Net flow rates versus non-performing consumer loans forecasting methodologies

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ABSTRACT

Roll rates and net flow rates can be seen as the evolution of ageing of accounts receivable and Markov chains. They are accepted methodologies to model the behavior of non-performing consumer loans by buckets and to predict losses, but we find that quite often they are wrongly used as interchangeable concepts, although roll rates track individual accounts across buckets in consecutive months and net flow rates just compare consecutive buckets in consecutive months. We determine their matrices of transition probabilities and analyze them in both stationary and steady-state conditions. Net flow rates have many advantages over roll rates, but a quite important finding for financial institutions and supervisors is that historical flow rates are not conservative for forecasting: when the level of new delinquencies soars, contemporary flow rates will tend to be lower than they would be in steady-state conditions, creating a feeling of false confidence and leading to the underestimation of future losses.

Net flow rates frente a roll rates como metodologías de predicción en préstamos al consumo morosos

RESUMEN

Las *roll rates* y las *net flow rates* pueden verse como la evolución de las *ageing of accounts receivable* y las cadenas de Markov. Son metodologías aceptadas para modelizar el comportamiento de los préstamos al consumo morosos por buckets y para predecir pérdidas, pero nos encontramos que con bastante frecuencia se usan incorrectamente como conceptos intercambiables, aunque las *roll rates* siguen cuentas individuales a través de los buckets en meses consecutivos y las *net flow rates* solo comparan buckets consecutivos en meses consecutivos. Determinamos sus matrices de probabilidades de transición y las analizamos en condiciones estacionarias y de estado estable. Las *net flow rates* tienen muchas ventajas sobre las *roll rates*, pero un hallazgo bastante importante para las instituciones financieras y los supervisores es que las *net flow rates* históricas no son conservadoras para hacer predicciones: cuando el nivel de nuevos impagos se dispara, las *net flow rates* contemporáneas tenderán a ser más bajas de lo que serían en condiciones de estado estable, creando un sentimiento de falsa confianza y llevando a la subestimación de las pérdidas futuras.

Palabras clave: roll rates; net flow rates; riesgo de crédito al consumo; morosidad; impagos; pérdida esperada.
Clasificación JEL: C53, E58, G21, G28.
MSC2010: 15B99, 60J20, 91G40.

1. Introduction

Consumer credit risk management requires predicting delinquency ageing and collections when delinquency occurs. Dynamic modeling using flows is a normal approach to predict how loans will behave, although there are several methods (Rosenberg & Gleit, 1994, pp. 603-604).

Ageing of Accounts Receivable could be considered the precursor of all the other methods. It worked in a very simple way and it was being used to manage accounts receivable in retail outlets in the 50s. It used to take all high-balance accounts and a sample of low-balance accounts and then divided their dollar value into five categories: 0 to 2 months, 2 to 4 months, 4 to 6 months, 6 to 10 months, and over 10 months old. This let them estimate the real figures in each category, and in turn, these figures allowed them to determine the amount of the allowance for uncollectibles using loss expectancy rates (similar to LGD) to each of the age groups (Cyert & Trueblood, 1957, pp. 185-190). These seemed sufficient in times where no computers and no electronic databases were available.

Later on, the ageing of accounts receivable method evolved towards Markov Chains thanks to the seminal work of Cyert, Davidson and Thompson (1962) and Cyert and Thompson (1968). As with the ageing of accounts receivable, this method was initially designed to estimate allowances for doubtful accounts. But it would soon be employed by the financial institutions to estimate losses for loans. There is a clear parallel between the customer's due balance and the outstanding of a loan, between a new purchase charge by a customer and a new installment of a loan. With this in mind, there are two possible approaches: the total balance method or the partial balance method (Cyert, Davidson, & Thompson, 1962, p. 290). The total balance method is the appropriate method to use for loans. We must always take the total outstanding of the loan (or loans of the borrower with that lender) with the oldest due date instead of taking the overdue amounts with their correspondent due dates separately.

A Multi-State Markov (MSM) model uses a discrete-time Markov chain which is a sequence of random transitions to a different state in which the probability of each transition depends only on the previous state. So, this process is assumed to be non path-dependent (i.e. has no memory of older states), but it can be dependent on covariates from individuals or management strategies. The states are either temporary or absorbent, depending on the possibility to further migrate to other states or not.

There is a lot of discretion to choose the set of possible states. The set of states proposed by Cyert, Davidson and Thompson (1962, pp. 288-290) included $n+2$ states for a given time i: n temporary states for n age categories (0 to express being "current", and j from 1 to n-1 to express j periods past due), and two absorbent states for "paid" and for "bad debt" (n or more periods past due). A simpler set may only include "current", "delinquent", "bad debt" and "paid". In any case, it would be advisable to add a new state called "prepaid", separate from the "paid" state, if we are dealing with loans (Stretton & Burra, 2011).

The matrix of transition probabilities (also known as "[Markovian probability] transition matrix", "migration matrix", "matrix of transition rates between statuses", "delinquency movement matrix" (DMM), "roll rate matrix" or "net flow matrix") compiles all transition probabilities among the states from time i to time $i+1$, and it is usually built with historical data, being assumed to be constant over time. Arranging firstly the two absorbing states and then the remaining transient states for the age categories (0, 1, …, n-1 periods past due), the transition matrix, P, can be partitioned (1962, p. 293) as:

$$P = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix} \qquad [1]$$

where I is 2×2 and Q is $n \times n$.

Additionally, N is the fundamental matrix of the absorbing Markov chain.

$$N = \sum_{k=0}^{\infty} Q^k = (I - Q)^{-1} \qquad [2]$$

given that:

$$\lim_{k \to \infty} Q^k = 0 \qquad [3]$$

Besides, B_i is the n-component vector at time i with the dollar amount in each age category.

And $N·R$ (which is $n \times 2$) gives "the probabilities of dollars in each of the age categories being paid" (first column) and "the probabilities of becoming bad debts" (second column), which are the loss expectancy rates. And $B·N·R$ is a 2-component vector with the expected payments and bad debts.

To all that has been said, we should add that in the case of a discrete-time period equal to the age categories period (e.g. the state is considered to change monthly and the age categories are measured in terms of months too), Q must be a triangular matrix plus one additional supradiagonal (and so does P), since it is obvious that accounts receivable (or loans) cannot be delayed faster than the passage of time; this is in theory, but in practice it could happen that a payment has been missed and the lender was not aware of it, or even a payment has been retroactively missed (e.g. direct debit late rejection), so some amounts may suddenly pop up in advanced states. On the other hand, using an identity matrix for the absorbing states means that any recovery since reaching an absorbing state will be obviated, which is not very realistic. Finally, all elements of the diagonal of Q (or P, in general) correspond to the stayers and all the other elements of the matrix to the movers (in the case of the elements of the additional supradiagonal they represent the forward-movers).

This method has some limitations. As a method based on historical data it ignores changing economic conditions. But it also has some flexibility. Cyert, Davidson and Thompson also pointed out that it is also possible to follow the transitions of accounts rather than account balances (1962, p. 290). And So and Thomas also suggested a different usage of Markov chain models taking the behavioral scores as the state space (2010, p. 96).

2. Theoretical formulation

Roll Rates and [Net] Flow Rates arise from the particular case of loans, taking the DPD as the age (if no payments are made by the borrower, DPD advance as time passes and at the same pace; if there are partial payments DPD can remain the same or decrease), and using the month (or a 30-day period) both as the discrete-time period and as the age category span for the MSM model. Remember that once a loan is delinquent, it can be classified in categories depending on the days past due grouped by the number of months, which also correspond to the number of missed payments in the case of installment loans. In this situation, the age categories are called buckets. All delinquent loans between 1 and 30 DPD are classified as bucket 1, between 31 and 60 DPD are bucket 2, and so on. And bucket 0 are all loans which are current (or open in good standing). So, each bucket comprises a set of loans every month, and these sets of loans suffer many changes from month to month, according to the different evolution of the DPD of each loan. Buckets are used in monetary units, if we measure balances, or they may be used in terms of number of borrowers or accounts. In any bucket, a loan can move forward (only 1 bucket) or backward, but once it is charged-off or written-off (an absorbing state) it will remain in that state. There are accounting differences between charge-off and write-off, but, here, we will use them as

synonyms. For what we want to explain in this section, the only important thing is that a certain bucket *n* is taken as an absorbing state for what it is considered as bad debt. For simplicity, it can be assumed that new balances enter through bucket 0 (new balances normally start as being current, although there could be other rare cases), that repayment occurs only from bucket 0 (this is not true, because full or partial payments from delinquent loans implies payments from a bucket other than bucket 0) and that there are no early charge-offs or write-offs (i.e. coming from a bucket other than the last transient bucket, bucket *n*-1) (again, this is not true, because an event could occur, such as bankruptcy, that implies collection of the debt being unlikely before arriving to the designated bucket *n* for charge-off). Some other assumptions made are that "interest, fees and others" are not considered as part of the balance, and that the "recoveries" (collections after charged-off or written-off) are not taken into account.

We define the new balances rate (*b*) as the new balances in a period in terms of bucket 0, and the repayment rate (*r*) as the fraction of bucket 0 that repays in a period. In the event we are working in currency terms, repayments must include not only attrition, but regular repayments of installments and full or partial payments of amounts due from delinquent loans. There are some limitations with these two rates, since the denominator is bucket 0. New balances or borrowers does not seem to be closely related to bucket 0, and the same could happen with payments of amounts due from delinquent loans, which would be more related to other buckets. Anyway, for simplicity, we will consider these rates as constants. Furthermore, given that both rates represent inflows and outflows of bucket 0, they might both be taken as a single variable netting.

$$b = \frac{monthly\ new\ balance\ (or\ borrowers)}{B_i(0)} \qquad [4]$$

$$r = \frac{monthly\ repaid\ amount\ (or\ borrowers)}{B_i(0)} \qquad [5]$$

Recall that B_i is the *n*-component vector at time *i* with the amounts in each bucket (from 0 to *n*-1). We also define the augmented vector *B'* as the *n*+1 component vector, including bucket *n*.

From now on, we will follow the same notation as Cyert, Davidson and Thompson (1962) and also use row-vectors and pre-multiplication of matrices by vectors (instead of column-vectors and post-multiplication).

Roll Rates are not always defined in the same way. For example, a simplified view states that the roll rates "analyze the migration of accounts from one billing cycle to the next", and they are expressed as "the percentage of balances, or accounts, that move from one delinquency stage to the next" or "become increasingly delinquent" (Hong Kong Monetary Authority, 2006, p. 38; PwC, 2015, p. 28). This is a simplified view because it discards the possibility of some other types of transitions. But the OCC has a more general view and states that the roll rates "measure the movement of accounts and balances from one payment status to another" so they may go from current to past due, "cure (return to current), remain in the same delinquency bucket, or improve to a less severe delinquency status", the first roll rate being the "percentage of accounts or dollars that were current last month rolling to 30 days past due this month" (Office of the Comptroller of the Currency, 2015; 2016). For our purposes, we will take this general view and we will consider that roll rates measure the movement of delinquent loans (accounts or balance) from one bucket to other buckets in the next month.

An Individual Roll Rate is the fraction of loans that roll-over from one specific bucket into a different specific bucket in two consecutive months. These individual roll rates can be classified according to the type of transition:

- Forward-movers (denoting the "Forward Roll Rates") are the fraction of loans in a bucket that pass from that bucket to the following one.

- Sitters (also called "stayers" or "pay-and-stay loans") are the fraction of loans in a bucket that pay only one loan installment each month, which is like paying the oldest missed payment and missing the most recent one, thus remaining in the same bucket. In other words, in one month, DPD increase by one month but also decrease by one month due to the effect of paying the oldest missed payment.

- [Other] Partial payers (denoting the "Reverse Roll Rates") are the fraction of loans in a bucket that pay more than one missed payment but not all of them, so they go back to a lower bucket. In other words, DPD decrease by more than one month.

- Curers are the fraction of loans in a bucket that pay all missed payments, so they come back to bucket 0 (current).

This classification was probably used or devised by Capital One, but no evidence was found in the literature.

Of course, loans can also roll over into an absorbing state: repaid or charge-off. If we assumed that there are no early charge-offs, being charged-off would be just a "forward roll" from the last transient bucket, which is bucket n-1, into bucket n.

Table 1. Types of individual roll rates. Bucket in month i (rows) and bucket in month i+1 (columns).

	0	1	2	3	4	5	6
0	S	FRR					
1	C	S	FRR				
2	C	PP	S	FRR			
3	C	PP	PP	S	FRR		
4	C	PP	PP	PP	S	FRR	
5	C	PP	PP	PP	PP	S	FRR

Source: Own elaboration.

Apart from these individual roll rates, it is common to use the 1-to-7 Sequential Roll Rate (1-to-7 sRR) and the 1-to-7 Coincidental Roll Rate (1-to-7 cRR) to measure the fraction of loans in bucket 1 that would end in bucket 7 (typically the bucket of charge-off) after 6 months (in the event the bad debt absorbing state, e.g. charge-off or write-off, is placed in a different bucket, say n, the appropriate roll rates to use would be the 1-to-n cRR and the 1-to-n sRR). The sequential roll rate always follows the same cohort of delinquent loans and takes the individual roll rates from consecutive months, while the coincidental roll rate takes all the individual roll rates from a single month, meaning that the roll rates pertain to different cohorts. Each one has its own advantages and disadvantages: sRR are highly dependent on the cohort and cRR are highly dependent on the performance in a specific month. Another compounded roll rate with special interest is the 0-to-d Roll Rate, where d is the designated bucket for default recognition (typically 4, i.e. more than 90 days), because it represents the PD in d months.

Table 2. 1-to-7 Coincidental Roll Rate vs 1-to-7 Sequential Roll Rate. Individual roll rates (rows) and month (columns).

	1	2	3	4	5	6	7
0-1							
1-2		RR					RR
2-3			RR				RR
3-4				RR			RR
4-5					RR		RR
5-6						RR	RR
6-7							RR

Source: Own elaboration.

Roll rates are a particularization of Markov chains because the Markov assumption is kept, although not all transitions are possible in this case, because it is not possible to move forward by more than one bucket per month. We also know, by definition, that any roll rate from a bucket j to a bucket k from time i to time $i+1$ must be between 0 and 1:

$$0 \leq RR_{i,i+1}(j,k) \leq 1 \tag{6}$$

$$\forall j, k \in \{0,1,\ldots,n-1\}: \ k > j+1 \Rightarrow RR_{i,i+1}(j,k) = 0 \tag{7}$$

$$0 \leq 1 - RR_{i,i+1}(j,k) \leq 1 \tag{8}$$

Thus, the matrix Q will be a lower triangular matrix with a single supradiagonal (representing the forward-movers). From now on, to avoid overloaded notation we may take $RR_{i,i+1}(j,k)$ just as $RR(j,k)$.

$$Q_{i,i+1}^{RR} = \begin{bmatrix} RR(0,0)+b & RR(0,1) & 0 & \cdots & \cdots & \cdots & 0 \\ RR(1,0) & RR(1,1) & RR(1,2) & \ddots & & & \vdots \\ RR(2,0) & RR(2,1) & RR(2,2) & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & & \ddots & RR(n-2,n-1) \\ RR(n-1,0) & RR(n-1,1) & RR(n-1,2) & \cdots & \cdots & \cdots & RR(n-1,n-1) \end{bmatrix} \tag{9}$$

satisfying that:

$$B_{i+1} = B_i \cdot Q_{i,i+1}^{RR} \tag{10}$$

Therefore, we can express each component of B_{i+1} as:

$$B_{i+1}(k=0) = B_i(0) \cdot b + \sum_{j=0}^{n-1} \Big(B_i(j) \cdot RR_{i,i+1}(j,0) \Big) \tag{11}$$

$$B_{i+1}(k \neq 0) = \sum_{j=k-1}^{n-1} \Big(B_i(j) \cdot RR_{i,i+1}(j,k) \Big) \tag{12}$$

And we can also say that:

$$\sum_{k=0}^{n} B'_{i+1}(k) = \sum_{j=0}^{n} B'_i(j) + (b - r) \cdot B'_i(0) \qquad [13]$$

But, in order to analyze roll rates as Markov chains, for now, we must assume there are no "sources", i.e. no new balances ($b = 0$). Remember from equation [1] that P is the matrix of transition probabilities and Q was the lower right submatrix. We said that the matrix of transition probabilities is sometimes referred as the roll-rate matrix. This is because its elements are really the roll rates, plus the transitions to absorbing states. Not surprisingly, there is a clear parallel between matrix Q and Table 1.

Most of the weight in Q is usually on the supradiagonal (forward-movers) and on the first column (curers). Elements on the diagonal correspond to sitters and the remaining non-zero elements correspond to the [other] partial payers.

Assuming stationary conditions, where the roll rates are not time-dependent, the fundamental matrix of the absorbing Markov chain can be derived from matrix Q according to equation [2]:

$$N = \sum_{k=0}^{\infty} (Q_{i,i+1}^{RR})^k = (I - Q_{i,i+1}^{RR})^{-1} \qquad [14]$$

given that:

$$\lim_{k \to \infty} (Q_{i,i+1}^{RR})^k = 0 \qquad [15]$$

Another submatrix of P was matrix R (which is $n \times 2$). This matrix would have only two elements other than zero, if we maintained the assumptions that all repayments are made only from bucket 0 and that there are no early charge-offs.

$$R = \begin{bmatrix} r & 0 \\ 0 & \vdots \\ \vdots & \vdots \\ \vdots & 0 \\ 0 & FR(n-1, n) \end{bmatrix} \qquad [16]$$

For the record, making those assumptions, we can move the corresponding row and column of the absorbing state of charge-off (bucket n) to the end, in order to create a matrix P containing all the possible roll rates ordered in a more meaningful way. Remember from equation [1] that matrix R has two columns, for the absorbing states "repaid" and "bad debt". We include b, but remember that Markov chains would require no new balances ($b = 0$).

$$P' = \begin{bmatrix} 1 & 0 & 0 \\ col_1 R & Q & col_2 R \\ 0 & 0 & 1 \end{bmatrix} \qquad [17]$$

$$P' = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ r & RR(0,0)+b & RR(0,1) & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & RR(1,0) & RR(1,1) & RR(1,2) & \ddots & & & \vdots & \vdots \\ 0 & RR(2,0) & RR(2,1) & RR(2,2) & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & RR(n-2,n-1) & 0 \\ 0 & RR(n-1,0) & RR(n-1,1) & RR(n-1,2) & \cdots & \cdots & \cdots & RR(n-1,n-1) & RR(n-1,n) \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix} \quad [18]$$

By definition, for any row of *P'*, all RR must sum 1 (or 1+*b*, if the source is in the row):

$$j = 0 : r + RR_{i,i+1}(0,0) + b + RR_{i,i+1}(0,1) = 1 + b \qquad [19]$$

$$\forall j \in \{1, \dots, n-1\} : \sum_{k=0}^{j+1} RR_{i,i+1}(j,k) = 1 \qquad [20]$$

If we use equation [11] and take apart the term corresponding to the sitters (*j* = *k* = 0), and then use equation [19] for this bucket *j* = 0, we can conclude what should be obvious: bucket 0 in time *i*+1 is equal to bucket 0 in time *i* plus curers and new balances less new delinquent loans and repayments.

$$B_{i+1}(0) = B_i(0) + \sum_{j=1}^{n-1}\left(B_i(j) \cdot RR_{i,i+1}(j,0)\right) + B_i(0) \cdot b - B_i(0) \cdot RR_{i,i+1}(0,1) - B_i(0) \cdot r \qquad [21]$$

On the other hand, Flow Rates compare stocks between consecutive buckets in consecutive months, instead of tracking how the individual loans roll across buckets. Since flow rates just compare stocks and not individual loans, it is agreed that charge-offs flow only from the last transient bucket and that repayments only flow from bucket 0 (with a repayment rate *r*), without a loss of generality. For roll rates this may be an assumption, but for flow rates it is just how it is defined. The same applies whenever there are new balances, we consider entering through bucket 0. With all this, we can say that, for any bucket, what is not going forward, is being cured (see Table 3). Thus, flow rates are simpler, faster to calculate and have no need to follow individual accounts. In addition, they have other advantages and some disadvantages over roll rates that we will address later. Higher flow rates, as higher forward roll rates, means fewer collections and higher risk costs.

Table 3. Individual flow rates. Bucket in month *i* (rows) and bucket in month *i*+1 (columns).

	0	1	2	3	4	5	6
0	1-FR-*r*+*b*	FR					
1	1-FR	-	FR				
2	1-FR	-	-	FR			
3	1-FR	-	-	-	FR		
4	1-FR	-	-	-	-	FR	
5	1-FR	-	-	-	-	-	FR

Source: Own elaboration.

Making an analogy, flow rates can be seen as a simplification of roll rates and a particularization of Markov chains, in which there is a set of imaginary loans with the same bucket

distribution, such that they are restricted to only two or three possible transitions for each given state: a transition to the state of the next bucket (or to bad debt, if it is the last bucket), to the state of current, or to the state of repaid (only if they are at bucket 0). In addition, this set of imaginary loans must meet these two conditions: first, matrix Q of the set of imaginary loans must yield the same bucket distribution for the next month, and second, matrix R of the set of imaginary loans must yield the same flows to absorbing states during the month.

$$B_{i+1} = B_i \cdot Q_{i,i+1}^{RR} = B_i \cdot Q_{i,i+1}^{FR} \qquad [22]$$

$$B_i \cdot R_{i,i+1}^{RR} = B_i \cdot R_{i,i+1}^{FR} \qquad [23]$$

Since most of the weight of collections is placed on the curers, it makes sense to simplify the view and obviate sitters and partial payers by netting the effects of all collections (Kellett, 2011, pp. 5-6), considerably reducing the number of parameters and making the matrix sparser and with orthogonal columns except for the first column. Note that in the event there were no partial payments (including sitters), forward roll rates and flow rates would obviously be the same.

This means that, here, the matrix Q would be almost empty except for the supradiagonal and for the first column. Matrix Q of flow rates compared to that of roll rates has this distinctiveness: it only considers forward mover or full payer transitions compared to the wider set of possible transitions the roll rates allow.

Considering again B_i as the n-component vector at time i with the amounts in each bucket, the flow rate between bucket j and bucket $j+1$ from month i to month $i+1$ can be expressed as follows:

$$FR_{i,i+1}(j,j+1) = \frac{B_{i+1}(j+1)}{B_i(j)} \qquad [24]$$

Although, we need to have a particular expression for the last flow rate (from the last transient bucket into the absorbing charge-off state):

$$FR_{i,i+1}(n-1,n) = \frac{monthly\ charged{-}off\ amount}{B_i(n-1)} = \frac{B'_{i+1}(n) - B'_i(n)}{B_i(n-1)} \qquad [25]$$

Flow rates, like roll rates, should be between 0 and 1, but they may occasionally be slightly above 1, although never on average and long term (if we want to use the analogy of Markov chains and if we want to keep the special properties of the matrices, we must ensure we are not using flow rates above 1):

$$0 \leq FR_{i,i+1}(j,j+1) \lesssim 1 \qquad [26]$$

$$0 \lesssim 1 - FR_{i,i+1}(j,j+1) \leq 1 \qquad [27]$$

With them, a sequential flow rate from bucket j to bucket k can be defined as:

$$\prod_{m=j}^{k-1} FR_{i+m-j,i+m-j+1}(m,m+1) = \frac{B_{i+k-j}(k)}{B_i(j)} \lesssim 1 \qquad [28]$$

Specifically, we define f as the **sequential current-to-charge-off flow rate** or just the 0-to-n sFR. In no way can f be seen as the probability of being charged-off in n months. That would be the 0-to-n RR. It can be seen as the probability of being charged-off in n months for one of

those imaginary loans we described earlier, but not for an average real loan. It can also be defined in a coincidental form. In a steady state, both forms, sequential and coincidental, would obviously be the same.

$$f = \prod_{j=0}^{n-1} FR(j, j+1) = \frac{B'_{i+n}(n) - B'_{i+n-1}(n)}{B_i(0)} \tag{29}$$

Note that the sum of r and f will normally be below 1. On one hand, all the factors of f should be below 1. On the other hand, the first factor of f, $FR(0,1)$, plus r, plus the sitters flow rate at bucket 0 $(1-FR(0,1)-r)$ should be 1. Even more, this sitters flow rate is expected to be positive and higher than the others. Anyway, the inequality might be temporarily broken in extreme cases.

$$r + f \lesssim 1 \tag{30}$$

Knowing this, the matrix Q for the flow rates will have the first column and a single supradiagonal with the flow rates as follow. Again, to avoid overloaded notation we will take $FR_{i,i+1}(j,j+1)$ just as $FR(j,j+1)$. We can even use just one parameter "j", but for clarity we still prefer to use "$j, j+1$".

$$Q_{i,i+1}^{FR} = \begin{bmatrix} 1 - FR(0,1) + (b-r) & FR(0,1) & 0 & \cdots & \cdots & \cdots & 0 \\ 1 - FR(1,2) & 0 & FR(1,2) & \ddots & & & \vdots \\ 1 - FR(2,3) & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & & \ddots & \ddots & 0 \\ \vdots & \vdots & & & & & \ddots & FR(n-2, n-1) \\ 1 - FR(n-1, n) & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix} \tag{31}$$

$$\det\left(Q_{i,i+1}^{FR}\right) = (-1)^{n-1} \cdot \left(\prod_{m=0}^{n-2} FR(m, m+1) - \prod_{m=0}^{n-1} FR(m, m+1) \right)$$
$$= (-1)^n \cdot (coincidental\ f) \cdot \left(1 - \frac{1}{FR(n-1, n)} \right) \tag{32}$$
$$= (-1)^n \cdot \frac{B_i(n-1) - (B'_{i+1}(n) - B'_i(n))}{B_i(0)}$$

Using equation [11] and [12] we can now express each component B_{i+1} as:

$$B_{i+1}(k = 0) = B_i(0) \cdot b + \sum_{j=0}^{n-1} \left(B_i(j) \cdot RR_{i,i+1}(j, 0) \right)$$
$$= B_i(0) \cdot (b - r) + \sum_{j=0}^{n-1} \left(B_i(j) \cdot \left(1 - FR_{i,i+1}(j, j+1) \right) \right) \tag{33}$$

$$B_{i+1}(k \neq 0) = \sum_{j=k-1}^{n-1} \left(B_i(j) \cdot RR_{i,i+1}(j, k) \right) = B_i(k-1) \cdot FR_{i,i+1}(k-1, k) \tag{34}$$

If we now solve for the flow rate variable, we obtain one of the most important equations that we will further discuss later:

$$\forall j \in \{0, \ldots, n-2\}: FR_{i,i+1}(j, j+1) = RR_{i,i+1}(j, j+1) + \sum_{m=j+1}^{n-1}\left(\frac{B_i(m)}{B_i(j)} \cdot RR_{i,i+1}(m, j+1)\right) \quad [35]$$

$$\forall j \in \{0, 1, \ldots, n-2\}: RR_{i,i+1}(j, j+1) \leq FR_{i,i+1}(j, j+1) \quad [36]$$

And we also know that:

$$j = n-1: FR_{i,i+1}(n-1, n) = RR_{i,i+1}(n-1, n) \quad [37]$$

Equation [35] shows the netting effect of partial payments on the flow rates. We can say that flow rates are like increased forward roll rates, due to new loans coming from higher (or the same) buckets with partial payments.

In order to analyze roll rates as Markov chains, for now, we must assume there are no "sources", i.e. no new balances ($b = 0$). Again, in stationary conditions, the fundamental matrix of the absorbing Markov chain can be calculated using matrix Q in the same way as with roll rates:

$$N = \sum_{k=0}^{\infty} (Q_{i,i+1}^{FR})^k = (I - Q_{i,i+1}^{FR})^{-1} \quad [38]$$

given that:

$$\lim_{k \to \infty} (Q_{i,i+1}^{FR})^k = 0 \quad [39]$$

We know that the inverse of a matrix can be calculated using Gauß-Jordan elimination or row reduction method (although we are working with row vectors and multiplying from the left, we can still use row reduction instead of column reduction because the inverse of a matrix is the same with both methods). So, we take the matrix and we augment it to the right with the identity matrix (i.e. $[I\text{-}Q \,|\, I]$ matrix), then we apply the method until we obtain the identity matrix on the left side, and what remains on the right side is the inverse matrix (i.e. $[I \,|\, (I\text{-}Q)^{-1}] = [I \,|\, N]$ matrix). Results are shown in Appendix A.

Since we agreed that repayment occurs from bucket 0 and that charge-offs occur from the last transient bucket, matrix R is the same one that we assumed for roll rates in equation [16].

Therefore, the matrix multiplication $N \cdot R$ results to be:

$$N \cdot R = \begin{bmatrix} \dfrac{r}{r+f} & \dfrac{f}{r+f} \\[2em] \dfrac{r}{r+f} \cdot \left[1 - \prod_{j=1}^{n-1} FR(j, j+1)\right] & \dfrac{f}{r+f} \cdot \left[1 - \prod_{j=1}^{n-1} FR(j, j+1)\right] + \prod_{j=1}^{n-1} FR(j, j+1) \\[1em] \vdots & \vdots \\[1em] \dfrac{r}{r+f} \cdot \left[1 - \prod_{j=n-1}^{n-1} FR(j, j+1)\right] & \dfrac{f}{r+f} \cdot \left[1 - \prod_{j=n-1}^{n-1} FR(j, j+1)\right] + \prod_{j=n-1}^{n-1} FR(j, j+1) \end{bmatrix} \quad [40]$$

This result is quite interesting. Remember that $N \cdot R$ represents "the probabilities of dollars in each of the age categories being paid" (first column) and "the probabilities of becoming bad debts" (second column). This should not be surprising, and it could have been deducted in another way. Let us think as follows:

- Repayment was agreed to happen only from bucket 0, and charge-off from bucket n-1.

- At any other bucket, a "loan" can only move forward or get back to bucket 0. Remember that flow rates are a simplification of roll rates. They work like roll rates but assuming there are no partial payments. Note that, in reality, there is not a set of real loans following exactly those flows (all of them at the same time).

- At bucket 0, there are four options for a loan each month: a. Repay (probability r); b. Stay at bucket 0 (probability $FR(0,0)$); c. Miss a payment but not going straight to charge-off (probability $FR(0,1)$-f); d. Miss a payment and going straight to charge-off (probability f). In no way can we interpret that there are real loans going straight to charge-off with f probability.

- Options b and c mean restarting again at bucket 0 sooner or later. Options a and d mean ending in one of the two absorbing states without going through bucket 0 anymore.

- Since the Markov assumption is kept (i.e. loans have no memory), at bucket 0 the odds of eventual repayment versus eventual charge-off are exactly r:f, so the probabilities are $r/(r+f)$ and $f/(r+f)$ respectively.

- At any other bucket, there are two options for a loan: a. Going straight to charge-off; b. Not going straight to charge-off, which necessarily means that it would eventually fall back to bucket 0.

- So, at any bucket other than bucket 0, the probability of eventual repayment is the probability of not going straight to charge-off multiplied by the probability of repayment when at bucket 0, and the probability of eventual charge-off is the probability of going straight to charge-off plus the probability of not going straight to charge-off multiplied by the probability of charge-off when at bucket 0.

If we now computed $B·N·R$, where B is the n-component vector, we would obtain the expected eventual repayments and charge-offs for a portfolio under stationary conditions.

Again, for the record, using equation [13], we can move the corresponding row and column of the absorbing state of charge-off (bucket n) to the end, in order to create a matrix P containing all the possible flow rates ordered in a more meaningful way. Remember that Markov chains would require no new balances ($b = 0$).

$$P' = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ r & 1-FR(0,1)+(b-r) & FR(0,1) & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 1-FR(1,2) & 0 & FR(1,2) & \ddots & & & \vdots & \vdots \\ 0 & 1-FR(2,3) & \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & & & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & & & & & \ddots & FR(n-2,n-1) & 0 \\ 0 & 1-FR(n-1,n) & 0 & \cdots & \cdots & \cdots & \cdots & 0 & FR(n-1,n) \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix} \quad [41]$$

Markov chains do not consider "sources" ($b \neq 0$) so, if we use them, equation [3] would no longer remain true and matrix N would not make sense. But in the event there were sources, there is no problem in considering monthly new balances as new cohorts or portfolios starting at bucket 0, and everything we have said in this section will remain valid for them. Another possibility is that, as long as monthly repayments remain higher than new balances, we could work with a net repayment ratio (r-b) as if they were only repayments and then obviating that there are new balances. This can be done forcing the Markov assumption of no memory, assuming new balances maintain the same behavior as the ones that still remain. If new balances are higher that repayments, we can still rely on most of what we obtained, but b-r must be lower than f in order

to assure convergence in the long run (i.e. inputs cannot be higher than outputs). Remember that *b* and *r* are defined in terms of a fraction of bucket 0, which allows convergence in an environment of diminishing bucket 0.

To clarify, we say that a portfolio is under stationary conditions when the RRs or FRs employed do not change over time. Stationary conditions for RRs come from the Markov assumption applied to each loan, although it may be a little hard to assume that loans have no memory, since it is known that ever-delinquent loans have higher propensity to relapse. What is more, there are other external factors that affect RRs, such as changing economic conditions. In the case of FRs, it is even harder to make the Markov assumption of stationary conditions as real flow rates depend on the bucket distribution, which in turn is time dependent, as we learned from equation [35].

On the other hand, we say that a portfolio is under steady-state conditions when it also happens that the *n*-component vector *B* remains invariant over time:

$$B_i = B_i \cdot Q_{i,i+1} \qquad [42]$$

Under steady-state or just stationary conditions, sequential and coincidental flow rates are the same. Although, in the case of just stationary conditions, it is hard to accept it, as we have said before (if RRs are constant and *B* changes over time, FRs must be changing).

Let us now explore eigenvalues and left eigenvectors of matrix Q^{FR}. Particularly, a steady state would be characterized by an eigenvalue of 1 ($\lambda = 1$).

$$Q_{i,i+1}^{FR} - \lambda I = \begin{bmatrix} 1 - FR(0,1) + (b-r) - \lambda & FR(0,1) & 0 & \cdots & \cdots & \cdots & 0 \\ 1 - FR(1,2) & -\lambda & FR(1,2) & \ddots & & & \vdots \\ 1 - FR(2,3) & 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & 0 \\ \vdots & \vdots & & & & \ddots & FR(n-2,n-1) \\ 1 - FR(n-1,n) & 0 & \cdots & \cdots & \cdots & 0 & -\lambda \end{bmatrix} \qquad [43]$$

$$\det(Q_{i,i+1}^{FR} - \lambda I)$$
$$= (-1)^n \cdot \lambda^n + (-1)^{n-1} \cdot \left(1 - FR(0,1) + (b-r)\right) \cdot \lambda^{n-1} + (-1)^{n-1}$$
$$\cdot \sum_{j=1}^{n-1} \left(\lambda^{n-j-1} \cdot \left(\prod_{m=0}^{j-1} FR(m,m+1) - \prod_{m=0}^{j} FR(m,m+1) \right) \right) \qquad [44]$$

If flow rates are not higher than 1, as we said in equation [27], and *b-r* is higher than $FR(0,1)$-1, which should be normal, then, the determinant equation, in terms of λ, has only one sign change in the sequence of polynomial coefficients and, according to the Descartes' rule of signs, this would mean that there is a unique positive root. We will now see that this positive root is 1, under a special condition.

Since we are working with row-vectors and multiplying from the left, in order to obtain the eigenvalues and left-eigenvectors, we perform a column reduction operation, summing all columns in the first column.

$$
\begin{bmatrix}
1-\lambda+(b-r) & FR(0,1) & 0 & \cdots & \cdots & \cdots & 0 \\
1-\lambda & -\lambda & FR(1,2) & \ddots & & & \vdots \\
1-\lambda & 0 & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & & & \ddots & \ddots & FR(n-2,n-1) \\
1-\lambda-FR(n-1,n) & 0 & \cdots & \cdots & \cdots & 0 & -\lambda
\end{bmatrix}
\qquad [45]
$$

If we now solve the determinant for $\lambda = 1$, we prove it can be 0 under a special condition:

$$
\det\big(Q^{FR}_{i,i+1} - I\big) = (-1)^{n-1} \cdot (b-r) + (-1)^{n} \cdot \prod_{j=0}^{n-1} FR(j,j+1)
\qquad [46]
$$

$$
\det\big(Q^{FR}_{i,i+1} - I\big) = 0 \Rightarrow (b-r) = f
\qquad [47]
$$

We can see this in a different and interesting way: under steady-state conditions, input and output flows of the portfolio must be the same, as shown in equation [49], and each bucket must keep a flow rate relationship with the previous one, as shown in equation [50]. If we successively replace buckets of equation [49] with equation [50], we will arrive at the same result.

$$
\begin{bmatrix}
-(b-r) & FR(0,1) & 0 & \cdots & \cdots & \cdots & 0 \\
0 & -1 & FR(1,2) & \ddots & & & \vdots \\
0 & 0 & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & & & \ddots & \ddots & FR(n-2,n-1) \\
-FR(n-1,n) & 0 & \cdots & \cdots & \cdots & 0 & -1
\end{bmatrix}
\qquad [48]
$$

$$
B(0) \cdot (b-r) = B(n-1) \cdot FR(n-1,n)
\qquad [49]
$$

$$
\forall j \in \{1, \ldots, n-1\} : B(j) = FR(j-1,j) \cdot B(j-1)
\qquad [50]
$$

A left-eigenvector for $\lambda = 1$ could easily be created with equations [49] and [50]. This left-eigenvector would represent a steady-state distribution of buckets (i.e. a row-vector B).

Similarly, for roll rates, if we compute $Q^{RR}\text{-}I$ ($\lambda = 1$) and then sum all columns in the first column, considering equations [19], [20] and [37], obviously we will also arrive at the same equation [49]. However, calculating its determinant, for a general n x n lower triangular matrix with a supradiagonal, is more complex, since it has many terms.

In steady-state conditions, equation [35] can be rewritten in a different way, using equation [28]:

$$
\forall j \in \{0, \ldots, n-2\} : FR(j,j+1) = RR(j,j+1) \cdot \frac{1}{1 - \sum_{m=j+1}^{n-1}\left(\prod_{l=j+1}^{m-1} FR(l,l+1) \cdot RR_{i,i+1}(m,j+1)\right)}
\qquad [51]
$$

In this equation [51] we can see that there is an adjusting factor in which roll rates of sitters and partial payers are discounted at a compounded flow rate down to the bucket in which they fall. It is like taking sitters and partial payers of a certain bucket as full payers at the expense of worsening the performance (increasing the flow rate) of lower buckets where those payers fall. It is expected that these effects will be lower, the higher the buckets. In fact, there is no effect at all for the highest bucket, as for equation [37].

Note that vector B_i is always the same for both roll rates and flow rates because it is given. But matrix R is the same in both cases only if roll rates assume what it was agreed by flow rates: that repayment occurs only from bucket 0 and that there are no early charge-offs. The repayment rate r is the same in both cases only if roll rates assume that repayment occurs only from bucket 0. Remember that equations [22] and [23] must be kept in any case.

Conversely, matrix N will be different in the two cases, because roll rates and flow rates can only be simultaneously constant over time in steady-state conditions but, in that case, the fundamental matrix N has no sense (in theory, there may be a very particular stationary, but not steady-state, scenario in which they are simultaneously constant over time). This means that when modelling in parallel with both RRs and FRs, we cannot get all the same results, although we can try to match some of them such as $B_i \cdot N \cdot R$ or a certain B_{i+1}. Consequently, if we wanted to assume stationary conditions, it would make more sense to assume first that RRs are constant and, then, that FRs change over time and follow equation [51].

Roll rates and flow rates work well for unsecured loans, since there is no collateral, but secured loans could end with a repossession or foreclosure that significantly reduces or compensates the amount due in certain buckets. Roll rates and flow rates can still be used in those cases but require handling such events properly. Sometimes repossession may be taken as an alternative absorbing state, instead of bad debt (Acenden, 2012). In that case, we assume they would then need an impairment study to assess the final loss.

As we have already said, roll rates and flow rates can be used in currency terms or in number of loans or accounts terms. A common notation precedes the acronyms with the currency symbol or a hash symbol respectively (e.g. €RR, $RR, #RR, €FR, $FR, #FR). In any case, they are always dimensionless and expressed as a fraction or a percentage.

One quite important thing is to always notice the nature of roll rates and flow rates that are being managed each time, so they are properly used:

- Real / Historical: calculated from actual performance of a portfolio in a certain month. Sometimes they are presented as an average for a period of months to offset seasonality.

- Theoretical: those used assuming stationary conditions (i.e. FRs or RRs do not change over time) or even steady conditions. They are obtained from historical data and then applied to estimate future performance. Most of what we have said in this section falls in this category.

- Modelled: used to forecast a future situation, they are usually first obtained from historical data and then optimized in a convenient way. They are usually devised as time-dependent and with further complexity.

- Implicit from provisions: those values that when applied to a real or theoretically bucket distribution would fit perfectly with a specific calendar of provisions established by a supervisor or central bank (i.e. the imposed provisions equal to the derived flow-to-loss rates, so there is no need to increase or reduce provisions in stationary conditions with no new balances and using the same time horizon). It is convenient to see the gap between real and provisional implicit flow-to-loss rates, to assess how accurate the calendar of provisions is.

- Budgeted: those used to control the risk cost and the performance of collections.

In addition, as we have just said, modelling of roll rates and flow rates can become very complex. Historical and vintage methods could be combined with roll rates (Office of the Comptroller of the Currency, 2004, p. 117) and the same applies for flow rates. When dealing with installment loans, it is normal to use different segmentations based on some covariates such as type of borrower, borrower quality, product category, origination vintage or any other

dimension (Office of the Comptroller of the Currency, 2004, p. 119). Incorporating new balances, repayments and attrition also increases complexity. Other important factors to consider are changing economic factors, bank policies, management strategies (including collection strategies) and other portfolio-specific variables (Hong Kong Monetary Authority, 2006; SAS, 2014, p. 5). Note that we can manage up to four dimensions related to time: calendar month, origination vintage, delinquency vintage and bucket, although buckets are usually taken in the part of independent variables (There is not a necessary relationship between delinquency vintage, bucket and calendar month. Buckets are formed from the due date of the oldest installment still pending to be fully collected, and this date could be rolling if partial payments are made). But many covariates depend on any of these time dimensions, such as economic factors (calendar month dependent), borrower profile and info (origination vintage), propensity to pay (combination of delinquency vintage and bucket (Frequent partial payers -oldest delinquency vintages in a bucket- behave differently)). The last one really challenges the Markov assumption of non-path dependency.

When calculating historical roll rates, it is expected to find a much lower sample in partial-payers roll rates than any other types of roll rates, so we can expect them to be more volatile over time and, thus, they may not be suitable for forecasting. Also, the 0-to-1 roll rate, the new delinquencies rate, is usually very volatile, regardless of the sample size.

Flow rates do not have this problem of low samples, given that they just focus on just n buckets plus the charge-off amount, instead of $n \cdot (n+1)/2$ independent roll rate transitions. But flow rates are very sensitive to stages of growth or decline of the portfolio, or any kind of waves of delinquent loans due to rapid changes in external conditions, because of their heavy dependence on the bucket distribution. Even more, assuming a constant performance of collections, when the level of new delinquencies soars, contemporary flow rates (i.e., most recent historical flow rates) will tend to be lower than they would be in steady-state conditions, creating a feeling of false confidence and leading to the underestimation of future losses. Hence, historical flow rates are not conservative for forecasting and must be use with extreme care.

As a starting point for forecasting, it would be very useful to compare (e.g. graphically) the theoretical steady-state distribution of buckets inferred from the real roll rates (eigenvector for $\lambda = 1$ of matrix Q^{RR} with historical average RRs) versus the real distribution of buckets, in order to detect whether the portfolio is on an increasing or diminishing delinquency scenario, or even under waves of delinquencies. In case of a mature and ongoing portfolio, FRs can be calculated directly from real buckets without concern, but only if external changing conditions are not anticipated, given that FRs are not conservative. In other cases, we can clearly benefit from the theoretical steady-state FRs.

In theory, without the low-sample problem, the course of action for neutral forecasting will be:

1. Compute the historical RRs values, without seasonality.

2. Ensure that the 0-to-1 RR has a steady-state average value.

3. Calculate the eigenvector of matrix Q^{RR} for $\lambda = 1$, to obtain the bucket distribution under theoretical steady-state conditions.

4. Calculate the derived steady-state FRs from the historical RRs and the eigenvector.

5. Calculate steady-state f with those derived FRs.

But, how can we overcome the low sample problem of RRs? One alternative can be to reduce the number of degrees of freedom of the RRs: smoothing and interpolating them with some

analytical expressions that are empirically meaningful, specially for partial-payers roll rates. Another alternative can be to use more historical data if available.

But, as another alternative, we propose to work just with historical forward roll rates, which have good properties (stable and a large sample), and to use adjusting factors α, knowing that FRs are lower bounded by the corresponding forward roll rate. Factors α can range from 0 (lower conservative) to any positive value, and they should be higher the lower the bucket, as for the equation [51].

$$RR(j, j+1) \leq FR(j, j+1) \tag{52}$$

$$adjusted\ FR(j, j+1) = \left(1 - \alpha_j\right) \cdot RR(j, j+1) + \alpha_j \cdot FR(j, j+1) \tag{53}$$

If for any circumstance, RRs are not available or they are not easy to calculate, we suggest to also use adjusting factors to increase those FRs that seem to be more underestimated (e.g. those from lower buckets).

In addition, we should bear in mind that RRs may also worsen over time due to external factors, such as deterioration of the economic environment. So, for conservative forecasting, an additional factor should be used to adjust and increase RRs other than curers RRs (RRs do not have such great cross effects as FRs, heavily dependent on the bucket. Therefore, in a first approach, a unique adjusting factor for all RR may be suitable). Given the nature of RR, a logistic transformation sounds appropriate. In any case, constant monitoring of RRs from lower buckets is quite necessary, in order to detect early a surge of new delinquents or deteriorating collections indicating a downturn.

The usage of roll rates to manage collections from delinquencies and to estimate provisions or forecast losses is quite common in financial institutions (So & Thomas, 2010, p. 96; Office of the Comptroller of the Currency, 2016, p. 27; Hong Kong Monetary Authority, 2006, p. 38), while some financial institutions use both net flow rates and roll rates as a tool to forecast the amount of the balance in default (Santander Consumer Finance E.F.C., 2016, pp. 111-112). Note that roll rates "forecast only average customer behavior" and they "do not help to identify the specific customers that are most likely to become delinquent" (Coffman & Chandler, 1983, pp. 3, 12). In contrast, flow rates align quite well with collection strategies divided by stage of delinquency.

According to FDIC, roll rate models are a valid methodology to predict losses and then estimate allowances for loan losses (FDIC, 2007, pp. 108-112). Expected Credit Losses (ECL) of a particular portfolio are always associated with a certain time horizon. When nothing is said about the time horizon, sometimes it is assumed that the ECL refers to the losses arising from just a straight-to-charge-off behavior of delinquent balances which, as we know from equation [36], will underestimate losses in the long term. Some common time horizons are 12-months-ECL or lifetime-ECL.

Following what FDIC says, roll-to-loss rates (also called "loss factors") are the multiplication of all the "roll-rates" (sic) from each delinquency bucket forward through loss. These roll-to-loss rates can then be multiplied by their corresponding bucket and then aggregated to determine the required allowance level. The only problem is that this method is not accurate enough when dealing with loans that are not delinquent, which are the majority (FDIC, 2007, p. 109). In other words, the first roll rate, from current to bucket 1 is the most volatile. The same arises when working with the current-to-loss factor which contemplates that loans that are current should also be provisioned. They are also known as provision rates (Anderson, 2007, p. 498; FDIC, 2007, p. 109) in an attempt to match loss expectancy and provisions. But FDIC is taking buckets where there are stayers or other partial payers. So, rather than saying they are using roll-to-loss rates, it would be more correct to say that they are using flow-to-loss rates. Surprisingly, the OCC, which do not explicitly use the flow rates but a simplification of the roll rates, use also

the concepts of migration-to-loss and flow-to-loss (Office of the Comptroller of the Currency, 2015, pp. 83, 90; Office of the Comptroller of the Currency, 2016, pp. 58, 83).

If we were to use a lifetime horizon, eventual-loss expectancy rates for each bucket category could be obtained from the multiplication of matrices $N \cdot R$, taking the column corresponding to the charge-off absorbing state, which represents the flow-to-eventual-loss rates under stationary conditions.

Expected loss (EL) factors (PD, EAD and LGD) have a direct relationship with roll rates and flow rates. Assuming we are calculating the EL of an ongoing portfolio under steady-state conditions (i.e. with a source), where d is the bucket designated for default and n is the bucket for charge-off or write-off, the PD would be the $\#FR(0,d)$, the LGD may be seen approximately as the $€FR(d,n)$, and the EAD would be similar to the quotient between $€FR(0,d)$ and $\#FR(0,d)$. As a result, not surprisingly, the n-month EL is the sequential current-to-charge-off flow rate f, in theory. In reality, things are more complex because the Markov assumption is not kept and there are many cross effects that generate biases in delinquent loans (e.g. loans with higher balances tend to have higher FRs, and so on). If we were to calculate the n-month EL of a portfolio under stationary conditions from origination (i.e. all initial balance at bucket 0), we would need to use €RRs instead of €FRs.

In our opinion, applying the prudence concept, lifetime-ECL should be recognized, as provisions, at the time of loan (or new balance) origination, using meaningful current-to-loss factors devised with multiple variables. In the cases of long duration or high-interest high-risk loans, it may be accepted to accrue these expenses according to the expected revenue stream. These provisions should be updated any time there is any changing external condition that suggests that roll rates may be varying. Obviously, this scheme should be for a neutral scenario. Prudence also suggests creating more provisions in case of unforeseen events.

Financial institutions with an information-based strategy and a test-vs-control methodology have a competitive advantage when dealing with collections (Clemons & Thatcher, 1998, p. 3). Coffman and Chandler state that it is worth focusing collection efforts on "customers most likely to remain delinquent or likely to become more seriously behind in their payments". But that is not necessarily true. The decisions should be based on the NPV of the different possible actions, that is to say, the incremental collections of the actions net of costs. In the case at hand, it means it is more worthwhile dedicating efforts to actions that can improve roll rates despite the level of the same. The most profitable decision should be chosen looking at the roll rates sensitivity to different actions.

Discussion

Roll rates and [net] flow rates are wrongly used as interchangeable concepts. The problem sometimes arises when the concept of "rolling" is constricted to just only rolling-over or rolling forward, because as we know, other types of "rolling's" are also possible due to partial paying. Further, roll rates follows individual accounts, while net flow rates do not track individual accounts and they just focus on aggregates related to buckets (total number of accounts or total outstanding amount in those buckets). And, by definition, roll rates can never be higher than 100%, whilst flow rates may be occasionally higher than 100%, but never on average and long term. For example, the OCC states that "for ease of calculation, roll rate analysis assumes all dollars at the end of a period flow from the prior period bucket" (Office of the Comptroller of the Currency, 2015, p. 146; Office of the Comptroller of the Currency, 2016, p. 145), which is really a flow rate analysis as we have already defined. FDIC also warns of little inaccuracies when working with buckets to compute forward "roll rates", that we know they are really flow rates (FDIC, 2007, p. 109). What they do not realize is that, despite being quite acceptable in normal

conditions, this simplification is not conservative at all, as we addressed before. To prevent confusion, other authors use the term "net roll rates" (Anderson, 2007, p. 498), separated from "roll rates", to really refer to flow rates, when they make the assumption that "accounts in each bucket either get worse, or are repaid in full [the due debt]" (Anderson, 2007, p. 499).

Table 4. Roll Rates vs Flow Rates definitions.

Roll Rates:	Track individual accounts across buckets in consecutive months.
[Net] Flow Rates:	Compare consecutive buckets in consecutive months.

Source: Own elaboration.

References

Acenden (2012). Overview of arrears roll rate. *UK and Ireland mortgage and property monthly, 6,* 1-3. https://www.acenden.com/docs/default-source/newsletter-uk-ireland-mortgage-property-monthly/uk-and-ireland-mortgage-and-property-monthly-issue-6-nov-2012.pdf.

Anderson, R. (2007). *The credit scoring toolkit: theory and practice for retail credit risk management and decision automation.* Oxford University Press.

Clemons, E., & Thatcher, M. (1998). Capital One: Exploiting an information-based strategy. In System Sciences (Ed.), *Thirty-First Hawaii International Conference, 6*, 311-320. https://doi.org/10.1109/HICSS.1998.654788

Coffman, J.Y., & Chandler, G.G. (1983). *Applications of performance scoring to accounts receivable management in consumer credit.* Krannert Graduate School of Management, Purdue University.
http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.197.9454&rep=rep1&type=pdf

Cyert, R.M., & Thompson, G.L. (1968). Selecting a portfolio of credit risks by Markov chains. *The Journal of Business, 41*(1), 39-46.

Cyert, R.M., & Trueblood, R.M. (1957). Statistical sampling techniques in the aging of accounts receivable in a department store. *Management Science, 3*(2), 185-195.

Cyert, R.M., Davidson, H.J., & Thompson, G.L. (1962). Estimation of the allowance for doubtful accounts by Markov chains. *Management Science, 8*(3), 287-303.

FDIC (2007). *Risk Management Examination Manual for Credit Card Activities.* Division of Supervision and Consumer Protection.
https://www.fdic.gov/regulations/examinations/credit_card/pdf_version/

Hong Kong Monetary Authority (2006). *Supervisory Policy Manual: Credit Card Business.* Hong Kong Monetary Authority.

Kellett, D. (2011). UK Credit Card Loss Forecasting Using Markov Chain Models. In Capital One (Ed.), *Credit Scoring and Credit Control XII conference.* Edinburgh.

Office of the Comptroller of the Currency (2004). *Retail Lending Examination Procedures. Comptroller's Handbook.* U.S. Department of the Treasury.

http://www.occ.gov/publications/publications-by-type/comptrollers-handbook/_pdf/retaillendingexaminationprocedures.pdf

Office of the Comptroller of the Currency (2015). *Credit Card Lending. Comptroller's Handbook.* U.S. Department of the Treasury.
https://www.occ.treas.gov/publications/publications-by-type/comptrollers-handbook/credit-card-lending/pub-ch-credit-card.pdf

Office of the Comptroller of the Currency (2016). *Installment Lending. Comptroller's Handbook.* U.S. Department of the Treasury. https://www.occ.treas.gov/publications/publications-by-type/comptrollers-handbook/installment-lending/pub-ch-installment-lending.pdf

PwC (2015). *IFRS 9: Credit Modelling and Implementation.* http://www.pwc.com/ca

Rosenberg, E., & Gleit, A. (1994). Quantitative methods in credit management: a survey. *Operations Research, 42*(4), 589-613.

Santander Consumer Finance E.F.C. (2016). *Securitization Fund Santander Consumer Spain Auto 2016-1.* [Issue Prospectus].
https://www.cnmv.es/Portal/Consultas/Folletos/FolletosEmisionOPV.aspx?isin=ES0305124002

SAS (2014). *CCAR: An Appraisal of Current Practices.*
https://www.sas.com/content/dam/SAS/en_us/doc/whitepaper1/ccar-appraisal-of-current-practices-107211.pdf

So, M.M., & Thomas, L.C. (2010). Modeling and model validation of the impact of the economy on the credit risk of credit card portfolios. *Journal of Risk Model Validation, 4*, 93-126.

Stretton, C., & Burra, P. (2011). *Financial instruments impairment: Adapting to change.* Johannesburg: Deloitte.
https://www.iasplus.com/de/binary/safrica/1107impairmentaccounting.pdf

Appendix A. Fundamental matrix of the absorbing Markov chain

Matrix N is the fundamental matrix of the absorbing Markov chain, and is defined as follows:

$$N = \sum_{k=0}^{\infty} Q^k = (I - Q)^{-1}$$

Matrix N of Flow rates

To avoid overloaded notation, we define $FR_j = FR(j, j+1)$

$$
N = \begin{bmatrix}
1 + \frac{1}{r+f} \cdot \left[1 - \prod_{j=0}^{n-1} FR_j\right] - \frac{r}{r+f} & \frac{\prod_{j=0}^{0} FR_j}{r+f} & \cdots & \cdots & \frac{\prod_{j=0}^{n-2} FR_j}{r+f} \\
\frac{1}{r+f} \cdot \left[1 - \prod_{j=1}^{n-1} FR_j\right] & 1 + \frac{\prod_{j=0}^{0} FR_j}{r+f} \cdot \left[1 - \prod_{j=1}^{n-1} FR_j\right] & \prod_{j=1}^{1} FR_j + \frac{\prod_{j=0}^{0} FR_j}{r+f} \cdot \left[1 - \prod_{j=1}^{n-1} FR_j\right] & \cdots & \prod_{j=1}^{n-2} FR_j + \frac{\prod_{j=0}^{n-2} FR_j}{r+f} \cdot \left[1 - \prod_{j=1}^{n-1} FR_j\right] \\
\vdots & \frac{\prod_{j=0}^{0} FR_j}{r+f} \cdot \left[1 - \prod_{j=2}^{n-1} FR_j\right] & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \prod_{j=n-2}^{n-2} FR_j + \frac{\prod_{j=0}^{n-2} FR_j}{r+f} \cdot \left[1 - \prod_{j=n-2}^{n-1} FR_j\right] \\
\frac{1}{r+f} \cdot \left[1 - \prod_{j=n-1}^{n-1} FR_j\right] & \frac{\prod_{j=0}^{0} FR_j}{r+f} \cdot \left[1 - \prod_{j=n-1}^{n-1} FR_j\right] & \cdots & \frac{\prod_{j=0}^{n-3} FR_j}{r+f} \cdot \left[1 - \prod_{j=n-1}^{n-1} FR_j\right] & 1 + \frac{\prod_{j=0}^{n-2} FR_j}{r+f} \cdot \left[1 - \prod_{j=n-1}^{n-1} FR_j\right]
\end{bmatrix}
$$

First element:

$$1 + \frac{1}{r+f} \cdot \left[1 - \prod_{j=row-1}^{n-1} FR_j \right] - \frac{r}{r+f}$$

First row elements, except the first element:

$$\frac{\prod_{j=0}^{col-2} FR_j}{r+f}$$

Upper right triangle elements, except the first row:

$$\prod_{j=row-1}^{col-2} FR_j + \frac{\prod_{j=0}^{col-2} FR_j}{r+f} \cdot \left[1 - \prod_{j=row-1}^{n-1} FR_j \right]$$

Diagonal elements, except the first element:

$$1 + \frac{\prod_{j=0}^{col-2} FR_j}{r+f} \cdot \left[1 - \prod_{j=row-1}^{n-1} FR_j \right]$$

Lower left triangle elements:

$$0 + \frac{\prod_{j=0}^{col-2} FR_j}{r+f} \cdot \left[1 - \prod_{j=row-1}^{n-1} FR_j \right]$$

col being the number of the column and *row* the row number.