LIE ALGEBROID CONNECTIONS, TWISTED HIGGS BUNDLES AND MOTIVES OF MODULI SPACES

DAVID ALFAYA AND ANDRÉ OLIVEIRA

Abstract. Let $\mathcal{L} = (L, [\cdot\,,\cdot], \delta)$ be an algebraic Lie algebroid over a smooth projective curve of genus $g \geq 2$ such that L is a line bundle whose degree is less than $2-2g$. Let r and d be coprime numbers. We prove that the motivic class (in the Grothendieck ring of varieties) of the moduli space of $\mathcal{L}$-connections of rank r and degree d over X does not depend on the Lie algebroid structure $[\cdot\,,\cdot]$ and δ of $\mathcal{L}$ and neither on the line bundle L itself, but only the degree of L (and of course on r, d, g and X). In particular it is equal to the motivic class of the moduli space of $K_X(D)$-twisted Higgs bundles of rank r and degree d, for D any divisor of positive degree. As a consequence, similar results (actually a little stronger) are obtained for the corresponding E-polynomials. Some applications of these results are then deduced.

Contents

1. Introduction 2
2. Moduli space of twisted Higgs bundles 5
3. $\mathcal{L}$-connections, Λ-modules and moduli spaces 7
3.1. Lie algebroids and $\mathcal{L}$-connections 7
3.2. Λ-modules and $\mathcal{L}$-connections 9
3.3. Moduli spaces of $\mathcal{L}$-connections and of Λ-modules 12
3.4. The $\mathcal{L}$-Hodge moduli spaces 14
4. Grothendieck ring of varieties, motives and E-polynomials 16
4.1. Grothendieck ring of varieties, motives and E-polynomials 16
4.2. The plan 17
5. Invariance of the motive and E-polynomial with respect to the algebroid structure 17
5.1. Bialynicki-Birula stratification 18
5.2. Semiprojectivity of the moduli space of Higgs bundles 20
5.3. Semiprojectivity of the moduli space of $\Lambda_{\mathcal{L}}^{\mathrm{red}}$-modules 21
5.4. Invariance of the motive and E-polynomial with respect to the algebroid structure 29
6. Motives of moduli spaces of twisted Higgs bundles 30
6.1. Variations of Hodge structure and chains 30
6.2. Independence of the motives of Higgs moduli on the twisting line bundle 32
6.3. Independence of motives and E-polynomials from the Lie algebroid structure 35
7. Applications 35
7.1. Topological properties of moduli spaces of Lie algebroid connections 35
7.2. Chow motives and Voevodsky motives 38
7.3. Motives of moduli spaces of irregular or logarithmic connections 39
7.4. Explicit motives and E-polynomials for rank 2 and 3 39
8. Motives of moduli spaces of $\mathcal{L}$-connections with fixed determinant 45

1991 *Mathematics Subject Classification.* 14D20, 14H60, 19E08, 14C35, 14C15.

Key words and phrases. Lie algebroid connections, Higgs bundles, moduli space, motive, virtual class, Hodge structure, E-polynomial.

8.1. Twisted Higgs bundles with fixed determinant 45
8.2. Moduli spaces of $\mathcal{L}$-connections with fixed determinant 47
References 51

1. Introduction

Consider the de Rham moduli space $\mathcal{M}^{dR}(r,0)$ of rank r algebraic connections over a smooth projective curve X and write K_X for the canonical bundle of X. So $\mathcal{M}^{dR}(r,0)$ parameterizes S-equivalence classes of pairs (E,∇) (all of them are semistable), with E a rank r and degree 0 algebraic vector bundle on X and $\nabla : E \to E \otimes K_X$ an algebraic connection on E. In general, the integrability condition $\nabla^2 = 0$ must be imposed, but in this case this is automatic because K_X is a line bundle. Consider also the closely-related Dolbeault moduli space $\mathcal{M}_{K_X}(r,0)$, of rank r and degree 0 semistable Higgs bundles (E,φ), with E as before, and $\varphi : E \to E \otimes K_X$ a morphism of $\mathcal{O}_X$-modules, called the Higgs field. The link between these two moduli spaces is provided by the non-abelian Hodge correspondence [Hit87, Sim92, Sim94, Sim95], which yields a homeomorphism between them and hence allows the study of the topology of the de Rham moduli, by studying the Dolbeault one.

Actually, more is true, as proved by Simpson in [Sim94]: both $\mathcal{M}_{K_X}(r,0)$ and $\mathcal{M}^{dR}(r,0)$ are, respectively, the fiber over 0 and 1 of an isosingular algebraic family over the affine line, such that any other fiber is isomorphic to $\mathcal{M}^{dR}(r,0)$. This family is obtained by taking the moduli space $\mathcal{M}_\lambda(r,0)$ of λ-connections (E,∇,λ) of rank r and degree 0, with $\lambda \in \mathbb{C}$. Here, the Leibniz rule for a λ-connection is obtained from the usual Leibniz rule by multiplying the summand with the de Rham differential by λ. If we consider the moduli space of semistable λ-connections (E,∇,λ), with λ varying in $\mathbb{C}$, then we obtain the so-called Hodge moduli space $\mathcal{M}^{Hod}(r,0)$. Then the obvious projection $\mathcal{M}^{Hod}(r,0) \to \mathbb{C}$ mapping to λ, produces the mentioned isosingular family.

The existence of singularities on the moduli spaces $\mathcal{M}_{K_X}(r,0)$ and $\mathcal{M}^{dR}(r,0)$ introduces, however, serious difficulties on the study of their topology and geometry. This is 'circumvented' if we consider instead d coprime with r and the corresponding moduli spaces $\mathcal{M}_{K_X}(r,d)$ of stable degree d and rank r Higgs bundles and $\mathcal{M}^{dR}(r,d)$ of stable logarithmic connections, hence with poles at some prescribed punctures on X, with fixed holonomy, depending on d, around the punctures. The non-abelian correspondence extends to this case, and $\mathcal{M}_{K_X}(r,d)$ and $\mathcal{M}^{dR}(r,d)$ are also homeomorphic. Moreover, a complete analogue of the picture of the preceding paragraph holds as well, by using the Hodge moduli space $\mathcal{M}^{Hod}(r,d)$, defined in the same way. The substantial difference is that all these moduli spaces (including all moduli $\mathcal{M}_\lambda(r,d)$ of logarithmic λ-connections of rank r and degree d) are now smooth.

Using $\mathcal{M}^{Hod}(r,d)$, and taking r and d coprime, Hausel and Thaddeus proved in [HT03] that $\mathcal{M}_{K_X}(r,d)$ and $\mathcal{M}^{dR}(r,d)$ actually share deeper geometrical invariants, other than just topological ones. Namely, their Hodge structure is pure and equal, so in particular both spaces share the same E-polynomial. The basic feature of their proof[1] is the use of the fact that, as proved in [Sim94, Sim97], $\mathcal{M}^{Hod}(r,d)$ is a smooth semiprojective variety in the sense of Hausel—Rodriguez-Villegas [HRV15] (which implies that it comes endowed with a $\mathbb{C}^*$-action), together with a surjective $\mathbb{C}^*$-equivariant submersion onto $\mathbb{C}$.

More recently, it was verified that the smoothness and the semiprojective structure on the Hodge moduli space $\mathcal{M}^{Hod}(r,d)$ implies the existence of another fundamental correspondence between $\mathcal{M}^{dR}(r,d)$ and $\mathcal{M}_{K_X}(r,d)$. Namely, in [HL19], Hoskins and Lehaleur established what they called

[1]This was actually done in [HT03] for the fixed determinant (and traceless) versions of these moduli spaces, but the argument also works in the non-fixed determinant case.

a "motivic non-abelian Hodge correspondence" by proving an equality of the Voevodsky motives of $\mathcal{M}^{dR}(r,d)$ and $\mathcal{M}_{K_X}(r,d)$; their result indeed holds for any characteristic zero field, not just $\mathbb{C}$. In fact, by considering $d=0$ and the stacky version of these moduli spaces, a similar result was proved before in [FSS18], for motivic classes in the Grothendieck ring of stacks, but through a completely different technique, namely point counting. This was recently generalized to the parabolic setting in [FSS20].

In the present paper, we prove the equality of the motivic class in the Grothendieck ring of varieties (and other invariants) between two generalizations of $\mathcal{M}_{K_X}(r,d)$ and $\mathcal{M}^{dR}(r,d)$, as we explain in the next paragraphs.

The Hodge moduli space $\mathcal{M}^{Hod}(r,d)$ is just a particular case of a much more general construction, by Simpson [Sim94], arising from the moduli space $\mathcal{M}_\Lambda(r,d)$ of Λ-modules (see Definition 3.8), where Λ is a sheaf of rings of differential operators on X. Now, as proved by Tortella in [Tor11, Tor12], there is an equivalence of categories between a certain subclass of such sheaves (consisting of the split and almost polynomial ones) and algebraic Lie algebroids on X. In turn, such equivalence induces an equivalence of categories between integrable $\mathcal{L}$-connections, where $\mathcal{L}$ is a Lie algebroid on X and $\Lambda_\mathcal{L}$-modules, with $\Lambda_\mathcal{L}$ the split almost polynomial sheaf of rings of differential operators corresponding to $\mathcal{L}$. This correspondence preserves semistability, hence one can think of $\mathcal{M}_{\Lambda_\mathcal{L}}(r,d)$ as the moduli space of $\mathcal{L}$-connections of rank r and degree d.

We now briefly recall what are these objects. Let $T_X = K_X^{-1}$ be the tangent bundle of X. An algebraic Lie algebroid $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$ on X consists of an algebraic vector bundle $V \to X$, equipped with a Lie bracket $[\cdot\,,\cdot] : V \otimes_\mathbb{C} V \to V$ and an anchor algebraic map $\delta : V \to T_X$, such that $[\cdot\,,\cdot]$ and δ are related by a Leibniz rule relation. Then an integrable (or flat) $\mathcal{L}$-connection of rank r and degree d on X is a pair $(E, \nabla_\mathcal{L})$, where $E \to X$ is an algebraic vector bundle of rank r and degree d, together with a $\mathbb{C}$-linear algebraic map $\nabla_\mathcal{L} : E \to E \otimes V^*$, satisfying a generalization of the Leibniz rule to the Lie algebroid setting, i.e. where the usual differential is replaced by an $\mathcal{L}$-differential $d_\mathcal{L}$, and so that the integrability condition $\nabla_\mathcal{L}^2 = 0$ holds. In this language, an algebraic connection on X is just a $\mathcal{T}_X$-connection, where $\mathcal{T}_X$ is the canonical Lie algebroid structure with underlying bundle T_X. Note also that if we consider the trivial algebroid $\mathcal{L} = (V, 0, 0)$, then an integrable $\mathcal{L}$-connection on X is simply a V^*-twisted Higgs bundle on X. i.e. so that the Higgs field φ is twisted by V^* rather than K_X and, furthermore, $\varphi \wedge \varphi = 0$.

More generally, we can consider integrable $(\lambda, \mathcal{L})$-connections $(E, \nabla_\mathcal{L}, \lambda)$ for each $\lambda \in \mathbb{C}$, these being the analogues of the above mentioned λ-connections (which are hence $(\lambda, \mathcal{T}_X)$-connections). As before, we see that for $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$, an integrable $(0, \mathcal{L})$-connection is just a V^*-twisted Higgs bundle on X and an integrable $(1, \mathcal{L})$-connection is an integrable $\mathcal{L}$-connection. Then, the generalized version of the above Hodge moduli space is the $\mathcal{L}$-Hodge moduli space $\mathcal{M}_{\Lambda_\mathcal{L}^{\mathrm{red}}}(r,d)$ which parameterizes S-equivalence classes of semistable integrable $(\lambda, \mathcal{L})$-connections on X. As before it comes with the natural map $\mathcal{M}_{\Lambda_\mathcal{L}^{\mathrm{red}}}(r,d) \to \mathbb{C}$, $(E, \nabla_\mathcal{L}, \lambda) \mapsto \lambda$, whose fiber over zero is hence the moduli space of rank r and degree d V^*-twisted Higgs bundles and every other fiber is isomorphic to the fiber over 1, namely the moduli space of integrable $\mathcal{L}$-connections of rank r and degree d.

The first main result of this paper is the following (see Theorem 5.15).

Theorem 1.1. *For any Lie algebroid $\mathcal{L} = (L, [\cdot\,,\cdot], \delta)$ such that L is a line bundle with $\deg(L) < 2-2g$, where $g \geq 2$ is the genus of X, and $(r,d) = 1$, the $\mathcal{L}$-Hodge moduli space $\mathcal{M}_{\Lambda_\mathcal{L}^{\mathrm{red}}}(r,d)$ is a smooth semiprojective variety. Moreover the projection $\mathcal{M}_{\Lambda_\mathcal{L}^{\mathrm{red}}}(r,d) \to \mathbb{C}$ is a surjective submersion.*

This generalizes the above mentioned result by Simpson for the Hodge moduli space for any Lie algebroid on the given conditions. The two equivalent interpretations of the "same object" — $\mathcal{L}$-connections and $\Lambda_\mathcal{L}$-modules — are actually required in this proof. For instance, in order to prove smoothness, we make use of the deformation theory for integrable $\mathcal{L}$-connections, developed in [Tor12], since deformation theory for Λ-modules is not yet well-understood in the required generality. On the other hand, the proof of the existence of limits of a natural $\mathbb{C}^*$-action on the $\mathcal{L}$-Hodge moduli (which is a condition for being semiprojective) is carried out by explicitly using $\Lambda_{\mathcal{L}}$-modules rather than $\mathcal{L}$-connections.

Let $K(\mathcal{V}ar_{\mathbb{C}})$ be the Grothendieck ring of varieties and let $\hat{K}(\mathcal{V}ar_{\mathbb{C}})$ be its completion with respect to the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1]$. Let $\mathcal{L} = (L, [\cdot\,,\cdot], \delta)$ be a Lie algebroid of rank 1. Then, using the previous semiprojectivity result, and making use of the Bialynicki-Birula decompositions of both the $\mathcal{L}$-Hodge moduli space and of the L^{-1}-twisted Higgs bundle moduli space $\mathcal{M}_{L^{-1}}(r,d)$, we prove the following result, concerning the motives of the moduli spaces $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ of $\mathcal{L}$-connections on X (note that integrability is automatic because L is a line bundle) and also concerning their Hodge structures and E-polynomials (cf. Theorem 6.7 and Theorem 7.4).

Theorem 1.2. *Suppose the genus of X is $g \geq 2$. Let $\mathcal{L} = (L, [\cdot\,,\cdot], \delta)$ and $\mathcal{L}' = (L', [\cdot\,,\cdot]', \delta')$ be any two Lie algebroids on X, such that L and L' are line bundles with $\deg(L) = \deg(L') < 2-2g$. Suppose $(r,d) = 1$. Then*

$$\mathcal{I}(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)) = \mathcal{I}(\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,d)).$$

where $\mathcal{I}(X)$ denotes one of the following:

1. *The virtual motive $[X] \in \hat{K}(\mathcal{V}ar_{\mathbb{C}})$;*
2. *The Voevodsky motive $M(X) \in \mathrm{DM}^{\mathrm{eff}}(\mathbb{C}, R)$ for any ring R. In this case, moreover, the motives are pure;*
3. *The Chow motive $h(X) \in \mathrm{Chow}^{\mathrm{eff}}(\mathbb{C}, R)$ for any ring R;*
4. *The Chow ring $\mathrm{CH}^\bullet(X, R)$ for any ring R.*

Moreover, the mixed Hodge structures of the moduli spaces are pure and if d' is any integer coprime with r, then

$$E(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)) = E(\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,d')).$$

Finally, if $L = L' = K(D)$ for some effective divisor D, then there is an actual isomorphism of pure mixed Hodge structures

$$H^\bullet(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)) \cong H^\bullet(\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,d')).$$

This is proved as follows. We prove it for $\hat{K}(\mathcal{V}ar_{\mathbb{C}})$ in Theorem 6.7. The results for the other motives (Theorem 7.4) follow by the same techniques using technical results of [HL19]. For Theorem 6.7, firstly we use the semiprojectivity of the $\mathcal{L}$-Hodge and $\mathcal{L}'$-Hodge moduli spaces described before to show that the classes of $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ and $\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,d)$ equals, respectively, that of the twisted Higgs bundles moduli spaces $\mathcal{M}_{L^{-1}}(r,d)$ and $\mathcal{M}_{L'^{-1}}(r,d)$, which correspond to the particular case of trivial algebroids; see Theorem 5.17. Notice that this is a generalization (for $k = \mathbb{C}$) of the "motivic non-abelian Hodge correspondence" of [HL19] to any Lie algebroid $\mathcal{L}$ on the given conditions. Secondly, by studying the Bialynicki-Birula decomposition of $\mathcal{M}_{L^{-1}}(r,d)$, we can prove that its motive only depends on the degree of the twisting line bundle L^{-1}, but not on the twisting itself; see Theorem 6.6.

Notice also that our setting includes, for example, the moduli spaces of logarithmic connections (without fixed residues on the poles), corresponding to the Lie subalgebroid $\mathcal{T}_X(-D) \subset \mathcal{T}_X$, and hence to the case $L^{-1} = K_X(D)$ for an effective and reduced divisor on X. It also includes the moduli spaces of wild connections (again without fixing the corresponding Stokes data), by taking $\mathcal{T}_X(-D') \subset \mathcal{T}_X$, thus $L^{-1} = K_X(D')$, where D' is an effective and non-reduced divisor on X. Hence, we have the following direct application of the above theorem (see Corollary 7.6):

Corollary 1.3. *The motive and E-polynomial of any moduli space of* irregular *connections on a smooth projective curve X of genus at least 2 equals that of any moduli space of* logarithmic *connections X, with singular divisor of the same degree.*

Moreover, using the above theorem, we provide the formulas for the motivic classes and for the E-polynomials of $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$, for any rank 1 Lie algebroid $\mathcal{L}$ of degree less than $2-2g$, when $r=2,3$ and d coprime with r; see Corollaries 7.7 and 7.9.

In addition, for a Lie algebroid $\mathcal{L}$ under the same conditions, we also compute the homotopy groups of the moduli spaces $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$, up to some bound depending on the rank of the $\mathcal{L}$-connection and on the genus of the curve X. See Theorem 7.2.

Our main results hold for Lie algebroids $\mathcal{L}$ whose underlying bundle is a line bundle, but we expect that at least some of them hold for higher rank Lie algebroids (see Remark 5.16). In addition, we prove some results in the higher rank setting, mostly concerning non-emptiness of the moduli spaces of $\mathcal{L}$-connections; see section 3.3.

Here is a brief description of the contents of the paper. In section 2 we give a quick introduction to V-twisted Higgs bundles. In Section 3 we introduce Lie algebroids, sheaves of rings of differential operators, $\mathcal{L}$-connections and Λ-modules and, following [Tor11, Tor12], give a short overview of the equivalence of categories between integrable $\mathcal{L}$-connections and $\Lambda_{\mathcal{L}}$-modules. This is done over any base variety. Then we introduce the moduli spaces of $\Lambda_{\mathcal{L}}$-modules / integrable $\mathcal{L}$-connections over a curve X, prove some non-emptiness results about them, and also introduce the $\mathcal{L}$-Hodge moduli spaces. The purpose of Section 4 is to introduce the Grothendieck ring of varieties and motives and to give an overview of our strategy to prove the main results. Sections 5 and 6 are the technical core of the paper. In Section 5 we prove that the motivic class of the moduli spaces of Lie algebroid connections equals that of the corresponding twisted Higgs bundle moduli (using the semiprojectivity of the $\mathcal{L}$-Hodge moduli, which is also proved) and in Section 6 we show that the motivic class of the twisted Higgs bundles moduli is independent of the twisting. In Section 7 we deduce some applications of the results proved in the previous sections, including the versions of the main theorem for Voevodsky motives, Chow motives and Chow rings. Finally, in Section 8 we show how to achieve similar results for the same moduli spaces of $\mathcal{L}$-connections $(E,\nabla_{\mathcal{L}})$ but where the determinant of E and the trace of $\nabla_{\mathcal{L}}$ are fixed.

Acknowledgments. This research was funded by MICINN (grants MTM2016-79400-P, PID2019-108936GB-C21 and "Severo Ochoa Programme for Centres of Excellence in R&D" SEV-2015-0554) and by CMUP – Centro de Matemática da Universidade do Porto – financed by national funds through FCT – Fundação para a Ciência e a Tecnologia, I.P., under the project with reference UIDB/00144/2020. The first author was also supported by a postdoctoral grant from ICMAT Severo Ochoa project. He would also like to thank the hospitality of CMUP during the research visits which took place in the course of the development of this work. Finally, we thank Vicente Muñoz, José Ángel González and Jaime Silva for helpful discussions.

2. Moduli space of twisted Higgs bundles

Throughout the paper we will only be considering algebraic objects (vector bundles, Lie algebroids, connections, etc.) on smooth projective varieties over $\mathbb{C}$, this being implicitly assumed whenever the corresponding adjective is missing. We will also always take the usual identification between algebraic vector bundles and locally free sheaves.

Let Y be a smooth complex projective variety and let V be an algebraic vector bundle on Y, thus a locally free $\mathcal{O}_Y$-module.

Definition 2.1. *A V-twisted* Higgs bundle *on Y is a pair (E,φ) consisting by an algebraic vector bundle E on Y and a map of $\mathcal{O}_Y$-modules*

$$\varphi : E \longrightarrow E\otimes V,$$

called the Higgs field*, such that $\varphi\wedge\varphi = 0 \in H^0(\mathrm{End}(E)\otimes\Lambda^2 V)$.*

If $\varphi \in H^0(\mathrm{End}(E)\otimes V)$, by $\varphi\wedge\varphi \in H^0(\mathrm{End}(E)\otimes \Lambda^2 V)$ in the previous definition we mean the following. Let $p : V\otimes V \to \Lambda^2 V$ be the quotient map. Then

$$\varphi\wedge\varphi = (\mathrm{Id}_E \otimes p)\circ(\varphi\otimes \mathrm{Id}_V)\circ\varphi.$$

A local version is given as follows. Take an open set $U\subset Y$ where both E and V are trivialized, and let $w_1,\dots,w_k$ be a local trivializing basis of V; then we can write $\varphi|_U = \sum_i G_i\otimes w_i$, with G_i a local section of $\mathrm{End}(E)$, so an $\mathcal{O}_X(U)$-valued matrix, and

$$(\varphi\wedge\varphi)|_U = \sum_{i<j}[G_i,G_j]\otimes w_i\wedge w_j.$$

A map between V-twisted Higgs bundles (E,φ) and (E',φ') is an algebraic map $f : E\to E'$ such that $(f\otimes \mathrm{Id}_V)\circ\varphi = \varphi'\circ f$, and if there is such an f which is an isomorphism, then (E,φ) and (E',φ') are *isomorphic*.

Observe that the integrability condition $\varphi\wedge\varphi = 0$, implies that providing a Higgs field on E is equivalent to endowing a $\mathrm{Sym}^\bullet(V^*)$-module structure on E compatible with the $\mathcal{O}_Y$-module structure.

Consider now the case where Y is a smooth projective curve X. This is going to be our base setting, by further assuming that the genus of X is at least 2, but we will also consider some particular cases where Y is any smooth projective variety. Let also K_X denote the canonical line bundle of X.

Higgs bundles were introduced by Hitchin in [Hit87] for $V = K_X$, in the context of gauge theory, namely as objects which are naturally associated to solutions to the now known as Hitchin equations.

Given a vector bundle E over X, let

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)}$$

be the *slope* of E. The vector bundle E is called *(semi)stable* if for every subbundle $0\neq F\subsetneq E$ we have

$$\mu(F) < \mu(E) \quad (\text{resp.}\ \le).$$

Similarly, a V-twisted Higgs bundle (E,φ) on X is *(semi)stable* if for every subbundle $0\neq F\subsetneq E$ preserved by φ (i.e., such that $\varphi(F)\subseteq F\otimes V$) we have

$$\mu(F) < \mu(E) \quad (\text{resp.}\ \le).$$

By *degree* of a V-twisted Higgs bundle (E,φ) on the curve X, we mean the degree of the underlying bundle E. Let $\mathcal{M}_V(r,d)$ be the moduli space of (S-equivalence classes of) semistable V-twisted Higgs bundles (E,φ) on X of rank r and degree d. When V is a line bundle this moduli space was first constructed, via GIT, by Nitsure [Nit91]. A gauge theoretic construction for $V = K_X$ is given in [Hit87]. In general, $\mathcal{M}_V(r,d)$ exists as a consequence of Simpson's GIT construction of the moduli space of Λ-modules [Sim94] (the general notion of Λ-module will be reviewed in Section 3). It is a quasi-projective complex algebraic variety and, moreover, we have the following Lemma which is a direct consequence of [BGL11, Proposition 3.3 and Theorem 1.2].

Lemma 2.2. *Suppose that $r\ge 1$ and d are coprime and that L is a line bundle with $\deg(L) > 2g-2$, where $g\ge 2$ is the genus of X. Then the moduli space $\mathcal{M}_L(r,d)$ is a smooth connected, so irreducible, quasi-projective variety of dimension $1 + r^2\deg(L)$.*

Remark 2.3. *There are choices of the twist V which do not satisfy the assumptions of the Lemma and for which $\mathcal{M}_V(r,d)$ is nonetheless a smooth variety. For instance, the classical case $V = K_X$ from [Hit87] and which was also considered in [BGL11,* Proposition 3.3*].*

However, a condition similar to the one presented in the previous lemma is to be expected in general since there exist many examples of low degree twists for which the moduli space is not smooth and even not irreducible.

For example, if $L = \mathcal{O}_X(x)$ for a curve X with genus at least 4 and $x \in X$, then $\mathcal{M}_{\mathcal{O}_X(x)}(r,d)$ is, in general, a singular variety. For generic stable vector bundles E, we have $H^0(\mathrm{End}(E)(x)) \cong \mathbb{C}$ by [BGM13, Lemma 2.2] *so $\mathcal{M}_{\mathcal{O}_X(x)}(r,d)$ contains an open subvariety $\mathcal{U} \subset \mathcal{M}_{\mathcal{O}_X(x)}(r,d)$ which is a line bundle over the open locus of such bundles in the moduli space $\mathcal{M}(r,d)$ of stable vector bundles. Nevertheless, suppose that we pick the curve X so that there exists a stable vector bundle F on X with a nonzero section $\varphi \in H^0(\mathrm{End}_0(F)(x))$, $\varphi \neq 0$. Then $(F,\varphi) \in \mathcal{M}_{\mathcal{O}_X(x)}(r,d)$ will not belong to the closure of $\mathcal{U}$, so the moduli space will not be irreducible, and the point $(F,0) \in \mathcal{M}_{\mathcal{O}_X(x)}(r,d)$ will belong to two different irreducible components, making the moduli space a singular variety.*

Consider a V-twisted Higgs bundle (E,φ) on X. Since $\varphi \wedge \varphi = 0$, the Higgs field $\varphi : E \to E \otimes V$ induces maps

$$\wedge^i \varphi : \Lambda^i E \longrightarrow \Lambda^i E \otimes S^i V,$$

for $i = 1, \dots, r$. If we define

$$s_i = \mathrm{tr}(\wedge^i \varphi) \in H^0(S^i V),$$

then this yields the *Hitchin map*

$$H : \mathcal{M}_V(r,d) \longrightarrow \bigoplus_{i=1}^{r} H^0(S^i V), \quad H(E,\varphi) = (s_1, \dots, s_r). \tag{2.1}$$

We call $W := \bigoplus_{i=1}^{r} H^0(S^i V)$ the *Hitchin base.*

Lemma 2.4. *The Hitchin map $H : \mathcal{M}_V(r,d) \longrightarrow W$ is proper.*

Proof. When V is a line bundle, this was proven by Nitsure [Nit91, Theorem 6.1]. In the case $V = K_X$, besides being proven by Hitchin [Hit87], it was also proven by Faltings [Fal93, Theorem I.3] following an argument by Langton, and the same argument works word by word for an arbitrary twisting V. ☐

3. $\mathcal{L}$-connections, Λ-modules and moduli spaces

3.1. **Lie algebroids and $\mathcal{L}$-connections.** Let Y be a smooth complex projective variety with tangent bundle T_Y. We will be mostly interested in the case Y is the smooth projective curve X, but we will actually need to consider a higher dimensional variety in sections 3.4 and 5.3.

Definition 3.1. *An algebraic* Lie algebroid *on Y is a triple $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$ consisting on*

- *an algebraic vector bundle V on Y,*
- *a $\mathbb{C}$-bilinear and skew-symmetric map $[\cdot\,,\cdot] : V \otimes_{\mathbb{C}} V \to V$, called the* Lie bracket,
- *a vector bundle map $\delta : V \to T_Y$, called the* anchor,

satisfying the following properties:

(1) $[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$ (Jacobi rule),
(2) $[u, fv] = f[u,v] + \delta(u)(f)v$ (Leibniz rule),

for any local sections u, v, w of V and any local function f in $\mathcal{O}_Y$.

The rank *of a Lie algebroid $\mathcal{L}$, denoted by $\mathrm{rk}(\mathcal{L})$, is the rank of the underlying vector bundle V.*

Example 3.2.

(1) The canonical example of Lie algebroid is the tangent bundle T_Y, together with the Lie bracket of vector fields and the identity anchor. Denote it as $\mathcal{T}_Y = (T_Y, [\cdot\,,\cdot]_{\mathrm{Lie}}, \mathrm{Id})$.

(2) More generally, an algebraic foliation $F_Y \subset T_Y$ gives also rise to a Lie algebroid $\mathcal{F}_Y$ simply by restricting the Lie bracket $[\cdot\,,\cdot]_{\mathrm{Lie}}$ and by taking the inclusion $F_Y \hookrightarrow T_Y$ as the anchor map.

(3) Any algebraic vector bundle can be seen as a Lie algebroid with the zero bracket and the zero map as the anchor; this is called a trivial algebroid*.*

The Lie bracket of a Lie algebroid $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$ endows V with a structure of a sheaf of $\mathbb{C}$-Lie algebras, which is not a sheaf of $\mathcal{O}_Y$-Lie algebras unless the anchor δ is zero.

A *Lie algebroid map* $f : \mathcal{L} \to \mathcal{L}'$ between $\mathcal{L} = (V, [\cdot\,,\cdot]_V, \delta_V)$ and $\mathcal{L}' = (V', [\cdot\,,\cdot]_{V'}, \delta_{V'})$ is a algebraic $\mathbb{C}$-Lie algebra bundle map $f : V \to V'$ such $\delta_{V'} \circ f = \delta_V$. For example, the anchor $\delta : V \to T_Y$ is a Lie algebroid map $\delta : \mathcal{L} \to \mathcal{T}_Y$. A *Lie algebroid isomorphism* is a Lie algebroid map which is an isomorphism of the underlying bundles; in that case, the Lie algebroids are said to be *isomorphic*.

Let $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$ be a Lie algebroid. We now define a differential on the complex of exterior powers $\Omega^\bullet_{\mathcal{L}} = \Lambda^\bullet V^*$,

$$d_{\mathcal{L}} : \Omega^k_{\mathcal{L}} \longrightarrow \Omega^{k+1}_{\mathcal{L}},$$

generalizing the classical de Rham complex $d : \Omega^k_Y \longrightarrow \Omega^{k+1}_Y$ on $\Omega^\bullet_Y = \Lambda^\bullet T^*_Y$. In degree 0, define $d_{\mathcal{L}} : \mathcal{O}_Y \to V^*$ as the composition of the canonical differential $d : \mathcal{O}_Y \to T^*_Y$ with the dual of the anchor, $\delta^t : T^*_Y \to V^*$. Thus, given $v \in V$ and f a local algebraic function on Y,

$$d_{\mathcal{L}}(f)(v) = df(\delta(v)) = \delta(v)(f). \tag{3.1}$$

The map $d_{\mathcal{L}} : \mathcal{O}_Y \to V^*$ is clearly a V^*-valued derivation. Now we extend it to higher order exterior powers through the usual recursive equation, but using the anchor map δ. For $\omega \in \Omega^n_{\mathcal{L}}$ and $v_1, \dots, v_{n+1}$ local sections of V, take

$$\begin{aligned} d_{\mathcal{L}}(\omega)(v_1, \dots, v_{n+1}) = &\sum_{i=1}^{n+1} (-1)^{i+1} \delta(v_i)(\omega(v_1, \dots, \hat{v}_i, \dots, v_{n+1})) \\ &+ \sum_{1 \le i < j \le n+1} (-1)^{i+j} \omega([v_i, v_j], v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{n+1}). \end{aligned} \tag{3.2}$$

This differential satisfies $d^2_{\mathcal{L}} = 0$, so $(\Omega^\bullet_{\mathcal{L}}, d_{\mathcal{L}})$ is a complex, called the *Chevalley–Eilenberg–de Rham complex* of $\mathcal{L}$. Note that $d_{\mathcal{L}} = 0$ for a trivial algebroid.

From $d_{\mathcal{L}}$ we now define a generalization of an algebraic connection.

Definition 3.3. *Let $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$ be a Lie algebroid on Y. An* $\mathcal{L}$-connection *on Y is a pair $(E, \nabla_{\mathcal{L}})$, where E is an algebraic vector bundle and where $\nabla_{\mathcal{L}}$ is a $\mathbb{C}$-linear algebraic vector bundle map*

$$\nabla_{\mathcal{L}} : E \longrightarrow E \otimes \Omega^1_{\mathcal{L}} = E \otimes V^*,$$

such that

$$\nabla_{\mathcal{L}}(fs) = f\nabla_{\mathcal{L}}(s) + s \otimes d_{\mathcal{L}}(f),$$

for s a local section of E and f a local algebraic function on Y. The rank *of an $\mathcal{L}$-connection is the rank of the underlying algebraic vector bundle.*

We will also refer to $\nabla_{\mathcal{L}} : E \to E \otimes \Omega^1_{\mathcal{L}}$ as an *$\mathcal{L}$-connection on the algebraic vector bundle E.*

Note that, for a trivial algebroid $\mathcal{L}$, the map $\nabla_{\mathcal{L}} : E \to E \otimes \Omega^1_{\mathcal{L}}$ is actually $\mathcal{O}_Y$-linear rather than just $\mathbb{C}$-linear.

Any $\mathcal{L}$-connection $(E, \nabla_{\mathcal{L}})$ can be extended to a map

$$\nabla_{\mathcal{L}} : E \otimes \Omega^\bullet_{\mathcal{L}} \longrightarrow E \otimes \Omega^{\bullet+1}_{\mathcal{L}},$$

as

$$\nabla_{\mathcal{L}}(s \otimes \omega) = \nabla_{\mathcal{L}}(s) \wedge \omega + s \otimes d_{\mathcal{L}}(\omega).$$

Definition 3.4. *Let $(E,\nabla_{\mathcal{L}})$ be an $\mathcal{L}$-connection on Y. The composition $\nabla^2_{\mathcal{L}} : E \to E\otimes \Omega^2_{\mathcal{L}}$ is called the* curvature *of $(E,\nabla_{\mathcal{L}})$. An $\mathcal{L}$-connection is* integrable *or* flat *if its curvature vanishes.*

Example 3.5.

(1) Recall the canonical Lie algebroid $\mathcal{T}_Y = (T_Y, [\cdot\,,\cdot]_{\mathrm{Lie}}, \mathrm{Id})$ given by the tangent bundle of Y. Then, a flat $\mathcal{T}_Y$-connection is just a flat algebraic connection in the usual sense.

(2) If $\mathcal{L}$ is a trivial algebroid, i.e. $\mathcal{L}=(V,0,0)$, then a flat $\mathcal{L}$-connection is simply a pair $(E,\nabla_{\mathcal{L}})$ formed by a vector bundle E and an $\mathcal{O}_Y$-linear map $\nabla_{\mathcal{L}} : E\to E\otimes V^*$ such that $\nabla_{\mathcal{L}}\wedge\nabla_{\mathcal{L}}=0$. This is precisely a V^*-twisted Higgs bundle from Definition 2.1.

Alternatively, we can think of $\mathcal{L}$-connections, for any Lie algebroid $\mathcal{L}=(V,[\cdot\,,\cdot],\delta)$, from a slightly different point of view. An $\mathcal{L}$-connection $\nabla_{\mathcal{L}} : E\to E\otimes\Omega^1_{\mathcal{L}}$ induces a $\mathcal{O}_Y$-linear map

$$\bar{\nabla}_{\mathcal{L}} : V \longrightarrow \mathrm{End}_{\mathbb{C}}(E), \qquad v\mapsto \nabla_{\mathcal{L},v}, \tag{3.3}$$

which, by (3.1), satisfies

$$\nabla_{\mathcal{L},v}(fs) = f\nabla_{\mathcal{L},v}(s) + \delta(v)(f)s. \tag{3.4}$$

In particular, identifying the local function $f\in\mathcal{O}_Y$ with the endomorphism $E\to E$, $s\mapsto fs$, we have that $\nabla_{\mathcal{L},v}\circ f - f\circ\nabla_{\mathcal{L},v}$ is $\mathcal{O}_Y$-linear,

$$\nabla_{\mathcal{L},v}\circ f - f\circ\nabla_{\mathcal{L},v} \in \mathrm{End}_{\mathcal{O}_Y}(E).$$

Hence, $\nabla_{\mathcal{L},v}$ is a section of the locally free sheaf $\mathrm{Diff}^1(E)$ of differentials of order at most 1. Note that, since $\mathrm{End}_{\mathbb{C}}(E)$ is a sheaf of associative algebras, the vector bundle $\mathrm{Diff}^1(E)$ inherits canonically a Lie algebroid structure $\mathcal{D}(E) = (\mathrm{Diff}^1(E), [\cdot\,,\cdot]_{\mathcal{D}(E)}, \delta_{\mathcal{D}(E)})$ through the commutator, by taking

$$[A,B]_{\mathcal{D}(E)} = AB - BA \in \mathrm{Diff}^1(E),$$

for each $A,B\in\mathrm{Diff}^1(E)$, and, by making use of the $\mathcal{O}_Y$-module structure of $\mathrm{Diff}^1(E)$,

$$\delta_{\mathcal{D}(E)}(A)(f) = Af - fA \in \mathcal{O}_Y \subseteq \mathrm{End}_{\mathcal{O}_Y}(E),$$

for every $f\in\mathcal{O}_Y$. Now, note that the $\mathcal{L}$-connection $\nabla_{\mathcal{L}}$ is integrable if and only if

$$[\nabla_{\mathcal{L},u},\nabla_{\mathcal{L},v}]_{\mathcal{D}(E)} = \nabla_{\mathcal{L},[u,v]},$$

for all $u,v\in V$. Furthermore, it follows by (3.4) that, for every v,

$$\delta_{\mathcal{D}(E)}(\nabla_{\mathcal{L},v}) = \delta(v).$$

We conclude that the map (3.3) can be canonically upgraded to a Lie algebroid map $\bar{\nabla}_{\mathcal{L}} : \mathcal{L}\to\mathcal{D}(E)$, sometimes called a *representation* of $\mathcal{L}$ in $\mathcal{D}(E)$. In addition, the correspondence $\nabla_{\mathcal{L}}\mapsto\bar{\nabla}_{\mathcal{L}}$ yields a bijective correspondence

$$\big\{\text{integrable } \mathcal{L}\text{-connections on } E\big\} \longleftrightarrow \big\{\text{representations of } \mathcal{L} \text{ in } \mathcal{D}(E)\big\}. \tag{3.5}$$

3.2. **Λ-modules and $\mathcal{L}$-connections.** In [Tor11, §3] and [Tor12, §4], Tortella proved that there is a very close relation between integrable $\mathcal{L}$-connections and Simpson's notion of Λ-modules [Sim94], to be introduced below. This will be important for us to deduce properties of the corresponding moduli spaces.

Let S be a smooth variety and let $\mathcal{X}\to S$ be a smooth variety over S. We start with the notion of sheaf of rings of differential operators as defined in [Sim94].

Definition 3.6. *A* sheaf of rings of differential operators *on $\mathcal{X}$ over S is a sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras Λ over $\mathcal{X}$, with a filtration by subalgebras $\Lambda_0\subseteq\Lambda_1\subseteq\cdots\subseteq\Lambda$, verifying the following properties:*

(1) $\Lambda=\bigcup_{i=0}^{\infty}\Lambda_i$ and $\Lambda_i\cdot\Lambda_j\subseteq\Lambda_{i+j}$, for every i,j;
(2) $\Lambda_0=\mathcal{O}_{\mathcal{X}}$;
(3) the image of $p^{-1}(\mathcal{O}_S)$ in $\mathcal{O}_{\mathcal{X}}$ lies in the center of Λ;

(4) the left and right $\mathcal{O}_{\mathcal{X}}$-module structures on $\mathrm{Gr}_i(\Lambda) := \Lambda_i/\Lambda_{i-1}$ are equal;
(5) the sheaves of $\mathcal{O}_{\mathcal{X}}$-modules $\mathrm{Gr}_i(\Lambda)$ are coherent;
(6) the morphism of sheaves

$$\mathrm{Gr}_1(\Lambda) \otimes \cdots \otimes \mathrm{Gr}_1(\Lambda) \to \mathrm{Gr}_i(\Lambda)$$

induced by the product is surjective.

Moreover, Λ is said to be polynomial *if $\Lambda \cong \mathrm{Sym}^\bullet(\mathrm{Gr}_1(\Lambda))$ and* almost polynomial *if $\mathrm{Gr}_\bullet(\Lambda) \cong \mathrm{Sym}^\bullet(\mathrm{Gr}_1(\Lambda))$.*

Remark 3.7. *Simpson's definition is a slightly more general in point (2), allowing Λ_0 to be a quotient of $\mathcal{O}_{\mathcal{X}}$ and, therefore, allowing Λ to be supported on a subscheme of $\mathcal{X}$. However, we will only be interested on sheafs of rings supported on the whole scheme $\mathcal{X}$.*

Definition 3.8. *Let Λ be a sheaf of rings of differential operators over $\mathcal{X}$, flat over S. A Λ-*module *is a pair (E, ∇_Λ) consisting of an algebraic vector bundle E on $\mathcal{X}$, flat on S, with an action $\nabla_\Lambda : \Lambda \otimes_{\mathcal{O}_{\mathcal{X}}} E \to E$ satisfying the usual module conditions. Moreover, E is required to be locally free as a Λ-module and its inherent $\mathcal{O}_{\mathcal{X}}$-module structure coincides with the $\mathcal{O}_{\mathcal{X}}$-module structure induced by the inclusion $\mathcal{O}_{\mathcal{X}} \subseteq \Lambda$. The* rank *of a Λ-module is the rank of the underlying bundle.*

The notions of maps and isomorphisms between Λ-modules over $\mathcal{X}$ are the obvious ones, as in the Lie algebroids and Higgs bundles cases.

Let us now briefly recall the relation between integrable $\mathcal{L}$-connections and Λ-modules. For details, see [Tor11, §3] and [Tor12, §4].

Fix a Lie algebroid $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$ on Y. Consider the associated Lie algebroid

$$\widehat{\mathcal{L}} = (\mathcal{O}_Y \oplus V, [\cdot\,,\cdot]_1, \delta_1)$$

given by

$$[(f,u),(g,v)]_1 = (\delta(u)(g) - \delta(v)(f), [u,v]), \qquad \delta_1(f,u) = \delta(u),$$

with $u, v \in V$ and $f, g \in \mathcal{O}_Y$. Note that this means that, if $\mathcal{O}_Y$ also denotes the trivial algebroid $(\mathcal{O}_Y, 0, 0)$, then we have a *split* short exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \widehat{\mathcal{L}} \longrightarrow \mathcal{L} \longrightarrow 0 \tag{3.6}$$

of Lie algebroids, where by this we mean $\widehat{\mathcal{L}} \cong \mathcal{O}_Y \oplus \mathcal{L}$ as Lie algebroids or, equivalently that there is a splitting $\zeta : \mathcal{L} \to \widehat{\mathcal{L}}$ of the sequence (3.6), which in this case is simply given by $\zeta(v) = (0, v)$, $v \in V$.

From the Lie algebroid $\widehat{\mathcal{L}}$, hence from $\mathcal{L}$, there is an associated an almost polynomial sheaf of rings of differential operators $\Lambda_{\mathcal{L}}$ whose weight 1 piece $\Lambda_{\mathcal{L},1} \subset \Lambda_{\mathcal{L}}$ in corresponding filtration is isomorphic to $\mathcal{O}_Y \oplus V$. This is roughly constructed as follows. The universal enveloping algebra of the Lie algebra $(\mathcal{O}_Y \oplus V, [\cdot\,,\,\cdot]_1)$ is the sheaf of $\mathcal{O}_Y$-algebras, defined by

$$U = T^\bullet(\mathcal{O}_Y \oplus V) / \langle x \otimes y - y \otimes x - [x,y]_1 \,|\, x, y \in \mathcal{O}_Y \oplus V \rangle ,$$

with $T^\bullet(\mathcal{O}_Y \oplus V)$ being the tensor algebra on $\mathcal{O}_Y \oplus V$. If $i : \mathcal{O}_Y \oplus V \hookrightarrow U$ is the canonical inclusion, then let $U^\dagger \subset U$ be the subalgebra generated by $i(\mathcal{O}_X \oplus V)$ and take

$$\Lambda_{\mathcal{L}} = U^\dagger / \left(i(f,0) \cdot i(g,v) - i(fg, fv) \,|\, f, g \in \mathcal{O}_Y, v \in V \right).$$

This is the *universal enveloping algebra of the Lie algebroid* $\mathcal{L}$. Then the graded structure from the tensor algebra $T^\bullet(\mathcal{O}_Y \oplus V)$ induces a filtered algebra structure on $\Lambda_{\mathcal{L}}$ which satisfies the axioms of an almost polynomial sheaf of rings of differential operators. We thus get the correspondence

$$\mathcal{L} \mapsto \Lambda_{\mathcal{L}}. \tag{3.7}$$

Since this construction started by taking the trivial extension of Lie algebroids in (3.6), the sheaf $\Lambda_{\mathcal{L}}$ is called a *split* almost polynomial sheaf of differential operators.

Conversely, take an almost polynomial sheaf of rings of differential operators Λ. Then the $\mathcal{O}_Y$-module $\Lambda_1 \subset \Lambda$ inherits a Lie algebroid structure in the following way [Tor11, Proposition 28] [Tor12, §4.1]. Given local sections u, v of Λ_1 and f of $\mathcal{O}_Y$, define

$$[u,v]_{\Lambda_1} = uv - uv \qquad \text{and} \qquad \delta_{\Lambda_1}(u)(f) = uf - fu, \tag{3.8}$$

using the product in Λ. Let us prove that $[u,v]_{\Lambda_1} \in \Lambda_1$ and that $\delta_{\Lambda_1}(u)(f) \in \mathcal{O}_Y$. As Λ is almost polynomial, then $\mathrm{Gr}_\bullet(\Lambda)$ is abelian. Let $\mathrm{sb} : \Lambda \to \mathrm{Gr}_\bullet(\Lambda)$ be the symbol map sending an element to the highest graded class where it is nonzero. One can check that sb is multiplicative. We have

$$\mathrm{sb}(uv) - \mathrm{sb}(vu) = \mathrm{sb}(u)\,\mathrm{sb}(v) - \mathrm{sb}(v)\,\mathrm{sb}(u) = 0 \in \mathrm{Gr}_\bullet(\Lambda)$$

As $uv, vu \in \Lambda_2$, we conclude that $uv \equiv vu \mod \Lambda_1$, that is, $uv - vu \in \Lambda_1$, proving the first claim. For the second claim, we use the fact that the left and right $\mathcal{O}_Y$-module structures on $\mathrm{Gr}_1(\Lambda) = \Lambda_1/\Lambda_0$ agree. Hence $uf - fu \in \Lambda_0 = \mathcal{O}_Y$ and the second claim follows. Now, using the associativity of Λ and the fact that elements of $\mathcal{O}_Y$ commute with all elements of Λ, one sees that $[\cdot\,,\cdot]_{\Lambda_1}$ is $\mathcal{O}_Y$-bilinear, skew-symmetric and verifies the Jacobi identity and that δ_{Λ_1} is so that Leibniz rule also holds. So (3.8) gives the mentioned Lie algebroid structure on Λ_1.

Now, it is clear that the induced Lie algebroid structure on $\mathcal{O}_Y = \Lambda_0 \subset \Lambda_1$ is trivial, thus (3.8) descends to give a Lie algebroid structure on $\mathrm{Gr}_1(\Lambda)$, which we denote by $\mathcal{L}_\Lambda$:

$$\mathcal{L}_\Lambda = (\mathrm{Gr}_1(\Lambda), [\cdot\,,\cdot]_{\mathrm{Gr}_1(\Lambda_1)}, \delta_{\mathrm{Gr}_1(\Lambda_1)}).$$

Moreover, the following short exact sequence of Lie algebroids

$$0 \longrightarrow (\mathcal{O}_X, 0, 0) \longrightarrow (\Lambda_1, [\cdot\,,\cdot]_{\Lambda_1}, \delta_{\Lambda_1}) \longrightarrow \mathcal{L}_\Lambda \longrightarrow 0,$$

hence the sheaf Λ is split. Thus, we have the correspondence

$$\Lambda \mapsto \mathcal{L}_\Lambda. \tag{3.9}$$

By [Tor12, Theorem 4.4], Λ is split almost polynomial if and only if it is isomorphic to the universal enveloping algebra of $\mathrm{Gr}_1(\Lambda)$. Hence, the upshot of this discussion is the following theorem.

Theorem 3.9. *The correspondences* (3.7) *and* (3.9) *induce inverse equivalences of categories:*

$$\left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{Lie algebroids on } Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{isomorphism classes of split} \\ \textit{almost polynomial sheaves of rings} \\ \textit{of differential operators on } Y \end{array} \right\}. \tag{3.10}$$
$$\mathcal{L} \longmapsto \Lambda_{\mathcal{L}}$$

Moreover, (3.10) induces a correspondence between integrable $\mathcal{L}$-connections and $\Lambda_{\mathcal{L}}$-modules. This goes roughly as follows (see again the above mentioned references for details).

Let $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$ be a Lie algebroid with corresponding sheaf $\Lambda_{\mathcal{L}}$. So we have a short exact sequence

$$0 \to \mathcal{O}_Y \to \Lambda_{\mathcal{L},1} \to \mathcal{L} \to 0$$

of Lie algebroids, with

$$\Lambda_{\mathcal{L},1} = \mathcal{O}_Y \oplus \mathcal{L} \subset \Lambda_{\mathcal{L}}.$$

Given a $\Lambda_{\mathcal{L}}$-module $\nabla_{\Lambda_{\mathcal{L}}} : \Lambda_{\mathcal{L}} \otimes E \to E$, define the $\mathcal{L}$-connection $(E, \nabla_{\mathcal{L}})$ by taking

$$\nabla_{\mathcal{L}} : E \to E \otimes V^*, \qquad \nabla_{\mathcal{L}}(s)(v) = \nabla_{\Lambda_{\mathcal{L}}}((0,v) \otimes s), \tag{3.11}$$

for s and v local sections of E and V respectively. This is a flat $\mathcal{L}$-connection.

Conversely, from a flat $\mathcal{L}$-connection $(E, \nabla_{\mathcal{L}})$, define

$$\nabla_{\Lambda_{\mathcal{L},1}} : \Lambda_{\mathcal{L},1} \otimes E \to E, \qquad \nabla_{\Lambda_{\mathcal{L},1}}((f,v) \otimes s) = fs + \nabla_{\mathcal{L}}(s)(v),$$

where $f \in \mathcal{O}_Y$, $v \in V$ and $s \in E$. By successive compositions, this defines a $(\Lambda_{\mathcal{L},1})^{\otimes \bullet}$-module structure on E, which descends to a $\Lambda_{\mathcal{L}}$-module structure

$$\nabla_{\Lambda_{\mathcal{L}}} : \Lambda_{\mathcal{L}} \otimes E \to E \tag{3.12}$$

because $\nabla_{\mathcal{L}}$ is integrable and satisfies Leibniz.

Now, from [Tor12, Proposition 5.3], we have the following.

Theorem 3.10. *The correspondences* (3.11) *and* (3.12) *induce inverse equivalences of categories:*

$$\left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{integrable } \mathcal{L}\textit{-connections on } Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \Lambda_{\mathcal{L}}\textit{-modules on } Y \end{array} \right\}. \tag{3.13}$$

Remark 3.11. *The results of* [Tor11] *and* [Tor12] *are actually more general, in the sense that there is no split condition on the almost polynomial sheaves of differential operators. The correspondence there includes, in both sides, a certain cohomology class Q, which is the obstruction to the existence of a Lie algebroid splitting of the above short exact sequence. The case stated above corresponds to $Q = 0$, which is actually the one of interest to us, since that is the only split case for which the corresponding moduli spaces (to be introduced in the next section) are non-empty.*

3.3. **Moduli spaces of $\mathcal{L}$-connections and of Λ-modules.** Let us return now to the case where Y is our fixed smooth complex projective curve X and where $\mathcal{X} \to S$ is an S-family of smooth complex projective curves over a scheme S. An $\mathcal{L}$-connection $(E, \nabla_{\mathcal{L}})$ is *(semi)stable* if for every subbundle $0 \neq F \subsetneq E$ preserved by $\nabla_{\mathcal{L}}$, i.e. such that $\nabla_{\mathcal{L}}(F) \subseteq F \otimes \Omega^1_{\mathcal{L}}$, we have

$$\mu(F) < \mu(E) \quad (\text{resp. } \leq)$$

and that a Λ-module (E, ∇_Λ) on a fiber X_s of $\mathcal{X} \to S$ is (semi)stable if for every $0 \neq F \subsetneq E$ preserved by ∇_Λ, i.e. such that $\nabla_\Lambda(\Lambda \otimes F) \subseteq F$ we have

$$\mu(F) < \mu(E) \quad (\text{resp. } \leq).$$

Note that if the vector bundle E is semistable, then any $\mathcal{L}$-connection and any Λ-module are also semistable. Clearly, an $\mathcal{L}$-connection is (semi)stable if and only if the corresponding $\Lambda_{\mathcal{L}}$-module is (semi)stable. Define the *degree* of an $\mathcal{L}$-connection or of a Λ-module over X to be the degree of the underlying vector bundle.

For any sheaf of rings of differential operators Λ on $\mathcal{X}$ over S, and any rank $r \geq 0$ and $d \in \mathbb{Z}$, Simpson [Sim94] proved that there exists a moduli space $\mathcal{M}_\Lambda(\mathcal{X}, r, d)$ of semistable Λ-modules of rank r and degree d and that it is a complex quasi-projective variety over S. If the family $\mathcal{X}$ is clear from the context, we denote the moduli space simply by $\mathcal{M}_\Lambda(r, d)$. If $s \in S$ is a point, let Λ_s be the pullback of Λ to the fiber X_s of $\mathcal{X} \to S$. A closed point of $\mathcal{M}_\Lambda(\mathcal{X}, r, d)$ over a point $s \in S$ represents a semistable Λ_s-module (E, ∇_{Λ_s}) over X_s.

Then, Theorem 3.10 also identifies $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r, d)$ with the moduli space of integrable $\mathcal{L}$-connections of rank r and degree d [Tor12].

Remark 3.12. *Alternatively, a moduli space of (possibly non-integrable) Lie algebroid connections was also constructed by Krizka* [Kri09] *using analytic tools. When $\mathcal{L}$ has rank 1, integrability is automatic and, therefore, this construction gives an alternative description for the moduli space of integtrable Lie algebroid connections. On the other hand, in an unpublished work* [Kri10]*, the same author works with flat connections and proposes a general analytical construction of such moduli space.*

We saw in Example 3.5 that when $\mathcal{L}$ has the trivial algebroid structure $\mathcal{L} = (V, 0, 0)$, an $\mathcal{L}$-connection is precisely the same thing as a V^*-twisted Higgs bundle. Moreover, it is obvious that (semi)stability is preserved under this identification. Hence, the moduli space of integrable $\mathcal{L}$-connections coincides with the moduli space of V^*-twisted Higgs bundles. Therefore, for a trivial Lie algebroid $\mathcal{L} = (V, 0, 0)$, we have the following canonical identification

$$\mathcal{M}_{\Lambda_{(V,0,0)}\mathcal{L}}(r, d) = \mathcal{M}_{V^*}(r, d)$$

between the moduli spaces of integrable $\mathcal{L}$-connections, $\Lambda_{\mathcal{L}}$-modules and V^*-twisted Higgs bundles over X, all of the same rank r and degree d.

Note that, depending on the choice of the rank, degree and Λ, the moduli space $\mathcal{M}_\Lambda(\mathcal{X}, r, d)$ may be singular or even empty. For instance, if Λ is the sheaf of differential operators $\mathcal{D}_X$ on a curve X, then $\mathcal{M}_{\mathcal{D}_X}(r, d)$ coincides with the moduli space of semistable ($\mathcal{T}_X$-)connections on X whose underlying vector bundle has rank r and degree d. But the flatness condition (which is automatic since $\mathcal{T}_X$ is a line bundle) implies that $d = 0$, so the moduli space is empty for $d \neq 0$. Moreover, for $d = 0$ and $r \geq 2$, the moduli space of flat connections is singular due to the existence of strictly semistable objects. In the remaining part of this section we will to prove sufficient conditions for these moduli spaces to be nonempty and smooth varieties.

The existence of algebraic connections (i.e. $\mathcal{T}_X$-connections in the language of Lie algebroids) on an algebraic vector bundle E over X has been studied by Atiyah in [Ati57]. There it was proved the existence of a class in $\mathrm{Ext}^1(T_X, \mathrm{End}(E))$ — now known as the Atiyah class of E — whose vanishing is equivalent to the existence of such a algebraic connection. A generalization of this picture to $\mathcal{L}$-connections was carried out in [Tor11, Section 2.4.4] [Tor12, §4]. Keep considering the bundle E and let $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$ be a Lie algebroid. Let $\mathfrak{a}(E) \in \mathrm{Ext}^1(T_X, \mathrm{End}(E))$ be the Atiyah class of E. Define the *$\mathcal{L}$-Atiyah class of E* as

$$\mathfrak{a}_{\mathcal{L}}(E) = \delta^*(\mathfrak{a}(E)) \in \mathrm{Ext}^1(V, \mathrm{End}(E)).$$

Proposition 3.13. [Tor11, Proposition 17] *An algebraic vector bundle E admits an algebraic $\mathcal{L}$-connection if and only if $\mathfrak{a}_{\mathcal{L}}(E) = 0$.*

From this we can exhibit some concrete examples of $\mathcal{L}$-connections.

Corollary 3.14. *Let E be a semistable vector bundle on the curve X. Let $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$ be a Lie algebroid such that the vector bundle V is semistable and $-\mu(V) > 2g - 2$. Then E admits an $\mathcal{L}$-connection. Moreover, if $\mathrm{rk}(\mathcal{L}) = 1$, then E admits an integrable $\mathcal{L}$-connection.*

Proof. The $\mathcal{L}$-Atiyah class $\mathfrak{a}(E)$ is an element of

$$\mathrm{Ext}^1(V, \mathrm{End}(E)) \cong H^1(\mathrm{End}(E) \otimes V^*) \cong H^0(\mathrm{End}(E) \otimes K_X \otimes V)^*,$$

by Serre duality. Since both E and V are semistable then $\mathrm{End}(E) \otimes K_X \otimes V$ is also semistable. Moreover

$$\mu(\mathrm{End}(E) \otimes K_X \otimes V) = 2g - 2 + \mu(V) < 0,$$

so $H^0(\mathrm{End}(E) \otimes K_X \otimes V) = 0$. Thus $\mathfrak{a}(E) = 0$, and the result follows from Proposition 3.13.

If in addition $\mathrm{rk}(\mathcal{L}) = 1$, then $\Omega^2_{\mathcal{L}} = \Lambda^2 V^* = 0$ thus any $\mathcal{L}$-connection on E is automatically flat. □

A Lie algebroid $\mathcal{L}$ is called *transitive* if the anchor δ is surjective and *intransitive* otherwise.

Corollary 3.15. *Let $\mathcal{L}$ be any intransitive Lie algebroid on X and let E be a semistable algebraic vector bundle. Then E admits an integrable $\mathcal{L}$-connection.*

Proof. As $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$ is intransitive, the anchor map $\delta : V \to T_X$ is not surjective. Then, as T_X is a line bundle, there exists a point $x \in X$ such that $\delta|_x = 0$. Thus, δ factors through $\bar{\delta} : V \to T_X(-x) \subset T_X$. Actually, the line bundle $T_X(-x)$ inherits a natural Lie algebroid structure, denoted by $\mathcal{T}(-x)$, from the one of $\mathcal{T} = (T_X, [\cdot\,,\cdot]_{\mathrm{Lie}}, \mathrm{Id})$, since the Lie bracket of two local vector fields which annihilate at x also annihilates at x. Thus, the anchor in $T_X(-x)$ is just the inclusion $T_X(-x) \hookrightarrow T_X$. Since we also know the anchor maps are also Lie algebroid maps, we have a commutative diagram of Lie algebroid maps

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\delta} & \mathcal{T} \\ & \searrow^{\bar{\delta}} & \uparrow \\ & & \mathcal{T}(-x). \end{array}$$

Now, $T_X(-x)$ is a line bundle such that $-\deg(T_X(-x)) = 2g-1 > 2g-2$, so the previous corollary shows that E admits an integrable $\mathcal{T}(-x)$-connection. Moreover, by (3.5) such connection is determined by a representation $\mathcal{T}(-x) \to \mathcal{D}(E)$. Pre-composing it with $\bar{\delta}$ yields a representation $\mathcal{L} \to \mathcal{D}(E)$ and, hence, again by (3.5), gives rise to an integrable $\mathcal{L}$-connection on E. □

The previous results can be now immediately used to prove, under certain conditions, non-emptiness of the moduli spaces of $\mathcal{L}$-connections of any rank and degree.

Proposition 3.16. *For any rank r and degree d and any Lie algebroid $\mathcal{L}$ such that either*

(1) $\mathrm{rk}(\mathcal{L}) = 1$ *and* $\deg(\mathcal{L}) < 2-2g$, *or*
(2) $\mathcal{L}$ *is intransitive.*

Then the moduli space $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ is nonempty.

Proof. Let $\mathcal{L}$ be any Lie algebroid verifying either of the two given conditions. Choose any semistable vector bundle E over X. By Corollary 3.14, if $\mathrm{rk}(\mathcal{L}) = 1$ and $\deg(\mathcal{L}) < 2-2g$, or by Corollary 3.15, if $\mathcal{L}$ is intransitive, we conclude that E admits an integrable $\mathcal{L}$-connection $\nabla_{\mathcal{L}}$. Moreover, since E is semistable, then so is $(E, \nabla_{\mathcal{L}})$, which therefore represents a point in $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$. □

3.4. **The $\mathcal{L}$-Hodge moduli spaces.** Let us now introduce the deformation of a split almost polynomial sheaf of rings of differential operators (cf. Definition 3.6) to its associated graded ring, as described in [Sim94, p. 86].

Let $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$ be a Lie algebroid over X and let $\Lambda_{\mathcal{L}}$ be its associated split almost polynomial sheaf of rings of differential operators. By construction, we know that

$$\mathrm{Gr}_\bullet(\Lambda_{\mathcal{L}}) \cong \mathrm{Sym}^\bullet(V).$$

We can associate to $\Lambda_{\mathcal{L}}$ a sheaf of rings of differential operators $\Lambda_{\mathcal{L}}^{\mathrm{red}}$ on $X \times \mathbb{C}$ over $\mathbb{C}$ whose fiber over 1 is $\Lambda_{\mathcal{L}}$ and whose fiber over 0 is isomorphic to its graded algebra $\mathrm{Sym}^\bullet(V)$. Let λ be the coordinate of $\mathbb{C}$ and let $p_X : X \times \mathbb{C} \to X$ be the projection. We define $\Lambda_{\mathcal{L}}^{\mathrm{red}}$ as the subsheaf of $p_X^*(\Lambda_{\mathcal{L}})$ generated by sections of the form $\sum_{i \geq 0} \lambda^i u_i$, where u_i is a local section of $\Lambda_{\mathcal{L},i} \subseteq \Lambda_{\mathcal{L}}$. This subsheaf is a sheaf of filtered algebras on $X \times \mathbb{C}$ over $\mathbb{C}$, which coincides with the Rees algebra construction of the sheaf of filtered algebras Λ and one can verify that it satisfies all properties of Definition 3.6, making it a sheaf of rings of differential operators on $X \times \mathbb{C}$ over $\mathbb{C}$.

On the other hand, given the Lie algebroid $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$ and $\lambda \in \mathbb{C}$, we can define another Lie algebroid $\mathcal{L}_\lambda$ as

$$\mathcal{L}_\lambda = (V, \lambda[\cdot\,,\cdot], \lambda\delta), \tag{3.14}$$

Then, the following can be checked through direct computation

$$\Lambda_{\mathcal{L}}^{\mathrm{red}}|_{X \times \{\lambda\}} \cong \Lambda_{\mathcal{L}_\lambda}. \tag{3.15}$$

Observe that if $\lambda \neq 0$ then multiplicacion by λ defines an isomorphism of Lie algebroids

$$\mathcal{L}_\lambda \xrightarrow{\sim} \mathcal{L}, \qquad v \mapsto \lambda v$$

and for $\lambda = 0$, we have that $\mathcal{L}_0 = (V, 0, 0)$ is the trivial algebroid over V. This implies the following properties of the fibers of $\Lambda_{\mathcal{L}}^{\mathrm{red}}$ over $\lambda \in \mathbb{C}$ which are also well known consequences of the Rees construction.

(1) For every $\lambda \neq 0$ we have $\Lambda_{\mathcal{L}}^{\mathrm{red}}|_{X\times\{\lambda\}} \cong \Lambda_{\mathcal{L}}^{\mathrm{red}}|_{X\times\{1\}} \cong \Lambda_{\mathcal{L}}$.
(2) If $\lambda = 0$ then $\Lambda_{\mathcal{L}}^{\mathrm{red}}|_{X\times\{0\}} \cong \mathrm{Gr}^\bullet(\Lambda_{\mathcal{L}}) \cong \mathrm{Sym}^\bullet(V)$.

Now, consider the moduli space $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ of $\Lambda_{\mathcal{L}}^{\mathrm{red}}$-modules of rank r and degree d. By Simpson's construction, it is a quasiprojective variety with a map

$$\pi : \mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d) \longrightarrow \mathbb{C} \tag{3.16}$$

and, by (3.15), it parametrizes S-equivalence classes of triples $(E, \nabla_{\mathcal{L}_\lambda}, \lambda)$ where $\lambda = \pi(E, \nabla_{\mathcal{L}_\lambda}, \lambda) \in \mathbb{C}$ and $(E, \nabla_{\mathcal{L}_\lambda})$ is a semistable integrable $\mathcal{L}_\lambda$-connection of rank r and degree d on X. By properties (1) and (2) above, it becomes clear that

$$\pi^{-1}(\lambda) = \mathcal{M}_{\Lambda_{\mathcal{L}_\lambda}}(r,d) \cong \mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d) = \pi^{-1}(1), \text{ for every } \lambda \in \mathbb{C}^* \quad \text{and} \quad \pi^{-1}(0) \cong \mathcal{M}_{V^*}(r,d). \tag{3.17}$$

so $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ is a variety which "interpolates" between the moduli space lf $\mathcal{L}$-connections and the moduli space of V^*-twisted Higgs bundles.

Let us explore more precisely what is an $\mathcal{L}_\lambda$-connection, for $\mathcal{L}_\lambda$ as in (3.14). By (3.1) and (3.2) we see that if $(\Omega_{\mathcal{L}}^\bullet, d_{\mathcal{L}})$ is the Chevalley–Eilenberg–de Rham complex of $\mathcal{L}$, then $\Omega_{\mathcal{L}_\lambda}^\bullet = \Omega_{\mathcal{L}}^\bullet$ and $(\Omega_{\mathcal{L}_\lambda}^\bullet, \lambda d_{\mathcal{L}})$ is the Chevalley–Eilenberg–de Rham complex of $\mathcal{L}_\lambda$. Hence, in an $\mathcal{L}_\lambda$-connection $(E, \nabla_{\mathcal{L}_\lambda})$, the map $\nabla_{\mathcal{L}_\lambda} : E \to E \otimes V^*$ verifies

$$\nabla_{\mathcal{L}_\lambda}(fs) = f\nabla_{\mathcal{L}_\lambda}(s) + \lambda s \otimes d_{\mathcal{L}}(f). \tag{3.18}$$

Motivated by the classical example of a λ-*connection* on an algebraic vector bundle E (see the example below), we also call a $\mathcal{L}_\lambda$-connection to be a $(\lambda, \mathcal{L})$-connection, and consider it as a triple $(E, \nabla_{\mathcal{L}}, \lambda)$, where $\nabla_{\mathcal{L}} : E \to E \otimes V^*$ verifies (3.18). Then we identify $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ with the moduli space of $(\lambda, \mathcal{L})$-connections and, in analogy with the terminology for the moduli space of λ-connections, we call it the $\mathcal{L}$-*Hodge moduli space.*

Example 3.17.

(1) A $(\lambda, \mathcal{T}_X)$-connection is a λ-connection *on an algebraic vector bundle E.*
(2) An $\mathcal{L}$-connection is a the same thing as a $(1, \mathcal{L})$-connection.
(3) For $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$, a $(0, \mathcal{L})$-connection is a V^*-twisted Higgs bundle.

The moduli space $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ is endowed with a $\mathbb{C}^*$-action scaling the $(\lambda, \mathcal{L})$-connection,

$$t \cdot (E, \nabla_{\mathcal{L}}, \lambda) = (E, t\nabla_{\mathcal{L}}, t\lambda), \ t \in \mathbb{C}. \tag{3.19}$$

Indeed, since $(E, \nabla_{\mathcal{L}})$ is flat and semistable, then so is $(E, t\nabla_{\mathcal{L}})$ for every t. Furthermore, $(E, t\nabla_{\mathcal{L}_\lambda})$ is a $(t\lambda, \mathcal{L})$-connection. Thus the map π is $\mathbb{C}^*$-equivariant with respect to this $\mathbb{C}^*$-action and the standard one on $\mathbb{C}$.

Note that for $\lambda = 0$, the action (3.19) restricts to the usual $\mathbb{C}^*$-action on the Higgs bundle moduli space by scaling the Higgs field (cf. (5.8)), and which will play an important role in section 6.

Finally, observe that since it is clear that $(E, t\nabla_{\mathcal{L}}, t)$ is semistable if and only if $(E, \nabla_{\mathcal{L}}, 1)$ is, so the $\mathbb{C}^*$-action (3.19) induces an isomorphism

$$\pi^{-1}(\mathbb{C}^*) \cong \pi^{-1}(1) \times \mathbb{C}^* = \mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d) \times \mathbb{C}^*$$

4. Grothendieck ring of varieties, motives and E-polynomials

4.1. **Grothendieck ring of varieties, motives and E-polynomials.** The main goal of this paper is to compare the class of the moduli spaces $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ in the Grothendieck ring of varieties by varying the Lie algebroid $\mathcal{L}$. In this brief section we recall such ring and its basic properties.

Denote by $\mathcal{V}ar_{\mathbb{C}}$ the category of quasi-projective varieties over $\mathbb{C}$. For each $Y \in \mathcal{V}ar_{\mathbb{C}}$, let $[Y]$ denote the corresponding isomorphism class. Consider the group obtained by the free abelian group on isomorphism classes $[Y]$, modulo the relation

$$[Y] = [Y'] + [Y \setminus Y'],$$

where $Y' \subset Y$ is a Zariski-closed subset. In particular, in such group,

$$[Y] + [Z] = [Y \sqcup Z],$$

where $\sqcup$ denotes disjoint union. If we define the product

$$[Y] \cdot [Z] = [Y \times Z],$$

in this quotient, then we obtain a commutative ring, known as the *Grothendieck ring of varieties* and denoted by $K(\mathcal{V}ar_{\mathbb{C}})$. Then $0 = [\emptyset]$ and $1 = [\mathrm{Spec}(\mathbb{C})]$ are the additive and multiplicative units of this ring.

The following is an extremely useful property of $K(\mathcal{V}ar_{\mathbb{C}})$, which follows directly from the definitions, and which we will repeatedly use without further notice.

Proposition 4.1. *If $\pi : Y \to B$ is an algebraic fiber bundle (thus Zariski locally trivial), with fiber F, then $[Y] = [F] \cdot [B]$.*

The class of the affine line, sometimes called the *Lefschetz object*, is denoted by

$$\mathbb{L} := [\mathbb{A}^1] = [\mathbb{C}].$$

Of course, $\mathbb{L}^n = [\mathbb{A}^n] = [\mathbb{C}^n]$. We will consider the localization $K(\mathcal{V}ar_{\mathbb{C}})[\mathbb{L}^{-1}]$, and then the dimensional completion

$$\hat{K}(\mathcal{V}ar_{\mathbb{C}}) = \left\{ \sum_{r\geq 0} [Y_r]\,\mathbb{L}^{-r} \,\middle|\, [Y_r] \in K(\mathcal{V}ar_{\mathbb{C}}) \text{ with } \dim Y_r - r \longrightarrow -\infty \right\}.$$

Notice that we have a map $K(\mathcal{V}ar_{\mathbb{C}}) \longrightarrow \hat{K}(\mathcal{V}ar_{\mathbb{C}})$. Observe also that $\mathbb{L}^n - 1$ is invertible in $\hat{K}(\mathcal{V}ar_{\mathbb{C}})$, for every n, with inverse equal to $-\sum_{k=0}^{\infty} \mathbb{L}^{-kn}$. This is the reason why we had to introduce the completion $\hat{K}(\mathcal{V}ar_{\mathbb{C}})$: there will be computations in which we will need to invert elements of the form $\mathbb{L}^n$ or $\mathbb{L}^n - 1$.

In this paper, by motive we mean the following.

Definition 4.2. *Let Y be a quasi-projective variety. The class $[Y]$ in $K(\mathcal{V}ar_{\mathbb{C}})$ or in $\hat{K}(\mathcal{V}ar_{\mathbb{C}})$ is called the* motive*, or* motivic class*, of Y.*

There are other notions of motive in different, but related, categories, such as Chow motive or Voevodsky motive. These will not appear anywhere in this paper, except in section 7.2.

The motive $[Y]$ is an important invariant of Y, from which it is possible to read of geometric information, such as the E-polynomial. If $Y \in \mathcal{V}ar_{\mathbb{C}}$ is d-dimensional variety, with pure Hodge structure, then its *E-polynomial* is defined as

$$E(Y) = E(Y)(u,v) = \sum_{i=0}^{d} h_c^{p,q}(Y) u^p v^q,$$

where $h_c^{p,q}(Y)$ stands for the dimension of the compactly supported cohomology groups $H_c^{p,q}(Y)$. For instance, $E(\mathbb{C}) = uv$. Actually, the E-polynomial can be seen as a ring map

$$E : \hat{K}(\mathcal{V}ar_{\mathbb{C}}) \longrightarrow \mathbb{Z}[u,v]\left[\left[\frac{1}{uv}\right]\right] \tag{4.1}$$

with values in the Laurent series in uv, which takes values in $\mathbb{Z}[u,v]$ when restricted to $K(\mathcal{V}ar_{\mathbb{C}})$. Hence two varieties with the same motive have the same E-polynomial. In particular, $Y' \subset Y$ is a closed subvariety, then $E(Y) = E(Y') + E(Y \setminus Y')$ and for an algebraic fiber bundle $Y \to B$ with fiber F, we have $E(Y) = E(F)E(B)$.

4.2. **The plan.** The goal of this paper is to prove that given any two rank 1 Lie algebroids $\mathcal{L}$ and $\mathcal{L}'$, over the genus g curve X, such that $\deg(\mathcal{L}) = \deg(\mathcal{L}') < 2-2g$, we have an equality of motivic classes

$$[\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)] = [\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,d)] \in \hat{K}(\mathcal{V}ar_{\mathbb{C}}),$$

where d is coprime with r. We will prove this in two steps.

Take any rank 1 Lie algebroid $\mathcal{L} = (L, [\cdot\,,\cdot], \delta)$ with $\deg(L) < 2-2g$ on X. Take $r \geq 1$ and $d \in \mathbb{Z}$ such that $(r,d) = 1$. We first prove that, under the given assumptions, the motive of the moduli space of $\mathcal{L}$-connections is invariant with respect to the Lie algebroid structure on L, i.e., that the motive of the moduli space of rank r and degree d flat $\mathcal{L}$-connections is the same as the motive of the moduli space of flat $\mathcal{L}_0 = (L,0,0)$-connections (i.e., of flat $(0,\mathcal{L})$-connections) of the same rank and degree, which just means the moduli space of L^{-1}-twisted Higgs bundles or rank r and degree d,

$$[\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)] = [\mathcal{M}_{L^{-1}}(r,d)].$$

Then, we prove that $[\mathcal{M}_{L^{-1}}(r,d)]$ is also invariant with respect to the choice of the twisting line bundle L^{-1} as long as we fix its degree. Thus, we prove that for any pair of line bundles L, L' with $\deg(L) = \deg(L') < 2-2g$ and any d coprime with r we have

$$[\mathcal{M}_{L^{-1}}(r,d)] = [\mathcal{M}_{L'^{-1}}(r,d)].$$

Combining the two results we conclude that

$$[\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)] = [\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,d)],$$

for every $\mathcal{L}$ and $\mathcal{L}'$ of degree less than $2-2g$ and every d and r coprime.

In particular, such equalities imply by (4.1) that the E-polynomials of these moduli spaces are also equal. Then, we can go further by using results by Maulik–Shen [MS20a] or of Groechening, Wyss and Ziegler [GWZ20] which imply that for every line bundle $N \to X$ such that $\deg(N) > 2g-2$, and every d and d' coprime with r, we have

$$E(\mathcal{M}_N(r,d)) = E(\mathcal{M}_N(r,d')).$$

Hence, this implies, together with the above equality of motivic classes, that

$$E(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)) = E(\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,d')),$$

for every $\mathcal{L}$ and $\mathcal{L}'$ of degree less than $2-2g$ and every d and d' coprime to r.

5. Invariance of the motive and E-polynomial with respect to the algebroid structure

As outlined in the previous section, our first objective is to prove that the motive (and, thus, the E-polynomial) of the moduli space of $\mathcal{L}$-connections is invariant with respect to the algebroid structure by proving that given a rank 1 algebroid $\mathcal{L} = (L, [\cdot\,,\cdot], \delta)$ such that $\deg(L) < 2-2g$ we have

$$[\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)] = [\mathcal{M}_{L^{-1}}(r,d)]$$

In order to prove it, we will generalize the strategy used by Hausel and Rodriguez-Villegas to show that the moduli spaces of Higgs bundles and certain logarithmic connections share the same E-polynomials [HRV15] by using the semiprojectivity of the Hodge moduli space (moduli space of λ-connections). In our case, we will show that for every algebroid $\mathcal{L}$ satisfying the given hypothesis the moduli space $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ is a smooth semiprojective variety over $\mathbb{C}$ which interpolates between $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ and $\mathcal{M}_{L^{-1}}(r,d)]$. The $\mathbb{C}^*$-action on this moduli space induces Bialynicki-Birula stratifications on $\mathcal{M}_{L^{-1}}(r,d)$ and $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ which allow us to decompose the corresponding motives and prove the desired motivic equality.

In order to follow this approach on motivic classes (Hausel and Rodriguez-Villegas results were explicit equalities of Hodge structures), we will first explore some results on motivic decompositions of semiprojective varieties. Then we will prove the regularity properties of each of the involved moduli spaces needed to guarantee that the interpolating scheme is a semiprojective variety inducing the desired motivic equality.

5.1. **Bialynicki-Birula stratification.** Let Y be a smooth quasi-projective variety endowed with an algebraic $\mathbb{C}^*$-action, denoted as $Y \to Y$, $x \mapsto t \cdot x$, $x \in Y$, $t \in \mathbb{C}^*$.

Definition 5.1. [HRV15, Definition 1.1] *The variety Y is* semiprojective *if the following conditions are satisfied:*

(1) for each $x \in Y$ the limit $\lim_{t\to 0} t \cdot x$ exists in Y;
(2) the fixed-point locus of the $\mathbb{C}^*$ action $Y^{\mathbb{C}^*}$ is proper.

Every semiprojective variety Y admits a canonical stratification as follows. Let

$$Y^{\mathbb{C}^*} = \bigcup_{\mu \in I} F_\mu$$

be the decomposition of the $\mathbb{C}^*$-fixed-point loci into connected components. Then, for each F_μ, we can consider the subsets

$$U_\mu^+ = \{x \in \mathcal{M} \mid \lim_{t\to 0} t \cdot x \in F_\mu\} \quad \text{and} \quad U_\mu^- = \{x \in \mathcal{M} \mid \lim_{t\to \infty} t \cdot x \in F_\mu\}.$$

By (1) of Definition 5.1, every point in Y belongs to exactly one of the subsets U_μ^+, hence there is a decomposition

$$Y = \bigcup_{\mu \in I} U_\mu^+,$$

called the *Bialynicki-Birula decomposition* of Y.

The arguments in [BB73, SS4], [Kir84] and [HRV15, SS1] prove the following lemma (see [HL19, Apendix A] for a compact complete proof).

Lemma 5.2. *Using the above notations, the following properties hold.*

(1) For every $\mu \in I$, the map $U_\mu^+ \to F_\mu$ defined by $x \mapsto \lim_{t\to 0} t \cdot x$ and the map $U_\mu^- \to F_\mu$ given by $x \mapsto \lim_{t\to\infty} t \cdot x$ are Zariski locally trivial fibrations in affine spaces.
(2) For every $\mu \in I$, U_μ^+ is a locally closed subset of Y.
(3) There exists an order of the components $\{\mu_i\}_{i=1}^n$ such that

$$0 \subset U_{\mu_1}^+ \subset \ldots \subset \bigcup_{i \leq j} U_{\mu_i}^+ \subset \ldots \subset \bigcup_{i=1}^n U_{\mu_i}^+ = Y$$

is a stratification of Y.

For each $p \in F_\mu$ the tangent space T_pY splits as follows

$$T_pY = T_p(U_\mu^+|_p) \oplus T_pF_\mu \oplus T_p(U_\mu^-|_p).$$

Define

$$N_\mu^+ = \dim T_p(U_\mu^+|_p), \quad N_\mu^0 = \dim T_pF_\mu \quad \text{and} \quad N_\mu^- = \dim T_p(U_\mu^-|_p). \tag{5.1}$$

Clearly $N_\mu^+ = \dim(U_\mu^+) - \dim(F_\mu)$ is the rank of the affine bundle $U_\mu^+ \to F_\mu$ and, as we assumed that Y is smooth,

$$N_\mu^+ + N_\mu^0 + N_\mu^- = \dim Y. \tag{5.2}$$

Lemma 5.3. *Let Y be a smooth complex semiprojective variety and consider the above notations. Then the motivic class of Y decomposes as*

$$[Y] = \sum_{\mu \in I} \mathbb{L}^{N_\mu^+}[F_\mu].$$

Proof. As Y is semiprojective, we have a Bialynicki-Birula decomposition which, in virtue of properties (2) and (3) of Lemma 5.2, forms a stratification

$$0 \subset U_{\mu_1}^+ \subset \ldots \subset \bigcup_{i \leq j} U_{\mu_i}^+ \subset \ldots \subset \bigcup_{i=1}^n U_{\mu_i}^+ = Y.$$

As the Grothendieck class is additive on closed subvarieties, we have

$$[Y] = \sum_{\mu \in I} [U_\mu^+]. \tag{5.3}$$

By property (1) of Lemma 5.2, each $\mu \in I$ is a Zariski locally trivial affine fibration over F_μ whose fiber has dimension N_μ^+, so we have

$$[U_\mu^+] = [\mathbb{C}^{N_\mu^+}][F_\mu] = \mathbb{L}^{N_\mu^+}[F_\mu]. \tag{5.4}$$

Now, (5.3) and (5.4) prove the lemma. □

On the other hand, we have the following proposition that is a generalization for motives of [HRV15, Corollary 1.3.3].

Proposition 5.4. *Let Y be a smooth complex semiprojective variety together with a surjective $\mathbb{C}^*$-equivariant submersion $\pi : Y \to \mathbb{C}$ covering the standard scaling action on $\mathbb{C}$. Then in $\hat{K}(\mathcal{V}ar_\mathbb{C})$ we have*

$$[\pi^{-1}(0)] = [\pi^{-1}(1)] \quad \text{and} \quad [Y] = \mathbb{L}[\pi^{-1}(0)].$$

Proof. Clearly, the fixed-point locus of Y is concentrated in $\pi^{-1}(0)$. As $\pi^{-1}(0)$ is a smooth closed subspace of Y, then $\pi^{-1}(0)$ is also a smooth semiprojective variety. Moreover, since π is $\mathbb{C}^*$-equivariant, then the fixed points of the $\mathbb{C}^*$-action on Y are precisely those of $\pi^{-1}(0)$. Let

$$Y^{\mathbb{C}^*} = \pi^{-1}(0)^{\mathbb{C}^*} = \bigcup_{\mu \in I} F_\mu$$

be the decomposition of the fixed-point locus into connected components. As we have discussed above, both Y and $\pi^{-1}(0)$ admit Bialynicki-Birula stratifications of the form

$$Y = \bigcup_{\mu \in I} \tilde{U}_\mu^+ \quad \text{and} \quad \pi^{-1}(0) = \bigcup_{\mu \in I} U_\mu^+,$$

where

$$\tilde{U}_\mu^+ = \left\{ p \in Y \,\middle|\, \lim_{t \to 0} t \cdot p \in F_\mu \right\} \quad \text{and} \quad U_\mu^+ = \left\{ p \in \pi^{-1}(0) \,\middle|\, \lim_{t \to 0} t \cdot p \in F_\mu \right\}.$$

Moreover, $\tilde{U}_\mu^+$ and U_μ^+ are affine bundles over F_μ of rank $\tilde{N}_\mu^+$ and N_μ^+ respectively. On the other hand, let

$$\tilde{U}_\mu^- = \left\{ p \in Y \,\middle|\, \lim_{t\to\infty} t\cdot p \in F_\mu \right\} \quad \text{and} \quad U_\mu^- = \left\{ p \in \pi^{-1}(0) \,\middle|\, \lim_{t\to\infty} t\cdot p \in F_\mu \right\}.$$

Then $\tilde{U}_\mu^-$ and U_μ^- are also affine bundles over F_μ, of rank $\tilde{N}_\mu^-$ and N_μ^- respectively, and we have the following decomposition of the tangent spaces T_pY and $T_p(\pi^{-1}(0))$ at each $p \in F_\mu$,

$$T_pY = T_p(\tilde{U}_\mu^+|_p) \oplus T_p(\tilde{U}_\mu^-|_p) \oplus T_pF_\mu \quad \text{and} \quad T_p(\pi^{-1}(0)) = T_p(U_\mu^+|_p) \oplus T_p(U_\mu^-|_p) \oplus T_pF_\mu.$$

Using the smoothness assumption, this yields

$$\dim Y = \tilde{N}_\mu^+ + \tilde{N}_\mu^- + \dim(F_\mu) \quad \text{and} \quad \dim \pi^{-1}(0) = N_\mu^+ + N_\mu^- + \dim(F_\mu). \tag{5.5}$$

Since $\mathbb{C}^*$-action contracts the points of Y to the 0 fibre of $\pi^{-1}(0)$, then, for each $\mu \in I$, all the points points p of Y such that $\lim_{t\to\infty} t\cdot p \in F_\mu$ must lie in $\pi^{-1}(0)$. Thus $\tilde{U}_\mu^- = U_\mu^-$ and we have $\tilde{N}_\mu^- = N_\mu^-$. On the other hand, as $\pi : Y \to \mathbb{C}$ is a submersion of smooth varieties, we have that $\dim Y = \dim \pi^{-1}(0) + 1$, so from (5.5) we conclude that for each μ we have

$$\tilde{N}_\mu^+ = N_\mu^+ + 1. \tag{5.6}$$

Thus, using the Bialynici-Birula decompositions of Y and $\pi^{-1}(0)$, we can apply Lemma 5.3 to decompose the corresponding motives as $[\pi^{-1}(0)] = \sum_{\mu\in I} \mathbb{L}^{N_\mu^+}[F_\mu]$, and

$$[Y] = \sum_{\mu\in I} \mathbb{L}^{\tilde{N}_\mu^+}[F_\mu] = \mathbb{L}[\pi^{-1}(0)]. \tag{5.7}$$

On the other hand, the $\mathbb{C}^*$-action yields an isomorphism $\pi^{-1}(\mathbb{C}^*) \cong \pi^{-1}(1) \times \mathbb{C}^*$, so we can write

$$[Y] = [\pi^{-1}(0)] + [\pi^{-1}(\mathbb{C}^*)] = [\pi^{-1}(0)] + (\mathbb{L}-1)[\pi^{-1}(1)],$$

and, by (5.7), this shows that $[\pi^{-1}(1)] = [\pi^{-1}(0)]$ in $\hat{K}(\mathcal{V}ar_{\mathbb{C}})$. □

5.2. **Semiprojectivity of the moduli space of Higgs bundles.** Let L be a line bundle over the genus g curve X. In this section we show the well-known fact that the moduli space $\mathcal{M}_L(r,d)$ of L-twisted Higgs bundles over X is a smooth semiprojective variety, under the usual conditions on the degree of L and on r and d. In the next section, we will prove the analogous result for the $\mathcal{L}$-Hodge moduli space and that will a more substantial amount of work.

The moduli $\mathcal{M}_L(r,d)$ admits a natural $\mathbb{C}^*$-action by scaling the Higgs field

$$t\cdot(E,\varphi) = (E, t\varphi). \tag{5.8}$$

Note that this is a particular case of (3.19).

Recall now the Hitchin map from (2.1). Then the Hitchin base $W = \bigoplus_{i=1}^r H^0(L^i)$ also admits a natural $\mathbb{C}^*$-action given by

$$t\cdot(s_1,\dots,s_r) = (ts_1, t^2 s_2, \dots, t^r s_r),$$

which makes the Hitchin map $H : \mathcal{M}_L(r,d) \to W$ a $\mathbb{C}^*$-equivariant map.

Let us first prove that the $\mathbb{C}^*$-action on $\mathcal{M}_L(r,d)$ verifies the first condition on Definition 5.1.

Lemma 5.5. *Let (E,φ) be a semistable Higgs bundle on X. Then the limit $\lim_{t\to 0}(E,t\varphi)$ exists in $\mathcal{M}_L(r,d)$.*

Proof. As $H : \mathcal{M}_L(r,d) \to W$ is $\mathbb{C}^*$-equivariant, we have

$$\lim_{t\to 0} H(E,t\varphi) = \lim_{t\to 0} t\cdot H(E,\varphi) = 0,$$

thus the map $\mathbb{C}^* \to W$ given by $t \mapsto H(E, t\varphi)$ extends to $\mathbb{C} \to W$. By Lemma 2.4, H is proper, so by the valuative criterion of properness the map $\mathbb{C}^* \to \mathcal{M}_L(r,d)$ given by $t \mapsto (E, t\varphi)$ must also extend to a map $\mathbb{C} \to \mathcal{M}_L(r,d)$, thus providing the desired limit. □

Now we consider the second condition on Definition 5.1.

Lemma 5.6. *The fixed-point set of the $\mathbb{C}^*$-action on $\mathcal{M}_V(r,d)$ is a proper scheme contained in $H^{-1}(0)$.*

Proof. Since the Hitchin map H is $\mathbb{C}^*$-equivariant, the fixed-point set $\mathcal{M}_L(r,d)^{\mathbb{C}^*}$ must be a closed subset of $H^{-1}(W^{\mathbb{C}^*}) = H^{-1}(0)$. By Lemma 2.4, H is proper, so $H^{-1}(0)$ is proper and hence so is $\mathcal{M}_L(r,d)^{\mathbb{C}^*}$. □

Proposition 5.7. *Suppose that r and d are coprime. Suppose the line bundle L is such that $\deg(L) > 2g-2$. Then the moduli space $\mathcal{M}_L(r,d)$ is a smooth complex semiprojective variety.*

Proof. By Lemma 2.2 the moduli space $\mathcal{M}_L(r,d)$ is a smooth complex variety. Then Lemmas 5.5 and 5.6 prove that the action $t\cdot(E,\varphi) = (E, t\varphi)$ satisfies the semiprojectivity conditions. □

5.3. **Semiprojectivity of the moduli space of $\Lambda_{\mathcal{L}}^{\mathrm{red}}$-modules.** Our aim in this section is to prove that the $\mathcal{L}$-Hodge moduli space $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ is also a smooth semiprojective variety, whenever $\mathrm{rk}(\mathcal{L}) = 1$, $\deg(\mathcal{L}) < 2-2g$ and r and d are coprime. This is going to take considerably more effort than the case of Higgs bundles from the previous section. In particular, we will need to explicitly use both interpretations, provided by Theorem 3.10, of the points parameterized by $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$, namely semistable $\Lambda_{\mathcal{L}}^{\mathrm{red}}$-modules and semistable $(\lambda,\mathcal{L})$-connections (which are automatically flat since $\mathrm{rk}(\mathcal{L}) = 1$). For instance, the proof that condition (1) of Definition 5.1 is goint to be proved by closely following an argument by Simpson, via Λ-modules, but all the arguments required to prove smoothness of the moduli will be carried out by taking the $\mathcal{L}$-connections point of view, because the deformation theory of such objects has been developed, contrary to the deformation theory of Λ-modules.

Recall the $\mathbb{C}^*$-action (3.19) on the $\mathcal{L}$-Hodge moduli space $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ by sending

$$t\cdot(E,\nabla_{\mathcal{L}},\lambda) \mapsto (E, t\nabla_{\mathcal{L}}, t\lambda).$$

We will show that with this $\mathbb{C}^*$-action $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ becomes a smooth semiprojective variety. Recall also the surjective map $\pi : \mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d) \to \mathbb{C}$ defined in (3.16).

Lemma 5.8. *Let $\mathcal{L}$ be any Lie algebroid on X. Let $(E,\nabla_{\mathcal{L}},\lambda) \in \mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ be any $(\lambda,\mathcal{L})$-connection. Then the limit $\lim_{t\to 0}(E, t\nabla_{\mathcal{L}}, t\lambda)$ exists in $\pi^{-1}(0) \subset \mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$.*

Proof. The proof is analogous to [Sim97, Corollary 10.2]. We will use (3.13) to consider the $\Lambda_{\mathcal{L}_\lambda}$-module $(E,\nabla_{\Lambda_{\mathcal{L}}},\lambda)$, with $\nabla_{\Lambda_{\mathcal{L}}} : \Lambda_{\mathcal{L}_\lambda}\otimes E \to E$, instead of the $(\lambda,\mathcal{L})$-connection (i.e. $\mathcal{L}_\lambda$-connection) $(E,\nabla_{\mathcal{L}},\lambda)$.

Consider the $\mathbb{C}^*$-flat family of relative $\Lambda_{\mathcal{L}}^{\mathrm{red}}|_{X\times\mathbb{C}^*}$-modules over $\pi_{\mathbb{C}} : X\times\mathbb{C} \to \mathbb{C}$, where $\pi_{\mathbb{C}}(x,t) = t\lambda$, given by

$$\left(\mathcal{E}, \nabla_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}\right) = \left(\pi_X^* E, t\pi_X^*\nabla_{\Lambda_{\mathcal{L}}}\right),$$

where $\pi_X : X\times\mathbb{C}^* \to X$ is the projection. For $t \neq 0$, the generic fibre of the family is semistable, as for any $t\neq 0$ we clearly have that the corresponding $(t\lambda,\mathcal{L})$-connection $(E, t\nabla_{\mathcal{L}}, t\lambda)$ is semistable if and only if $(E,\nabla_{\mathcal{L}},\lambda)$ is semistable. By [Sim97, Theorem 10.1], there exists a family $(\overline{\mathcal{E}}, \overline{\nabla_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}})$ of $\Lambda_{\mathcal{L}}^{\mathrm{red}}$-modules over $\pi_{\mathbb{C}} : X\times\mathbb{C} \to \mathbb{C}$, flat over $\mathbb{C}$, such that $(\overline{\mathcal{E}}, \overline{\nabla_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}})|_{X\times\mathbb{C}^*} \cong (\pi_X^*E, t\nabla_{\Lambda_{\mathcal{L}}})$ and such that $(\overline{\mathcal{E}}, \overline{\nabla_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}})|_{X\times\{0\}}$ is semistable. Thus, $(\overline{\mathcal{E}}, \overline{\nabla_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}})|_{X\times\{0\}} \in \pi^{-1}(0)$ is the limit at $t=0$ of the $\mathbb{C}^*$-orbit of $(E,\nabla_{\mathcal{L}},\lambda)$ in $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$. □

Now we will focus on the regularity of the $\mathcal{L}$-Hodge moduli space. Now we will use $\mathcal{L}$-connections in our study. We start with a simple lemma.

Lemma 5.9. *Let $\mathcal{L}$ be any Lie algebroid on the curve X, and let $(E,\nabla_{\mathcal{L}})$ and $(E',\nabla'_{\mathcal{L}})$ be semistable $\mathcal{L}$-connections.*

(1) *If $\mu(E)>\mu(E')$ then $\operatorname{Hom}((E,\nabla_{\mathcal{L}}),(E',\nabla'_{\mathcal{L}}))=0$.*
(2) *Suppose $(E,\nabla_{\mathcal{L}})$ and $(E',\nabla'_{\mathcal{L}})$ are stable and $\mu(E)=\mu(E')$. Let $\psi\in\operatorname{Hom}((E,\nabla_{\mathcal{L}}),(E',\nabla'_{\mathcal{L}}))$ be a non-zero map. Then it is an isomorphism.*
(3) *If $(E,\nabla_{\mathcal{L}})$ is stable, then its the only endomorphisms are the scalars, i.e. $\operatorname{End}(E,\nabla_{\mathcal{L}})\cong\mathbb{C}$.*

Proof. The proof is classical. (1) and (2) are completely analogous to [BGL11, Lemma 3.2]. To prove (3), let $\alpha:(E,\nabla_{\mathcal{L}})\to(E,\nabla_{\mathcal{L}})$ be any endomorphism. Choose any point $x\in X$. Then α induces an endomorphism of the fiber E_x. Let $\lambda\in\mathbb{C}$ be an eigenvalue of such morphism. As $\nabla_{\mathcal{L}}$ is $\mathbb{C}$-linear, then $\alpha-\lambda\operatorname{Id}\in\operatorname{End}(E,\nabla_{\mathcal{L}})$. By (2), we know that this map is either zero or an isomorphism. Nevertheless, we know that λ is an eigenvalue of α_x, so $\alpha-\lambda\operatorname{Id}$ has a nontrivial kernel at the fiber over x and, therefore, it cannot be an isomorphism. Thus, $\alpha-\lambda\operatorname{Id}=0$, so $\alpha=\lambda\operatorname{Id}$. □

Given any Lie algebroid $\mathcal{L}$, the deformation theory of flat $\mathcal{L}$-connections was studied in Chapter 5 of [Tor11]. In particular, it follows from Theorem 47 of loc. cit. that that the Zariski tangent space to the moduli space at an integrable $\mathcal{L}$-connection $(E,\nabla_{\mathcal{L}})$ is isomorphic to $\mathbb{H}^1(X,C^\bullet(E,\nabla_{\mathcal{L}}))$, where $C^\bullet(E,\nabla_{\mathcal{L}})$ is the complex

$$C^\bullet(E,\nabla_{\mathcal{L}})\,:\,\operatorname{End}(E)\xrightarrow{[-,\nabla_{\mathcal{L}}]}\operatorname{End}(E)\otimes\Omega^1_{\mathcal{L}}\xrightarrow{[-,\nabla_{\mathcal{L}}]}\dots\xrightarrow{[-,\nabla_{\mathcal{L}}]}\operatorname{End}(E)\otimes\Omega^{\operatorname{rk}(\mathcal{L})}_{\mathcal{L}} \tag{5.9}$$

and that the obstruction for the deformation theory lies in $\mathbb{H}^2(X,C^\bullet(E,\nabla_{\mathcal{L}}))$.

In the next lemma we only consider rank 1 Lie algebroids.

Lemma 5.10. *Let $\mathcal{L}$ be Lie algebroid of rank 1 on X and let $(E,\nabla_{\mathcal{L}})$ be a stable $\mathcal{L}$-connection of rank r and degree d. Then the dimension of the Zariski tangent space to $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ at $(E,\nabla_{\mathcal{L}})$ is given by*

$$\dim T_{(E,\nabla_{\mathcal{L}})}\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)=1-r^2\deg(\mathcal{L})+\dim\left(\mathbb{H}^2(C^\bullet(E,\nabla_{\mathcal{L}}))\right).$$

Proof. Since $\operatorname{rk}(\mathcal{L})=1$, then the deformation complex (5.9) only has two terms,

$$C^\bullet(E,\nabla_{\mathcal{L}})\,:\,\operatorname{End}(E)\xrightarrow{[-,\nabla_{\mathcal{L}}]}\operatorname{End}(E)\otimes\Omega^1_{\mathcal{L}},$$

thus the hypercohomology of the complex $C^\bullet(E,\nabla_{\mathcal{L}})$ fits in the following exact sequence

$$\begin{aligned}0\longrightarrow\mathbb{H}^0(C^\bullet(E,\nabla_{\mathcal{L}}))\longrightarrow H^0(\operatorname{End}(E))\longrightarrow H^0(\operatorname{End}(E)\otimes\Omega^1_{\mathcal{L}})\longrightarrow\\ \mathbb{H}^1(C^\bullet(E,\nabla_{\mathcal{L}}))\longrightarrow H^1(\operatorname{End}(E))\longrightarrow H^1(\operatorname{End}(E)\otimes\Omega^1_{\mathcal{L}})\longrightarrow\mathbb{H}^2(C^\bullet(E,\nabla_{\mathcal{L}}))\longrightarrow 0.\end{aligned}$$

Therefore,

$$\dim(\mathbb{H}^1(C^\bullet(E,\nabla_{\mathcal{L}})))=\dim(\mathbb{H}^0(C^\bullet(E,\nabla_{\mathcal{L}})))+\dim(\mathbb{H}^2(C^\bullet(E,\nabla_{\mathcal{L}})))+\chi(\operatorname{End}(E)\otimes\Omega^1_{\mathcal{L}})-\chi(\operatorname{End}(E)).$$

We can compute each term in the previous expression working in an analogous way to [BGL11, Proposition 3.3]. By construction, $\mathbb{H}^0(C^\bullet(E,\nabla_{\mathcal{L}}))$ corresponds to sections of $\operatorname{End}(E)$ belonging to the kernel of the commutator $[-,\nabla_{\mathcal{L}}]$, so $\mathbb{H}^0(C^\bullet(E,\nabla_{\mathcal{L}}))\cong H^0(\operatorname{End}(E,\nabla_{\mathcal{L}}))$. By stability of $(E,\nabla_{\mathcal{L}})$, point (3) of Lemma 5.9 shows that $\dim(\mathbb{H}^0(C^\bullet(E,\nabla_{\mathcal{L}})))=\dim(H^0(\operatorname{End}(E,\nabla_{\mathcal{L}})))=1$. On the other hand, $\chi(\operatorname{End}(E))=r^2(1-g)$ and $\chi(\operatorname{End}(E)\otimes\Omega^1_{\mathcal{L}})=-r^2\deg(\mathcal{L})+r^2(1-g)$. Hence

$$\dim T_{(E,\nabla_{\mathcal{L}})}\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)=\dim(\mathbb{H}^1(C^\bullet(E,\nabla_{\mathcal{L}})))=1-r^2\deg(\mathcal{L})+\dim\left(\mathbb{H}^2(C^\bullet(E,\nabla_{\mathcal{L}}))\right),$$

as claimed. □

Let $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$ be any Lie-algebroid on X, so no constrains on the algebraic vector bundle V. Recall the associated Lie algebroid $\mathcal{L}_\lambda$ given by (3.14). Then $\mathcal{L}_0 = (V, 0, 0)$ is the trivial algebroid with underlying bundle V. Now we aim to study the first order deformations of a semistable integrable $\mathcal{L}_0$-connection $(E, \nabla_{\mathcal{L}_0})$ of rank r and degree d (i.e. a semistable V^*-twisted Higgs bundle) inside the not just of $\pi^{-1}(0) = \mathcal{M}_{\Lambda_{\mathcal{L}_0}}(r,d) = \mathcal{M}_{V^*}(r,d)$ but, rather insider the $\mathcal{L}$-Hodge moduli space $\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)$. So we allow deformations of $(E, \nabla_{\mathcal{L}_0})$ not only along $\pi^{-1}(0)$ but also to $\pi^{-1}(\lambda)$ for some $\lambda \neq 0$. Recall that here $\pi : \mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d) \to \mathbb{C}$ is the projection (3.16).

Lemma 5.11. *Let $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$ be any Lie algebroid. Then the Zariski tangent space to the $\mathcal{L}$-Hodge moduli space $\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)$ at a point $(E, \nabla_{\mathcal{L}}, 0)$ lying over the 0 fiber is*

$$T_{(E,\nabla_{\mathcal{L}},0)}\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d) \cong \frac{\left\{ (c, C, \lambda_\varepsilon) \in \begin{pmatrix} C^1(\mathcal{U}, \mathrm{End}(E)) \times \\ C^0(\mathcal{U}, \mathrm{End}(E) \otimes \Omega_{\mathcal{L}}) \times \mathbb{C} \end{pmatrix} \,\middle|\, \begin{array}{l} \partial c = 0 \\ \partial C = \tilde{\nabla}_{\mathcal{L}} c + \lambda_\varepsilon \omega \\ \tilde{\nabla}_{\mathcal{L}} C = -\lambda_\varepsilon d_{\mathcal{L}}(\nabla_{\mathcal{L}}) \end{array} \right\}}{\left\{ (\partial \eta, \tilde{\nabla}_{\mathcal{L}} \eta, 0) \,\middle|\, \eta \in C^0(\mathcal{U}, \mathrm{End}(E)) \right\}}$$

where $\mathcal{U} = \{U_\alpha\}$ is an open cover of X such that E is trivial over each open subset U_α, where $\omega \in C^1(\mathcal{U}, \mathrm{End}(E) \otimes \Omega_{\mathcal{L}})$ is some 1-cocycle. Moreover, if $\pi : \mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d) \longrightarrow \mathbb{C}$ is the map sending $(E, \nabla_{\mathcal{L}}, \lambda)$ to λ then its differential $d\pi : T_{(E,\nabla_{\mathcal{L}},0)}\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d) \longrightarrow \mathbb{C}$ is just $[(c, C, \lambda_\varepsilon)] \mapsto \lambda_\varepsilon$.

Proof. We will proceed analogously to [Tor11, §5.2]. Let $(E, \nabla_{\mathcal{L}}, 0) \in \pi_\lambda^{-1}(0)$. Then, by definition, $(E, \nabla_{\mathcal{L}})$ is a V^*-twisted semistable Higgs bundle. Fix an open cover $\mathcal{U} = \{U_\alpha\}$ of X such that E is trivial over each open subset U_α. We will use the usual notation $U_{\alpha\beta} := U_\alpha \cap U_\beta$, etc. to denote the intersections of the open subsets. For each α and β, let $g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{GL}(r, \mathbb{C})$ be the transition functions of E, and let G_α be the matrix valued function representing the Higgs field $\nabla_{\mathcal{L}}$ in the local coordinates over U_α.

The first order deformations of $(E, \nabla_{\mathcal{L}})$ are given by families of $\Lambda^{\mathrm{red}}_{\mathcal{L}}$-module over each $\mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2) \longrightarrow \mathrm{Spec}(\mathbb{C}[\lambda])$. The possible maps $\mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2) \longrightarrow \mathrm{Spec}(\mathbb{C}[\lambda])$ are given by the choice of the image of ε, which must be of the form $\lambda_\varepsilon \lambda$ for some $\lambda_\varepsilon \in \mathbb{C}$. Fix the value λ_ε. Then a family over $\mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2) \longrightarrow \mathrm{Spec}(\mathbb{C}[\lambda])$ for that parameter λ_ε is a triple $(E', \nabla'_{\mathcal{L}}, \lambda_\varepsilon \varepsilon)$ such that E' is a vector bundle over $X \times \mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$ and $\nabla'_{\mathcal{L}}$ is a $(\lambda_\varepsilon \varepsilon, \mathcal{L})$-connection over E'. Relative to the open cover $\{U_\alpha \times \mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)\}$, we can write the transition functions of E' as

$$g'_{\alpha\beta} = g_{\alpha\beta} + \varepsilon g^1_{\alpha\beta},$$

where $g^1_{\alpha\beta} \in \mathcal{O}_X(U_{\alpha\beta}) \times \mathfrak{gl}_r$. Similarly, we can write locally $\nabla'_{\mathcal{L}}$ over U_α as

$$\nabla'_{\mathcal{L},\alpha} = \lambda_\varepsilon \varepsilon d_{\mathcal{L}} + G_\alpha + \varepsilon G^1_\alpha,$$

where $G^1_\alpha \in \Omega_{\mathcal{L}}(U_\alpha) \otimes \mathfrak{gl}_r$.

Moreover, define the 1-cocycle $c \in C^1(\mathcal{U}, \mathrm{End}(E))$ in the following way. Given an isomorphism For each $U_{\alpha\beta}$, let

$$(c_{\alpha\beta})^{(\alpha)} = g^1_{\alpha\beta} g_{\beta\alpha}, \tag{5.10}$$

where we use the notation $(-)^{(\alpha)}$ to denote the matrix with respect to the basis given by the trivialization over U_α.

Since E and E' are vector bundles, the following equations must be satisfied:

$$g_{\alpha\beta} g_{\beta\alpha} = 1, \quad g'_{\alpha\beta} g'_{\beta\alpha} = 1, \quad g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \quad \text{and} \quad g'_{\alpha\beta} g'_{\beta\gamma} g'_{\gamma\alpha} = 1.$$

A direct computation with the first two equations yields $g^1_{\beta\alpha} = -g_{\beta\alpha} g^1_{\alpha\beta} g_{\beta\alpha}$, thus

$$c^{(\alpha)}_{\beta\alpha} = g_{\alpha\beta} c^{(\beta)}_{\beta\alpha} g_{\beta\alpha} = g_{\alpha\beta} g^1_{\beta\alpha} = -g^1_{\alpha\beta} g_{\beta\alpha} = -c^{(\alpha)}_{\alpha\beta}. \tag{5.11}$$

On the other hand, the last couple of equations imply $g^1_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}+g_{\alpha\beta}g^1_{\beta\gamma}g_{\gamma\alpha}+g_{\alpha\beta}g_{\beta\gamma}g^1_{\gamma\alpha}=0$. We can rewrite each summand of the last equation in terms of the cocycle c as follows

$$g^1_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}=g^1_{\alpha\beta}g_{\beta\alpha}=c^{(\alpha)}_{\alpha\beta},\quad g_{\alpha\beta}g^1_{\beta\gamma}g_{\gamma\alpha}=g_{\alpha\beta}g^1_{\beta\gamma}g_{\gamma\beta}g_{\beta\alpha}=c^{(\alpha)}_{\beta\gamma}\quad\text{and}\quad g_{\alpha\beta}g_{\beta\gamma}g^1_{\gamma\alpha}=g_{\alpha\gamma}g^1_{\gamma\alpha}=c^{(\alpha)}_{\gamma\alpha}.$$

Thus, we obtain

$$c^{(\alpha)}_{\alpha\beta}+c^{(\alpha)}_{\beta\gamma}+c^{(\alpha)}_{\gamma\alpha}=0, \tag{5.12}$$

so $c\in C^1(\mathcal{U},\mathrm{End}(E))$ in (5.10) is a closed 1-cocycle.

On the other hand, as $(E,\nabla_{\mathcal{L}})$ is a V^*-twisted Higgs bundle and $(E',\nabla'_{\mathcal{L}},\lambda_\varepsilon\varepsilon)$ is a $(\lambda_\varepsilon\varepsilon,\mathcal{L})$-connection, then, on $U_{\alpha\beta}$, we must have $\nabla_{\mathcal{L},\beta}=g_{\beta\alpha}\nabla_{\mathcal{L},\alpha}g_{\alpha\beta}$ and $\nabla'_{\mathcal{L},\beta}=g_{\beta\alpha}\nabla'_{\mathcal{L},\alpha}g_{\alpha\beta}$. Expanding each side of the last expression and taking into account that $\varepsilon^2=0$ we obtain

$$\begin{aligned}\lambda_\varepsilon\varepsilon d_{\mathcal{L}}+G_\beta+\varepsilon G^1_\beta&=g'_{\beta\alpha}(\lambda_\varepsilon\varepsilon d_{\mathcal{L}}+G_\alpha+\varepsilon G^1_\alpha)g'_{\alpha\beta}\\&=\lambda_\varepsilon\varepsilon d_{\mathcal{L}}+\lambda_\varepsilon\varepsilon g_{\beta\alpha}d_{\mathcal{L}}g_{\alpha\beta}+g_{\beta\alpha}G_\alpha g_{\alpha\beta}+\varepsilon\left(g^1_{\beta\alpha}G_\alpha g_{\alpha\beta}+g_{\beta\alpha}G^1_\alpha g_{\alpha\beta}+g_{\beta\alpha}G_\alpha g^1_{\alpha\beta}\right),\end{aligned}$$

hence we conclude that

$$G^1_\beta=g^1_{\beta\alpha}G_\alpha g_{\alpha\beta}+g_{\beta\alpha}G^1_\alpha g_{\alpha\beta}+g_{\beta\alpha}G_\alpha g^1_{\alpha\beta}+\lambda_\varepsilon g_{\beta\alpha}d_{\mathcal{L}}g_{\alpha\beta}. \tag{5.13}$$

Define the 0-cocycle $C\in C^0(\mathcal{U},\mathrm{End}(E)\otimes\Omega_{\mathcal{L}})$ by taking

$$C^{(\alpha)}_\alpha=G^1_\alpha,$$

for each α. Then, the equality (5.13) written in terms of the cocycles c and C, reads as

$$\begin{aligned}C^{(\alpha)}_\beta&=g_{\alpha\beta}G^1_\beta g_{\beta\alpha}\\&=g_{\alpha\beta}g^1_{\beta\alpha}G_\alpha+G^1_\alpha+G_\alpha g^1_{\alpha\beta}g_{\beta\alpha}+\lambda_\varepsilon(d_{\mathcal{L}}g_{\alpha\beta})g_{\beta\alpha}\\&=-c^{(\alpha)}_{\alpha\beta}G_\alpha+C^{(\alpha)}_\alpha+G_\alpha c^{(\alpha)}_{\alpha\beta}+\lambda_\varepsilon(d_{\mathcal{L}}g_{\alpha\beta})g_{\beta\alpha}\\&=C^{(\alpha)}_\alpha+\left[G_\alpha,c^{(\alpha)}_{\alpha\beta}\right]+\lambda_\varepsilon(d_{\mathcal{L}}g_{\alpha\beta})g_{\beta\alpha}.\end{aligned}$$

Let us finally consider the 1-cocycle $\omega\in C^1(\mathcal{U},\mathrm{End}(E)\otimes\Omega_{\mathcal{L}})$ defined as

$$\omega^{(\alpha)}_{\alpha\beta}=(d_{\mathcal{L}}g_{\alpha\beta})g_{\beta\alpha}, \tag{5.14}$$

for each α,β. Observe that

$$(d_{\mathcal{L}}g_{\alpha\beta})g_{\beta\alpha}+g_{\alpha\beta}(d_{\mathcal{L}}g_{\beta\alpha})=d_{\mathcal{L}}(g_\alpha\beta g_\beta\alpha)=d_{\mathcal{L}}(1)=0$$

so $d_{\mathcal{L}}g_{\beta\alpha}=-g_{\beta\alpha}(d_{\mathcal{L}}g_{\alpha\beta})g_{\beta\alpha}$, and we get

$$\omega^{(\alpha)}_{\beta\alpha}=g_{\alpha\beta}(d_{\mathcal{L}}g_{\beta\alpha})g_{\alpha\beta}g_{\beta\alpha}=g_{\alpha\beta}(d_{\mathcal{L}}g_{\beta\alpha})=-(d_{\mathcal{L}}g_{\alpha\beta})g_{\beta\alpha}=-\omega^{(\alpha)}_{\alpha\beta},$$

thus

$$C^{(\alpha)}_\beta-C^{(\alpha)}_\alpha=[G_\alpha,c^{(\alpha)}_{\alpha\beta}]+\lambda_\varepsilon\omega^{(\alpha)}_{\alpha\beta}. \tag{5.15}$$

On the other hand

$$\begin{aligned}0&=d_{\mathcal{L}}(1)\\&=d_{\mathcal{L}}(g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha})\\&=(d_{\mathcal{L}}g_{\alpha\beta})g_{\beta\gamma}g_{\gamma\alpha}+g_{\alpha\beta}(d_{\mathcal{L}}g_{\beta\gamma})g_{\gamma\alpha}+g_{\alpha\beta}g_{\beta\gamma}(d_{\mathcal{L}}g_{\gamma\alpha})\\&=(d_{\mathcal{L}}g_{\alpha\beta})g_{\beta\alpha}+g_{\alpha\beta}(d_{\mathcal{L}}g_{\beta\gamma})g_{\gamma\beta}g_{\beta\alpha}+g_{\alpha\gamma}(d_{\mathcal{L}}g_{\gamma\alpha})g_{\alpha\gamma}g_{\gamma\alpha}\\&=\omega^{(\alpha)}_{\alpha\beta}+\omega^{(\alpha)}_{\beta\gamma}+\omega^{(\alpha)}_{\gamma\alpha}.\end{aligned}$$

ω is hence a closed 1-cocycle.

Finally, flatness of $\nabla_{\mathcal{L}}$ and $\nabla'_{\mathcal{L}}$ implies that for each α we have $0=\nabla^2_{\mathcal{L},\alpha}=G_\alpha\wedge G_\alpha$ and

$$0=(\nabla'_{\mathcal{L},\alpha})^2=(\lambda_\varepsilon\varepsilon d_{\mathcal{L}}+G_\alpha+\varepsilon G^1_\alpha)^2=\lambda_\varepsilon\varepsilon d_{\mathcal{L}}(G_\alpha)+G_\alpha\wedge G_\alpha+\varepsilon G_\alpha\wedge G^1_\alpha+\varepsilon G^1_\alpha\wedge G_\alpha.$$

Denote by $d_{\mathcal{L}}(\nabla_{\mathcal{L}})\in C^0(\mathcal{U},\mathrm{End}(E)\otimes\Omega^2_L)$ the 0-cocycle defined locally as $d_{\mathcal{L}}(\nabla_{\mathcal{L}})^{(\alpha)}=d_{\mathcal{L}}(G_\alpha)$. Then, we can write the previous equation in terms of C and $d_{\mathcal{L}}(\nabla_{\mathcal{L}})$ as follows. The flatness equations yield $G_\alpha\wedge G^1_\alpha+G^1_\alpha\wedge G_\alpha=-\lambda_\varepsilon d_{\mathcal{L}}(G_\alpha)$, so

$$\tilde{\nabla}_{\mathcal{L}}C^{(\alpha)}_\alpha=-\lambda_\varepsilon d_{\mathcal{L}}(\nabla_{\mathcal{L}})^{(\alpha)}. \tag{5.16}$$

where $\tilde{\nabla}_{\mathcal{L}}=[-,\nabla_{\mathcal{L}}]:\mathrm{End}(E)\to\mathrm{End}(E)\otimes\Omega_{\mathcal{L}}$ is the induced map on $\mathrm{End}(E)$ by $\nabla_{\mathcal{L}}$. We can express equations (5.11), (5.12), (5.15) and (5.16) globally as follows. Each deformation of $(E,\nabla_{\mathcal{L}},0)$ is given by a triple

$$(c,C,\lambda_\varepsilon),$$

with $c\in C^1(\mathcal{U},\mathrm{End}(E))$, $C\in C^0(\mathcal{U},\mathrm{End}(E)\otimes\Omega_{\mathcal{L}})$ and $\lambda_\varepsilon\in\mathbb{C}$, such that

$$\begin{cases}\partial c=0\\ \partial C=\tilde{\nabla}_{\mathcal{L}}c+\lambda_\varepsilon\omega\\ \tilde{\nabla}_{\mathcal{L}}C=-\lambda_\varepsilon d_{\mathcal{L}}(\nabla_{\mathcal{L}}).\end{cases} \tag{5.17}$$

On the other hand, two such triples $(c,C,\lambda_\varepsilon)$ and $(\bar{c},\bar{C},\bar{\lambda}_\varepsilon)$ give rise to equivalent deformations $(E',\nabla'_{\mathcal{L}},\lambda_\varepsilon)$ and $(\overline{E}',\overline{\nabla}'_{\mathcal{L}},\bar{\lambda}_\varepsilon)$ of $(E,\nabla_{\mathcal{L}},0)$ if and only if $\lambda_\varepsilon=\bar{\lambda}_\varepsilon$ and there exists a 0-cocycle of local automorphisms $\xi_\alpha:U_\alpha\times\mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)\to\mathrm{GL}_r$ of the form $\xi_\alpha=\mathrm{Id}+\varepsilon\eta_\alpha$ with $\eta_\alpha:U_\alpha\to\mathfrak{gl}_r$ such that

$$\overline{g}'_{\alpha\beta}(g'_{\alpha\beta})^{-1}=\xi_\beta\xi_\alpha^{-1} \tag{5.18}$$

and

$$\overline{\nabla}'_{\mathcal{L},\alpha}=\xi_\alpha^{-1}\nabla'_{\mathcal{L},\alpha}\xi_\alpha. \tag{5.19}$$

Here, following the previous notation, we write $\overline{E}'$ and $\overline{\nabla}'_{\mathcal{L}}$ locally in the corresponding trivialization over $\mathcal{U}$ as

$$\overline{g}'_{\alpha\beta}=\overline{g}_{\alpha\beta}+\varepsilon\overline{g}^1_{\alpha\beta}=g_{\alpha\beta}+\varepsilon\overline{g}^1_{\alpha\beta}\quad\text{and}\quad\overline{\nabla}'_{\mathcal{L},\alpha}=\bar{\lambda}_\varepsilon\varepsilon d_{\mathcal{L}}+\overline{G}_\alpha+\varepsilon\overline{G}^1_\alpha=\lambda_\varepsilon\varepsilon d_{\mathcal{L}}+G_\alpha+\varepsilon\overline{G}^1_\alpha.$$

We have that $\xi_\beta\xi_\alpha^{-1}=(\mathrm{Id}+\varepsilon\eta_\beta)(\mathrm{Id}-\varepsilon\eta_\alpha)=\mathrm{Id}+\varepsilon\left(\eta_\beta-\eta_\alpha\right)$ and

$$\overline{g}'_{\alpha\beta}(g'_{\alpha\beta})^{-1}=(g_{\alpha\beta}+\varepsilon\overline{g}^1_{\alpha\beta})(g_{\beta\alpha}+\varepsilon g^1_{\beta\alpha})=\mathrm{Id}+\varepsilon(g_{\alpha\beta}g^1_{\beta\alpha}+\overline{g}^1_{\alpha\beta}g_{\beta\alpha})=\mathrm{Id}+\varepsilon(\overline{c}_{\alpha\beta}-c_{\alpha\beta}).$$

From (5.18), we obtain $\overline{c}_{\alpha\beta}-c_{\alpha\beta}=\eta_\beta-\eta_\alpha$, so $\overline{c}-c=\partial\eta$. On the other hand,

$$\begin{aligned}\xi_\alpha^{-1}\nabla'_{\mathcal{L},\alpha}\xi_\alpha&=\lambda_\varepsilon\varepsilon d_{\mathcal{L}}+\lambda_\varepsilon\varepsilon(\mathrm{Id}-\varepsilon\eta_\alpha)d_{\mathcal{L}}(\mathrm{Id}+\varepsilon\eta_\alpha)+(\mathrm{Id}-\varepsilon\eta_\alpha)G_\alpha(\mathrm{Id}+\varepsilon\eta_\alpha)+\varepsilon(\mathrm{Id}-\varepsilon\eta_\alpha)G^1_\alpha(\mathrm{Id}+\varepsilon\eta_\alpha)\\&=\lambda_\varepsilon\varepsilon d_{\mathcal{L}}+G_\alpha+\varepsilon\left(G_\alpha\eta_\alpha-\eta_\alpha G_\alpha+G^1_\alpha\right).\end{aligned}$$

Hence, (5.19) yields $\overline{G}^1_\alpha=[G_\alpha,\eta_\alpha]+G^1_\alpha$ or, equivalently, $\overline{C}_\alpha-C_\alpha=[G_\alpha,\eta_\alpha]=\tilde{\nabla}_{\mathcal{L}}\eta_\alpha$, and thus, $\overline{C}-C=\tilde{\nabla}_{\mathcal{L}}\eta$. We finally conclude that the deformation space of $\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)$ at $(E,\nabla_{\mathcal{L}},0)$ is

$$T_{(E,\nabla_{\mathcal{L}},0)}\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)\cong\frac{\left\{(c,C,\lambda_\varepsilon)\in\begin{pmatrix}C^1(\mathcal{U},\mathrm{End}(E))\times\\ C^0(\mathcal{U},\mathrm{End}(E)\otimes\Omega_{\mathcal{L}})\times\mathbb{C}\end{pmatrix}\,\middle|\,\begin{array}{l}\partial c=0\\ \partial C=\tilde{\nabla}_{\mathcal{L}}c+\lambda_\varepsilon\omega\\ \tilde{\nabla}_{\mathcal{L}}C=-\lambda_\varepsilon d_{\mathcal{L}}(\nabla_{\mathcal{L}})\end{array}\right\}}{\left\{(\partial\eta,\tilde{\nabla}_{\mathcal{L}}\eta,0)\,\middle|\,\eta\in C^0(\mathcal{U},\mathrm{End}(E))\right\}}$$

and the map $d\pi:T_{(E,\nabla_{\mathcal{L}},0)}\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)\longrightarrow\mathbb{C}$ is just $[(c,C,\lambda_\varepsilon)]\mapsto\lambda_\varepsilon$. □

Remark 5.12. *While the deformation theory of $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ is governed by a nice deformation complex (as expectable for this type of deformation problems), we have not been able to provide, in general, a natural cohomological interpretation for the deformation theory of $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$.*

This can, however, be achieved in certain cases. For instance, suppose $\mathrm{rk}(\mathcal{L})=1$ and suppose E is a stable vector bundle over X. By Corollary 3.14, E admits an integrable $\mathcal{L}$-connection $\nabla_{\mathcal{L},0}: E \to E\otimes \Omega_{\mathcal{L}}$. Let us consider the family $(\pi_X^*E, \lambda\pi_X^*\nabla_{\mathcal{L},0},\lambda)$ over $X\times\mathbb{C}$, where $\pi_X : X\times \mathbb{C}\to X$ is the projection, and consider the infinitesimal family over $\mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$ around 0 given by $(\pi_X^*E,\varepsilon\pi_X^*\nabla_{\mathcal{L},0},\varepsilon)$. We can now express the family locally in a similar way to the previous Lemma. Given an open cover $\{U_\alpha\times \mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$, let $g_{\alpha\beta}: U_{\alpha\beta}\to \mathrm{GL}(r,\mathbb{C})$ be the transition functions of E. As $\varepsilon\nabla_{\mathcal{L},0}$ is an $(\varepsilon,\mathcal{L})$-connection, we can express it locally over U_α as

$$\varepsilon\nabla_{\mathcal{L},0,\alpha}=\varepsilon d_{\mathcal{L}}+\varepsilon G_\alpha^1$$

for some $G_\alpha^1\in\Omega_{\mathcal{L}}(U_\alpha)\otimes\mathfrak{gl}_r$. As G_α^1 comes from an actual $\mathcal{L}$-connection, we must have

$$\varepsilon\nabla_{\mathcal{L},0,\beta}=g_{\beta\alpha}\varepsilon\nabla_{\mathcal{L},0,\alpha}g_{\alpha\beta}$$

on the overlaps $U_{\alpha\beta}$. Plugging in the local representation of $\nabla_{\mathcal{L}}$ yields

$$\varepsilon d_{\mathcal{L}}+\varepsilon G^1_\beta=\varepsilon d_{\mathcal{L}}+\varepsilon g_{\beta\alpha}d_{\mathcal{L}}g_{\alpha\beta}+\varepsilon g_{\beta\alpha}G^1_\alpha g_{\alpha\beta}.$$

Thus,

$$g_{\alpha\beta}G^1_\beta g_{\beta\alpha}=d_{\mathcal{L}}g_{\alpha\beta}g_{\beta\alpha}+G^1_\alpha=\omega_{\alpha\beta}^{(\alpha)}+G^1_\alpha.$$

Let $\Omega\in C^0(\mathcal{U},\mathrm{End}(E)\otimes\Omega_L)$ be the 0-cocycle defined as $\Omega_\alpha^{(\alpha)}=G^1_\alpha$. Then $\omega^{(\alpha)}_{\alpha\beta}=\Omega^{(\alpha)}_\beta-\Omega^{(\alpha)}_\alpha$, hence

$$\omega=\partial\Omega.$$

Now, since $\mathcal{L}$ has rank 1, the integrability condition is automatic and so

$$T_{(E,\nabla_{\mathcal{L}},0)}\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)\cong\frac{\left\{(c,C,\lambda_\varepsilon)\in\left(\begin{array}{c}C^1(\mathcal{U},\mathrm{End}(E))\times\\ C^0(\mathcal{U},\mathrm{End}(E)\otimes\Omega_{\mathcal{L}})\times\mathbb{C}\end{array}\right)\middle|\begin{array}{l}\partial c=0\\ \partial C=\tilde{\nabla}_{\mathcal{L}}c+\lambda_\varepsilon\partial\Omega\end{array}\right\}}{\left\{(\partial\eta,\tilde{\nabla}_{\mathcal{L}}\eta,0)\,\middle|\,\eta\in C^0(\mathcal{U},\mathrm{End}(E))\right\}}.$$

Then, the map $(c,C,\lambda_\varepsilon)\mapsto(c,C-\lambda_\varepsilon\Omega,\lambda_\varepsilon)$ induces an isomorphism

$$\begin{aligned}T_{(E,\nabla_{\mathcal{L}},0)}\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)&\cong\frac{\left\{(c,D,\lambda_\varepsilon)\in\left(\begin{array}{c}C^1(\mathcal{U},\mathrm{End}(E))\times\\ C^0(\mathcal{U},\mathrm{End}(E)\otimes\Omega_{\mathcal{L}})\times\mathbb{C}\end{array}\right)\middle|\begin{array}{l}\partial c=0\\ \partial D=\tilde{\nabla}_{\mathcal{L}}c\end{array}\right\}}{\left\{(\partial\eta,\tilde{\nabla}_{\mathcal{L}}\eta,0)\,\middle|\,\eta\in C^0(\mathcal{U},\mathrm{End}(E))\right\}}\\ &\cong\mathbb{H}^1(C^\bullet(E,\nabla_{\mathcal{L}}))\times\mathbb{C},\end{aligned}$$

yielding the desired cohomological interpretation of the deformation space. It is clear in this case that $\mathbb{H}^1(C^\bullet(E,\nabla_{\mathcal{L}}))$ parameterizes the deformations along the fiber $p_\lambda^{-1}(0)$ of the projection $p_\lambda:\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)\to\mathbb{C}$ from (3.16), and $\mathbb{C}$ parameterizes the deformations of λ i.e. along the target of p_λ.

However, a vector bundle E need not admit an integrable $\mathcal{L}$-connection and, therefore, the associated cocycle ω in (5.14) may not be exact. Moreover, for higher rank $\mathcal{L}$, the presence of the integrability condition breaks the previous trivialization of the deformation theory and the corresponding cohomological description.

We believe that the somehow unnatural presentation of the deformation theory of Lemma 5.11 is a reflection of the fact that the moduli space of Higgs bundles admits a broader range of deformations than the ones considered in this section, as suggested in [Tor11, Section 7.3]. *More precisely, for each family of Lie algebroid structures over V, $\mathcal{L}\longrightarrow X\times T$ on T, we obtain a moduli space $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)\longrightarrow T$ over T. Each family going through the trivial Lie algebroid $(V,0,0)$ gives rise* *to a deformation of the moduli space $\mathcal{M}_{V^*}(r,d)$. More generally, the infinitesimal deformations of the trivial algebroid structure give rise to deformations of $\mathcal{M}_{V^*}(r,d)$. We expect that if we considered the whole space of such infinitesimal deformations, then that space would get indeed a natural cohomological interpretation.*

In Lemma 5.11 we are only considering "radial sections" of such deformation space, corresponding to families over $\mathbb{C}$ of the form (3.14), therefore obtaining "sections" or "cuts" of the whole deformation space, and these deformations do not seem to exhibit a cohomological description anymore.

The preceding Lemmas 5.10 and 5.11 now allow us to show that the $\mathcal{L}$-Hodge moduli space under the following conditions.

Lemma 5.13. *Let $r \geq 1$ and d be coprime and $\mathcal{L} = (L, [\cdot,\cdot], \delta)$ be a Lie algebroid such that $\mathrm{rk}(\mathcal{L}) = 1$ and $\deg(\mathcal{L}) < 2-2g$. Then:*

(1) $\mathcal{M}_{\Lambda_\mathcal{L}}(r,d)$ is a smooth variety, whose connected components have all dimension $1-r^2\deg(L)$;
(2) $\mathcal{M}_{\Lambda_\mathcal{L}^{\mathrm{red}}}(r,d)$ is a smooth variety, whose connected components have all dimension $2-r^2\deg(L)$. Moreover, the map $\pi : \mathcal{M}_{\Lambda_\mathcal{L}^{\mathrm{red}}}(r,d) \to \mathbb{C}$ from (3.16) is a smooth submersion.

Proof. Let us start by proving that $\mathcal{M}_{\Lambda_\mathcal{L}}(r,d)$ is a smooth variety of dimension $1-r^2\deg(L)$. Let $(E,\nabla_\mathcal{L}) \in \mathcal{M}_{\Lambda_\mathcal{L}}(r,d)$. Consider the map $\mathbb{C}^* \to \mathcal{M}_{\Lambda_\mathcal{L}^{\mathrm{red}}}(r,d)$ given by $t \mapsto (E, t\nabla_\mathcal{L}, t)$. By Lemma 5.8, the limit of the $\mathbb{C}^*$-action at zero exists, so this map extends to a curve $\gamma : \mathbb{C} \to \mathcal{M}_{\Lambda_\mathcal{L}^{\mathrm{red}}}(r,d)$, which is a section of the map $\pi : \mathcal{M}_{\Lambda_\mathcal{L}^{\mathrm{red}}}(r,d) \to \mathbb{C}$. Let $(E_0, \nabla_{\mathcal{L},0}, 0) := \gamma(0)$.

Consider the map $\rho : \mathbb{C} \to \mathbb{Z}$ given by

$$\rho(\lambda) = \dim \left(\gamma^* T\mathcal{M}_{\Lambda_\mathcal{L}^{\mathrm{red}}}(r,d)\right)\Big|_\lambda .$$

We know from (3.17) that the $\mathbb{C}^*$-action produces an isomorphism between any nonzero fiber of π and $\pi^{-1}(1)$, yielding an isomorphism

$$\pi^{-1}(\mathbb{C}^*) \cong \mathcal{M}_{\Lambda_\mathcal{L}}(r,d) \times \mathbb{C}^*.$$

Therefore, for every $\lambda \neq 0$ we have

$$\rho(\lambda) = \dim \left(\gamma^* T\mathcal{M}_{\Lambda_\mathcal{L}^{\mathrm{red}}}(r,d)\right)\Big|_\lambda = \dim T_{(E,\nabla_\mathcal{L})}\mathcal{M}_{\Lambda_\mathcal{L}}(r,d) + 1$$

The map ρ is upper semicontinuous, so applying Lemma 5.10, we get

$$\rho(0) \geq \dim T_{(E,\nabla_\mathcal{L})}\mathcal{M}_{\Lambda_\mathcal{L}}(r,d) + 1 = 2 - r^2\deg(L) + \dim\left(\mathbb{H}^2(C^\bullet(E,\nabla_\mathcal{L}))\right) \geq 2 - r^2\deg(L). \tag{5.20}$$

Note that this is where we used the fact that r and d are coprime, because in such a case every semistable $\mathcal{L}$-connection is actually stable, so Lemma 5.10 applies at every point of $\mathcal{M}_{\Lambda_\mathcal{L}}(r,d)$.

On the other hand, we have that

$$\dim T_{(E_0,\nabla_{\mathcal{L},0},0)}\mathcal{M}_{\Lambda_\mathcal{L}^{\mathrm{red}}}(r,d) = \dim\ker d\pi + \dim \mathrm{Im}\, d\pi \leq \dim\ker d\pi + 1.$$

The kernel of $d\pi|_{(E_0,\nabla_{\mathcal{L},0},0)}$ can be computed explicitly through our formula for the Zariski tangent space given in Lemma 5.11

$$\begin{aligned}\ker d\pi|_{(E_0,\nabla_{\mathcal{L},0},0)} &\cong \frac{\left\{ (c,C,0) \in \left(\begin{array}{c} C^1(\mathcal{U},\mathrm{End}(E_0))\times \\ C^0(\mathcal{U},\mathrm{End}(E_0)\otimes\Omega_{\mathcal{L}})\times\mathbb{C} \end{array}\right) \middle| \begin{array}{l} \partial c = 0 \\ \partial C = \tilde{\nabla}_{\mathcal{L},0} c \\ \tilde{\nabla}_{\mathcal{L}_0} C = 0 \end{array} \right\}}{\left\{ (\partial\eta, \tilde{\nabla}_{\mathcal{L},0}\eta, 0) \,\middle|\, \eta\in C^0(\mathcal{U},\mathrm{End}(E_0)) \right\}} \\ &\cong \mathbb{H}^1\left(\mathrm{End}(E_0) \stackrel{[-,\nabla_{\mathcal{L},0}]}{\longrightarrow} \mathrm{End}(E_0)\otimes\Omega^1_{\mathcal{L}} \right) \\ &\cong T_{(E_0,\nabla_{\mathcal{L},0})}\mathcal{M}_{L^{-1}}(r,d).\end{aligned}$$

Using Lemma 2.2, we know that $\mathcal{M}_{L^{-1}}(r,d)$ is smooth of dimension $1-r^2\deg(L)$, so

$$\dim\ker d\pi|_{(E_0,\nabla_{\mathcal{L},0},0)} = 1-r^2\deg(L)$$

and, therefore,

$$\rho(0) = \dim T_{(E_0,\nabla_{\mathcal{L},0},0)}\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d) \le \dim\ker d\pi|_{(E_0,\nabla_{\mathcal{L},0},0)} + 1 \le 2-r^2\deg(L). \tag{5.21}$$

From (5.20) and (5.21), we conclude that

$$\rho(0) = 2-r^2\deg(L),$$

that is, $\mathbb{H}^2(C^\bullet(E,\nabla_{\mathcal{L}}))=0$ for all $(E,\nabla_{\mathcal{L}})\in\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$.

The upshot is that the deformation theory at $(E,\nabla_{\mathcal{L}})$ is unobstructed and the dimension of the Zariski tangent space $T_{(E,\nabla_{\mathcal{L}})}\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ is $1-r^2\deg(L)$ for each $(E,\nabla_{\mathcal{L}})$. As a consequence, by [FM98] the moduli space $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ is a smooth variety of dimension $1-r^2\deg(L)$, completing the proof of (1).

Let us now consider point (2), i.e. the regularity of the $\mathcal{L}$-Hodge moduli space and the map π. First note that we have the isomorphism

$$\pi^{-1}(\mathbb{C}^*) \cong \mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)\times\mathbb{C}^*,$$

and by (1), $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ is smooth, so $\pi^{-1}(\mathbb{C}^*)$ is also smooth and the map $\pi|_{\pi^{-1}(\mathbb{C}^*)}:\pi^{-1}(\mathbb{C}^*)\to\mathbb{C}^*$ is clearly a smooth submersion. Therefore, it is enough to study the deformation of the elements in the zero fiber of π and then check that the dimension of the corresponding Zariski tangent spaces coincides with the expected one and that the differential of the map π is surjective at those points.

Let us consider the subvariety

$$\overline{\pi^{-1}(\mathbb{C}^*)} \subset \mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$$

given by the closure of $\pi^{-1}(\mathbb{C}^*)$ in the $\mathcal{L}$-Hodge moduli. The $\mathbb{C}^*$-flow through any point of $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ has a limit at 0 in $\pi^{-1}(0)$, due to Lemma 5.8, so

$$\pi^{-1}(0)\cap\overline{\pi^{-1}(\mathbb{C}^*)}\neq\emptyset.$$

By (1) and (3.17), we have $\dim\pi^{-1}(\lambda)=1-r^2\deg(L)$ for every $\lambda\neq 0$. Hence, by semicontinuity, each component of $\pi^{-1}(0)\cap\overline{\pi^{-1}(\mathbb{C}^*)}$ has dimension at least $1-r^2\deg(L)$. By Lemma 2.2 the variety $\pi^{-1}(0)=\mathcal{M}_{L^{-1}}(r,d)$ is smooth and connected of dimension $1-r^2\deg(L)$, so we conclude that

$$\pi^{-1}(0)\cap\overline{\pi^{-1}(\mathbb{C}^*)}=\pi^{-1}(0)$$

and thus $\overline{\pi^{-1}(\mathbb{C}^*)}=\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$.

As $\pi^{-1}(\mathbb{C}^*)\cong\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)\times\mathbb{C}^*$, then we know that for any $(E',\nabla'_{\mathcal{L}},\lambda')\in\pi^{-1}(\mathbb{C}^*)$ we have

$$\dim T_{(E',\nabla'_{\mathcal{L}},\lambda')}\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d) = \dim T_{(E',\nabla'_{\mathcal{L}}/\lambda)}\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)+1 = 2-r^2\deg(L)$$

so, by semicontinuity, for each $(E,\nabla_{\mathcal{L}},0)\in\pi^{-1}(0)\cap\overline{\pi^{-1}(\mathbb{C}^*)}=\pi^{-1}(0)$ we have

$$2-r^2\deg(L)\le\dim T_{(E,\nabla_{\mathcal{L}},0)}\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)=1-r^2\deg(L)+\dim\operatorname{Im}d\pi\le 2-r^2\deg(L).$$

Hence, we have

$$\dim T_{(E,\nabla_{\mathcal{L}},0)}\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)=2-r^2\deg(L)\quad\text{ and }\quad\dim\operatorname{Im}d\pi|_{(E,\nabla_{\mathcal{L}},0)}=1,$$

for each $(E,\nabla_{\mathcal{L}},0)\in\pi^{-1}(0)$. So the map π is a smooth submersion with equidimensional fibers and the dimension of the Zariski tangent space of the moduli space $\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)$ at any point is constant and coincides with the dimension of the scheme which is therefore smooth. □

Remark 5.14. *We will see in Theorem 7.2 below that, under the stated conditions, $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ is actually connected, hence so is $\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)$.*

Combining the previous result yields the desired semiprojectivity and regularity of the $\mathcal{L}$-Hodge moduli space.

Theorem 5.15. *Let X be a smooth projective curve of genus $g\ge 2$. Let $\mathcal{L}=(L,[\cdot\,,\cdot],\delta)$ be a rank Lie algebroid on X such that $\operatorname{rk}(\mathcal{L})=1$ and $\deg(\mathcal{L})<2-2g$. Then the moduli space $\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)$, with the $\mathbb{C}^*$-action $t\cdot(E,\nabla_{\mathcal{L}},\lambda)=(E,t\nabla_{\mathcal{L}},t\lambda)$, is a semiprojective variety.*

If, moreover, r and d are coprime and $r\ge 1$, then it is a smooth semiprojective variety and the map $\pi:\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)\to\mathbb{C}$ from (3.16) is a surjective $\mathbb{C}^*$-equivariant submersion covering the standard action on $\mathbb{C}$.

Proof. By the GIT construction of [Sim94], $\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)$ is a complex quasi-projective variety. Lemma 5.8 ensures that for every $(E,\nabla_{\mathcal{L}},\lambda)\in\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)$ the limit $\lim_{t\to 0}(E,t\nabla_{\mathcal{L}},t\lambda)$ exists. Moreover, the fixed-point set corresponds to the fixed-point set of the $\mathbb{C}^*$-action in $\pi^{-1}(0)$, which coincides with the moduli space of L^{-1}-twisted Higgs bundles. Then Lemma 5.6 implies that $\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)^{\mathbb{C}^*}$ is proper. So the $\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)$ is a semiprojective variety. In the coprime case, the smoothness claim follows from Lemma 5.13. □

Remark 5.16. *We expect that the above results still hold true for higher rank Lie algebroids $\mathcal{L}=(V,[\cdot\,,\cdot],\delta)$ with V polystable such that $\mu(V)<2-2g$. Indeed most of the above arguments go through immediately in this situation, except in two related steps. First, Lemma 5.10 really requires rank 1 Lie algebroids, because it is only in that setting that the deformation complex (5.9) has only two terms. If $\operatorname{rk}(\mathcal{L})=n$, then (5.9) has n terms, and it is not clear how to proceed to compute the dimension of $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$. Similarly, Lemma 2.2 also requires the twisting to be a line bundle, and the corresponding result for higher rank twistings is not yet known, by similar reasons (notice that the infinitesimal study carried out in [BR94] is done for any twisting, but it does not take into account the integrability condition on the Higgs field).*

5.4. **Invariance of the motive and E-polynomial with respect to the algebroid structure.** We continue with our fixed base curve X, of genus $g\ge 2$. Now that we have established the required regularity conditions and the semiprojectivity of the $\mathcal{L}$-Hodge moduli space, for $\mathcal{L}=(L,[\cdot\,,\cdot],\delta)$ of rank 1 and degree less than $2-2g$, we can address the invariance of the motive with respect to the algebroid structure of $\mathcal{L}$ by keeping L fixed, and when we vary λ in $\mathbb{C}$. Hence the variation on the Lie algebroid structure we are considering is the one given by (3.14). By (3.17), this is clearly true if one changes the Lie algebroid structure by varying from $\lambda\in\mathbb{C}^*$ to $\lambda'\in\mathbb{C}^*$, so the main point is that the motivic class remains unchanged when we go to the trivial algebroid structure, thus $\lambda=0$. This is one of the contents of the next theorem.

Theorem 5.17. *Let $\mathcal{L} = (L, [\cdot\,,\cdot], \delta)$ be Lie algebroid on X such that L is a line bundle with $\deg(L) < 2-2g$. If r and d are coprime, then the following equalities hold in $\hat{K}(\mathcal{V}ar_{\mathbb{C}})$*

$$[\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)] = [\mathcal{M}_{L^{-1}}(r,d)], \qquad [\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)] = \mathbb{L}[\mathcal{M}_{L^{-1}}(r,d)]$$

and we have an isomorphism of Hodge structures

$$H^\bullet(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)) \cong H^\bullet(\mathcal{M}_{L^{-1}}(r,d))$$

In particular,

$$E(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)) = E(\mathcal{M}_{L^{-1}}(r,d)), \qquad E(\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)) = uvE(\mathcal{M}_{L^{-1}}(r,d)).$$

Moreover, both $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ and $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ have pure mixed Hodge structures.

Proof. By Theorem 5.15, the moduli space $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ is a smooth semiprojective variety for the $\mathbb{C}^*$-action (3.19). Moreover, the map π from (3.16) is a surjective $\mathbb{C}^*$-equivariant submersion covering the standard $\mathbb{C}^*$-action on $\mathbb{C}$. Then Proposition 5.4 gives the desired motivic equalities,

$$[\mathcal{M}_{L^{-1}}(r,d)] = [\pi^{-1}(0)] = [\pi^{-1}(1)] = [\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)] \quad \text{and} \quad [\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)] = \mathbb{L}[\pi^{-1}(0)] = \mathbb{L}[\mathcal{M}_{L^{-1}}(r,d)],$$

which yield the corresponding equalities of E-polynomials,

$$E(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)) = E(\mathcal{M}_{L^{-1}}(r,d)) \quad \text{and} \quad E(\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)) = uvE(\mathcal{M}_{L^{-1}}(r,d)).$$

Moreover, by [HRV15, Corollary 1.3.3], the fibers $\mathcal{M}_{L^{-1}}(r,d) = \pi^{-1}(0)$ and $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d) = \pi^{-1}(1)$ have isomorphic cohomology supporting pure mixed Hodge structures. Finally, as $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ is also smooth and semiprojective, its cohomology is also pure by [HRV15, Corollary 1.3.2]. □

6. Motives of moduli spaces of twisted Higgs bundles

Continuing with the plan outlined in Section 4, after proving that we can reduce the computation of the motivic classes of the moduli spaces of $\mathcal{L}$-connections to the computation of the motivic classes of moduli spaces of twisted Higgs bundles, the next step is to analyze the structure of the motive of the latter moduli space.

In this section we will prove that, under certain assumptions, such motive is independent on the twisting line bundle, up to its degree, and we will provide tools to decompose the motive of the moduli space that will be useful later on, in section 7, to compute explicitly the motives and E-polynomials of the moduli spaces in low ranks. In order to do this, it will be useful to introduce the notion of variation of Hodge structure.

6.1. Variations of Hodge structure and chains.

Let L be a line bundle over the curve X. An *L-twisted variation of Hodge structure* of *type* $\overline{r} = (r_1, \ldots, r_k)$ and *multidegree* $\overline{d} = (d_1, \ldots, d_k)$ is an L-twisted Higgs bundle (E, φ) of the form

$$(E_\bullet, \varphi_\bullet) = \left(\bigoplus_{i=1}^k E_i, \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \varphi_1 & 0 & \cdots & 0 & 0 \\ 0 & \varphi_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi_{k-1} & 0 \end{pmatrix} \right), \tag{6.1}$$

where E_i are vector bundles on X, with $\mathrm{rk}(E_i) = r_i$ and $\deg(E_i) = d_i$, and $\varphi_i : E_i \to E_{i+1} \otimes L$ for each $i = 1, \ldots, k$, with $\varphi_k = 0$.

Let $r = \sum_{i=1}^k r_i$ and $d = \sum_{i=1}^k d_i$ and denote by

$$\mathrm{VHS}_L(\overline{r}, \overline{d}) \subset \mathcal{M}_L(r,d)$$

the subscheme of the moduli space of L-twisted Higgs bundles corresponding to semistable variations of Hodge structure of type $\overline{r}$ and multi-degree $\overline{d}$.

On the other hand, recall that an algebraic *chain* on X is a quiver bundle of type A, hence a sequence of algebraic vector bundles $(E_1, \dots, E_k)$, together with maps $\varphi_i : E_i \to E_{i+1}$. We denote a chain by the symbol $(\tilde{E}_\bullet, \tilde{\varphi}_\bullet)$.

Given real numbers $\alpha = (\alpha_1, \dots, \alpha_k)$, we define the α-*degree* of the chain $(\tilde{E}_i, \tilde{\varphi}_i)$ as

$$\deg_\alpha(\tilde{E}_\bullet, \tilde{\varphi}_\bullet) = \sum_{i=1}^{k} (\deg(E_i) + \mathrm{rk}(E_i)\alpha_i) \tag{6.2}$$

and the α-*slope* as

$$\mu_\alpha(\tilde{E}_\bullet, \tilde{\varphi}_\bullet) = \frac{\deg_\alpha(\tilde{E}_\bullet, \tilde{\varphi}_\bullet)}{\sum_{i=1}^{k} \mathrm{rk}(E_i)}.$$

We say that $(\tilde{E}_\bullet, \tilde{\varphi}_\bullet)$ is of type $\overline{r} = (r_1, \dots, r_k)$ if $\mathrm{rk}(E_i) = r_i$ for each $i = 1, \dots, k$ and we call $\overline{d} = (\deg(E_1), \dots, \deg(E_k))$ its *multidegree*.

A *subchain* $(\tilde{F}_\bullet, \tilde{\varphi}_\bullet)$ of $(\tilde{E}_\bullet, \tilde{\varphi}_\bullet)$ is a collection $(F_1, \dots, F_k)$ of subbundles of $(E_1, \dots, E_k)$, i.e. $F_i \subset E_i$, such that $\varphi_i(F_i) \subset F_{i+1}$, so that $(\tilde{F}_\bullet, \tilde{\varphi}_\bullet|_{F_\bullet})$ is itself a chain. An algebraic chain $(\tilde{E}_\bullet, \tilde{\varphi}_\bullet)$ is *(semi)stable* if for any subchain $(\tilde{F}_\bullet, \tilde{\varphi}_\bullet|_{F_\bullet}) \subset (\tilde{E}_\bullet, \tilde{\varphi}_\bullet)$, we have

$$\mu_\alpha(\tilde{F}_\bullet, \tilde{\varphi}_\bullet|_{F_\bullet}) < \mu_\alpha(\tilde{E}_\bullet, \tilde{\varphi}_\bullet) \quad (\text{resp. } \leq).$$

Denote by

$$\mathrm{HC}^\alpha(\overline{r}, \overline{d})$$

the moduli space of α-semistable algebraic chains on X of type $\overline{r}$ and multidegree $\overline{d}$.

Now, given a variation of Hodge structure $(E_\bullet, \varphi_\bullet)$ of type $\overline{r} = (r_1, \dots, r_k)$ and multidegree $\overline{d} = (d_1, \dots, d_k)$, an algebraic chain $(\tilde{E}_\bullet, \tilde{\varphi}_\bullet)$ can be constructed as follows. Take

$$\tilde{E}_i = E_i \otimes L^{i-k}.$$

Then φ_i induces a map

$$\tilde{\varphi_i} = \varphi_i \otimes \mathrm{Id}_{L^{i-k}} : \tilde{E}_i \longrightarrow \tilde{E}_{i+1},$$

thus $(\tilde{E}_\bullet, \tilde{\varphi}_\bullet)$ is a chain of type $\overline{r}$ and multidegree $\overline{d}_L = (d_1 + r_1(1-k)\deg(L), \dots, d_k)$. This construction is reversible, giving a variation of Hodge structure from an algebraic chain, and hence giving a bijection between these two kinds of objects.

It turns out that their (semi)stability conditions also match, if one chooses a particular set of real numbers α. Indeed, if $\alpha_L = ((k-1)\deg(L), \dots, \deg(L), 0)$, then the α_L-degree (6.2) of any chain $(\tilde{E}_\bullet, \tilde{\varphi}_\bullet)$, is

$$\deg_{\alpha_L}(\tilde{E}_\bullet, \tilde{\varphi}_\bullet) = \sum_{i=1}^{k} \Big(\deg(E_i \otimes L^{i-k}) + \mathrm{rk}(E_i)(k-i)\deg(L) \Big) = \sum_{i=1}^{k} \deg(E_i)$$

so

$$\mu_\alpha(\tilde{E}_\bullet, \tilde{\varphi}_\bullet) = \mu(E_\bullet, \varphi_\bullet),$$

where $(E_\bullet, \varphi_\bullet)$ is the corresponding variation of Hodge structure.

The proof of the next lemma follows by the exact same argument as in [ÁCGP01, Proposition 3.5], by replacing the canonical line bundle K_X by L.

Proposition 6.1. *A variation of Hodge structure $(E_\bullet, \varphi_\bullet)$ is (semi)stable (as an L-twisted Higgs bundle) if and only if for every choice of subbundles $F_i \subset E_i$ with $\varphi_i(F_i) \subset F_{i+1}$ we have*

$$\mu(F_\bullet, \varphi_\bullet|_{F_\bullet}) < \mu(E_\bullet, \varphi_\bullet) \quad (\textit{resp. } \leq).$$

Hence $(E_\bullet, \varphi_\bullet)$ is (semi)stable if and only if the corresponding chain $(\tilde{E}_\bullet, \tilde{\varphi}_\bullet)$ is α_L-(semi)stable.

So the following corollary is immediate.

Corollary 6.2. *Fix an algebraic line bundle L over X. Let $\overline{r}=(r_1,\dots,r_k)$, $\overline{d}=(d_1,\dots,d_k)$ and $\overline{d}_L=(d_1+r_1(1-k)\deg(L), d_2+r_2(2-k)\deg(L),\dots,d_k)$. The previously described correspondence between chains and L-twisted variations of Hodge structure induces an isomorphism,*

$$\mathrm{VHS}_L(\overline{r},\overline{d})\cong \mathrm{HC}^{\alpha_L}(\overline{r},\overline{d}_L),$$

for $\alpha_L=((k-1)\deg(L),\dots,\deg(L),0)$.

6.2. **Independence of the motives of Higgs moduli on the twisting line bundle.** We will use Corollary 6.2 to show that the motivic class of the moduli spaces of L-twisted Higgs bundles for coprime rank and degree only depend on the degree of the twisting line bundle L, whenever it is big enough.

Recall from (5.8) that the moduli space $\mathcal{M}_L(r,d)$ has a natural $\mathbb{C}^*$-action given by scalling the Higgs field. Suppose that d is coprime with $r\geq 2$ and that L is a line bundle with $\deg(L)>2g-2$. Then by Proposition 5.7, the moduli space $\mathcal{M}_L(r,d)$ is a smooth semiprojective variety. Accordingly, it admits a Bialynicki-Birula stratification

$$\mathcal{M}_L(r,d)=\bigcup_{\mu\in I}U_\mu^+,$$

which, by Lemma 5.3, induces the decomposition

$$[\mathcal{M}_L(r,d)]=\sum_{\mu\in I}\mathbb{L}^{N_\mu^+}[F_\mu] \tag{6.3}$$

of its motivic class, where $N_\mu^+=\dim(U_\mu^+)-\dim(F_\mu)$ is the rank of the affine bundle $U_\mu^+\to F_\mu$, corresponding to those Higgs bundles $(E,\varphi)\in\mathcal{M}_L(r,d)$ such that $\lim_{t\to 0}(E,t\varphi)$ lies in the $\mathbb{C}^*$-fixed point set F_μ. The characterization of the fixed points under the $\mathbb{C}^*$-action carried out by Simpson in [Sim92, §4] also applies to the L-twisted case, obtaining the following lemma.

Lemma 6.3. *Let (E,φ) be any L-twisted Higgs bundle such that $(E,\varphi)\cong(E,t\varphi)$ for some $t\in\mathbb{C}^*$ which is not a root of unity. Then E has the structure of an L-twisted variation of Hodge structure (6.1). Reciprocally, any L-twisted variation of Hodge structure is a fixed point of the $\mathbb{C}^*$-action.*

Given any multirank $\overline{r}=(r_1,\dots,r_k)$ and multidegree $\overline{d}=(d_1,\dots,d_k)$, define

$$|\overline{r}|=\sum_{j=1}^k r_j,\quad |\overline{d}|=\sum_{j=1}^k d_j\quad\text{and}\quad \Delta_L=\{(\overline{r},\overline{d})\mid \mathrm{VHS}_L(\overline{r},\overline{d})\neq\emptyset\}. \tag{6.4}$$

The previous lemma says that the semistable $\mathbb{C}^*$-fixed points are precisely those in $\mathrm{VHS}_L(\overline{r},\overline{d})\subset\mathcal{M}_L(r,d)$ for each suitable choice of $\overline{r}$ and $\overline{d}$. Thus, we rewrite (6.3) as

$$[\mathcal{M}_L(r,d)]=\sum_{\substack{(\overline{r},\overline{d})\in\Delta_L\\ |\overline{r}|=r,\,|\overline{d}|=d}}\mathbb{L}^{N^+_{L,\overline{r},\overline{d}}}[\mathrm{VHS}_L(\overline{r},\overline{d})], \tag{6.5}$$

where $N^+_{L,\overline{r},\overline{d}}$ is the notation for N_μ^+ in this case. We will also use the notations $N^-_{L,\overline{r},\overline{d}}$ for N_μ^- and $N^0_{L,\overline{r},\overline{d}}$ for N_μ^0; see (5.1).

Next, we will focus on the computation and invariance with respect to L of the ranks $N^\pm_{L,\overline{r},\overline{d}}$ in (6.5). Suppose that r and d are coprime, so that the moduli space $\mathcal{M}_L(r,d)$ is smooth. Following [Hit87, Kir84] and working as in [BGL11], we will proceed by analyzing the Bialynicki-Birula stratification from a Morse-theoretic point of view. The moduli space $\mathcal{M}_L(r,d)$ has a Kähler structure which is preserved by the action of $S^1 \subset \mathbb{C}^*$. Therefore, the $\mathbb{C}^*$-action induces a Hamiltonian action of S^1, with moment map

$$\mu : \mathcal{M}_L(r,d) \longrightarrow \mathbb{R}, \quad \mu(E,\varphi) = \frac{1}{2}\|\varphi\|^2,$$

where the L_2-norm is given with respect to the (harmonic) metric solving the Hitchin equations corresponding to the stable Higgs bundle (E,φ) under the Hitchin-Kobayashi correspondence; cf. [Hit87].

By [Fra59] the map μ becomes a perfect Morse-Bott function in $\mathcal{M}_L(r,d)$ and we have the following lemma.

Lemma 6.4. *Suppose that r and d are coprime and that $\deg(L) > 2g-2$. Let $\overline{r} = (r_1,\dots,r_k)$ and $\overline{d} = (d_1,\dots,d_k)$ be such that $r = r_1+\cdots+r_k$, $d = d_1+\cdots+d_k$ and $\mathrm{VHS}_L(\overline{r},\overline{d})$ is non-empty. Then $\mathrm{VHS}_L(\overline{r},\overline{d})$ is a component of the critical point set of μ and if $M_{L,\overline{r},\overline{d}}$ is its Morse index, then $M_{L,\overline{r},\overline{d}} = 2N^-_{L,\overline{r},\overline{d}}$. In particular,*

$$N^+_{L,\overline{r},\overline{d}} + N^0_{L,\overline{r},\overline{d}} + M_{L,\overline{r},\overline{d}}/2 = \dim(\mathcal{M}_L(r,d)) = 1 + r^2\deg(L).$$

Proof. $\mathcal{M}_L(r,d)$ is smooth by Lemma 2.2. Then, by [Kir84, Theorem 6.18, Example 9.4 and Corollary 13.2], we conclude that, for each $(\overline{r},\overline{d})$ in the given conditions, the component $\mathrm{VHS}_L(\overline{r},\overline{d})$ of the fixed-point locus $\mathcal{M}_L(r,d)^{\mathbb{C}}$ is a component the critical point set of μ and that the affine bundle $U^-_{\overline{r},\overline{d}} \longrightarrow \mathrm{VHS}_L(\overline{r},\overline{d})$ coincides with the downwards Morse flow of μ. Then, for each point $p \in \mathrm{VHS}_L(\overline{r},\overline{d})$, we have

$$N^-_{L,\overline{r},\overline{d}} = \dim\left(T_p\left(U^-_{\overline{r},\overline{d}}|_p\right)\right) = \frac{1}{2}\dim_{\mathbb{R}}\left(U^-_{\overline{r},\overline{d}}|_p\right) = \frac{1}{2}M_{L,\overline{r},\overline{d}}.$$

The last statement follows from (5.2) and again Lemma 2.2. □

Lemma 6.5. *Let L and L' be line bundles on X such that $\deg(L) = \deg(L') > 2g-2$. Suppose that r and d are coprime. Then the Morse index $M_{L,\overline{r},\overline{d}}$ of $\mathrm{VHS}_L(\overline{r},\overline{d}) \subset \mathcal{M}_L(r,d)$ is the same as the Morse index $M_{L',\overline{r},\overline{d}}$ of $\mathrm{VHS}_{L'}(\overline{r},\overline{d}) \subset \mathcal{M}_{L'}(r,d)$.*

Proof. Either by [BR94, Theorem 2.3] or by [Tor11, Theorem 47], the tangent space to the moduli space $\mathcal{M}_L(r,d)$ of L-Higgs bundles at a point (E,φ) is isomorphic to $\mathbb{H}^1(X, C^\bullet(E,\varphi))$, where $C^\bullet(E,\varphi)$ is the following complex

$$C^\bullet(E,\varphi) \,:\, \mathrm{End}(E) \xrightarrow{[-,\varphi]} \mathrm{End}(E)\otimes L.$$

At a variation of Hodge structure $(E_\bullet,\varphi_\bullet) \in \mathrm{VHS}_L(\overline{r},\overline{d})$, this deformation complex decomposes as

$$C^\bullet(E,\varphi) = \bigoplus_{l=-k+1}^{k-1} C^\bullet_l(E,\varphi) \tag{6.6}$$

where

$$C^\bullet_l(E,\varphi) : \bigoplus_{j-i=l}\mathrm{Hom}(E_i,E_j) \xrightarrow{[-,\varphi]} \bigoplus_{j-i=l+1}\mathrm{Hom}(E_i,E_j)\otimes L$$

and, thus, the tangent space decomposes as

$$\mathbb{H}^1(C^\bullet(E,\varphi)) = \bigoplus_{l=-k+1}^{k-1}\mathbb{H}^1(C^\bullet_l(E_\bullet,\varphi_\bullet)).$$

Then, the computations in [BGL11, §5] show that if r and d are coprime and $\deg(L) > 2g-2$, then the Morse index of $\mathrm{VHS}_L(\overline{r},\overline{d}) \subset \mathcal{M}_L(r,d)$ is

$$M_{L,\overline{r},\overline{d}} = 2\sum_{l=1}^{k-1} \dim(\mathbb{H}^1(C_l^\bullet(E_\bullet,\varphi_\bullet))) = -2\sum_{l=1}^{k-1}\chi(C_l^\bullet(E_\bullet,\varphi_\bullet)), \tag{6.7}$$

where, for each $l = 1,\dots,k-1$,

$$\begin{aligned}
-\chi(C_l^\bullet(E_\bullet,\varphi_\bullet)) &= \sum_{i=1}^{k-l-1}\chi(\mathrm{Hom}(E_i,E_{i+l+1})\otimes L) - \sum_{i=1}^{k-l}\chi(\mathrm{Hom}(E_i,E_{i+l}))\\
&= \sum_{i=1}^{k-l-1}(-r_{i+l+1}d_i + r_i d_{i+l+1} + r_i r_{i+l}\deg(L) + r_i r_{i+l+1}(1-g))\\
&\quad -\sum_{i=1}^{k-l}(-r_{i+l}d_i + r_i d_{i+l} + r_i r_{i+l}(1-g)).
\end{aligned} \tag{6.8}$$

Thus $\chi(C_l^\bullet(E_\bullet,\varphi_\bullet))$ depends on the degree of L, but not on L itself, and hence the same is true for the Morse index. □

Theorem 6.6. *Let X be a smooth complex projective curve of genus $g \geq 2$. Let L and L' be line bundles over X such that $\deg(L) = \deg(L') > 2g-2$. Assume that the rank r and degree d are coprime. Then the virtual motives of the corresponding moduli spaces $[\mathcal{M}_L(r,d)]$ and $[\mathcal{M}_{L'}(r,d)]$ are equal in $K(\mathcal{V}ar_{\mathbb{C}})$. Moreover, if d' is any integer coprime with r, then $E(\mathcal{M}_L(r,d)) = E(\mathcal{M}_{L'}(r,d'))$. Finally, if $L = L' = K(D)$ for some effective divisor D, then there is an actual isomorphism of pure mixed Hodge structures $H^\bullet(\mathcal{M}_L(r,d)) \cong H^\bullet(\mathcal{M}_{L'}(r,d'))$.*

Proof. By Corollary 6.2, for each $k = 1,\dots,r$ and each $\overline{r} = (r_1,\dots,r_k)$ and $\overline{d} = (d_1,\dots,d_k)$ with $|\overline{r}| = r$ and $|\overline{d}| = d$ (recall (6.4)), we have

$$\mathrm{VHS}_L(\overline{r},\overline{d}) \cong \mathrm{HC}^{\alpha_L}(\overline{r},\overline{d}_L),$$

where $\alpha_L = ((k-1)\deg(L),\dots,\deg(L),0)$ and $\overline{d}_L = (d_{L,i})$ with $d_{L,i} = d_i + r_i(i-k)\deg(L)$. As $\deg(L) = \deg(L')$ we have $\alpha_L = \alpha_{L'}$ and $\overline{d}_L = \overline{d}'_L$, so we obtain an isomorphism

$$\mathrm{VHS}_L(\overline{r},\overline{d}) \cong \mathrm{HC}^{\alpha_L}(\overline{r},\overline{d}_L) = \mathrm{HC}^{\alpha_{L'}}(\overline{r},\overline{d}_{L'}) \cong \mathrm{VHS}_{L'}(\overline{r},\overline{d}). \tag{6.9}$$

In particular, we have $\Delta_L = \Delta_{L'}$; cf. (6.4).

On the other hand, as $\mathcal{M}_L(r,d)$ is smooth of dimension $1 + r^2\deg(L)$, then for each $(\overline{r},\overline{d}) \in \Delta_L = \Delta_{L'}$, we have, using Lemma 6.4,

$$N^+_{L,\overline{r},\overline{d}} = 1 + r^2\deg(L) - \dim(\mathrm{VHS}_L(\overline{r},\overline{d})) - M_{L,\overline{r},\overline{d}}/2. \tag{6.10}$$

By Lemma 6.5 and (6.9) we have $N^+_{L,\overline{r},\overline{d}} = N^+_{L',\overline{r},\overline{d}}$.

Therefore, using (6.5), we conclude that

$$[\mathcal{M}_L(r,d)] = \sum_{k=1}^{r}\sum_{\substack{(\overline{r},\overline{d})\in\Delta_L\\ |\overline{r}|=r,\,|\overline{d}|=d}} \mathbb{L}^{N^+_{L,\overline{r},\overline{d}}}[\mathrm{VHS}_L(\overline{r},\overline{d})] = \sum_{k=1}^{r}\sum_{\substack{(\overline{r},\overline{d})\in\Delta_{L'}\\ |\overline{r}|=r,\,|\overline{d}|=d}} \mathbb{L}^{N^+_{L',\overline{r},\overline{d}}}[\mathrm{VHS}_{L'}(\overline{r},\overline{d})] = [\mathcal{M}_{L'}(r,d)].$$

Then

$$E(\mathcal{M}_L(r,d)) = E(\mathcal{M}_{L'}(r,d)) \tag{6.11}$$

is direct from (4.1).

Take now any d' also coprime with r. To prove that $\mathcal{M}_L(r,d)$ and $\mathcal{M}_{L'}(r,d')$ have the same E-polynomial, we proceed as follows. By [MS20a, Theorem 0.1] (see also [GWZ20, Theorem 7.15]),

$$E(\mathcal{M}_{K_X(D)}(r,d)) = E(\mathcal{M}_{K_X(D)}(r,d')), \tag{6.12}$$

where K_X denotes the canonical bundle of X and D is an effective divisor. To prove it for any twisting line bundles L, L' of the same degree (greater than $2g-2$), take any point $x_0 \in X$ and let $m = \deg(L) + 2 - 2g > 0$. Then $\deg(K_X(mx_0)) = \deg(L) = \deg(L')$ so, applying (6.11) and (6.12), yields

$$E(\mathcal{M}_L(r,d)) = E(\mathcal{M}_{K_X(mx_0)}(r,d)) = E(\mathcal{M}_{K_X(mx_0)}(r,d')) = E(\mathcal{M}_{L'}(r,d')),$$

as claimed.

Finally, if $L = L' = K(D)$ with D effective, the isomorphism $H^\bullet(\mathcal{M}_L(r,d)) \cong H^\bullet(\mathcal{M}_{L'}(r,d'))$ of Hodge structures follows immediately from [MS20a, Theorem 0.1]. □

6.3. **Independence of motives and E-polynomials from the Lie algebroid structure.** We can combine all the previous invariance results to prove our main theorem. Given a Lie algebroid, $\mathcal{L}$ on the curve X, recall that $\mathcal{M}_{\Lambda_\mathcal{L}}(r,d)$ denotes the moduli space of semistable integrable $\mathcal{L}$-connections of rank r and degree d or, equivalently, of semistable $\Lambda_\mathcal{L}$-modules, where $\Lambda_\mathcal{L}$ is the split almost polynomial sheaf of rings of differential operators associated to $\mathcal{L}$, under the equivalence provided by Theorem 3.9. If $\mathrm{rk}(\mathcal{L}) = 1$, then every $\mathcal{L}$-connection is automatically integrable, so in that case $\mathcal{M}_{\Lambda_\mathcal{L}}(r,d)$ is the moduli space of all semistable $\mathcal{L}$-connections of rank r and degree d.

Theorem 6.7. *Let X be a smooth projective curve of genus $g \geq 2$ and let $\mathcal{L}$ and $\mathcal{L}'$ be any two Lie algebroids on X such that $\mathrm{rk}(\mathcal{L}) = \mathrm{rk}(\mathcal{L}') = 1$ and $\deg(\mathcal{L}) = \deg(\mathcal{L}') < 2-2g$. Suppose that r and d are coprime. Then $[\mathcal{M}_{\Lambda_\mathcal{L}}(r,d)] = [\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,d)]$ in $\hat{K}(\mathcal{V}ar_\mathbb{C})$. Moreover, if d' is any integer coprime with r, then $E(\mathcal{M}_{\Lambda_\mathcal{L}}(r,d)) = E(\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,d'))$. Finally, if $L = L' = K(D)$ for some effective divisor D, then there is an actual isomorphism of pure mixed Hodge structures $H^\bullet(\mathcal{M}_{\Lambda_\mathcal{L}}(r,d)) \cong H^\bullet(\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,d'))$.*

Proof. This follows directly from Theorems 5.17 and 6.6, and by (4.1). □

Hence to compute the motivic class or the E-polynomial of $\mathcal{M}_{\Lambda_\mathcal{L}}(r,d)$, it is enough to do it for the moduli space $\mathcal{M}_{K_X(D)}(r,1)$ of $K_X(D)$-twisted Higgs bundles of rank r and degree 1, for some divisor D of the appropriate (positive) degree.

7. Applications

Now, let us analyze some consequences of the preceding results. In the next section we deduce some topological properties of the moduli spaces of $\mathcal{L}$-connections. These properties are not obtained by using the motivic results proved before, but rather the Bialynicki-Birula stratification of the $\mathcal{L}$-Hodge moduli space. In the subsequent sections, we will give a direct application of Theorem 6.7 related to the moduli spaces of logarithmic and irregular connections on X, and we will also provide explicit formulas for the motivic classs and E-polynomials, of Theorem 6.7, for $r = 2, 3$.

7.1. **Topological properties of moduli spaces of Lie algebroid connections.** Similarly to how we used the smoothness of the moduli space of twisted Higgs bundles to prove the smoothness of the moduli space of $\mathcal{L}$-connections back in section 5.3, we can also use the regularity properties and the Bialynicki-Birula stratification of $\mathcal{M}_{\Lambda_\mathcal{L}^{\mathrm{red}}}(r,d)$ to transfer other known properties of the moduli spaces of twisted Higgs bundles to moduli spaces of $\mathcal{L}$-connections. As an example, in this section we prove that, under certain conditions, the moduli space of $\mathcal{L}$-connections is irreducible and compute some its homotopy groups, by showing that they are isomorphic to the ones of the moduli space of vector bundles.

Let $\mathbf{M}(r,d)$ denote the moduli space of semistable vector bundles of rank r and degree d on the curve X.

Lemma 7.1. *Let X be a smooth projective curve of genus $g \geq 2$ and let $\mathcal{L}$ be a rank 1 Lie algebroid on X such that $\deg(\mathcal{L}) < 2-2g$. Suppose that $r \geq 2$ and d are coprime. Then the loci in $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ corresponding to semistable $\mathcal{L}$-connections whose underlying vector bundle is not stable has codimension at least $(g-1)(r-1)$. In particular, with the given bounds on g and r it has codimension at least 1.*

Proof. Let $\mathcal{L} = (L, [\cdot,\cdot].\delta)$. By Proposition 5.7 and Theorem 5.15 we know that the moduli spaces $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ and $\pi^{-1}(0) = \mathcal{M}_{L^{-1}}(r,d) \subset \mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ are smooth semiprojective varieties for the $\mathbb{C}^*$-action (3.19) and its restriction (5.8) to $\mathcal{M}_{L^{-1}}(r,d)$ respectively. Recall that here π is the map (3.16).

The fixed-point locus of this action is concentrated in $\mathcal{M}_{L^{-1}}(r,d)$ and, by Lemma 6.3, it corresponds to the subset of variations of Hodge structure (recall (6.4)),

$$\mathcal{M}_{L^{-1}}(r,d)^{\mathbb{C}^*} = \mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)^{\mathbb{C}^*} = \bigcup_{\substack{(\overline{r},\overline{d})\in\Delta_{L^{-1}} \\ |\overline{r}|=r,\, |\overline{d}|=d}} \mathrm{VHS}_{L^{-1}}(\overline{r},\overline{d}).$$

In this decomposition there is a distinguished component, namely the one for which $\overline{r} = \{r\}$ and $\overline{d} = \{d\}$. It parameterizes points of the form $(E,0,0)$ with E stable (because $(r,d)=1$), and it is therefore isomorphic to the moduli space $\mathbf{M}(r,d)$. Let

$$\mathcal{M}_{L^{-1}}(r,d) = \bigcup_{\substack{(\overline{r},\overline{d})\in\Delta_{L^{-1}} \\ |\overline{r}|=r,\, |\overline{d}|=d}} U^+_{L^{-1},\overline{r},\overline{d}} \quad \text{and} \quad \mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d) = \bigcup_{\substack{(\overline{r},\overline{d})\in\Delta_{L^{-1}} \\ |\overline{r}|=r,\, |\overline{d}|=d}} \tilde{U}^+_{L^{-1},\overline{r},\overline{d}}$$

be the corresponding Bialynicki-Birula decompositions, hence where

$$U^+_{L^{-1},\overline{r},\overline{d}} = \left\{ (E,\nabla_{\mathcal{L}},0) \in \mathcal{M}_{L^{-1}}(r,d) \,\middle|\, \lim_{t\to 0}(E,t\nabla_{\mathcal{L}}) \in \mathrm{VHS}_{L^{-1}}(\overline{r},\overline{d}) \right\},$$

$$\tilde{U}^+_{L^{-1},\overline{r},\overline{d}} = \left\{ (E,\nabla_{\mathcal{L}},\lambda) \in \mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d) \,\middle|\, \lim_{t\to 0}(E,t\nabla_{\mathcal{L}},t\lambda) \in \mathrm{VHS}_{L^{-1}}(\overline{r},\overline{d}) \right\}$$

are affine bundles over $\mathrm{VHS}_{L^{-1}}(\overline{r},\overline{d})$ of rank $N^+_{L^{-1},\overline{r},\overline{d}}$ and $\tilde{N}^+_{L^{-1},\overline{r},\overline{d}}$ respectively.

Let us write $U^+ = U^+_{L^{-1},\{r\},\{d\}}$ and $\tilde{U}^+ = \tilde{U}^+_{L^{-1},\{r\},\{d\}}$ the affine bundles lying over $\mathbf{M}(r,d)$. Let S and $\tilde{S}$ denote the subsets of $\mathcal{M}_{L^{-1}}(r,d)$ and $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ respectively corresponding to triples $(E,\nabla_{\mathcal{L}},\lambda)$ with E not stable. If E is a stable vector bundle then for every $(E,\nabla_{\mathcal{L}},\lambda) \in \mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)$ we have

$$\lim_{t\to 0}(E,t\nabla_{\mathcal{L}},t\lambda) = (E,0,0) \in \mathbf{M}(r,d) \subset \mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)^{\mathbb{C}^*},$$

so $S \subset \mathcal{M}_{L^{-1}}(r,d)\backslash U$ and $\tilde{S} \subset \mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)\backslash \tilde{U}$. Actually, by [BGL11, Proposition 5.1], $S = \mathcal{M}_{L^{-1}}(r,d)\backslash U$ (the proof is given for the moduli space with fixed determinant, but the proof also works for fixed degree) and in [BGL11, Proposition 5.4] it is proven that $\operatorname{codim}(S) \geq (g-1)(r-1)$ by showing that, if $U^+_{L^{-1},\overline{r},\overline{d}} \neq U^+$, then

$$\operatorname{codim}(U^+_{L^{-1},\overline{r},\overline{d}}) \geq (g-1)(r-1).$$

On the other hand, we know $\tilde{N}_{L^{-1},\overline{r},\overline{d}} = N_{L^{-1},\overline{r},\overline{d}}+1$ (see (5.6)), hence $\dim \tilde{U}^+_{L^{-1},\overline{r},\overline{d}} = \dim U^+_{L^{-1},\overline{r},\overline{d}}+1$. By Lemma 5.13, $\dim \mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d) = \dim(\mathcal{M}_{L^{-1}}(r,d))+1$ so we conclude that

$$\operatorname{codim}(\tilde{U}^+_{L^{-1},\overline{r},\overline{d}}) = \operatorname{codim}(U^+_{L^{-1},\overline{r},\overline{d}}) \geq (g-1)(r-1).$$

Finally, define $S' = \tilde{S} \cap \pi^{-1}(1)$. As $\mathbb{C}^*$ preserves the stability of the underlying bundle, then $\tilde{S}$ is $\mathbb{C}^*$-invariant and the restriction of this action to $\tilde{S}$ gives an isomorphism

$$\tilde{S} \cap \pi^{-1}(\mathbb{C}^*) \cong S' \times \mathbb{C}^*.$$

Therefore,

$$\dim(S') = \dim(\tilde{S} \cap \pi^{-1}(\mathbb{C}^*)) - 1 \leq \dim(\tilde{S}) - 1.$$

Finally, as $\dim \mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d) = \dim \mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d) - 1$, we conclude that

$$\operatorname{codim}(S') \geq \operatorname{codim}(\tilde{S}) \geq (g-1)(r-1),$$

completing the proof. □

Consider an $\mathcal{L}$-connection $(E, \nabla_{\mathcal{L}})$ of rank r and degree d on X. Let $\mathbb{E}$ be the C^∞ vector bundle on X underlying the algebraic vector bundle E. Note that $\mathbb{E}$ is independent of the choice of the $\mathcal{L}$-connection, as long as the rank and degree are still r and d. Let $\mathcal{G}(\mathbb{E})$ be the unitary gauge group for a fixed Hermitian metric on $\mathbb{E}$. In other words, $\mathcal{G}(\mathbb{E}) = \Omega^0(\mathfrak{u}(\mathbb{E}))$, where $\Omega^0(\mathfrak{u}(\mathbb{E}))$ stands for the space of C^∞-sections of the C^∞-bundle of unitary endomorphisms of $\mathbb{E}$.

Theorem 7.2. *Let X be a smooth complex curve of genus $g \geq 2$ and let $\mathcal{L}$ be a algebroid on X such that $\operatorname{rk}(\mathcal{L}) = 1$ and $\deg(\mathcal{L}) < 2 - 2g$. Suppose that r and d are coprime and that $r \geq 2$. Then, $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ is connected, hence irreducible. If, moreover, $(r,g) \neq (2,2)$, then its higher homotopy groups are given as follows:*

- $\pi_1(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)) \cong H_1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$;
- $\pi_2(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)) \cong \mathbb{Z}$;
- $\pi_k(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)) \cong \pi_{k-1}(\mathcal{G}(\mathbb{E}))$, *for every* $k = 3, \ldots, 2(g-1)(r-1) - 2$.

Proof. Let $\mathcal{L} = (L, [\cdot,\cdot].\delta)$. By Lemma 5.13 we know that the moduli space $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ is a smooth variety whose components are all of the same dimension $1 - r^2 \deg(\mathcal{L})$. Let $S' \subset \mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ be the subspace of $\mathcal{L}$-connections $(E, \nabla_{\mathcal{L}})$ with E not stable. Define $U' = \mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d) \backslash S'$.

It turns out that the $\mathbb{C}^*$-flow provides a deformation retraction from U' to $\mathbf{M}(r,d)$. To be precise, consider the forgetful map $\pi_{\mathbf{M}} : U' \longrightarrow \mathbf{M}(r,d)$, given by $\pi_{\mathbf{M}}(E, \nabla_{\mathcal{L}}) = E$. By Corollary 3.14 the map is surjective and we have the following explicit description of each fiber

$$\pi_{\mathbf{M}}^{-1}(E) = \{\nabla_{\mathcal{L}} : E \to E \otimes L^* \mid \nabla_{\mathcal{L}}(fs) = f\nabla_{\mathcal{L}}(s) + s \otimes d_{\mathcal{L}}(f),\ \forall s \in E, \forall f \in \mathcal{O}_X\}.$$

Observe that if $\nabla_{\mathcal{L}}, \nabla'_{\mathcal{L}} \in \pi_E^{-1}(E)$, then $\nabla_{\mathcal{L}} - \nabla'_{\mathcal{L}} \in H^0(\operatorname{End}(E) \otimes L^*)$, so $\pi_{\mathbf{M}}^{-1}(E)$ is an affine space on $H^0(\operatorname{End}(E) \otimes L^*)$. Moreover,

$$H^1(\operatorname{End}(E) \otimes L^*) \cong H^0(\operatorname{End}(E) \otimes K \otimes L)^* = 0$$

because, since E is stable, $\operatorname{End}(E)$ is semistable and so $\operatorname{End}(E) \otimes K_X \otimes L$ is a semistable vector bundle with $\deg(\operatorname{End}(E) \otimes K_X \otimes L) = r^2(\deg(K_X) + \deg(L)) < 0$. Thus, by Riemann-Roch, for each $E \in \mathbf{M}(r,d)$,

$$\dim(\pi_{\mathbf{M}}^{-1}(E)) = \dim H^0(\operatorname{End}(E) \otimes L^*) = 1 - r^2 \deg(L) - g$$

is constant and thus the map $\pi_{\mathbf{M}}$ is equidimensional. Let $\mathcal{E} \to X \times \mathbf{M}(r,d)$ be the universal bundle over $\mathbf{M}(r,d)$ (i.e., the bundle whose fiber over $X \times \{E\}$ is isomorphic to E); it exists since $(r,d) = 1$. Let $\pi_X : X \times \mathbf{M}(r,d) \to X$ be the projection. Then we conclude that U' is a torsor for the vector bundle

$$R(\pi_X)_*(\operatorname{End}(\mathcal{E}) \otimes \pi_X^* L^*) \longrightarrow \mathbf{M}(r,d).$$

It follows that, the homotopy groups of U' verify $\pi_k(U') \cong \pi_k(\mathbf{M}(r,d))$, for every $k \geq 0$.

By Lemma 7.1, $\operatorname{codim}(S') \geq (g-1)(r-1)$. Since $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ is smooth, this implies that

$$\pi_k(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)) \cong \pi_k(U') \cong \pi_k(\mathbf{M}(r,d)),$$

for every $k = 0, \dots, 2(g-1)(r-1)-2$. The moduli $\mathbf{M}(r,d)$ is connected, hence so is $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$, and thus irreducible because it is smooth. As for the higher homotopy groups, the results follow from [DU95, Theorem 3.1]. □

Remark 7.3. *By taking the trivial Lie algebroid $\mathcal{L} = (L,0,0)$ in the above theorem, one gets the results for the moduli space of $\mathcal{M}_{L^{-1}}(r,d)$ of L^{-1}-twisted Higgs bundles. The irreducibility conclusion was proved in [BGL11] by the same arguments, and actually the higher homotopy groups of $\mathcal{M}_{L^{-1}}(r,d)$ would also follow directly from [BGL11] by the same argument as above.*

On the other hand, improvements on the bound of k for which the isomorphism $\pi_k(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)) \cong \pi_{k-1}(\mathcal{G}(\mathbb{E}))$ holds have been achieved, for twisted Higgs bundles, in certain particular situations (cf. [Hau98] and [ZnR18]), hence we might expect that such isomorphism also holds, in this generality, for higher values of k.

7.2. **Chow motives and Voevodsky motives.** Theorem 6.7 shows that, under the stated conditions, there is an equality of motives

$$[\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)] = [\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,d)] \in \hat{K}(\mathcal{V}ar_{\mathbb{C}})$$

in the (completed) Grothendieck ring of varieties.

Nevertheless, the techniques that we use to prove this equality (namely, semiprojectivity of the $\mathcal{L}$-Hodge moduli space and the exposed relations between the Bialynicki-Birula decompositions of the corresponding moduli spaces of twisted Higgs bundles) also allow us to obtain isomorphisms for other types of invariants. Given a complex scheme X and a ring R, let us consider the following.

- Let $M(X) \in \mathrm{DM}^{\mathrm{eff}}(\mathbb{C},R)$ denote the Voevodsky motive of X, where $\mathrm{DM}^{\mathrm{eff}}(\mathbb{C},R)$ is the category of effective geometric motives as defined by Voevodsky in [Voe00].
- Let $h(X) \in \mathrm{Chow}^{\mathrm{eff}}(\mathbb{C},R)$ be the Chow motive of X, where $\mathrm{Chow}^{\mathrm{eff}}(\mathbb{C},R)$ is the category of effective Chow motives; see for example [Man68, Sch94, dBn01].
- Let $CH^\bullet(X,R)$ denote the chow ring of X with coefficients in R.

Moreover, recall that we say that X has a pure Voevodsky motive if $M(X)$ belongs to the heart of $\mathrm{DM}^{\mathrm{eff}}(\mathbb{C},R)$ which is equivalent to $\mathrm{Chow}^{\mathrm{eff}}(\mathbb{C},R)$ through Voevodsky's embedding (c.f. [HL19, Section 6.3]).

Theorem 7.4. *Let X be a smooth projective curve of genus $g \geq 2$ and let $\mathcal{L}$ and $\mathcal{L}'$ be any two Lie algebroids on X such that $\mathrm{rk}(\mathcal{L}) = \mathrm{rk}(\mathcal{L}') = 1$ and $\deg(\mathcal{L}) = \deg(\mathcal{L}') < 2-2g$. Suppose that r and d are coprime. Then, for every ring R, the Voevodsky motive of the moduli space $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ is pure and we have*

$$\begin{aligned} M(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)) &\cong M(\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,d)) \in \mathrm{DM}^{\mathrm{eff}}(\mathbb{C},R), \\ h(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)) &\cong h(\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,d)) \in \mathrm{Chow}^{\mathrm{eff}}(\mathbb{C},R), \\ \mathrm{CH}^\bullet(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)) &\cong \mathrm{CH}^\bullet(\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,d)). \end{aligned}$$

Proof. The proof is analogous to the one in Theorem 5.17 and Theorem 6.7, but we now use the technical theorems from Appendices A and B of [HL19] to perform the necessary computations in $\mathrm{DM}^{\mathrm{eff}}(\mathbb{C},R)$ instead of $\hat{K}(\mathcal{V}ar_{\mathbb{C}})$. By Theorem 5.15, for every rank one algebroid $\mathcal{L} = (L,[\cdot,\cdot],\delta)$ satisfying the hypothesis of the theorem the moduli space $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ is a smooth quasiprojective semiprojective variety with a $\mathbb{C}^*$-equivariant submersion $\pi : \mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d) \longrightarrow \mathbb{C}$ such that $\pi^{-1}(0) = \mathcal{M}_{L^*}(r,d)$ and $\pi^{-1}(1) = \mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$. Then [HL19, Theorem B.1] and [HL19, Corollary B.2] yield isomorphisms

$$M(\mathcal{M}_{L^{-1}}(r,d)) = M(\pi^{-1}(0)) \cong M(\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)) \cong M(\pi^{-1}(1)) = M(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d))$$

$$\mathrm{CH}^\bullet(\mathcal{M}_{L^{-1}}(r,d)) = \mathrm{CH}^\bullet(\pi^{-1}(0)) \cong \mathrm{CH}^\bullet(\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,d)) \cong \mathrm{CH}^\bullet(\pi^{-1}(1)) = \mathrm{CH}^\bullet(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d))$$

As $\mathcal{M}_{L^{-1}}(r,d)$ is also smooth and semiprojective by Proposition 5.7, its motive is pure by [HL19, Corollary A.5], so the motive of $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ is also pure and, therefore, the above isomorphism of Voevodsky motives induces an isomorphism of the Chow motives. Thus, we can assume without loss of generality that the algebroid structures for $\mathcal{L}$ and $\mathcal{L}'$ are trivial (i.e., that we have moduli spaces of Higgs bundles). Furthermore, purity of the Voevodsky motives and representability of the Chow groups as certain spaces of morphisms in $\mathrm{DM}^{\mathrm{eff}}(\mathbb{C},R)$ (c.f. [HL19, Corollary B.2]) imply that it is enough to prove that the Voevodsky motives are isomorphic to conclude the desired isomorphisms between Chow motives or Chow rings. The Bialynicki-Birula decomposition of the moduli spaces of Higgs bundles yield the following motivic decompositions [HL19, Theorem A.4], in which we follow the notation from Section 6.

$$M(\mathcal{M}_{L^{-1}}(r,d)) \cong \bigoplus_{k=1}^{r} \bigoplus_{\substack{(\overline{r},\overline{d})\in\Delta_{L^{-1}} \\ |\overline{r}|=r,\,|\overline{d}|=d}} M(\mathrm{VHS}_{L^{-1}}(\overline{r},\overline{d}))\{N^{-}_{L^{-1},\overline{r},\overline{d}}\}$$

$$M(\mathcal{M}_{(L')^{-1}}(r,d)) \cong \bigoplus_{k=1}^{r} \bigoplus_{\substack{(\overline{r},\overline{d})\in\Delta_{(L')^{-1}} \\ |\overline{r}|=r,\,|\overline{d}|=d}} M(\mathrm{VHS}_{(L')^{-1}}(\overline{r},\overline{d}))\{N^{-}_{(L')^{-1},\overline{r},\overline{d}}\}$$

By Corollary 6.2 we have $\Delta_{L^{-1}} = \Delta_{(L')^{-1}}$. Calling this set Δ, then Corollary 6.2 and Lemma 6.5 imply that for each $(\overline{r},\overline{d})\in\Delta$, we have $\mathrm{VHS}_{L^{-1}}(\overline{r},\overline{d}) \cong \mathrm{VHS}_{(L')^{-1}}(\overline{r},\overline{d})$ and $N^{-}_{L^{-1},\overline{r},\overline{d}} = N^{-}_{(L')^{-1},\overline{r},\overline{d}}$, so we obtain a term-by-term isomorphism of the previous Voevodsky motives. □

This result can be considered as an extension to moduli spaces of Lie algebroid connections (over $\mathbb{C}$) of [HL19, Theorem 4.2], in which it is proved that there exists an isomorphism between the Voevodsky motives and Chow rings of the de Rham and K-twisted Higgs moduli spaces.

7.3. **Motives of moduli spaces of irregular or logarithmic connections.** Recall the canonical Lie algebroid $\mathcal{T}_X = (T_X, [\cdot\,,\cdot]_{\mathrm{Lie}}, \mathrm{Id})$ on our smooth projective curve X. Consider an effective divisor $D=\sum_{i=1}^n k_i x_i$ on X, with $k_i \geq 1$. Let $\mathcal{M}_{\mathrm{conn}}(D,r,d)$ be the moduli space of rank r and degree d semistable *singular* ($\mathcal{T}_X$-)connections, with poles of order at most k_i over each $x_i\in D$. These connections are irregular if $k_i>1$ for some i.

Take the Lie subalgebroid $\mathcal{T}_X(-D)\subset\mathcal{T}_X$, thus with underlying bundle $T_X(-D)\subset T_X$, the induced Lie bracket of vector fields, and the inclusion anchor map. Then $\mathcal{M}_{\Lambda_{\mathcal{T}_X(-D)}}(r,d) = \mathcal{M}_{\mathrm{conn}}(D,r,d)$, hence we have the following direct corollary of Theorem 6.7.

Corollary 7.5. *If D and D' are any two effective divisors on X with $\deg(D)=\deg(D')$ and r and d are coprime, then $[\mathcal{M}_{\mathrm{conn}}(D,r,d)] = [\mathcal{M}_{\mathrm{conn}}(D',r,d)] \in \hat{K}(\mathcal{V}ar_{\mathbb{C}})$ and $E(\mathcal{M}_{\mathrm{conn}}(D,r,d)) = E(\mathcal{M}_{\mathrm{conn}}(D',r,d))$.*

In particular, by taking D' to be a simple divisor, we conclude the following.

Corollary 7.6. *The motivic class and E-polynomial of any moduli space of* irregular *connections on a smooth projective curve X of genus at least 2 equals that of any moduli space of* logarithmic *connections X, with singular divisor of the same degree.*

7.4. **Explicit motives and E-polynomials for rank 2 and 3.** Fix a rank 1 Lie algebroid $\mathcal{L} = (L,[\cdot\,,\cdot],\delta)$ on the curve X, such that $\deg(L)<2-2g$. In this section, we provide explicit formulae for the motivic classes and E-polynomials of the moduli spaces of $\mathcal{L}$-connections of rank 2 and 3 and coprime degree. Theorem 6.7 allows us to perform all computations by just considering the trivial Lie algebroid $(L,0,0)$, that is, the moduli space of L^{-1}-twisted Higgs bundles of corresponding rank and degree.

7.4.1. *Recollection of properties of motives.* We first need to introduce some notation and recall, without proof, some facts on the theory of motivic classes in $\hat{K}(\mathcal{V}ar_{\mathbb{C}})$. For details, see for example [Hei07, Kap00].

The symmetric product of a variety gives rise to the λ-*operator* defined, for each $n \geq 0$, as

$$\lambda^n : \hat{K}(\mathcal{V}ar_{\mathbb{C}}) \to \hat{K}(\mathcal{V}ar_{\mathbb{C}}), \quad \lambda^n([Y]) = [\mathrm{Sym}^n(Y)]. \tag{7.1}$$

For example $\lambda^n(\mathbb{L}^k) = \mathbb{L}^{nk}$. With these operators, $\hat{K}(\mathcal{V}ar_{\mathbb{C}})$ acquires the structure of a λ-*ring*. In particular, the relation

$$\lambda^n([Y] + [Z]) = \sum_{i+j=n} \lambda^i([Y])\lambda^j([Z]), \tag{7.2}$$

holds.

The motive of our fixed genus $g \geq 2$ curve X splits as $[X] = 1 + h^1(X) + \mathbb{L}$, where $h^1(X) \in \hat{K}(\mathcal{V}ar_{\mathbb{C}})$ is such that the motive of the Jacobian of X is given by

$$[\mathrm{Jac}(X)] = \sum_{i=0}^{2g} \lambda^i(h^1(X)). \tag{7.3}$$

Define the $\hat{K}(\mathcal{V}ar_{\mathbb{C}})$-valued polynomial

$$P_X(x) = \sum_{i=0}^{2g} \lambda^i(h^1(X))x^i \in \hat{K}(\mathcal{V}ar_{\mathbb{C}})[[x]] \tag{7.4}$$

and note that $P_X(1) = [\mathrm{Jac}(X)]$. Consider also the *zeta function* of X, defined as

$$Z(X, x) = \sum_{k \geq 0} \lambda^k([X])x^k \in \hat{K}(\mathcal{V}ar_{\mathbb{C}})[[x]].$$

Using that $\lambda^n(h^1(X)) = \mathbb{L}^{n-g}\lambda^{2g-n}(h^1(X))$, if $n = 0, \ldots, 2g$, and that $\lambda^n(h^1(X)) = 0$ if $n > 2g$, it follows that

$$\lambda^n([X]) = \operatorname*{coeff}_{x^0} \frac{Z(X, x)}{x^n} = \operatorname*{coeff}_{x^0} \frac{P_X(x)}{(1-x)(1-\mathbb{L}x)x^n}.$$

7.4.2. *Motives of $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r, d)$ for $r = 2, 3$.* Now we move on to the motives of moduli spaces. We want to compute the motive of $\mathcal{M}_{L^{-1}}(r, d)$, for $r = 2, 3$ and d coprime with r. This will be done by using the formula (6.5), and so we will need to consider the moduli space of rank r and degree d vector bundles (which we think of consisting of Higgs bundles which are variations of Hodge structure of type (r)) and then variations of Hodge structure of type $(1, 1)$ for $r = 2$ and type $(1, 2)$, $(2, 1)$ and $(1, 1, 1)$ for $r = 3$. We will not fill the full details of the computations, and leave them to the reader.

In this section, we use the notation d_L for the degree of the line bundle L, so that $d_L < 2 - 2g$.

Recall that $\mathbf{M}(r, d)$ denotes the moduli space of stable vector bundles of rank r and degree d over the curve X.

Let us start with rank 2 case. Let d be odd. By Example 3.4 of [GPHS14] or equation (3.9), page 41 of [Sán14], the motivic class of $\mathbf{M}(2, d)$ is given by

$$[\mathbf{M}(2, d)] = \frac{[\mathrm{Jac}(X)]P_X(\mathbb{L}) - \mathbb{L}^g[\mathrm{Jac}(X)]^2}{(\mathbb{L}-1)(\mathbb{L}^2-1)}, \tag{7.5}$$

where P_X is the polynomial given in (7.4). Notice that this formula is obtained by the one in [GPHS14] by multiplying by $\mathbb{L} - 1$ because in loc. cit., the stated formula stands for the stack of stable vector bundles, which is is $\mathbb{C}^*$-gerbe over $\mathbf{M}(2, d)$.

We move on the motivic class of subvarieties of $\mathcal{M}_{L^{-1}}(2,d)$ corresponding to variations of Hodge structure of type $(1,1)$. An L^{-1}-twisted-Higgs bundle (E,φ) lies in $\mathrm{VHS}_{L^{-1}}((1,1),(d_1,d-d_1))$ if it is stable and

$$E = E_1 \oplus E_2, \quad \varphi = \left(\begin{smallmatrix} 0 & 0 \\ \varphi_1 & 0 \end{smallmatrix}\right), \tag{7.6}$$

with E_1, E_2 line bundles of degree d_1 and $d-d_1$ respectively and $\varphi_1 : E_1 \to E_2 \otimes L^{-1}$ nonzero. The fact that $\varphi_1 \neq 0$ and stability (E_2 is φ-invariant) impose conditions on the degree d_1 and, indeed,

$$\mathrm{VHS}_{L^{-1}}((1,1),(d_1,d-d_1)) \neq \emptyset \iff d/2 < d_1 \leq (d-d_L)/2.$$

In such a case, the map $\left(E_1 \oplus E_2, \left(\begin{smallmatrix} 0 & 0 \\ \varphi_1 & 0 \end{smallmatrix}\right)\right) \mapsto (\mathrm{div}(\varphi_1), E_2)$, where $\mathrm{div}(\varphi_1)$ denotes the divisor of the section $\varphi_1 \in H^0(E_1^{-1}E_2L^{-1})$, yields the isomorphism

$$\mathrm{VHS}_{L^{-1}}((1,1),(d_1,d-d_1)) \cong \mathrm{Sym}^{d-2d_1-d_L}(X) \times \mathrm{Jac}^{d-d_1}(X). \tag{7.7}$$

Hence, since the 'Jacobian' of degree $d-d_1$ line bundles on X is isomorphic to the Jacobian $\mathrm{Jac}(X)$ of (degree 0 line bundles on) X,

$$[\mathrm{VHS}_{L^{-1}}((1,1),(d_1,d-d_1))] = \lambda^{d-2d_1-d_L}([X])[\mathrm{Jac}(X)], \tag{7.8}$$

where we are using the λ-operations defined in (7.1).

Now we consider the rank 3 case. Fix d coprime with 3, so that every semistable L^{-1}-twisted Higgs bundle is stable. We have

$$\begin{aligned}[\mathbf{M}(3,d)] = &\frac{[\mathrm{Jac}(X)]}{(\mathbb{L}-1)(\mathbb{L}^2-1)^2(\mathbb{L}^3-1)}\Big(\mathbb{L}^{3g-1}(1+\mathbb{L}+\mathbb{L}^2)[\mathrm{Jac}(X)]^2 \\ &- \mathbb{L}^{2g-1}(1+\mathbb{L})^2[\mathrm{Jac}(X)]P_X(\mathbb{L}) + P_X(\mathbb{L})P_X(\mathbb{L}^2)\Big).\end{aligned} \tag{7.9}$$

by [GPHS14, Remark 3.5] or [Sán14, Theorem 4.7]. As in the $r=2$ case, this is obtained by the stated formula in [GPHS14] by multiplying by $\mathbb{L}-1$.

Consider now variations of Hodge structure of type $(1,2)$ in $\mathcal{M}_{L^{-1}}(3,d)$. A stable L^{-1}-twisted Higgs bundle (E,φ) lies in $\mathrm{VHS}_{L^{-1}}((1,2),(d_1,d-d_1))$ if it is of the form (7.6), with the only difference that now E_2 has rank 2. Let $I \subset E_2$ be the line bundle such that the saturation of the image of φ_1 equals IL^{-1}. Then both E_2 and $E_1 \oplus I$ are φ-invariant subbundles of E. Checking the stability for them imposes conditions of d_1, and actually we have that

$$\mathrm{VHS}_{L^{-1}}((1,2),(d_1,d-d_1)) \neq \emptyset \iff d/3 < d_1 < d/3 - d_L/2. \tag{7.10}$$

Moreover, by Corollary 6.2, there is an isomorphism with the moduli space of $\alpha_{L^{-1}} = (-d_L, 0)$-stable chains

$$\mathrm{VHS}_{L^{-1}}((1,2),(d_1,d-d_1)) \cong \mathrm{HC}^{\alpha_{L^{-1}}}((1,2),(d_1+d_L,d-d_1)). \tag{7.11}$$

Since d is coprime with 3, then $\alpha_{L^{-1}}$ is not a critical value (i.e. a value of the stability parameter where semistability changes), so the motive of $\mathrm{HC}^{\alpha_{L^{-1}}}((1,2),(d_1+d_L,d-d_1))$ can be read off from [GPHS14, Example 6.4], by adapting the computation to the L^{-1}-twisting setting, or, perhaps more directly, from Theorem 3.2 of [Sán14], where the author considers the moduli space of $-d_L$-stable triples of type $((2,1)(d-d_1-2d_L,d_1))$ (cf. [BGPG04]), which is isomorphic to the moduli space of $\alpha_{L^{-1}}$-stable chains. From this, we conclude that

(7.12)
$$\begin{aligned}[\mathrm{VHS}_{L^{-1}}((1,2),(d_1,d-d_1))] &= \frac{[\mathrm{Jac}(X)]^2}{\mathbb{L}-1}\Big(\mathbb{L}^{2\lfloor d/3\rfloor-d+d_1+g+1}\lambda^{d-\lfloor d/3\rfloor-2d_1-d_L-1}([X]+\mathbb{L}^2)\\ &\quad -\lambda^{d-\lfloor d/3\rfloor-2d_1-d_L-1}([X]\mathbb{L}+1)\Big)\\ &= \frac{[\mathrm{Jac}(X)]^2}{\mathbb{L}-1}\Big(\mathbb{L}^{2\lfloor d/3\rfloor-d+d_1+g+1}\\ &\quad \times \sum_{i=0}^{d-\lfloor d/3\rfloor-2d_1-d_L-1}\lambda^i([X])(\mathbb{L}^{2d-2\lfloor d/3\rfloor-4d_1-2d_L-2-2i}-\mathbb{L}^i)\Big).\end{aligned}$$

The motives of the subvarieties of $\mathcal{M}_{L^{-1}}(3,d)$ corresponding to variations of Hodge structure of type $(2,1)$ are directly obtained from the ones of type $(1,2)$ by making use of the isomorphism

$$\mathrm{VHS}_{L^{-1}}((2,1),(d_1,d-d_1)) \cong \mathrm{VHS}_{L^{-1}}((1,2),(d_1-d,-d_1)) \tag{7.13}$$

arising from duality.

Finally, we deal with variations of Hodge structure of type $(1,1,1)$ in $\mathcal{M}_{L^{-1}}(3,d)$. Similarly to the previous cases, it follows that

$$\mathrm{VHS}_{L^{-1}}((1,1,1),(d_1,d_2,d-d_1-d_2)) \neq \emptyset \iff (d_1,d_2)\in\Delta_{-d_L}(d),$$

where

$$\Delta_{-d_L}(d) = \left\{(a,b)\in\mathbb{Z}^2 \mid a-b\leq -d_L,\ a+2b-d\leq -d_L,\ a>d/3,\ a+b>2d/3\right\}, \tag{7.14}$$

and in that case, we have the following isomorphism

(7.15)
$$\mathrm{VHS}_{L^{-1}}((1,1,1),(d_1,d_2,d-d_1-d_2)) \cong \mathrm{Sym}^{-d_1+d_2-d_L}(X)\times\mathrm{Sym}^{d-d_1-2d_2-d_L}(X)\times\mathrm{Jac}^{d-d_1-d_2}(X).$$

Thus,

$$[\mathrm{VHS}_{L^{-1}}((1,1,1),(d_1,d_2,d-d_1-d_2))] = \lambda^{-d_1+d_2-d_L}([X])\lambda^{d-d_1-2d_2-d_L}([X])[\mathrm{Jac}(X)]. \tag{7.16}$$

Now we have the promised corollary of Theorem 6.7.

Corollary 7.7. *Let X be a smooth projective curve of genus $g\geq 2$ and let $\mathcal{L}$ be a rank 1 Lie algebroid on X. Write $d_L=\deg(\mathcal{L})$ and suppose that $d_L<2-2g$. Then,*

(1) if $(2,d)=1$,

$$\begin{aligned}[\mathcal{M}_{\Lambda_{\mathcal{L}}}(2,d)] &= \frac{\mathbb{L}^{-4d_L+4-4g}\Big([\mathrm{Jac}(X)]P_X(\mathbb{L})-\mathbb{L}^g[\mathrm{Jac}(X)]^2\Big)}{(\mathbb{L}-1)(\mathbb{L}^2-1)}\\ &\quad +\mathbb{L}^{-3d_L+2-2g}[\mathrm{Jac}(X)]\sum_{d_1=\lfloor d/2\rfloor+1}^{\lfloor\frac{d-d_L}{2}\rfloor}\lambda^{d-2d_1-d_L}([X]).\end{aligned}$$

(2) if $(3,d)=1$,

$$\begin{aligned}
[\mathcal{M}_{\Lambda_{\mathcal{L}}}(3,d)] = & \frac{\mathbb{L}^{-9d_L+9-9g}[\mathrm{Jac}(X)]}{(\mathbb{L}-1)(\mathbb{L}^2-1)^2(\mathbb{L}^3-1)}\Big(\mathbb{L}^{3g-1}(1+\mathbb{L}+\mathbb{L}^2)[\mathrm{Jac}(X)]^2 \\
& - \mathbb{L}^{2g-1}(1+\mathbb{L})^2[\mathrm{Jac}(X)]P_X(\mathbb{L}) + P_X(\mathbb{L})P_X(\mathbb{L}^2)\Big) \\
& + \frac{\mathbb{L}^{-7d_L+5-5g}[\mathrm{Jac}(X)]^2}{\mathbb{L}-1}\sum_{d_1=\lfloor d/3\rfloor+1}^{\lfloor \frac{d}{3}-\frac{d_L}{2}\rfloor}\Big(\mathbb{L}^{2\lfloor d/3\rfloor-d+d_1+g+1}\lambda^{d-\lfloor d/3\rfloor-2d_1-d_L-1}([X]+\mathbb{L}^2) \\
& - \lambda^{d-\lfloor d/3\rfloor-2d_1-d_L-1}([X]\mathbb{L}+1)\Big) \\
& + \frac{\mathbb{L}^{-7d_L+5-5g}[\mathrm{Jac}(X)]^2}{\mathbb{L}-1}\sum_{d_1=\lfloor 2d/3\rfloor+1}^{\lfloor \frac{2d}{3}-\frac{d_L}{2}\rfloor}\Big(\mathbb{L}^{2\lfloor -d/3\rfloor+d_1+g+1}\lambda^{d-\lfloor -d/3\rfloor-2d_1-d_L-1}([X]+\mathbb{L}^2) \\
& - \lambda^{d-\lfloor -d/3\rfloor-2d_1-d_L-1}([X]\mathbb{L}+1)\Big) \\
& + \mathbb{L}^{-6d_L+3-3g}[\mathrm{Jac}(X)]\sum_{(d_1,d_2)\in\Delta_{-d_L}(d)}\lambda^{-d_1+d_2-d_L}([X])\lambda^{d-d_1-2d_2-d_L}([X]),
\end{aligned}$$

with $\Delta_{-d_L}(d)$ *defined in* (7.14).

Remark 7.8. *It is easy to see that the motivic class* $[\mathcal{M}_{\Lambda_{\mathcal{L}}}(3,d)]$ *is indeed the same by replacing* d *for* $-d$ *(for the last sum, one should use the bijection between* $\Delta_{-d_L}(d)$ *and* $\Delta_{-d_L}(-d)$ *given by* $(a,b)\mapsto(-d+a+b,-b)$*). Of course, this had to occur since duality yields an isomorphism between the moduli spaces* $\mathcal{M}_{\Lambda_{\mathcal{L}}}(3,d)$ *and* $\mathcal{M}_{\Lambda_{\mathcal{L}}}(3,-d)$ *(and, of course, the mentioned bijection* $\Delta_{-d_L}(d)\simeq\Delta_{-d_L}(-d)$ *is provided by duality).*

Proof. As we are considering cases in which the rank and the degree are coprime, we can apply Theorem refthm:equalMotive, and assume without loss of generality that $\mathcal{L}$ has the trivial Lie algebroid structure $(L,0,0)$. Therefore, $\mathcal{M}_{\mathcal{L}}(r,d)$ corresponds to the moduli space of L^{-1}-twisted Higgs bundles of rank r and degree d.

Then, everything follows from the decomposition (6.5), using the formula (6.10) (with L replaced by L^{-1}),

$$N^{+}_{L^{-1},\overline{r},\overline{d}} = 1 - r^2\deg(L) - \dim(\mathrm{VHS}_{L^{-1}}(\overline{r},\overline{d})) - M_{L^{-1},\overline{r},\overline{d}}/2$$

for the exponents of $\mathbb{L}$ in each summand. All of them are straightforward, using (6.7) and (6.8). We leave the details of the computations for the reader. In rank 2, use (7.7) followed by (7.5) and (7.8).

In rank 3, use (7.11) and (7.15), knowing that, in the $(1,2)$-type,

$$\begin{aligned}
\dim(\mathrm{VHS}_{L^{-1}}(X,(1,2),(d_1,d-d_1))) &= \dim(\mathrm{HC}^{\alpha_{L^{-1}}-ss}((1,2),(d_1+d_L,d-d_1))) \\
&= 3g-2-3d_1+d-2d_L
\end{aligned}$$

by Theorem A (2) of [BGPG04] and that the dimension of $\dim(\mathrm{VHS}_{L^{-1}}(X,(1,2),(d_1,d-d_1)))$ is computed similarly using (7.13). Then, (6.5) becomes

$$\begin{aligned}[\mathcal{M}_{\Lambda_{\mathcal{L}}}(3,d)] = \mathbb{L}^{-9d_L+9-9g}[\mathbf{M}(3,d)] \\ + \mathbb{L}^{-7d_L+5-5g} \sum_{d_1=\lfloor \frac{d}{3} \rfloor+1}^{\lfloor \frac{d}{3}-\frac{d_L}{2} \rfloor} [\mathrm{VHS}_{L^{-1}}((1,2),(d_1,d-d_1))] \\ + \mathbb{L}^{-7d_L+5-5g} \sum_{d_1=\lfloor \frac{2d}{3} \rfloor+1}^{\lfloor \frac{2d}{3}-\frac{d_L}{2} \rfloor} [\mathrm{VHS}_{L^{-1}}((2,1),(d_1,d-d_1))] \\ + \mathbb{L}^{-6d_L+3-3g} \sum_{(d_1,d_2)\in\Delta_{-d_L}(d)} [\mathrm{VHS}_{L^{-1}}((1,1,1),(d_1,d_2,d-d_1-d_2))],\end{aligned}$$

and we obtain the result from (7.9)–(7.16). □

7.4.3. *E-polynomials of $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ for $r=2,3$.* We now apply the E-polynomial map (4.1) to the formulas of the preceding result.

If X is again our smooth projective curve of genus g, it is well-known that $E(\mathrm{Jac}(X)) = (1+u)^g(1+v)^g$, thus

$$E(\lambda^n([X])) = \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^n}.$$

In addition, from (7.3) and (7.4), we have that, for any k,

$$\begin{aligned}E(P_X(\mathbb{L}^k)) &= \sum_{i=0}^{2g} E(\lambda^i(h^1(X)))(u^kv^k)^i = \sum_{i=0}^{2g}\sum_{p+q=i} h^{p,q}(\mathrm{Jac}(X))u^pv^q(u^kv^k)^i \\ &= \sum_{i=0}^{2g}\sum_{p+q=i} h^{p,q}(\mathrm{Jac}(X))(u^{k+1}v^k)^p(u^kv^{k+1})^q \\ &= E(\mathrm{Jac}(X))(u^{k+1}v^k, u^kv^{k+1}) = (1+u^{k+1}v^k)^g(u^kv^{k+1})^g.\end{aligned}$$

In particular,

$$E(P_X(\mathbb{L})) = (1+u^2v)^g(1+uv^2)^g \quad \text{and} \quad E(P_X(\mathbb{L}^2)) = (1+u^3v^2)^g(1+u^2v^3)^g, \tag{7.17}$$

Corollary 7.9. *Let X be smooth projective curve of genus g and let $\mathcal{L}$ be a rank 1 Lie algebroid on X. Write $d_L = \deg(\mathcal{L})$ and suppose that $d_L < 2-2g$.*

(1) Let d be odd. Then,

$$\begin{aligned}E(\mathcal{M}_{\Lambda_{\mathcal{L}}}(2,1)) = (uv)^{-4d_L+4-4g}E(\mathbf{M}(2,d)) \\ + (uv)^{-3d_L+2-2g}(1+u)^g(1+v)^g \operatorname{coeff}_{x^0}\left(\frac{(1+ux)^g(1+vx)^gx^{d_L+1}}{(1-x^2)(1-x)(1-uvx)}\right),\end{aligned}$$

where

$$E(\mathbf{M}(2,d)) = \frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^{2g}(1+v)^{2g}}{(uv-1)((uv)^2-1)}.$$

(2) Let d be coprime with 3. Then,

$$\begin{aligned}
E(\mathcal{M}_{\Lambda_{\mathcal{L}}}(3,1)) &= (uv)^{-9d_L+9-9g}E(\mathbf{M}(3,d))\\
&+\frac{(1+u)^{2g}(1+v)^{2g}(uv)^{-7d_L+6-4g}}{uv-1}\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{d_L+2}}{(1-x)(1-uvx)(1-(uv)^2x)(1-uvx^2)}\\
&-\frac{(1+u)^{2g}(1+v)^{2g}(uv)^{-8d_L+6-5g}}{uv-1}\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{d_L+2}}{(1-x)(1-uvx)(uv-x)((uv)^2-x^2)}\\
&+\frac{(1+u)^{2g}(1+v)^{2g}(uv)^{-7d_L+5-4g}}{uv-1}\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{d_L+1}}{(1-x)(1-uvx)(1-(uv)^2x)(1-uvx^2)}\\
&-\frac{(1+u)^{2g}(1+v)^{2g}(uv)^{-8d_L+7-5g}}{uv-1}\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{d_L+1}}{(1-x)(1-uvx)(uv-x)((uv)^2-x^2)}\\
&+(1+u)^g(1+v)^g(uv)^{-6d_L+3-3g}\cdot\\
&\cdot\operatorname*{coeff}_{x^0y^0}\frac{(1+ux)^g(1+vx)^g(1+uy)^g(1+vy)^g x^{2d_L+2}y^{2d_L+1}(x^{-2d_L}-y^{-d_L})(y^{-2d_L}-x^{-d_L})}{(1-x)(1-uvx)(1-y)(1-uvy)(x-y^2)(y-x^2)},
\end{aligned}$$

where

$$\begin{aligned}
E(\mathbf{M}(3,d)) &= \frac{(1+u)^g(1+v)^g}{(uv-1)((uv)^2-1)^2((uv)^3-1)}\Big((uv)^{3g-1}(1+uv+(uv)^2)(1+u)^{2g}(1+v)^{2g}\\
&-(uv)^{2g-1}(1+uv)^2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g\\
&+(1+u^2v)^g(1+uv^2)^g(1+u^2v^3)^g(1+u^3v^2)^g\Big).
\end{aligned}$$

Proof. This follows from Corollary 7.7 and from computations which are now standard. We leave the details for the reader, who may see for example [Got94], [GPHS14], [Ben10] (especially Theorems 3.1.4 and 3.5.7) and [Sán14] for techniques on these computations. Note however that the ones in [Ben10] contain some slight inaccuracies (so that the final results stated there, in Theorems 3.1.4 and 3.5.7, are not correct). The formulas for the E-polynomials of the appropriate powers of the λ-operations of $[X]+\mathbb{L}^2$ and of $[X]\mathbb{L}+1$, which appear in Corollary 7.7, may be found in equation (3.3) of page 39 of [Sán14] and the ones concerning $P_X(\mathbb{L})$ and $P_X(\mathbb{L}^2)$ are given in (7.17). ☐

Remark 7.10. *The E-polynomials of the moduli space of vector bundles of rank 2 and 3 and coprime degree d were first computed recursively in* [EK00, Theorem 1]. *The one for rank 3 was also explicitly obtained with different techniques, in Theorem 1.2 of* [Muñ08] *(even though the formula there has a minor inaccuracy on a sign and the one on Theorem 7.1 – not in Theorem 1.2 – has an extra $(1+u)^g(1+v)^g$ term which should not be there).*

8. Motives of moduli spaces of $\mathcal{L}$-connections with fixed determinant

So far we have considered moduli spaces of flat $\mathcal{L}$-connections (or Λ-modules) with fixed rank and degree, but the techniques presented in the previous sections also allow us to obtain analogues of the previous results for moduli spaces of flat $\mathcal{L}$-connections with fixed determinant. In this section we will define the moduli space of flat $\mathcal{L}$-connections with fixed determinant and show several results proving the invariance of its motivic class and E-polynomial regarding the Lie algebroid structure, providing the necessary changes in the previously exposed arguments to treat the fixed determinant scenario.

8.1. **Twisted Higgs bundles with fixed determinant.** Let ξ be an algebraic line bundle over X of degree d coprime with r. Let $\mathcal{M}_L(r,\xi)\subset\mathcal{M}_L(r,d)$ be the moduli space of traceless L-twisted

Higgs bundles with fixed determinant ξ, i.e., the moduli space of pairs (E,φ) with $\deg(E) \cong \xi$ and $\varphi \in H^0(\mathrm{End}_0(E)\otimes L)$, where $\mathrm{End}_0(E)$ denotes the endomorphisms of E with trace 0.

By [BGL11, Theorem 1.2], if $g \geq 2$ and $\deg(L) > 2g-2$, $\mathcal{M}_L(r,\xi)$ is a smooth irreducible $\mathbb{C}^*$-invariant closed subvariety of $\mathcal{M}_L(r,d)$, so it is a smooth semiprojective variety. Hence the digression held about the structure of the Bialynicki-Birula stratification of $\mathcal{M}_L(r,d)$ applies to $\mathcal{M}_L(r,\xi)$ as well. The fixed-point locus of the $\mathbb{C}^*$-action on $\mathcal{M}_L(r,\xi)$ clearly corresponds to the intersection of the fixed-point locus of the $\mathbb{C}^*$-action on $\mathcal{M}_L(r,d)$ with $\mathcal{M}_L(r,\xi)$. In section 6 we described a decomposition of the fixed-point locus as

$$\mathcal{M}_L(r,d)^{\mathbb{C}^*} = \bigcup_{\substack{(\overline{r},\overline{d})\in\Delta_L \\ |\overline{r}|=r,\, |\overline{d}|=d}} \mathrm{VHS}_L(\overline{r},\overline{d}),$$

thus we have a decomposition

$$\mathcal{M}_L(r,\xi)^{\mathbb{C}^*} = \bigcup_{\substack{(\overline{r},\overline{d})\in\Delta_L \\ |\overline{r}|=r,\, |\overline{d}|=d}} \mathrm{VHS}_L(\overline{r},\overline{d},\xi),$$

where $\mathrm{VHS}_L(\overline{r},\overline{d},\xi) = \mathrm{VHS}_L(\overline{r},\overline{d}) \cap \mathcal{M}_L(r,\xi)$. By construction all variations of Hodge structure have traceless Higgs fields, so

$$\mathrm{VHS}_L(\overline{r},\overline{d},\xi) = \left\{ (E_\bullet,\varphi_\bullet) \in \mathrm{VHS}_L(\overline{r},\overline{d}) \,\middle|\, \bigotimes_{i=1}^k \det(E_i) \cong \xi \right\}.$$

On the other hand, we can consider algebraic chains with fixed "total determinant" in the following sense. For $\overline{r}$ and $\overline{d}$ such that $|\overline{r}| = r$ and $|\overline{d}| = d$, define

$$\mathrm{HC}^\alpha(\overline{r},\overline{d},\xi) = \left\{ (E_\bullet,\varphi_\bullet) \in \mathrm{HC}^\alpha(\overline{r},\overline{d}) \,\middle|\, \bigotimes_{i=1}^k \det(E_i) \cong \xi \right\}.$$

Lemma 8.1. *Given $\overline{r} = (r_1,\ldots,r_k)$ and a degree d line bundle ξ, consider the line bundle*

$$\xi_L = \xi \otimes L^{\otimes(\sum_{i=1}^k (i-k)r_i)}.$$

Then the isomorphism described in Corollary 6.2 induces an isomorphism

$$\mathrm{VHS}_L(\overline{r},\overline{d},\xi) \cong \mathrm{HC}^{\alpha_L}(\overline{r},\overline{d}_L,\xi_L).$$

Proof. Given $(E_\bullet,\varphi_\bullet) \in \mathrm{VHS}_L(\overline{r},\overline{d})$, the underlying bundles of its corresponding algebraic chain are $\tilde{E}_i = E_i \otimes L^{i-k}$. So if $(E_\bullet,\varphi_\bullet) \in \mathrm{VHS}_L(\overline{r},\overline{d},\xi)$, then

$$\bigotimes_{i=1}^k \det(\tilde{E}_i) = \bigotimes_{i=1}^k \left(\det(E_i)\otimes L^{(i-k)r_i}\right) = \left(\bigotimes_{i=1}^k \det(E_i)\right) \otimes L^{\otimes(\sum_{i=1}^k (i-k)r_i)} \cong \xi_L.$$

The converse is analogous. □

Lemma 8.2. *Fix $\overline{r} = (r_1,\ldots,r_k)$ and $\overline{d} = (d_1,\ldots,d_k)$. Let ξ and ξ' be two line bundles over X of degree $\sum_{i=1}^k d_i$. Then*

$$\mathrm{HC}^\alpha(\overline{r},\overline{d},\xi) \cong \mathrm{HC}^\alpha(\overline{r},\overline{d},\xi').$$

Proof. Let $r = \sum_{i=1}^k r_i$. As ξ an ξ' have the same degree, $\xi' \otimes \xi^{-1}$ has degree zero, so there exists a line bundle ψ such that $\psi^{\otimes r} \cong \xi' \otimes \xi^{-1}$. Given $(E_\bullet,\varphi_\bullet) \in \mathrm{HC}^\alpha(\overline{r},\overline{d},\xi)$, consider the algebraic chain

$$(E_\bullet \otimes \psi, \varphi_\bullet \otimes \mathrm{Id}_\psi).$$

As ψ has degree zero, tensoring by ψ gives an α-slope-preserving correspondence between subchains of $(E_\bullet, \varphi_\bullet)$ and those of $(E_\bullet \otimes \psi, \varphi_\bullet \otimes \mathrm{Id}_\psi)$. Thus, $(E_\bullet \otimes \psi, \varphi_\bullet \otimes \mathrm{Id}_\psi)$ is α-(semi)stable and clearly

$$\bigotimes_{i=1}^{k} \det(E_\bullet \otimes \psi) = \bigotimes_{i=1}^{k} (\det(E_i) \otimes \psi^{r_i}) = \left(\bigotimes_{i=1}^{k} \det(E_i) \right) \otimes \psi^{\sum_{i=1}^{r} r_i} \cong \xi \otimes \psi^r \cong \xi',$$

and then tensorization by ψ yields the desired isomorphism. □

Corollary 8.3. *Let ξ and ξ' be line bundles on X of degree d. Let L and L' be line bundles with $\deg(L) = \deg(L')$. Then*

$$\mathrm{VHS}_L(\overline{r}, \overline{d}, \xi) \cong \mathrm{VHS}_{L'}(\overline{r}, \overline{d}, \xi')$$

Proof. This follows from Lemmas 8.1 and 8.2, using the fact that $\alpha_L = \alpha_{L'}$, $\overline{d}_L = \overline{d}_{L'}$. □

Finally, consider the Bialynicki-Birula docomposition of $\mathcal{M}_L(r, \xi)$

$$\mathcal{M}_L(r, \xi) = \bigcup_{\substack{(\overline{r},\overline{d}) \in \Delta_L \\ |\overline{r}|=r,\, |\overline{d}|=d}} U^+_{\overline{r},\overline{d},\xi}$$

where, clearly, $U^+_{\overline{r},\overline{d},\xi} = U_{\overline{r},\overline{d},\xi} \cap \mathcal{M}_L(r, \xi)$. We know that $U^+_{\overline{r},\overline{d},\xi} \to \mathrm{VHS}_L(\overline{r}, \overline{d}, \xi)$ is an affine bundle of rank $N^+_{L,\overline{r},\overline{d},\xi}$ and, therefore, we have the analogue of equation (6.5),

$$[\mathcal{M}_L(r, \xi)] = \sum_{\substack{(\overline{r},\overline{d}) \in \Delta_L \\ |\overline{r}|=r,\, |\overline{d}|=d}} \mathbb{L}^{N^+_{L,\overline{r},\overline{d},\xi}} [\mathrm{VHS}_L(\overline{r}, \overline{d}, \xi)], \tag{8.1}$$

which can be used in an analogous way to prove the following invariance property of the motivic class of the moduli space of L-twisted Higgs bundles with fixed determinant.

Theorem 8.4. *Let X be a smooth projective curve of genus $g \geq 2$. Let L and L' be line bundles over X such that $\deg(L) = \deg(L') > 2g - 2$. Assume that ξ and ξ' are line bundles of degree d coprime with the rank r. Then the motives of the corresponding moduli spaces $[\mathcal{M}_L(r, \xi)]$ and $[\mathcal{M}_{L'}(r, \xi')]$ are equal in $K(\mathcal{V}ar_{\mathbb{C}})$. Moreover, if d'' is any integer coprime with r, and ξ'' is any line bundle of degree d'', then $E(\mathcal{M}_L(r, \xi)) = E(\mathcal{M}_{L'}(r, \xi''))$.*

Proof. The argument is completely analogous to the one which lead to Theorem 6.7. One just has to use the corresponding fixed determinant versions of the objects involved. Note that the appropriate deformation complex is $C^\bullet_0(E, \varphi) : \mathrm{End}_0(E) \xrightarrow{[-,\varphi]} \mathrm{End}_0(E) \otimes L$, which decomposes as $C^\bullet_0(E, \varphi) = \bigoplus_{l=-k+1}^{k-1} C^\bullet_{0,l}(E, \varphi)$, just like $C^\bullet(E, \varphi)$ in (6.6), but $C^\bullet_{0,l}(E, \varphi) = C^\bullet_l(E, \varphi)$, so the Morse index for the fixed determinant case equals the non-fixed determinant case. Moreover, the equality of the E-polynomials is also precisely the same argument, but here one has to refer to [MS20b, Theorem 0.5] (see also [GWZ20, Corollary 7.17]), instead of the references stated in the proof of Theorem 6.7. The details are left to the reader.

□

8.2. **Moduli spaces of $\mathcal{L}$-connections with fixed determinant.** Let $\mathcal{L}$ be any Lie algebroid. Let E be a rank r vector bundle with determinant $\xi = \det(E) = \Lambda^r E$ and let $\nabla_{\mathcal{L}} : E \to E \otimes \Omega^1_{\mathcal{L}}$ be an integrable $\mathcal{L}$-connection on E. Then $\nabla_{\mathcal{L}}$ induces a map

$$\mathrm{tr}(\nabla_{\mathcal{L}}) : \xi \longrightarrow \xi \otimes \Omega^1_{\mathcal{L}},$$

defined as follows. For local sections $s_1, \ldots, s_r$ of E,

$$\operatorname{tr}(\nabla_{\mathcal{L}})(s_1 \wedge \ldots \wedge s_r) = \sum_{i=1}^{r} s_1 \wedge \ldots \wedge \nabla_{\mathcal{L}}(s_i) \wedge \ldots \wedge s_r.$$

Observe that if $v_1, \ldots, v_r$ is a local trivializing basis of E over some open subset of X, then if we write $\nabla_{\mathcal{L}}$ in that basis as $\nabla_{\mathcal{L}} = d_{\mathcal{L}} + G$ with $G = (g_{ij})$, we get

$$\operatorname{tr}(\nabla_{\mathcal{L}})(v_1 \wedge \ldots \wedge v_r) = \sum_{i=1}^{r} v_1 \wedge \ldots \wedge G v_i \wedge \ldots \wedge v_r = \sum_{i=1}^{r} v_1 \wedge \ldots \wedge g_{ii} v_i \wedge \ldots \wedge v_r = \operatorname{tr}(G) v_1 \wedge \ldots \wedge v_r,$$

justifying the notation "$\operatorname{tr}(\nabla_{\mathcal{L}})$".

Lemma 8.5. *Let X be a smooth projective curve and let $\mathcal{L}$ be a Lie algebroid on X. Let $(E, \nabla_{\mathcal{L}})$ be an integrable $\mathcal{L}$-connection with $\det(E) \cong \xi$. Then $\operatorname{tr}(\nabla_{\mathcal{L}})$ is an integrable $\mathcal{L}$-connection on ξ.*

Proof. It is clear by construction that $\operatorname{tr}(\nabla_{\mathcal{L}})$ is $\mathbb{C}$-linear, so we need to prove that it satisfies the Leibniz rule and that it is integrable. Let $s_1, \ldots s_r$ be local sections E and f a local algebraic function on X. Then, for each $j = 1, \ldots, r$, we have

$$\begin{aligned}
\operatorname{tr}(\nabla_{\mathcal{L}})(s_1 \wedge \ldots \wedge f s_j \wedge \ldots \wedge s_r) &= \sum_{i \neq j} s_1 \wedge \ldots \wedge \nabla_{\mathcal{L}}(s_i) \wedge \ldots \wedge f s_j \wedge \ldots \wedge s_r \\
&\quad + s_1 \wedge \ldots \wedge f \nabla_{\mathcal{L}}(s_j) \wedge \ldots \wedge s_r + s_1 \wedge \ldots \wedge s_j \otimes d_{\mathcal{L}}(f) \wedge \ldots \wedge s_r \\
&= f \sum_{i=1}^{r} s_1 \wedge \ldots \wedge \nabla_{\mathcal{L}}(s_i) \wedge \ldots \wedge s_r + s_1 \wedge \ldots \wedge s_r \otimes d_{\mathcal{L}}(f) \\
&= f \operatorname{tr}(\nabla_{\mathcal{L}})(s_1 \wedge \ldots \wedge s_r) + s_1 \wedge \ldots \wedge s_r \otimes d_{\mathcal{L}}(f),
\end{aligned}$$

so $(\xi, \operatorname{tr}(\nabla_{\mathcal{L}}))$ is an $\mathcal{L}$-connection.

Let us now prove that it is integrable. Suppose $\mathcal{L} = (V, [\cdot\,,\cdot], \delta)$, with $\operatorname{rk}(V) = k$. We will prove it via a local representation of $\operatorname{tr}(\nabla_{\mathcal{L}})$. Let $U \subset X$ be an open subset such that E and V are trivial bundles over U. Write $\nabla_{\mathcal{L}}$ locally over U as $\nabla_{\mathcal{L}} = d_{\mathcal{L}} + G$, where G is an V^*-valued $r \times r$ matrix. Let $w_1, \ldots, w_k$ be a trivializing basis of V^* over U. Then we can write

$$G = \sum_{i=1}^{k} G_i \otimes w_i$$

where G_i is an $\mathcal{O}_X(U)$-valued matrix. Now, we have that, over U, translates into

$$\nabla_{\mathcal{L}}^2 = d_{\mathcal{L}} + G \wedge G = d_{\mathcal{L}}(G) + \sum_{i,j=1}^{k} G_i G_j \otimes w_i \wedge w_j = d_{\mathcal{L}}(G) + \sum_{i<j} [G_i, G_j] \otimes w_i \wedge w_j, \tag{8.2}$$

and, since $\operatorname{tr}(\nabla_{\mathcal{L}}) = d_{\mathcal{L}} + \operatorname{tr}(G) = d_{\mathcal{L}} + \sum_{i=1}^{k} \operatorname{tr}(G_i) \otimes w_i$,

$$\operatorname{tr}(\nabla_{\mathcal{L}})^2 = d_{\mathcal{L}}(\operatorname{tr}(G)) + \sum_{i<j} (\operatorname{tr}(G_i)\operatorname{tr}(G_j) - \operatorname{tr}(G_j)\operatorname{tr}(G_i)) \otimes w_i \wedge w_j = d_{\mathcal{L}}(\operatorname{tr}(G)) = \operatorname{tr}(d_{\mathcal{L}}(G)).$$

From the integrability of $\nabla_{\mathcal{L}}$, it follows from (8.2) that $d_{\mathcal{L}}(G) = -\sum_{i<j} [G_i, G_j] \otimes w_i \wedge w_j$, hence

$$\operatorname{tr}(d_{\mathcal{L}}(G)) = -\sum_{i<j} \operatorname{tr}([G_i, G_j]) \otimes w_i \wedge w_j = 0,$$

proving that $(\xi, \operatorname{tr}(\nabla_{\mathcal{L}}))$ is integrable. ∎

Let $(E,\nabla_{\mathcal{L}})\in\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$. As ξ is a line bundle, $(\xi,\mathrm{tr}(\nabla_{\mathcal{L}}))$ is automatically stable, so $(\xi,\mathrm{tr}(\nabla_{\mathcal{L}}))\in\mathcal{M}_{\Lambda_{\mathcal{L}}}(1,d)$. As the determinant construction can be clearly done in families, then it defines the following map,

$$\det:\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)\longrightarrow\mathcal{M}_{\Lambda_{\mathcal{L}}}(1,d),\quad \det(E,\nabla_{\mathcal{L}})=(\det(E),\mathrm{tr}(\nabla_{\mathcal{L}})).$$

Let $(\xi,\delta)\in\mathcal{M}_{\Lambda,\mathcal{L}}(1,d)$ be an integrable $\mathcal{L}$-connection of rank 1 and degree d. Define

$$\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,\xi,\delta)=\det{}^{-1}(\xi,\delta)\subset\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$$

as the moduli space of $\mathcal{L}$-connections with fixed determinant (ξ,δ).

For example, if $\mathcal{L}=(L,0,0)$is trivial, with L a line bundle, then

$$\mathcal{M}_{\Lambda_{(L,0,0)}}(r,\xi,0)=\mathcal{M}_{L^{-1}}(r,\xi).$$

If $\mathcal{T}_X$ is the canonical Lie algebroid on X, then $\mathcal{M}_{\Lambda_{\mathcal{T}_X}}(r,\mathcal{O}_X,0)$ is the moduli space of $\mathrm{SL}(r,\mathbb{C})$-connections on X.

The determinant map extends to the $\mathcal{L}$-Hodge moduli space, obtaining a map

$$\det:\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)\longrightarrow\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(1,d),$$

over $\mathbb{C}$, by taking $\det(E,\nabla_{\mathcal{L}},\lambda)=(\det(E),\mathrm{tr}(\nabla_{\mathcal{L}}),\lambda)$. Moreover, this map is $\mathbb{C}^*$-equivariant for action (3.19). For each $(\xi,\delta,\lambda)\in\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(1,d)$, define

$$\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,\xi,\delta)=\det{}^{-1}(\mathbb{C}\cdot(\xi,\delta,\lambda))$$

as the *$\mathcal{L}$-Hodge moduli space with fixed determinant* (ξ,δ). Here $\mathbb{C}\cdot(\xi,\delta,\lambda)$ denotes the closure of the $\mathbb{C}^*$-orbit of (ξ,δ,λ) in $\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(1,d)$ which, since ξ is a line bundle, is just the set of elements of the form $(\xi,t\delta,t\lambda)$, with $t\in\mathbb{C}$. Then $\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,\xi,\delta)$ is clearly a $\mathbb{C}^*$-invariant closed subvariety of the $\mathcal{L}$-Hodge moduli space $\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,d)$ and if $\pi:\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,\xi,\delta)\to\mathbb{C}$ is the restriction of the map (3.16), we have

- $\pi^{-1}(0)\cong\mathcal{M}_{L^{-1}}(r,\xi)$;
- $\pi^{-1}(1)\cong\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,\xi,\delta)$;
- $\pi^{-1}(\mathbb{C}^*)\cong\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,\xi,\delta)\times\mathbb{C}^*$.

The deformation theory for this moduli space is very similar to the deformation for the moduli space of $\mathcal{L}$-connections with fixed degree computed in [Tor11, Theorem 47].

Lemma 8.6. *The Zariski tangent space to the moduli space $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,\xi,\delta)$ at a point $(E,\nabla_{\mathcal{L}})$ is isomorphic to $\mathbb{H}^1(C_0^\bullet(E,\nabla_{\mathcal{L}}))$, where $C_0^\bullet(E,\nabla_{\mathcal{L}})$ is the complex*

$$C_0^\bullet(E,\nabla_{\mathcal{L}})\ :\ \mathrm{End}_0(E)\xrightarrow{[-,\nabla_{\mathcal{L}}]}\mathrm{End}_0(E)\otimes\Omega^1_{\mathcal{L}}\xrightarrow{[-,\nabla_{\mathcal{L}}]}\dots\xrightarrow{[-,\nabla_{\mathcal{L}}]}\mathrm{End}_0(E)\otimes\Omega^{\mathrm{rk}(\mathcal{L})}_{\mathcal{L}},$$

and the obstruction for the deformation theory lies in $\mathbb{H}^2(C_0^\bullet(E,\nabla_{\mathcal{L}}))$.

Proof. The deformations of $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,\xi,\delta)$ are precisely the deformations of $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ which preserve the determinant and trace. Following the same notation as the one used in Lemma 5.11, let $\mathcal{U}=\{U_\alpha\}$ be a covering of X such that E is trivial over U_α. Fix a trivialization of E over $\mathcal{U}$ and for each α and β, let $g_{\alpha\beta}:U_{\alpha\beta}\to\mathrm{GL}(r,\mathbb{C})$ be the transition functions for E and let $\nabla_{\mathcal{L},\alpha}=d_{\mathcal{L}}+G_\alpha$ be the local representation of $\nabla_{\mathcal{L}}$ over U_α.

Let $(E',\nabla'_{\mathcal{L}})$ be a deformation of $(E,\nabla_{\mathcal{L}})$ over $X\times\mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$ such that the transition functions of E' are

$$g'_{\alpha\beta}=g_{\alpha\beta}+\varepsilon g^1_{\alpha\beta}\quad\text{ and }\quad\nabla'_{\mathcal{L},\alpha}=\varepsilon d_{\mathcal{L}}+G_\alpha+\varepsilon G^1_\alpha.$$

By [Tor11, Theorem 47], $g'_{\alpha\beta}$ and G^1_α correspond to a deformation of $(E,\nabla_{\mathcal{L}})$ in $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,d)$ if and only if the cocycles $c\in C^1(\mathcal{U},\mathrm{End}(E))$ and $C\in C^0(\mathcal{U},\mathrm{End}(E)\otimes\Omega^1_{\mathcal{L}})$ defined by $c^{(\alpha)}_{\alpha\beta}=g^1_{\alpha\beta}g_{\beta\alpha}$ and $C^{(\alpha)}_\alpha=G^1_\alpha$ satisfy $\partial c=0$, $\partial C=\tilde{\nabla}_{\mathcal{L}}c$ and $\tilde{\nabla}_{\mathcal{L}}C=0$.

We will prove that (c,C) defines a deformation of $(E,\nabla_{\mathcal{L}})$ in $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,\xi,\delta)$ if and only if $(c,C)\in C^1(\mathcal{U},\mathrm{End}_0(E))\times C^0(\mathcal{U},\mathrm{End}_0(E)\otimes\Omega^1_{\mathcal{L}})$. We have $\mathrm{tr}(\nabla'_{\mathcal{L}})=\delta=\mathrm{tr}(\nabla_{\mathcal{L}})$ if and only if

$$\mathrm{tr}(G_\alpha)=\mathrm{tr}(G^1_\alpha)=\mathrm{tr}(G_\alpha)+\varepsilon\,\mathrm{tr}(G^1_\alpha),$$

so $\mathrm{tr}(G^1_\alpha)=0$ and, therefore, $C\in C^1(\mathcal{U},\mathrm{End}_0(E)\otimes\Omega^1_{\mathcal{L}})$. On the other hand, $\det(E')=\xi$ if and only if

$$\det(g_{\alpha\beta})=\det(g^1_{\alpha\beta})=\det(g_{\alpha\beta}+\varepsilon g^1_{\alpha\beta})=\det(g_{\alpha\beta})+\varepsilon\sum_{i=1}^r\det\left((g_{\alpha\beta})_1|\cdots|(g^1_{\alpha\beta})_i|\cdots|(g_{\alpha\beta})_r\right),$$

since $\varepsilon^2=0$. Here, $((g_{\alpha\beta})_1|\cdots|(g^1_{\alpha\beta})_i|\cdots|(g_{\alpha\beta})_r)$ denotes the $r\times r$ matrix whose i-th column equals the i-th column of $g^1_{\alpha\beta}$ and the other columns are the corresponding ones of $g_{\alpha\beta}$. Set

$$D_i=\det\left((g_{\alpha\beta})_1|\cdots|(g^1_{\alpha\beta})_i|\cdots(g_{\alpha\beta})_r\right)$$

so that $\det(E')=\xi$ if and only if $\sum_{i=1}^r D_i=0$. Let A be such that $g_{\alpha\beta}A=g^1_{\alpha\beta}$. By Cramer's rule, $D_i=A_{ii}\det(g_{\alpha\beta})$, thus

$$\sum_{i=1}^r D_i=\det(g_{\alpha\beta})\,\mathrm{tr}(A)=\det(g_{\alpha\beta})\,\mathrm{tr}(g^{-1}_{\alpha\beta}g^1_{\alpha\beta})=\det(g_{\alpha\beta})\,\mathrm{tr}(c^{(\alpha)}_{\alpha\beta}),$$

and so $\det(E')=\xi$ if and only if $\mathrm{tr}(c)=0$, i.e., if $c\in C^1(\mathcal{U},\mathrm{End}_0(E))$.

The rest of the proof is exactly the same as the one of [Tor11, Theorem 47]. □

Proposition 8.7. *Let X be a smooth projective curve of genus $g\geq 2$. Let $\mathcal{L}$ be a Lie algebroid with $\mathrm{rk}(\mathcal{L})=1$ and $\deg(\mathcal{L})<2-2g$. Take r and d is coprime. Then, for each $(\xi,\delta)\in\mathcal{M}_{\Lambda_{\mathcal{L}}}(1,d)$, the moduli space $\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,\xi,\delta)$ is a smooth semiprojective variety for the $\mathbb{C}^*$-action $t\cdot(E,\nabla_{\mathcal{L}},\lambda)=(E,t\nabla_{\mathcal{L}},t\lambda)$. Furthermore, the map $\pi:\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,\xi,\delta)\to\mathbb{C}$, $\pi(E,\nabla_{\mathcal{L}},\lambda)=\lambda$ is a surjective submersion and $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,\xi,\delta)$ is a smooth variety of dimension $\deg(\mathcal{L})(1-r^2)$.*

Proof. The argument is exactly the same as the one carried on in section 5.3. The only difference is that the computation of the dimension of the tangent bundle done in Lemma 5.10 now becomes

$$\dim T_{(E,\nabla_{\mathcal{L}})}\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,\xi,\delta)=\deg(\mathcal{L})(1-r^2)+\dim\left(\mathbb{H}^2(C^\bullet_0(E,\nabla_{\mathcal{L}}))\right).$$

Notice that here we have to take trace-free endomorphisms, hence by point (3) of Lemma 5.9, $\mathbb{H}^0(C^\bullet_0(E,\nabla_{\mathcal{L}}))=0$. Taking into account Lemma 8.6, the deformation theory computed in Lemma 5.11 becomes now

$$T_{(E,\nabla_{\mathcal{L}},0)}\mathcal{M}_{\Lambda^{\mathrm{red}}_{\mathcal{L}}}(r,\xi,\delta)\cong\frac{\left\{(c,C,\lambda_\varepsilon)\in\begin{pmatrix}C^1(\mathcal{U},\mathrm{End}(E))\times\\ C^0(\mathcal{U},\mathrm{End}(E)\otimes\Omega_{\mathcal{L}})\times\mathbb{C}\end{pmatrix}\,\middle|\,\begin{array}{l}\partial c=0\\ \partial C=\tilde{\nabla}_{\mathcal{L}}c+\lambda_\varepsilon\omega\\ \tilde{\nabla}_{\mathcal{L}}C=-\lambda_\varepsilon d_{\mathcal{L}}(\nabla_{\mathcal{L}})\\ \mathrm{tr}(c)=0\\ \mathrm{tr}(C)=\lambda_\varepsilon\delta\end{array}\right\}}{\left\{(\partial\eta,\tilde{\nabla}_{\mathcal{L}}\eta,0)\,\middle|\,\eta\in C^0(\mathcal{U},\mathrm{End}_0(E))\right\}}$$

with $d\pi([(c,C,\lambda_\varepsilon)])=\lambda_\varepsilon$. Then, clearly

$$\ker d\pi\cong\frac{\left\{(c,C,0)\in\begin{pmatrix}C^1(\mathcal{U},\mathrm{End}_0(E))\times\\ C^0(\mathcal{U},\mathrm{End}_0(E)\otimes\Omega_{\mathcal{L}})\times\mathbb{C}\end{pmatrix}\,\middle|\,\begin{array}{l}\partial c=0\\ \partial C=\tilde{\nabla}_{\mathcal{L}}c+\lambda_\varepsilon\omega\\ \tilde{\nabla}_{\mathcal{L}}C=-\lambda_\varepsilon d_{\mathcal{L}}(\nabla_{\mathcal{L}})\end{array}\right\}}{\left\{(\partial\eta,\tilde{\nabla}_{\mathcal{L}}\eta,0)\,\middle|\,\eta\in C^0(\mathcal{U},\mathrm{End}_0(E))\right\}}\cong T_{(E,\nabla_{\mathcal{L}})}\mathcal{M}_{L^{-1}}(r,\xi)$$

and the proof proceeds exactly as in Lemma 5.13 and Theorem 5.15. □

Theorem 8.8. *Let $\mathcal{L} = (L, [\cdot\,,\cdot], \delta)$ be Lie algebroid on X such that L is a line bundle with $\deg(L) < 2-2g$. If r and d are coprime, then, for each $(\xi,\delta) \in \mathcal{M}_{\Lambda_{\mathcal{L}}}(1,d)$, we have*

$$[\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,\xi,\delta)] = [\mathcal{M}_{L^{-1}}(r,\xi,\delta)], \quad [\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,\xi,\delta)] = \mathbb{L}[\mathcal{M}_{L^{-1}}(r,\xi,\delta)]$$

and we have an isomorphism of Hodge structures

$$H^\bullet(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,\xi,\delta)) \cong H^\bullet(\mathcal{M}_{L^{-1}}(r,\xi,\delta))$$

In particular,

$$E(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,\xi,\delta)) = E(\mathcal{M}_{L^{-1}}(r,\xi,\delta)), \quad E(\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,\xi,\delta)) = uvE(\mathcal{M}_{L^{-1}}(r,\xi,\delta)).$$

Moreover, both $\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,\xi,\delta)$ and $\mathcal{M}_{\Lambda_{\mathcal{L}}^{\mathrm{red}}}(r,\xi,\delta)$ have pure mixed Hodge structures.

Proof. The proof is completely analogous to that of Theorem 5.17. The details are left to the reader. □

Finally, combining this result with Theorem 8.4 and working analogously to Theorem 7.4, yields the fixed-determinant version of Theorems 6.7 and 7.4.

Theorem 8.9. *Let X be a smooth projective curve of genus $g \geq 2$ and let $\mathcal{L}$ and $\mathcal{L}'$ be any Lie algebroids on X such that $\mathrm{rk}(\mathcal{L}) = \mathrm{rk}(\mathcal{L}') = 1$ and $\deg(\mathcal{L}) = \deg(\mathcal{L}') < 2-2g$. Suppose that r and d are coprime. Let $(\xi,\delta) \in \mathcal{M}_{\Lambda_{\mathcal{L}}}(1,d)$ and $(\xi',\delta') \in \mathcal{M}_{\Lambda_{\mathcal{L}'}}(1,d)$. Then*

$$\mathcal{I}(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,\xi,\delta)) = \mathcal{I}(\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,\xi',\delta'))$$

where $\mathcal{I}(X)$ denotes one of the following

(1) *The virtual motive $[X] \in \hat{K}(\mathcal{V}ar_{\mathbb{C}})$;*
(2) *The Voevodsky motive $M(X) \in \mathrm{DM}^{\mathrm{eff}}(\mathbb{C}, R)$ for any ring R. In this case, moreover, the motives are pure;*
(3) *The Chow motive $h(X) \in \mathrm{Chow}^{\mathrm{eff}}(\mathbb{C}, R)$ for any ring R;*
(4) *The Chow ring $\mathrm{CH}^\bullet(X,R)$ for any ring R.*

Moreover, the mixed Hodge structures of the moduli spaces are pure and if d'' is any integer coprime with r and $(\xi'',\delta'') \in \mathcal{M}_{\Lambda_{\mathcal{L}'}}(1,d'')$, then

$$E(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,\xi,\delta)) = E(\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,\xi'',\delta'')).$$

Finally, if $L = L' = K(D)$ for some effective divisor D, then we have an actual isomorphism of pure mixed Hodge structures

$$H^\bullet(\mathcal{M}_{\Lambda_{\mathcal{L}}}(r,\xi,\delta)) \cong H^\bullet(\mathcal{M}_{\Lambda_{\mathcal{L}'}}(r,\xi'',\delta'')).$$

References

[ÁCGP01] Luis Álvarez-Cónsul and Oscar García-Prada. Dimensional reduction, SL(2, ℂ)-equivariant bundles and stable holomorphic chains. *International Journal of Mathematics*, 12(02):159–201, 2001.

[Ati57] Michael F. Atiyah. Complex analytic connections in fibre bundles. *Trans. Amer. Math. Soc.*, 85:181–207, 1957.

[BB73] A. Bialynicki-Birula. Some theorems on actions of algebraic groups. *Annals of Mathematics*, 98(3):480–497, 1973.

[Ben10] Sandra Bento. *Topologia do Espaço Moduli de Fibrados de Higgs Torcidos*. PhD thesis, Faculdade de Ciências da Universidade do Porto, 2010.

[BGL11] I. Biswas, P. B. Gothen, and M. Logares. On moduli spaces of hitchin pairs. *Mathematical Proceedings of the Cambridge Philosophical Society*, 151:441–457, 2011.

[BGM13] I. Biswas, T. Gómez, and V. Muñoz. Automorphisms of moduli spaces of vector bundles over a curve. 31:73–86, 2013.

[BGPG04] Steven B. Bradlow, Oscar García-Prada, and Peter B. Gothen. Moduli spaces of holomorphic triples over compact Riemann surfaces. *Math. Ann.*, 328(1–2):299–351, 2004.

[BR94] I. Biswas and S. Ramanan. An infinitesimal study of the moduli of Hitchin pairs. *Journal of the London Mathematical Society*, 49(2):219–231, 1994.

[dBn01] Sebastian del Baño. On the Chow motive of some moduli spaces. *J. Reine Angew. Math.*, 532:105–132, 2001.

[DU95] Georgios D. Daskalopoulos and Karen K. Uhlenbeck. An application of transversality to the topology of the moduli space of stable bundles. *Topology*, 34(1):203–215, 1995.

[EK00] Richard Earl and Frances Kirwan. The Hodge numbers of the moduli spaces of vector bundles over a Riemann surface. *The Quarterly Journal of Mathematics*, 51(4):465–483, 12 2000.

[Fal93] Gerd Faltings. Stable g-bundles and projective connections. *J. Algebaric Geometry*, 2:507–568, 1993.

[FM98] Barbara Fantechi and Marco Manetti. Obstruction calculus for functors of Artin rings. I. *J. Algebra*, 202(2):541–576, 1998.

[Fra59] Theodore Frankel. Fixed points and torsion on Kähler manifolds. *Ann. of Math. (2)*, 70:1–8, 1959.

[FSS18] Roman Fedorov, Alexander Soibelman, and Yan Soibelman. Motivic classes of moduli of Higgs bundles and moduli of bundles with connections. *Commun. Number Theory Phys.*, 12(4):687–766, 2018.

[FSS20] Roman Fedorov, Alexander Soibelman, and Yan Soibelman. Motivic Donaldson-Thomas invariants of parabolic Higgs bundles and parabolic connections on a curve. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 16:Paper No. 070, 49, 2020.

[Got94] Peter B. Gothen. The Betti numbers of the moduli space of stable rank 3 Higgs bundles on a Riemann surface. *Internat. J. Math.*, 5(6):861–875, 1994.

[GPHS14] Oscar García-Prada, Jochen Heinloth, and Alexander Schmitt. On the motives of moduli of chains and Higgs bundles. *Journal of the European Mathematical Society*, 16:2617–2668, 2014.

[GWZ20] Michael Groechenig, Dimitri Wyss, and Paul Ziegler. Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration. *Inventiones mathematicae*, 221:505–596, 2020.

[Hau98] Tamas Hausel. *Geometry of the moduli space of Higgs bundles*. PhD thesis, Cambridge University, 1998.

[Hei07] Franziska Heinloth. A note on functional equations for zeta functions with values in Chow motives. *Ann. Inst. Fourier (Grenoble)*, 57(6):1927–1945, 2007.

[Hit87] N. J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc. Third Series*, 55(1):59–126, 1987.

[HL19] Victoria Hoskins and Simon Pepin Lehalleur. On the Voevodsky motive of the moduli space of higgs bundles on a curve. *arXiv:1910.04440*, 2019.

[HRV15] Tamás Hausel and F Rodriguez-Villegas. Cohomology of large semiprojective hyperkähler varieties. *Asterisque*, 2015(370):113–156, 2015.

[HT03] Tamás Hausel and Michael Thaddeus. Mirror symmetry, Langlands duality, and the Hitchin system. *Invent. Math.*, 153(1):197–229, 2003.

[Kap00] M. Kapranov. The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups. *arXiv:math/0001005*, 2000.

[Kir84] F.C. Kirwan. *Cohomology of Quotients in Symplectic and Algebraic Geometry. (MN-31), Volume 31*, volume 104. Princeton University Press, 1984.

[Kri09] Libor Krizka. Moduli spaces of lie algebroid connections. *Archivum Mathematicum*, 44(5):403–418, 2009.

[Kri10] Libor Krizka. Moduli spaces of flat lie algebroid connections. *arXiv:1012.3180*, 2010.

[Man68] Ju. I. Manin. Correspondences, motifs and monoidal transformations. *Mat. Sb. (N.S.)*, 77 (119):475–507, 1968.

[MS20a] Davesh Maulik and Junliang Shen. Cohomological χ-independence for moduli of one-dimensional sheaves and moduli of Higgs bundles. *arXiv:2012.06627*, 2020.

[MS20b] Davesh Maulik and Junliang Shen. Endoscopic decompositions and the Hausel–Thaddeus conjecture. *arXiv:2008.08520*, 2020.

[Muñ08] Vicente Muñoz. Hodge polynomials of the moduli spaces of rank 3 pairs. *Geometriae Dedicata*, 136(1):17–46, 2008.

[Nit91] N. Nitsure. Moduli space of semistable pairs on a curve. *Proc. London Math. Soc.*, 62(3):275–300, 1991.

[Sán14] Jonathan Sánchez. *Motives of moduli spaces of pairs and applications*. PhD thesis, Universidad Complutense, Madrid, 2014.

[Sch94] A. J. Scholl. Classical motives. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 163–187. Amer. Math. Soc., Providence, RI, 1994.

[Sim92] Carlos T. Simpson. Higgs bundles and local systems. *Publications Mathématiques de l'IHÉS*, 75:5–95, 1992.

[Sim94] Carlos T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety I. *Publi. Math. I.H.E.S.*, 79:47–129, 1994.

[Sim95] Carlos T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety II. *Publi. Math. I.H.E.S.*, 80:5–79, 1995.

[Sim97] Carlos Simpson. The Hodge filtration on nonabelian cohomology. *Proceedings of Symposia in Pure Mathematics*, 62.2:217–284, 1997.

[Tor11] Pietro Tortella. Λ-*modules and holomorphic Lie algebroids*. PhD thesis, Scuola Internazionale Superiore di Studi Avanzati, 2011.

[Tor12] Pietro Tortella. Λ-modules and holomorphic lie algebroid connections. *Cent. Eur. J. of Math.*, 10(4):1422 – 1441, 2012.

[Voe00] Vladimir Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 188–238. Princeton Univ. Press, Princeton, NJ, 2000.

[ZnR18] Ronald A. Zúñiga Rojas. Stabilization of the homotopy groups of the moduli spaces of k-Higgs bundles. *Rev. Colombiana Mat.*, 52(1):9–31, 2018.

Department of Applied Mathematics and Institute for Research in Technology, ICAI School of Engineering, Comillas Pontifical University, C/Alberto Aguilera 25, 28015 Madrid, Spain
E-mail address: dalfaya@comillas.edu

Centro de Matemática da Universidade do Porto, CMUP, Faculdade de Cincias, Universidade do Porto, Rua do Campo Alegre 587, 4169-007 Porto, Portugal
E-mail address: andre.oliveira@fc.up.pt