \tag{80}$$

Similarly, owing to (70) and (77) and since $a > 0$, the following estimate is satisfied in $\mathcal{N}_{\frac{\delta}{2}}$ for each $K > 0$

$$\begin{array}{rcl} -\Delta \bar{u} - \lambda \bar{u} + a(x)\bar{u}^p & = & K\eta_\delta \left(\sigma_1^{\mathcal{N}_\delta}[-\Delta, \mathcal{D}] - \lambda + a(x)K^{p-1}\eta_\delta^{p-1}\right) \\ & \geq & K\eta_\delta \left(\sigma_1^{\mathcal{N}_\delta}[-\Delta, \mathcal{D}] - \lambda\right) > 0\,. \end{array} \tag{81}$$

Also, owing to (78) and (79) there exists $\kappa_1 > 0$ such that for any $K \geq \kappa_1 > 0$ the following estimate is satisfied in $\Omega \setminus \left(\Omega_{\frac{\delta}{2}} \cup \mathcal{N}_{\frac{\delta}{2}}\right)$

$$\begin{array}{rcl} -\Delta \bar{u} - \lambda \bar{u} + a(x)\bar{u}^p & \geq & K\left(-\Delta \xi_\delta^\varepsilon - \lambda \xi_\delta^\varepsilon + a(x)\left(\xi_\delta^\varepsilon\right)^p K^{p-1}\right) \\ & \geq & K\left(-\Delta \xi_\delta^\varepsilon - \lambda \xi_\delta^\varepsilon + \underline{a}_\delta \mu_\delta^p \kappa_1^{p-1}\right) \\ & \geq & K\left(-\|\Delta \xi_\delta^\varepsilon + \lambda \xi_\delta^\varepsilon\|_{L^\infty} + \underline{a}_\delta \mu_\delta^p \kappa_1^{p-1}\right) > 0\,. \end{array} \tag{82}$$

As for the boundary conditions, on Γ_1 we will distinguish two different subregions, $\Gamma_1^0 \cup \Gamma_1^{(0,\varepsilon]}$ and $\Gamma_1^+ \setminus \Gamma_1^{(0,\varepsilon]}$. Since

$$V_\varepsilon(x) = 0, \qquad b(x) \geq 0 \qquad \forall\, x \in \Gamma_1^0 \cup \Gamma_1^{(0,\varepsilon]},$$

we find that by construction the following estimate is satisfied for any $K > 0$ on $\Gamma_1^0 \cup \Gamma_1^{(0,\varepsilon]}$

$$\begin{array}{rcl} \partial_\nu \bar{u} + b(x)\bar{u}^q & = & K\partial_\nu \varphi_\delta^\varepsilon + b(x)K^q\left(\varphi_\delta^\varepsilon\right)^q \\ & = & -KV_\varepsilon \varphi_\delta^\varepsilon + b(x)K^q\left(\varphi_\delta^\varepsilon\right)^q \\ & = & b(x)K^q\left(\varphi_\delta^\varepsilon\right)^q \geq 0\,. \end{array} \tag{83}$$

Also, since $\varphi_\delta^\varepsilon$ is strongly positive in Ω_δ yields

$$m_\delta^\varepsilon := \min_{x\in\Gamma_1} \varphi_\delta^\varepsilon(x) > 0\,. \tag{84}$$

Then, owing to the fact that $V_\varepsilon(x) > 0$ for all $x \in \Gamma_1^+ \setminus \Gamma_1^{(0,\varepsilon]}$ and (71) and (84) hold, we find that there exists $\kappa_2 \geq \kappa_1 > 0$ such that the following estimate is satisfied on $\Gamma_1^+ \setminus \Gamma_1^{(0,\varepsilon]}$ for each $K \geq \kappa_2 > 0$

$$\begin{array}{rcl} \partial_\nu \bar{u} + b(x)\bar{u}^q & = & K\varphi_\delta^\varepsilon \left[-V_\varepsilon(x) + b(x)K^{q-1}\left(\varphi_\delta^\varepsilon\right)^{q-1}\right] \\ & \geq & K\varphi_\delta^\varepsilon \left[-\|V_\varepsilon\|_{L^\infty(\Gamma_1)} + \beta_\varepsilon \kappa_2^{q-1}\left(m_\delta^\varepsilon\right)^{q-1}\right] > 0\,. \end{array} \tag{85}$$

Finally, by construction

$$\bar{u}|_{\Gamma_0} = K\eta_\delta|_{\Gamma_0} = 0\,. \tag{86}$$

Then, (80)-(82) and (83)-(86) prove that, under condition (62), $\bar{u}$ provides us with a positive strict supersolution of (1) for each $K \geq \kappa_2 > 0$, which by construction is strongly positive in Ω.

This completes the proof of the result under condition (62).

Now, let assume that (63) holds. Then, pick up $\delta > 0$, let denote

$$\tilde{\Omega} := \Omega \cup B_\delta(\Gamma_0)\,, \qquad \tilde{\Gamma}_0 := \partial\tilde{\Omega} \setminus \Gamma_1\,.$$

where $B_\delta(\Gamma_0) \subset \mathbb{R}^N$ stands for a δ-neighborhood of Γ_0, let consider the auxiliary potential

$$\tilde{a} = \begin{cases} 1 & \text{in } \tilde{\Omega} \setminus \Omega\,, \\ a & \text{in } \Omega\,, \end{cases}$$

the auxiliary boundary operator

$$\tilde{\mathfrak{B}}(b) := \begin{cases} \mathcal{D} & \text{on } \tilde{\Gamma}_0\,, \\ \partial_\nu + b & \text{on } \Gamma_1\,, \end{cases}$$

and the associated boundary value problem

$$\begin{cases} -\Delta u = \lambda u - \tilde{a}(x) u^p & \text{in } \tilde{\Omega}\,, \\ u = 0 & \text{on } \tilde{\Gamma}_0\,, \\ \partial_\nu u = -b(x) u^q & \text{on } \Gamma_1\,. \end{cases} \tag{87}$$

By construction

$$\tilde{\Omega}_0 = \Omega_0\,, \quad \tilde{\Gamma}_0 \cap \bar{\tilde{\Omega}}_0 = \emptyset\,, \quad \sigma_1^{\tilde{\Omega}_0}[-\Delta, \mathfrak{B}^*\left(\Gamma_1^0\right)] = \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*\left(\Gamma_1^0\right)] = \sigma_0^*$$

and (60) becomes

$$\lambda < \sigma_1^{\tilde{\Omega}_0}[-\Delta, \mathfrak{B}^*(\Gamma_1^0)]\,. \tag{88}$$

Then, (87) satisfies condition (62), and hence, since (88) holds, we find by the above arguments that (87) possesses a positive strict supersolution $\tilde{u}$ arbitrarily large and strongly positive in $\tilde{\Omega}$ for each λ satisfying (60). Now, it is straightforward to prove that the function

$$\bar{u} := \tilde{u}|_{\bar{\Omega}}$$

provides us with, for each λ satisfying (60), a positive strict supersolution of (1) under condition (63), which is arbitrarily large and strongly positive in Ω.

This completes the proof of the result under condition *i*).

We now prove the result under condition *ii*). Let us denote $\sigma_0 := \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]$ and let λ be satisfying (61). Whithout lost of generality we will assume that (62) holds. On the contrary we would argue as in case *i*) when (63) holds. Owing to (33) and (37) we have that

$$\sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(n)] < \sigma_0 \quad \forall\, n \in \mathbb{N}\,, \qquad \lim_{n\uparrow\infty} \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(n)] = \sigma_0\,. \tag{89}$$

Then, owing to (61) and (89), there exists $n \in \mathbb{N}$ large enough such that

$$\lambda < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(n)] < \sigma_0 \,. \tag{90}$$

Fix $n \in \mathbb{N}$ satisfying (90). Now, let consider the δ-neighborhoods Ω_δ and $\mathcal{N}_\delta$ defined in (72). Using the same arguments than in case i), it follows the existence of $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0)$ (73) holds and moreover

$$\sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(n)] < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(n)]$$

and

$$\lim_{\delta \downarrow 0} \sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(n)] = \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(n)] \,.$$

Thus, taking into account (90), there exists $\delta_1 \in (0, \delta_0)$ such that for each $\delta \in (0, \delta_1)$ the following hold

$$\lambda < \sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(n)] < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(n)] < \sigma_0 \,. \tag{91}$$

Also, arguing as in case i), there exists $\delta_2 \in (0, \delta_1)$ such that for each $\delta \in (0, \delta_2)$

$$\sigma_1^{\mathcal{N}_\delta}[-\Delta, \mathcal{D}] > \sigma_0 \,. \tag{92}$$

Now, fix $\delta \in (0, \delta_2)$ satisfying (91) and (92), let φ_δ^n and η_δ denote the principal eigenfunctions associated with the principal eigenvalues $\sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(n)]$ and $\sigma_1^{\mathcal{N}_\delta}[-\Delta, \mathcal{D}]$ normalized so that $\|\varphi_\delta^n\|_{L^\infty(\Omega_\delta)} = 1$ and $\|\eta_\delta\|_{L^\infty(\mathcal{N}_\delta)} = 1$ Now, let consider the positive function

$$\bar{u} = K\Phi \,,$$

where $K > 0$ is a sufficiently large constant to be determined later and $\Phi : \bar{\Omega} \to [0, \infty)$ is defined by

$$\Phi := \begin{cases} \varphi_\delta^n & \text{in } \bar{\Omega}_{\frac{\delta}{2}} \,, \\ \eta_\delta & \text{in } \bar{\mathcal{N}}_{\frac{\delta}{2}} \,, \\ \xi_\delta^n & \text{in } \bar{\Omega} \setminus \left(\bar{\Omega}_{\frac{\delta}{2}} \cup \bar{\mathcal{N}}_{\frac{\delta}{2}} \right) \,, \end{cases}$$

being ξ_δ^n any regular positive extension of φ_δ^n and η_δ from $\bar{\Omega}_{\frac{\delta}{2}} \cup \bar{\mathcal{N}}_{\frac{\delta}{2}}$ to $\bar{\Omega}$ which is bounded away from zero. Now, taking into account (91) and (92) and the fact that φ_δ^n is strongly positive in Ω_δ and arguing as in case i), it is not hard to prove that there exists $\tilde{\kappa} > 0$ such that for each $K \geq \tilde{\kappa}$, $\bar{u}$ is a positive strict supersolution of (1), which by construction is strongly positive in Ω.

This completes the proof of the result under condition ii).

Now, taking the notations of the previous cases, the proof of the result under condition iii) follows arguing as in cases i) and ii), taking $\bar{u} = K\Phi$, where $K > 0$ is sufficiently large and

$$\Phi := \begin{cases} \eta_\delta & \text{in } \bar{\mathcal{N}}_{\frac{\delta}{2}}\,, \\ \varphi_\delta & \text{in } \bar{\Omega}_{\frac{\delta}{2}}\,, \\ \psi_\delta & \text{in } \bar{\mathcal{A}}_{\frac{\delta}{2}} \cap \Omega\,, \\ \xi_\delta & \text{in } \bar{\Omega} \setminus \left(\bar{\mathcal{N}}_{\frac{\delta}{2}} \cup \bar{\Omega}_{\frac{\delta}{2}} \cup (\bar{\mathcal{A}}_{\frac{\delta}{2}} \cap \Omega)\right), \end{cases}$$

being $\Omega_\delta := \bar{\Omega}_0 + B_\delta$ and $\mathcal{A}_\delta := \Gamma_1 + B_\delta$, for $\delta > 0$ small enough so that

$$\bar{\Omega}_\delta \cap \bar{\mathcal{N}}_\delta = \emptyset\,, \qquad \bar{\Omega}_\delta \cap \bar{\mathcal{A}}_\delta = \emptyset\,, \qquad \bar{\mathcal{A}}_\delta \cap \bar{\mathcal{N}}_\delta = \emptyset\,,$$

and where ψ_δ stands for the principal eigenfunction associated to $\sigma_1^{\mathcal{A}_\delta}[-\Delta, \mathcal{D}]$, normalized with L^∞-norm equals 1 in its domain, and ξ_δ is any positive regular extension of η_δ, φ_δ and ψ_δ from $\bar{\mathcal{N}}_{\frac{\delta}{2}} \cup \bar{\Omega}_{\frac{\delta}{2}} \cup (\bar{\mathcal{A}}_{\frac{\delta}{2}} \cap \Omega)$ to $\bar{\Omega}$, bounded away from zero.

Finally, taking the notations of the previous cases, the proof of the result under condition *iv*) follows arguing in a similar way taking $\bar{u} = K\Phi$ for $K > 0$ sufficiently large and

$$\Phi := \begin{cases} \eta_\delta & \text{in } \bar{\mathcal{N}}_{\frac{\delta}{2}}\,, \\ \varphi_\delta & \text{in } \bar{\Omega}_{\frac{\delta}{2}}\,, \\ \xi_\delta & \text{in } \bar{\Omega} \setminus \left(\bar{\mathcal{N}}_{\frac{\delta}{2}} \cup \bar{\Omega}_{\frac{\delta}{2}}\right), \end{cases}$$

where now ξ_δ is any positive regular extension of η_δ and φ_δ from $\bar{\mathcal{N}}_{\frac{\delta}{2}} \cup \bar{\Omega}_{\frac{\delta}{2}}$ to $\bar{\Omega}$ bounded away from zero, satisfying

$$\partial_\nu \xi_\delta(x) \geq 0 \qquad \forall\, x \in \Gamma_1\,, \tag{93}$$

whose existence is guaranteed by construction. This completes the proof. □

We now prove Theorem 1.1

Proof of Theorem 1.1: We are going to prove *i*). Indeed, let u_λ be a positive solution of (1) for the value λ of the parameter. Then, taking into account the definition of $\sigma_0^*[b, \Omega_0]$ (cf.(9)), the necessary condition (12) for the existence of positive solution follows from (45), (47) and (48).

To prove the sufficient condition (12) for the existence of positive solution of (1) we will use the sub-supersolution method (cf. [1]). Let λ be satisfying (12). Then owing to Proposition 3.2, (1) possesses a positive strict subsolution $\underline{u}_\lambda$, arbitrarily small and strongly positive in Ω. On the other hand, it follows from Proposition 3.3 that for each λ satisfying (12), (1) possesses a positive strict supersolution $\bar{u}_\lambda$, arbitrarily large and strongly positive in Ω.

Then, since both of them, the subsolution $\underline{u}_\lambda$ and the supersolution $\bar{u}_\lambda$, are strongly positive in Ω, it is possible to take them satisfying $\underline{u}_\lambda < \bar{u}_\lambda$ in Ω, and owing to the sub-supersolution method we find that (1) possesses a positive solution u_λ, with $\underline{u}_\lambda < u_\lambda < \bar{u}_\lambda$, for each λ satisfying (12).

The proof of the uniqueness of positive solution, if it exists, is obtained following the same arguments than in [9, Theorem 3.1].

The fact that any positive solution u_λ of (1) is strongly positive in Ω and that (13) holds, follow from (46).

We now prove *ii*) The results about the structure of $\Lambda_L(\Omega_0)$ in (14), (15) and (18) follow from [14, Theorem 3.5] and [6, Theorem 3.4]. The results about the structure of $\Lambda_{NL}(\Omega_0, b)$ in (14), (16) and (19) follow from (12), taking into account the definition of $\sigma_0^*[\Omega_0, b]$. Finally, (17) and (20) follow from (33) and (42).

We now prove *iii*) Let u_λ be a positive solution of (1) for the value λ of the parameter. Then, (12) and (44) hold and differentiating (1) with respect to λ we find that $\dot{u}_\lambda := \frac{du_\lambda}{d\lambda}$ satisfies the following problem

$$\begin{cases} \left(-\Delta + pa(x)u_\lambda^{p-1} - \lambda\right)\dot{u}_\lambda = u_\lambda > 0 & \text{in } \Omega\,, \\ \mathfrak{B}\left(qb(x)u_\lambda^{q-1}\right)\dot{u}_\lambda = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{94}$$

Also, since $a > 0$ and $u_\lambda > 0$ in Ω, $b > 0$ on Γ_1 and $p, q > 1$, owing to (31) and (32) it follows from (44) that

$$\sigma_1^\Omega\left[-\Delta + pau_\lambda^{p-1} - \lambda, \mathfrak{B}(qbu_\lambda^{q-1})\right] > \sigma_1^\Omega\left[-\Delta + au_\lambda^{p-1} - \lambda, \mathfrak{B}(bu_\lambda^{q-1})\right] = 0\,, \tag{95}$$

that is, u_λ is linearly asymptotically stable. Moreover, owing to the Characterization of the Strong Maximum Principle given by H. Amann and J. López-Gómez in [3, Theorem 2.4], it follows from (95) that (94) satisfies the strong maximum principle, and hence (21) holds.

We now prove *iv*). Let $I = [\alpha, \beta]$ be, with $\beta > \sigma_1$, a compact interval with $I \subset [\sigma_1, \sigma_0^*)$ and let u_β the unique positive solution of (1) for $\lambda = \beta$, whose existence and uniqueness are guaranteed by *i*). Then, owing to (21) we have that $u_\lambda \le u_\beta$ for all $\lambda \in I$ and therefore, $\|u_\lambda\|_{L^\infty(\Omega)} \le \|u_\beta\|_{L^\infty(\Omega)}$ for all $\lambda \in I$ which proves *iv*).

We now prove *v*) The fact that $\lambda = \sigma_1$ is the unique bifurcation value to positive solutions of (1) from the trivial branch $(\lambda, u) = (\lambda, 0)$, and the existence of a differentiable continuum $\mathfrak{C}(\sigma_1)$ of solutions of (1) emanating from the trivial branch at the value $\lambda = \sigma_1$, follow from [8, Theorem 1.1]. Now, let denote by $\mathfrak{C}^+(\sigma_1)$ the maximal subcontinuum of $\mathfrak{C}(\sigma_1)$ constituted by the positive solutions of (1) emanating from the trivial branch at $(\lambda, u) = (\sigma_1, 0)$ and $\mathcal{P}_\lambda\left(\mathfrak{C}^+(\sigma_1)\right)$ its projection on the λ axis. The fact that $\mathfrak{C}^+(\sigma_1)$ emanates supercritically from the trivial branch follows from (12) or [8, Theorem 1.1].

Let φ_1 denote the principal eigenfunction associated to the principal eigenvalue σ_1, normalized such that $\|\varphi_1\|_{L^\infty(\Omega)} = 1$. It is known that $\varphi_1 \gg 0$ in Ω. Since $\lambda = \sigma_1$ is a simple eigenvalue of the linearization of (1) around $(\lambda, u) = (\sigma_1, 0)$ and owing to the fact that $\lambda = \sigma_1$ is the unique bifurcation value to positive solutions of (1) from the trivial branch, it follows from the updated version of the Global Alternative of P.H. Rabinowitz [23, Theorem 1.27] given by J. López-Gómez in [19, Theorem 6.4.3] that either $\mathfrak{C}^+(\sigma_1)$ is unbounded in $\mathbb{R} \times \mathcal{C}^1_{\Gamma_0}$, or it contains a pair $(\tilde{\lambda}, \tilde{u})$ with $\tilde{u}$ strongly positive in Ω satisfying $\int_\Omega \tilde{u}\varphi_1 = 0$, which is impossible since φ_1 is strongly positive in Ω. Then, we get that $\mathfrak{C}^+(\sigma_1)$ is unbounded in $\mathbb{R} \times \mathcal{C}^1_{\Gamma_0}(\bar{\Omega})$ and since (12) holds, we find that $\mathfrak{C}^+(\sigma_1)$ is bounded in $\mathbb{R}$ and unbounded in $L^\infty(\Omega)$. Now, the existence of uniform $L^\infty(\Omega)$ bounds for the positive solutions of (1) in compact intervals of values of λ contained in $[\sigma_1, \sigma_0^*)$, guaranteed by *iv*), joint with the fact that $\mathfrak{C}^+(\sigma_1)$ is unbounded in $L^\infty(\Omega)$, imply that $\lambda = \sigma_0^*$ is the unique bifurcation value to positive solutions of (1) from infinity and that $\mathfrak{C}^+(\sigma_1)$ bifurcates from infinity to positive solutions at $\lambda = \sigma_0^*$. In particular, owing to (12) and since $\mathfrak{C}^+(\sigma_1)$ is connected and it bifurcates to positive solutions from the trivial branch at $\lambda = \sigma_1$ and from infinity at $\lambda = \sigma_0^*$, we find that (22) holds.

Finally, since (12) and (22) hold and taking into account the structure of $\mathfrak{C}^+(\sigma_1)$, the fact that any positive solution of (1) belongs to $\mathfrak{C}^+(\sigma_1)$ follows from the uniqueness of positive solution of (1) for any λ satisfying (12). The fact that $\mathfrak{C}^+(\sigma_1)$ is increasing in $\|\cdot\|_{L^\infty(\Omega)}$ with respect to the λ-parameter follows from (21) and the uniqueness of positive solution of (1) for each λ satisfying (12).

We now prove *vi*) Since (24) holds, the existence and uniqueness of u_i, $i = 1, 2$, follow from (24) and *i*). Owing to (44) the following holds

$$\sigma_1^\Omega[-\Delta - \lambda + a(x)u_i^{p-1}, \mathfrak{B}(b_i u_i^{q-1})] = 0\,, \qquad i = 1, 2\,. \tag{96}$$

Now, let denote $\Theta = u_1 - u_2$. By construction, Θ satisfies the following problem

$$\begin{cases} (-\Delta - \lambda + a(x)F(x))\,\Theta = 0 & \text{in } \Omega\,, \\ \Theta = 0 & \text{on } \Gamma_0\,, \\ (\partial_\nu + b_1(x)G(x))\,\Theta = (b_2 - b_1)u_2^q > 0 & \text{on } \Gamma_1\,, \end{cases} \tag{97}$$

where $F \in \mathcal{C}(\bar{\Omega})$ and $G \in \mathcal{C}(\Gamma_1)$ are defined by

$$F(x) := \begin{cases} \dfrac{u_1(x)^p - u_2(x)^p}{u_1(x) - u_2(x)} & \text{if } u_1(x) \neq u_2(x)\,, \\ p u_1^{p-1}(x) & \text{if } u_1(x) = u_2(x)\,, \end{cases}$$

and

$$G(x) := \begin{cases} \dfrac{u_1(x)^q - u_2(x)^q}{u_1(x) - u_2(x)} & \text{if } u_1(x) \neq u_2(x)\,, \\ q u_1^{q-1}(x) & \text{if } u_1(x) = u_2(x)\,. \end{cases}$$

By construction, and since $p, q > 1$ and $u_1 \gg 0$ in Ω we have that

$$F(x) > u_1^{p-1}\,, \qquad G(x) > u_1^{q-1}$$

and hence, since $a > 0$, $b_1 > 0$ and owing to (31), (32) and (96) we find that

$$\begin{aligned}\sigma_1^{\Omega}[-\Delta - \lambda + a(x)F(x), \mathfrak{B}(b_1 G(x))]&\\ > \sigma_1^{\Omega}[-\Delta - \lambda + a(x)u_1^{p-1}, \mathfrak{B}(b_1 u_1^{q-1})] = 0\,. \quad (98)\end{aligned}$$

Then, owing to the Characterization of the Strong Maximum Principle [3, Theorem 2.4] it follows from (98) that (97) satisfies the strong maximum principle and therefore $\Theta := u_1 - u_2 \gg 0$ in Ω, which proves (25) and completes the proof of *vi*).

We now prove *vii*). The existence and uniqueness of u_0 follows from the fact that $\lambda \in \Lambda_L(\Omega_0)$ and [6, Theorem 1.5]. The existence of $\tilde{u}$ follows from the fact that owing to *ii*), we have that $\lambda \in \Lambda_L(\Omega_0) \subset \Lambda_{NL}(\Omega_0, b)$. The uniqueness of $\tilde{u}$ follows from *i*). Finally (26) follows arguing exactly as in *vi*), taking into account that $b > 0$ instead of (23).

We now prove *viii*). Since (4) holds and $b_i \in \mathcal{C}^+(\Gamma_1^+)$, $i = 1, 2$, it follows from (19) that

$$\Lambda_{NL}(\Omega_0, b_1) = \Lambda_{NL}(\Omega_0, b_2) = (\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*\left(\Gamma_1^0\right)]\,,$$

and hence,

$$\lambda \in \Lambda_{NL}(\Omega_o, b_i)\,, \quad i = 1, 2\,. \tag{99}$$

Then, the existence and uniqueness of u_i, $i = 1, 2$ is guaranteed by (99) and *i*). Now the result follows from *vi*).

We now prove *ix*). The result follows from *vi*), arguing as in *viii*), taking into account that now

$$\Lambda_{NL}(\Omega_0, b_1) = \Lambda_{NL}(\Omega_0, b_2) = \Big(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]\Big)\,.$$

This completes the proof. □

4. The amplitude of the nonlinearity in the boundary conditions as bifurcation parameter

In order to complete the exposition, in this section we consider, under the general assumptions of this paper, the bi-parameter logistic elliptic problem

$$\begin{cases} -\Delta u = \lambda u - a(x)u^p & \text{in } \Omega\,, \quad p > 1\,, \\ u = 0 & \text{on } \Gamma_0\,, \\ \partial_\nu u = \gamma b(x) u^q & \text{on } \Gamma_1\,, \quad q > 1\,, \end{cases} \tag{100}$$

where λ, $\gamma \in \mathbb{R}$, $a > 0$ in Ω and $b > 0$ on Γ_1. The main goal of this section is to ascertain the global structure of the set of positive solutions of (100) considering γ as the bifurcation-continuation parameter, for some λ fixed in a suitable interval. We focus in the particular case when $\bar{\Omega}_0 \subset \Omega$ and (3) holds, and we denote

$$\sigma_1 := \sigma_1^{\Omega}[-\Delta, \mathfrak{B}(0)]\,, \qquad \tilde{\sigma}_0 := \sigma_1^{\Omega}[-\Delta, \mathcal{D}]\,, \qquad \sigma_0^* := \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]\,,$$

where $\sigma_1 < \tilde{\sigma}_0 < \sigma_0^*$.

The following result collects the main findings about the global structure of the set of positive solutions of (100) considering γ as bifurcation-continuation parameter, for some λ fixed in a suitable interval. It is one of the main results of [11]. We include it without proof and we remit to [11, Theorem 1.2] for the details of its proof.

THEOREM 4.1. *Assume $\lambda \in (\sigma_1, \sigma_0^*)$, and let u_0 denote the unique positive solution of (100) for $\gamma = 0$. Then:*

i) For each $\gamma \leq 0$, (100) possesses a unique positive solution u_γ, which is linearly asymptotically stable. Moreover, the map

$$\begin{array}{ccc} (-\infty, 0] & \longrightarrow & \mathcal{C}^1_{\Gamma_0}(\bar{\Omega}) \\ \gamma & \rightarrow & u_\gamma \end{array}$$

is differentiable and $\dot{u}_\gamma := \dfrac{du_\gamma}{d\gamma} \gg 0$ in Ω, and in particular, the map

$$\begin{array}{ccc} (-\infty, 0] & \mapsto & \mathcal{C}_{\Gamma_0}(\bar{\Omega}) \\ \gamma & \mapsto & u_\gamma \end{array}$$

is strictly increasing.

ii) If (6) holds and $\lambda \in (\sigma_1, \tilde{\sigma}_0)$, then there exists $D(\lambda) > 0$ such that

$$\|u_\gamma\|_{L^\infty(\Omega)} \leq D(\lambda) \left(\frac{1}{\underline{b}\tilde{\gamma}}\right)^{\frac{1}{q-1}}, \qquad \text{for all } \gamma < 0\,.$$

In particular

$$\lim_{\gamma \downarrow -\infty} \|u_\gamma\|_{L^\infty(\Omega)} = 0\,,$$

that is, the problem exhibits bifurcation to positive solutions from the trivial branch $(\gamma, u) = (\gamma, 0)$ when $\gamma \downarrow -\infty$.

iii) There exists $\varepsilon_0 > 0$ and a differentiable map

$$\begin{array}{cccc} u: & (-\varepsilon_0, \varepsilon_0) & \rightarrow & \mathcal{C}^1_{\Gamma_0}(\bar{\Omega}) \\ & \gamma & \mapsto & u^*_\gamma \end{array}$$

such that $u^*_\gamma = u_\gamma$ for all $\gamma \in (-\varepsilon_0, 0]$, there exists a neighborhood $\mathfrak{U}$ of $(\gamma, u) = (0, u_0)$ in $(-\varepsilon_0, \varepsilon_0) \times \mathcal{C}^1_{\Gamma_0}(\bar\Omega)$ such that if $(\gamma, \tilde u) \in \mathfrak{U}$ is a positive solution of (100), then $\tilde u = u^*_\gamma$, and in addition (γ, u^*_γ) is a positive linearly asymptotically stable solution of (100) for all $\gamma \in (-\varepsilon_0, \varepsilon_0)$. Moreover,

$$\dot u^*_\gamma := \frac{du^*_\gamma}{d\gamma} \gg 0 \qquad \text{in } \Omega\,,$$

and in particular the map

$$\begin{array}{ccc} (-\varepsilon_0, \varepsilon_0) & \to & \mathcal{C}_{\Gamma_0}(\bar\Omega) \\ \gamma & \mapsto & u^*_\gamma \end{array}$$

is strictly increasing.

iv) Any positive solution $\hat u_\gamma$ of (100) for $\gamma > 0$ satisfies $\hat u_\gamma \gg u_0$.

v) If $p > 2q - 1$, then the following hold:

a) For each $\gamma > 0$, (100) possesses at least a positive solution.

b) For each $\gamma > 0$, (100) possesses a minimal positive solution u^{min}_γ satisfying $u^{min}_\gamma \gg u_0$ and $u^{min}_\gamma = u^*_\gamma$ for $\gamma \in (0, \varepsilon_0)$, where ε_0 and u^*_γ are defined by iii).

c) There exist uniform $L^\infty(\Omega)$ bounds for the positive solutions of (100) in compact intervals of values of γ.

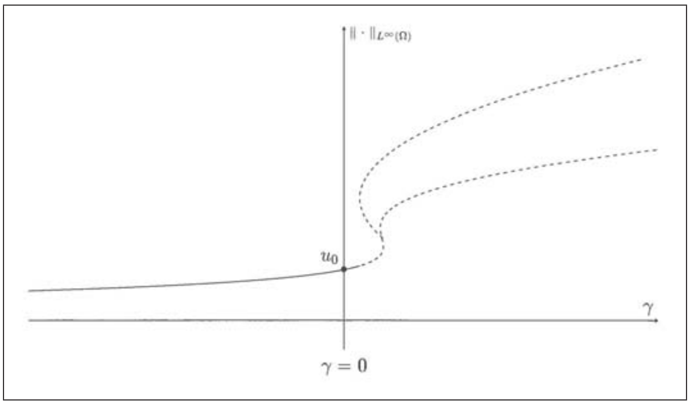

Figure 5: Global bifurcation diagram of positive solutions of (100) in the γ parameter ($\lambda \in (\sigma_1, \sigma^*_0)$).

Theorem 4.1 establishes that, for each fixed $\lambda \in (\sigma_1, \sigma_0^*)$, the global bifurcation diagram in the γ-parameter of the positive solutions of (100) should be like shown by Figure 5, where the continuous line stands for the exact structure of the set of positive solutions for $\gamma < \varepsilon_0$ and the dashed line stands for a possible configuration of the set of positive solutions of (100) for $\gamma > \varepsilon_0$.

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