C Dynamic Models of Electric Machines

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C.1 INTRODUCTION
This appendix provides dynamic models of electric machines for power system stability studies. Models of both induction and synchronous machines, including simplified models for stability and fault analysis are presented and discussed.

The appendix starts with the model of electric machines’ rotor dynamics. The electromagnetic model of the induction machine is then presented and discussed, based on the space vector approach. The model of the synchronous machine is presented as an extension of the induction machine model, incorporating the representation of the field winding.

C.2 ROTOR DYNAMICS MODEL OF ELECTRIC MACHINES
The equation that describes the dynamics of the rotor of an electric machine is the motion equation of a rigid body. In power system stability studies, the rigid body comprises not only the rotor of the electric machine but also the rotor of the generators’ prime mover or the motors’ mechanical load. For other studies, such as subsynchronous resonance, each rotating mass is represented individually.

Assuming that the electric machine is a generator, the motion equation of the generator rotor is

\[ \frac{d\Omega}{dt} = T_m - T_e - T_d = T_m - T_e - K_d(\Omega - \Omega_0) \]  
(C.1)
where

\[ J \] is the inertia momentum of the rotating masses expressed in Nm\( \cdot \)s\(^2\)

\[ \Omega \] is the mechanical angular speed expressed in rad/s

\[ T_m \] is the mechanical torque expressed in Nm

\[ T_e \] is the electrical torque expressed in Nm

\[ T_d \] is the damping torque expressed in Nm, which is assumed to be proportional to the speed deviation from the synchronous speed

\[ K_d \] is the damping coefficient expressed in Nm/s

\[ \Omega_0 \] is the synchronous mechanical angular speed expressed in rad/s

Equation C.1 can be expressed in per unit (pu) by dividing it by the base torque \( T_{\text{base}} \):

\[
\frac{J}{T_{\text{base}}} \frac{d\Omega}{dt} = \frac{T_m}{T_{\text{base}}} - \frac{T_e}{T_{\text{base}}} - \frac{K_d}{T_{\text{base}}} (\Omega - \Omega_0) \tag{C.2}
\]

where the base torque can be expressed in terms of the apparent power base \( S_{\text{base}} \) and the mechanical angular speed base \( \Omega_{\text{base}} \) (the synchronism speed \( \Omega_0 \)) as follows:

\[ T_{\text{base}} = \frac{S_{\text{base}}}{\Omega_{\text{base}}} = \frac{S_{\text{base}}}{\Omega_0} \]

Hence, Equation C.2 becomes

\[
\frac{J \Omega_0^2}{S_{\text{base}} \Omega_0} \frac{d\Omega}{dt} = \tau_m - \tau_e - \frac{K_d \Omega_0^2}{S_{\text{base}} \Omega_0} (\Omega - \Omega_0) \tag{C.3}
\]

where \( \tau_m \) and \( \tau_e \) are, respectively, the mechanic and electric torques in pu.

In terms of the inertia constant \( H \) and the damping factor \( D \), Equation C.3 can be written as

\[
\frac{2H}{\Omega_0} \frac{d\Omega}{dt} = \tau_m - \tau_e - \frac{D \Omega_0}{\Omega_0} (\Omega - \Omega_0) \tag{C.4}
\]

where the inertia constant \( H \) is defined as the pu rotating kinetic energy at base speed with respect to the base power, that is,

\[ H = \frac{E_c}{S_{\text{base}}} = \frac{1}{2} \frac{J \Omega_0^2}{S_{\text{base}}} \]

The units of \( H \) are s, since the rotating kinetic energy units are Ws and the base power is in VA. On the other hand, the damping factor \( D \) is defined as

\[ D = \frac{K_d \Omega_0^2}{S_{\text{base}}} \]

If the mechanical angular velocities \( \Omega \) and \( \Omega_0 \) are expressed in terms of the electrical angular velocities \( \omega \) and \( \omega_0 \), respectively, and the number of poles \( p \) (\( \omega = p/2\Omega \), and similarly for \( \omega_0 \)), Equation C.4 becomes

\[
\frac{2H}{\omega_0} \frac{d\omega}{d\omega_0} = \tau_m - \tau_e - \frac{D}{\omega_0} (\omega - \omega_0) \tag{C.5}
\]
C.3 DYNAMIC MODELS OF INDUCTION MACHINES

C.3.1 EQUATIONS IN PHASE VARIABLES

Figure C.1 depicts the equivalent windings of each phase of a two-pole, wound-rotor, three-phase induction machine, with the rotor rotating at $\omega_r$. At any given time, the angle between the magnetic axes of the stator phase $a$ and rotor phase $a$ is defined as $\theta$. Figure C.2 illustrates the currents and voltages of each winding, assuming motor operation and a Y connection for both stator and rotor windings. Applying Kirchoff’s voltage law to each winding results in

\[ v_{sa} = R_s i_s + \frac{d\psi_{sa}}{dt} \]
\[ v_{sb} = R_s i_s + \frac{d\psi_{sb}}{dt} \]

FIGURE C.1 Induction machine windings.

FIGURE C.2 Induction machine stator and rotor circuits.
\[ v_{sc} = R_s i_{sc} + \frac{d\psi_{sc}}{dt} \quad (C.8) \]
\[ v_{ta} = R_t i_{ta} + \frac{d\psi_{ta}}{dt} \quad (C.9) \]
\[ v_{tb} = R_t i_{tb} + \frac{d\psi_{tb}}{dt} \quad (C.10) \]
\[ v_{tc} = R_t i_{tc} + \frac{d\psi_{tc}}{dt} \quad (C.11) \]

where \( R_s \) and \( R_t \) are the resistance of stator and rotor windings, respectively, and

\[
\begin{bmatrix}
\psi_{sa} \\
\psi_{sb} \\
\psi_{sc} \\
\psi_{ta} \\
\psi_{tb} \\
\psi_{tc}
\end{bmatrix} =
\begin{bmatrix}
L_{ss} & L_m \cos \frac{2\pi}{3} & L_m \cos \frac{4\pi}{3} \\
L_m \cos \frac{2\pi}{3} & L_{ss} & L_m \cos \frac{2\pi}{3} \\
L_m \cos \frac{4\pi}{3} & L_m \cos \frac{2\pi}{3} & L_{ss}
\end{bmatrix}
\begin{bmatrix}
\psi_{sa} \\
\psi_{sb} \\
\psi_{sc} \\
\psi_{ta} \\
\psi_{tb} \\
\psi_{tc}
\end{bmatrix}
\]

\[
\begin{bmatrix}
i_{sa} \\
i_{sb} \\
i_{sc} \\
i_{ta} \\
i_{tb} \\
i_{tc}
\end{bmatrix} =
\begin{bmatrix}
L_m \cos \theta & L_m \cos \left( \theta + \frac{2\pi}{3} \right) & L_m \cos \left( \theta + \frac{4\pi}{3} \right) \\
L_m \cos \left( \theta + \frac{2\pi}{3} \right) & L_m \cos \theta & L_m \cos \left( \theta + \frac{2\pi}{3} \right) \\
L_m \cos \left( \theta + \frac{4\pi}{3} \right) & L_m \cos \left( \theta + \frac{2\pi}{3} \right) & L_m \cos \theta
\end{bmatrix}
\begin{bmatrix}
i_{sa} \\
i_{sb} \\
i_{sc} \\
i_{ta} \\
i_{tb} \\
i_{tc}
\end{bmatrix}
\]  

with \( L_m \) representing the mutual (magnetizing) inductance of both stator and rotor windings, and \( L_{ss} \) and \( L_{tt} \) representing the stator and the rotor self-inductances, respectively, which result from the sum of the corresponding leakage and magnetizing inductances:

\[
L_{ss} = L_s + L_m \\
L_{tt} = L_t + L_m
\]

### C.3.2 Equations in Complex Form

Equations C.6 through C.11 are a set of linear, time-varying differential equations, due to the variation of the angle \( \theta \) with time \((d\theta/dt = \omega_t)\). These equations can be converted into a set of linear, time-invariant differential equations applying an appropriate variable transformation, that is, Park’s transformation, which is the approach followed in classical books on power system dynamics. In this appendix, we use a transformation more commonly used in modern textbooks on vector control of electric machines [1,2].

The transformation used here is characterized by the fact that the resulting variables, the so-called space vectors, are complex magnitudes. Thus, a three-phase system of either stator or rotor variables
(voltages, currents, and fluxes) can be represented respectively by space vectors with respect to a reference frame (see Figure C.3) according to

\[
\vec{x}_s = ke^{j\theta_{ref}} \left( x_{sa} + x_{sb} e^{j\frac{2\pi}{3}} + x_{sb} e^{j\frac{4\pi}{3}} \right) \\
\vec{x}_r = ke^{j(\theta_{ref} - \theta)} \left( x_{ra} + x_{rb} e^{j\frac{2\pi}{3}} + x_{rb} e^{j\frac{4\pi}{3}} \right)
\]

where \( k \) is a scaling factor.

If Equations C.6 through C.8 are multiplied respectively by \( ke^{-j\theta_{ref}} \), \( ke^{-j(\theta_{ref} - \theta)} e^{j\frac{2\pi}{3}} \), and \( ke^{-j(\theta_{ref} - \theta)} e^{j\frac{4\pi}{3}} \), and then added up, the following complex equation is obtained:

\[
\vec{v}_s = R_s \vec{i}_s + e^{-j\theta_{ref}} \left( e^{j\theta_{ref}} \vec{\psi}_s \right) dt = R_s \vec{i}_s + d\vec{\psi}_s dt + j \frac{d\theta_{ref}}{dt} \vec{\psi}_s
\]

This equation shows two components of the induced stator-winding voltages: the speed voltages \( j \frac{d\theta_{ref}}{dt} \vec{\psi}_s \), and the “transformer” voltages \( d\vec{\psi}_s dt \). On the other hand, if Equations C.9 through C.11 are multiplied respectively by \( ke^{-j(\theta_{ref} - \theta)} \), \( ke^{-j(\theta_{ref} - \theta)} e^{j\frac{2\pi}{3}} \), and \( ke^{-j(\theta_{ref} - \theta)} e^{j\frac{4\pi}{3}} \), and then added up, the following complex equation is obtained for the rotor voltages:

\[
\vec{v}_r = R_r \vec{i}_r + e^{-j(\theta_{ref} - \theta)} \left( e^{-j(\theta_{ref} - \theta)} \vec{\psi}_r \right) dt = R_r \vec{i}_r + d\vec{\psi}_r dt + j \frac{d(\theta_{ref} - \theta)}{dt} \vec{\psi}_r
\]

Expressions of the space vectors of stator and rotor flux linkages can be obtained in a similar way, resulting in the following matrix equation:

\[
\begin{bmatrix}
\vec{\psi}_s \\
\vec{\psi}_r
\end{bmatrix} =
\begin{bmatrix}
L_{ss} & L_m \\
L_m & L_{rr}
\end{bmatrix}
\begin{bmatrix}
\vec{i}_s \\
\vec{i}_r
\end{bmatrix}
\]

Assuming that the reference frame is rotating at synchronous speed, that is, \( \frac{d\theta_{ref}}{dt} = \omega_s \), Equations C.15 through C.17 can be represented as the equivalent circuit depicted in Figure C.4. This circuit
C.3.3 ELECTROMAGNETIC TORQUE

The rotor output power expressed in terms of the space vectors of rotor voltage and current is

\[
P_r = -\text{Re} \left\{ \vec{\dot{v}}_r \vec{i}_r^* \right\}
\]  

(C.18)

If the expression of \( \vec{v}_r \) in Equation C.16 is incorporated into this equation, it results in

\[
P_r = -\text{Re} \left\{ R_r \vec{\dot{i}}_r \vec{i}_r^* + \frac{d}{dt} \vec{\dot{\psi}}_r \vec{i}_r^* + j(\omega_s - \omega_r) \vec{\dot{\psi}}_r \vec{i}_r^* \right\}
\]  

(C.19)

Three terms can be identified here: the rotor losses \(-R_r \vec{\dot{i}}_r \vec{i}_r^*\), the variation of the magnetic energy \(\text{Re}\left\{-d \vec{\dot{\psi}}_r / dt \vec{i}_r^* \right\}\), and the electromagnetic power \(\text{Re}\left\{-j(\omega_s - \omega_r) \vec{\dot{\psi}}_r \vec{i}_r^* \right\}\). The electromagnetic torque can be simply obtained dividing the electromagnetic power by the rotor speed minus the speed of the rotating frame, that is,

\[
\tau_e = \frac{P_e}{\omega_s - \omega_r} = -\text{Re} \left\{ j \vec{\dot{\psi}}_r \vec{i}_r^* \right\} = -\text{Im} \left\{ \vec{\dot{\psi}}_r \vec{i}_r^* \right\}
\]  

(C.20)
Table C.1
Eigenvalues and Participation Factors for the Detailed Fifth-Order Linear Model of an Induction Machine

| µ<sub>1,2</sub> | -1.2839 ± j314.1109 | µ<sub>2,3</sub> | -7.0598 ± j12.3228 | µ<sub>5</sub> | -11.2618 |
|---|---|---|---|---|
| ψ<sub>sd</sub> | 0.5001 | 0.0000 | 0.0002 | 0.5001 | 0.0000 |
| ψ<sub>sq</sub> | 0.0001 | 0.6002 | 0.0959 | 0.0001 | 0.0000 |
| ψ<sub>rd</sub> | 0.0001 | 0.6002 | 0.0959 | 0.0001 | 0.0000 |
| s = ω<sub>s</sub> − ω<sub>r</sub> | 0.0000 | 0.5535 | 0.0670 | 0.0000 | 0.0000 |

C.3.4 Model for Power System Stability Studies

The model of an induction machine for power system stability studies neglects the dynamics of the stator transients; in other words, the derivative of the stator flux space vector is assumed to be equal to zero. This assumption can be justified by means of an eigenvalue analysis of the linearized equations of the detailed model of an induction generator connected to an infinite bus. Table C.1 shows the eigenvalues and corresponding participation factors (see Chapter 10) obtained for this system model.<sup>*</sup> Two distinct dynamic patterns can be observed here [3]: one close to the fundamental frequency associated with the stator fluxes and characterized by eigenvalues µ<sub>1,2</sub>, and another in the frequency range of stability studies and characterized by eigenvalues µ<sub>3,4</sub> and µ<sub>5</sub>; the complex pair µ<sub>3,4</sub> corresponds to the rotor dynamics in the d-axis (excitation flux and slip s), whereas the real µ<sub>5</sub> corresponds to the rotor dynamics in the q-axis. Observe the little effect of the stator fluxes on the rotor variables, and vice versa.

The aforementioned assumption yields the following set of differential-algebraic equations (DAE) for the rotor dynamics and the electromagnetic model of the induction machine (third-order approximate model):

\[
\frac{d\omega_r}{dt} = \frac{\omega_0}{2H}(τ_e − τ_m) \tag{C.21}
\]

\[
\frac{d\tilde{\psi}_r}{dt} = -R_r\tilde{i}_r - j(\omega_s - \omega_r)\tilde{\psi}_r \tag{C.22}
\]

\[
0 = -\tilde{v}_s + R_s\tilde{i}_s + j\omega_s\tilde{\psi}_s \tag{C.23}
\]

\[
\begin{bmatrix}
0 \\
0 \\
\end{bmatrix} = \begin{bmatrix}
\tilde{\psi}_s \\
\tilde{\psi}_r \\
\end{bmatrix} + \begin{bmatrix}
L_{ss} & L_{sr} \\
L_{sr} & L_{rr} \\
\end{bmatrix} \begin{bmatrix}
\tilde{i}_s \\
\tilde{i}_r \\
\end{bmatrix} \tag{C.24}
\]

\[
0 = -τ_e - \text{Im}\{ -\tilde{\psi}_r^*\tilde{i}_r\} \tag{C.25}
\]

The incorporation of an induction machine model into a power system stability simulation tool requires considering the stator voltages and currents as “interface” variables. Thus, Equations C.22

<sup>*</sup>The machine is assumed to be operating at rated conditions, with parameters: \(R_s = 0.001\) pu, \(L_s = 0.10\) pu, \(L_m = 4\) pu, \(R_r = 0.01\) pu, \(L_r = 0.15\) pu, \(H = 3\) s.
through C.25 can be rewritten in a more compact form as

\[ \dot{x} = f(x, z, u, \tilde{v}_s) \]  \hspace{1cm} (C.26)

\[ 0 = g(x, z, u, \tilde{v}_s) \]  \hspace{1cm} (C.27)

\[ \tilde{i}_s = h(x, z, u, \tilde{v}_s) \]  \hspace{1cm} (C.28)

where

\[ x^T = [\omega_r \, \vec{\psi}_r] \]

\[ z^T = [\vec{i}_r \, \tau_e] \]

\[ u^T = \tau_m \]

Figure C.6 shows a comparison of the dynamic response of the detailed fifth-order and the reduced third-order models of an induction machine operating as a generator for a 100 ms three-phase solid fault at its terminals. Machine slip and electromagnetic torque are displayed; the solid line corresponds to the fifth-order model, whereas the dash line corresponds to the third-order model. Observe that the reduced model does not contain the fundamental-frequency oscillations that the detailed model shows; however, the general dynamic response of the machine is adequately approximated by the simplified model.

**FIGURE C.6** Comparison of the detailed fifth-order and reduced third-order models of an induction machine for a 100 ms three-phase solid fault at its terminals.
Dynamic Models of Electric Machines

C.4 DYNAMIC MODELS OF SYNCHRONOUS MACHINES

C.4.1 EQUATIONS IN \(d\) AND \(q\)-AXIS VARIABLES

The synchronous machine model is developed here from the aforementioned induction machine model. This is possible since a synchronous machine contains only an additional winding in the rotor with respect to the induction machine, that is, the field winding, which is located in the \(d\)-axis; the damper windings are basically the short-circuited windings on the machine’s rotor. Thus, the induction machine model represented by the \(d\)- and \(q\)-axis equivalent circuits in Figure C.5 are modified assuming

- The reference frame is considered to be on the rotor, that is, \(d\theta_{\text{ref}}/dt = \omega_r\). Hence, no speed voltages are induced in the rotor.
- A new branch is added to the \(d\)-axis equivalent circuit corresponding to the field winding.
- The reluctance of the magnetic paths on the \(d\) and \(q\) axes are different due to pole saliency. Therefore, different values of the magnetizing inductance \(L_m\) are used in the \(d\)- and the \(q\)-axis equivalent circuits, that is, \(L_{md}\) and \(L_{mq}\).
- The resistance and the leakage inductances of the rotor windings are different on the \(d\) and \(q\) axes.

Figure C.7 shows the equivalent circuits of the synchronous machine for both \(d\) and \(q\) axes; these circuits result from the previous considerations, assuming generator operation. From these equivalent circuits, the voltage equations of the synchronous machine model are

\[
\begin{align*}
v_{sd} &= -R_{sd}i_{sd} + \frac{d\psi_{sd}}{dt} - \omega_r\psi_{sq} \quad (C.29) \\
v_{sq} &= -R_{sq}i_{sq} + \frac{d\psi_{sq}}{dt} + \omega_r\psi_{sd} \quad (C.30) \\
e_{td} &= R_{td}i_{td} + \frac{d\psi_{td}}{dt} \quad (C.31) \\
0 &= R_{kd}i_{kd} + \frac{d\psi_{kd}}{dt} \quad (C.32) \\
0 &= R_{kq}i_{kq} + \frac{d\psi_{kq}}{dt} \quad (C.33)
\end{align*}
\]

where

\[
\begin{bmatrix}
\psi_{sd} \\
\psi_{td} \\
\psi_{kd} \\
\psi_{sq} \\
\psi_{kq}
\end{bmatrix} =
\begin{bmatrix}
L_s + L_{md} & L_{md} & L_{md} & 0 & 0 \\
L_{md} & L_{sd} + L_{md} & L_{md} & 0 & 0 \\
L_{md} & L_{md} & L_{kd} + L_{md} & 0 & 0 \\
0 & 0 & 0 & L_s + L_{mq} & L_{mq} \\
0 & 0 & 0 & L_{mq} & L_{kq} + L_{mq}
\end{bmatrix}
\begin{bmatrix}
-i_{sd} \\
i_{td} \\
i_{kd} \\
i_{sq} \\
i_{kq}
\end{bmatrix}
\]

(C.34)

On this detailed model of the synchronous machine, the field and the damper rotor windings are represented as two circuits on the \(d\) and \(q\) axes. Damper windings can be represented with different degrees of modeling detail. In the IEEE Standard [4], a number of alternatives are discussed, depending on the number of circuits represented on each axis; the model presented here corresponds to Model 2.1 in this Standard.

The detailed models presented here for both induction and synchronous machines are in terms of the circuit parameters. The circuit parameters of induction machines are usually provided by manufacturers. However, manufacturers of synchronous machines usually provide the transient and
FIGURE C.7 Equivalent circuits of a synchronous machine for the $d$ and $q$ axes.

subtransient reactances and time constants instead of leakage and magnetizing reactances and resistances. Section C.5 presents and discusses the two simplified models of the synchronous machine from which these transient and subtransient reactances can be obtained. The interested reader can further refer to Ref. 5 for the expressions of the time constants.

C.4.2 ELECTROMAGNETIC TORQUE

The expression of the electromagnetic torque of the induction machine has been presented above in terms of rotor quantities. In contrast, the expression of the electromagnetic torque of the synchronous machine is given here in terms of stator quantities. Thus, the stator input power expressed in terms of the space vectors of stator voltage and current can be written as

$$P_s = \text{Re} \left\{ \overline{\vec{v}_s \vec{i}_s^*} \right\}$$

$$= \text{Re} \left\{ -R_s \vec{i}_s \vec{i}_s^* + j \omega_r \vec{\psi}_s \vec{i}_s^* + \frac{d}{dt} \vec{\psi}_s \vec{i}_s^* \right\}$$

(C.35)

The electromagnetic power term can be easily identified in this equation; hence, the electromagnetic torque is computed dividing this power by the rotor speed:

$$\tau_e = \frac{\text{Re} \left\{ j \omega_r \vec{\psi}_s \vec{i}_s^* \right\}}{\omega_r} = \text{Im} \left\{ \vec{\psi}_s \vec{i}_s^* \right\}$$

(C.36)
Table C.2  
Eigenvalues and Participation Factors for the Detailed Seventh-Order Linear Model of a Synchronous Machine

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\psi_{sd}$</th>
<th>$\psi_{sq}$</th>
<th>$\psi_{fd}$</th>
<th>$\psi_{kq}$</th>
<th>$\omega_r$</th>
<th>$\delta_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_{1,2}$</td>
<td>$-5.7189 \pm j313.8063$</td>
<td>$0.0042$</td>
<td>$0.0001$</td>
<td>$0.0080$</td>
<td>$0.0003$</td>
<td>$0.0009$</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>$-45.1916$</td>
<td>$0.0003$</td>
<td>$0.0000$</td>
<td>$0.0058$</td>
<td>$0.0007$</td>
<td>$0.0001$</td>
</tr>
<tr>
<td>$\mu_4$</td>
<td>$-46.6577$</td>
<td>$0.0002$</td>
<td>$0.0005$</td>
<td>$0.0104$</td>
<td>$0.0028$</td>
<td>$0.0002$</td>
</tr>
<tr>
<td>$\mu_{5,6}$</td>
<td>$-0.8950 \pm j9.4084$</td>
<td>$0.1948$</td>
<td>$0.0017$</td>
<td>$0.0682$</td>
<td>$0.0009$</td>
<td>$0.0579$</td>
</tr>
<tr>
<td>$\mu_7$</td>
<td>$-0.4011$</td>
<td>$0.0369$</td>
<td>$0.0017$</td>
<td>$0.0009$</td>
<td>$0.0009$</td>
<td>$0.0579$</td>
</tr>
</tbody>
</table>

C.4.3 Rotor Angle

The rotor angle $\delta_r$ is defined as the angle between the terminal voltage phasor and the $q$-axis. Hence, the components of the terminal voltage in the $d$ and $q$ axes rotating at rotor speed are

\[ v_{sd} = v_s \sin \delta_r \quad (C.37) \]
\[ v_{sq} = v_s \cos \delta_r \quad (C.38) \]

The rotor angle is related to the rotor speed as follows:

\[ \frac{d\delta_r}{dt} = \omega - \omega_0 \quad (C.39) \]

C.4.4 Model for Power System Stability Studies

As in the case of the induction machine, the model of a synchronous machine for power system stability also neglects the dynamics of the stator transients. This assumption can be justified as well by means of an eigenvalue analysis of the linear set of equations corresponding to the detailed model of a synchronous generator connected to an infinite bus. The participation factors and eigenvalues shown in Table C.2 also present two distinct dynamic patterns: one close to fundamental frequency associated with the stator fluxes and characterized by eigenvalues $\mu_{1,2}$, and another in the frequency range of stability studies and characterized by eigenvalues $\mu_4 - \mu_7$; the complex pair $\mu_{5,6}$ corresponds to the rotor dynamics, and the real eigenvalues $\mu_4$ and $\mu_7$ are associated with the dynamics of the field and the damper windings.

If the rotor speed deviations with respect to the synchronous speed are assumed small, which is typically the case in stability studies, the speed voltages $j\omega_0 \psi_s$ can be approximated by $j\omega_0 \psi_s$.

---

* The machine operates at unity power factor and rated conditions, with parameters: $R_s = 0.005 \text{ pu}$, $L_s = 0.1 \text{ pu}$, $L_d = 1.05 \text{ pu}$, $L_q = 0.7 \text{ pu}$, $L_d' = 0.35 \text{ pu}$, $L_q' = 0.25 \text{ pu}$, $L_d'' = 0.5 \text{ pu}$, $T_{d0} = 5 \text{ s}$, $T_{d0}' = 0.03 \text{ s}$, $T_{d0}'' = 0.05 \text{ s}$, $H = 3 \text{ s}$. 
From this and the above-mentioned assumptions, a fifth-order approximate model represented by the following equations can be obtained:

\[
\frac{d\delta_r}{dt} = \omega_r - \omega_0 \tag{C.40}
\]

\[
\frac{d\omega_r}{dt} = \frac{\omega_0}{2H}(\tau_m - \tau_e) \tag{C.41}
\]

\[
\frac{d\psi_{fd}}{dt} = -R_{fd}i_{fd} + e_{fd} \tag{C.42}
\]

\[
\frac{d\psi_{kd}}{dt} = -R_{kd}i_{kd} \tag{C.43}
\]

\[
\frac{d\psi_{fq}}{dt} = -R_{fq}i_{fq} \tag{C.44}
\]

\[
\frac{d\psi_{sd}}{dt} = \omega_s \psi_s q - \omega_s \psi_s d \tag{C.45}
\]

\[
\frac{d\psi_{sq}}{dt} = \omega_s \psi_s d - \omega_s \psi_s q \tag{C.46}
\]

\[
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \psi_{sd} \\ \psi_{fd} \\ \psi_{kd} \\ \psi_{fq} \\ \psi_{sq} \end{bmatrix}
\]

\[
\begin{bmatrix} L_s + L_{md} & L_{md} & L_{md} & 0 & 0 \\ L_{md} & L_{id} + L_{md} & L_{md} & 0 & 0 \\ L_{md} & L_{md} & L_{kd} + L_{md} & 0 & 0 \\ 0 & 0 & 0 & L_s + L_{mq} & L_{mq} \\ 0 & 0 & 0 & L_{mq} & L_{kq} + L_{mq} \end{bmatrix}
\]

\[
\begin{bmatrix} -i_{sd} \\ i_{fd} \\ i_{kd} \\ -i_{sq} \\ i_{kq} \end{bmatrix} \tag{C.47}
\]

\[
0 = -\tau_e + \psi_{sd} i_{sd} - \psi_{sq} i_{sq} \tag{C.48}
\]

All variables are referred to a reference frame rotating at synchronous speed. Stator voltage and current components in the rotating \(d\)- and \(q\)-axis reference frame are then transformed to a synchronously rotating frame \(R\)-\(I\) (complex phasor plane) using the rotor angle \(\delta_r\), which is the angle between the real \(q\)-axis and the \(R\)-axis. This transformation is accomplished by means of the following equations:

\[
\begin{bmatrix} V_R \\ V_I \end{bmatrix} = \begin{bmatrix} \sin \delta_r & \cos \delta_r \\ -\cos \delta_r & \sin \delta_r \end{bmatrix} \begin{bmatrix} v_{sd} \\ v_{sq} \end{bmatrix} \tag{C.49}
\]

\[
\begin{bmatrix} I_R \\ I_I \end{bmatrix} = \begin{bmatrix} \sin \delta_r & \cos \delta_r \\ -\cos \delta_r & \sin \delta_r \end{bmatrix} \begin{bmatrix} i_{sd} \\ i_{sq} \end{bmatrix} \tag{C.50}
\]

The aforementioned machine equation can be written in a DAE form as follows:

\[
\dot{x} = f(x, z, u, V) \tag{C.51}
\]

\[
0 = g(x, z, u, V) \tag{C.52}
\]

\[
I = h(x, z, u, V) \tag{C.53}
\]
where

\[ x^T = \begin{bmatrix} \delta & \omega & \psi_{id} & \psi_{kd} & \psi_{dq} \end{bmatrix} \]

\[ z^T = \begin{bmatrix} \psi_{sd} & \psi_{sq} & i_{sd} & i_{id} & i_{kd} & i_{iq} & v_{sd} & v_{sq} & \tau_e \end{bmatrix} \]

\[ u^T = \begin{bmatrix} \tau_m & e_{id} \end{bmatrix} \]

\[ V^T = \begin{bmatrix} V_R & V_I \end{bmatrix} \]

\[ I^T = \begin{bmatrix} I_R & I_I \end{bmatrix} \]

Figure C.8 illustrates a comparison of the dynamic response of the detailed seventh-order and reduced fifth-order models of a synchronous machine working as a generator for a 100 ms three-phase solid fault at its terminals. Machine speed and electromagnetic torque are displayed; the solid line corresponds to the seventh-order model, whereas the dash line depicts the fifth-order model response. Observe that the more detailed model contains fundamental-frequency oscillations that are not found in the reduced model; however, the latter adequately approximates the overall dynamic trend of the machine.

### C.5 SYNCHRONOUS MACHINE SIMPLIFIED MODELS

This section discusses the subtransient and transient models of the synchronous machine, which are often used in stability studies. Both models assume that rotor fluxes are constant. The subtransient model considers two circuits on each axis, whereas the transient model assumes one circuit on each axis; the subtransient model corresponds to Model 2.2 in the IEEE Standard [4], while the transient model corresponds to Model 1.1.
C.5.1 SUBTRANSIENT MODEL

The stator equations of the synchronous machine in the $d$ and $q$ axes are

$$v_{sd} = -R_s i_{sd} - \omega_s \psi_{sq}$$ \hspace{1cm} (C.54)

$$v_{sq} = -R_s i_{sq} + \omega_s \psi_{sd}$$ \hspace{1cm} (C.55)

Assuming that the rotor fluxes remain constant, the following expressions of the stator fluxes in the $d$ and $q$ axes can be obtained from the equivalent circuits depicted in Figure C.9:

$$\psi_{sd} = -\left( L_s + \frac{1}{L_{mq}} + \frac{1}{L_{sd}} + \frac{1}{L_{kd}} \right) i_{sd}$$

$$\psi_{sd} = -\frac{1}{L_{mq}} + \frac{1}{L_{sd}} + \frac{1}{L_{kd}} \left( \frac{\psi_{td}}{L_{td}} + \frac{\psi_{kd}}{L_{kd}} \right)$$ \hspace{1cm} (C.56)

$$\psi_{sq} = -\left( L_s + \frac{1}{L_{mq}} + \frac{1}{L_{sq1}} + \frac{1}{L_{sq2}} \right) i_{sq}$$

$$\psi_{sq} = -\frac{1}{L_{mq}} + \frac{1}{L_{sq1}} + \frac{1}{L_{sq2}} \left( \frac{\psi_{kq1}}{L_{kq1}} + \frac{\psi_{kq2}}{L_{kq2}} \right)$$ \hspace{1cm} (C.57)

If Equations C.56 and C.57 are substituted in Equations C.54 and C.55, one obtains

$$v_{sd} = -R_s i_{sd} + X''_d i_{sd} + E''_d$$ \hspace{1cm} (C.58)

$$v_{sq} = -R_s i_{sq} - X''_q i_{sq} + E''_q$$ \hspace{1cm} (C.59)
where $X''_d$ and $X''_q$ are, respectively, the subtransient reactances on the $d$ and $q$ axes. If rotor saliency is neglected, that is, $X''_d = X''_q$, Equations C.58 and C.59 can be rewritten as the single complex equation:

$$
\mathcal{E}''_s = (R_s + jX'')I_s + U_s
$$

(C.60)

where $X''$ is the subtransient reactance, and $\mathcal{E}''_s$ is the voltage behind the subtransient impedance.

Subtransient models are typically used to represent synchronous machines in short-circuit studies.

### C.5.2 Transient Model

Assuming that the rotor fluxes remain constant, expressions of the stator fluxes in the $d$ and $q$ axes can be obtained from the equivalent circuits of Figure C.10 as follows:

$$
\psi_{sd} = -(L_s + \frac{1}{L_{ad} + \frac{1}{L_{dd}}})i_{sd} - \frac{1}{L_{ad} + \frac{1}{L_{dd}}} \psi_{ld}
$$

(C.61)

$$
\psi_{sq} = (L_s + \frac{1}{L_{aq} + \frac{1}{L_{kq}}})i_{sq} + \frac{1}{L_{aq} + \frac{1}{L_{kq}}} \psi_{kq}
$$

(C.62)
If Equations C.61 and C.62 are substituted in Equations C.54 and C.55, the following expressions can be obtained:

\[ v_{sd} = -R_s i_{sd} - X'_q i_{sq} + E'_d \]  
(C.63)

\[ v_{sq} = -R_s i_{sq} + X'_d i_{sd} + E'_q \]  
(C.64)

where \( X'_d \) and \( X'_q \) are, respectively, the transient reactances on the \( d \) and \( q \) axes. If rotor saliency is neglected, that is, \( X'_d = X'_q \), Equations C.63 and C.64 yield the following single complex equation:

\[ E'_s = (R_s + jX') I_s + U_s \]  
(C.65)

where \( X' \) is the transient reactance, and \( E'_s \) is the voltage behind the transient impedance.

Transient models are used to represent synchronous machines in simplified power system stability studies.

REFERENCES