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Markoff m-Triples with k-Fibonacci Components

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Abstract. We classify all solution triples with k-Fibonacci components to the equation $x^2 + y^2 + z^2 = 3xyz + m$, where m is a positive integer and $k \geq 2$. As a result, for $m = 8$, we have the Markoff triples with Pell components $(F_2(2), F_2(2n), F_2(2n+2))$, for $n \geq 1$. For all other m there exists at most one such ordered triple, except when $k = 3$, a is odd, b is even and $b \geq a + 3$, where $(F_3(a), F_3(b), F_3(a+b))$ and $(F_3(a+1), F_3(b-1), F_3(a+b))$ share the same m.

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1. Introduction

In the realm of number theory, Markoff m-triples represent an interesting area of exploration. These triples are positive integer solutions to the Markoff m-equation

$$x^2 + y^2 + z^2 = 3xyz + m, \tag{1.1}$$

where m is a positive integer. The case $m = 0$ corresponds to the original equation studied by Markoff [1,2], where it was proved that all the solution triples are distributed in a unique tree. Some of its branches are interesting families of numbers: Fibonacci, Pell, etc. Many authors studied generalizations of this equation [3–5] and noticed that, depending on m, there could exist one, multiple trees or none at all. In particular, in [5] it is proved that the number of trees, for every $m > 0$, is equal to the number of Markoff m-triples (x, y, z) that are minimal, that is to say, those that satisfy the inequality

$$z \geq 3xy. \tag{1.2}$$

In this paper, we study Markoff m-triples with k-Fibonacci components, i.e. solutions of the Markoff m-equation (1.1), such that all its components

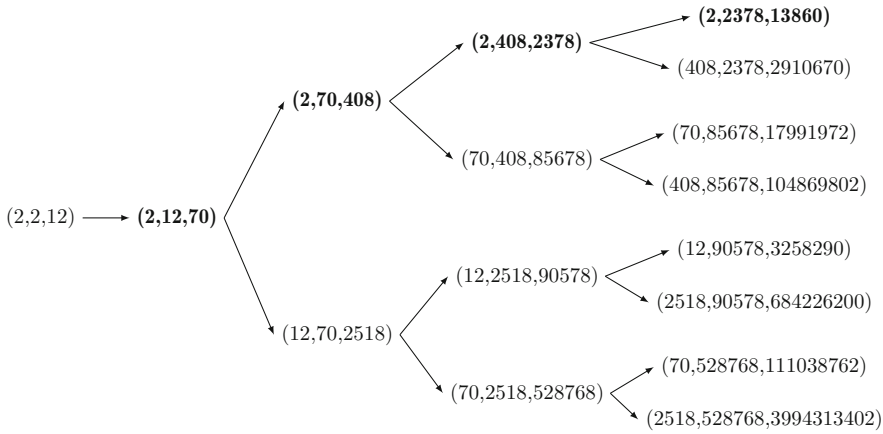

Figure 1. Beginning of the Markoff 8-tree with minimal triple $(2, 2, 12)$. The sequence of non-minimal 8-Markoff triples with 2-Fibonacci components (Pell components) is represented in bold

are k-Fibonacci numbers. These numbers are defined recursively for every positive integer k as follows

$$\begin{cases} F_k(0) = 0 \\ F_k(1) = 1 \\ F_k(n) = kF_k(n-1) + F_k(n-2), \quad \forall n \geq 2. \end{cases} \tag{1.3}$$

When $k = 1$, the sequence corresponds to the classic Fibonacci numbers, and for $k = 2$, it yields Pell numbers. Some particular cases of Markoff m-triples with k-Fibonacci components have already been studied: ($k = 1$, $m = 0$), was studied in [6]; ($k = 2$, $m = 0$), was examined in [7]; ($k > 1$, $m = 0$), was treated in [8]; the case $m = 0$, with Lucas sequences in [9,10] and, finally, the case ($k = 1$, $m > 0$) was dealt with in [11]. Because of this, henceforth, we will assume that $m > 0$ and $k \geq 2$.

In this work, we classify all Markoff m-triples with k-Fibonacci components, dividing our analysis first into non-minimal triples and then into minimal ones. Specifically, our main results are the following.

Theorem 1.1. *Every non-minimal Markoff m-triple with k-Fibonacci components and $m > 0$ is a Markoff 8-triple of the form $(F_2(2), F_2(2n), F_2(2n+2))$, for $n \geq 2$.*

In particular, the non-minimal Markoff m-triples with k-Fibonacci components are situated on the upper branch of the 8-tree with minimal triple $(2, 2, 12)$. The triples in this branch are composed of Pell numbers, as shown in Fig. 1.

Theorem 1.2. *If $m > 0$ admits a minimal Markoff m-triple with k-Fibonacci components, then it is unique, except for $k = 3$ and all pairs of triples $(F_3(a), F_3(b), F_3(a + b)), (F_3(a + 1), F_3(b - 1), F_3(a + b))$, for a odd and b even with $b \geq a + 3$.*

The paper is structured as follows. Section 2 provides certain identities and inequalities satisfied by k-Fibonacci numbers which will be useful in the next sections. Although most of them are well known [12–14], we have included proofs for some of them for the sake of completeness. In Sect. 3, we prove Theorem 1.1 and in Sect. 4, Theorem 1.2. The strategy to obtain uniqueness in minimal Markoff m-triples $(F_k(a), F_k(b), F_k(c))$ except in the case $k = 3, c = a + b, a$ odd, b even and $b \geq a + 3$ involves proving that any pair of such triples which share the same m must have the same third component c, and the sum $a + b$ should be constant (see Lemmas 4.4 and 4.6). These two lemmas, in turn, follow from Lemma 4.1, which computes a lower bound for the m associated with an m-triple $(F_k(a), F_k(b), F_k(c))$ in terms of k and c.

2. Some Preliminary Results on k-Fibonacci Numbers

For any $k > 0$ and $n \geq 0$ the n-th term of the sequence of k-Fibonacci numbers, defined in Eq. (1.3), can be obtained using Binet's formula

$$F_k(n) = \frac{\alpha_k^n - \bar{\alpha}_k^n}{D_k}, \tag{2.1}$$

where α_k and $\bar{\alpha}_k$ are the roots of the characteristic polynomial of the recurrence $\alpha^2 - k\alpha - 1 = 0$ and $D_k = \alpha_k - \bar{\alpha}_k$. Concretely,

$$\alpha_k = \frac{k + \sqrt{k^2+4}}{2}, \quad \bar{\alpha}_k = \frac{k - \sqrt{k^2+4}}{2}, \quad D_k = \alpha_k - \bar{\alpha}_k = \sqrt{k^2+4}.$$

The above formula is well known; for a proof the reader may consult Theorem 7.4 of [13]. It is a consequence of the fact that any k-Fibonacci number is defined by recurrence relation (1.3) and it is a solution of the corresponding second-order finite difference equation. Notice that $\alpha_k\bar{\alpha}_k = -1$. In particular, for $k = 1$, $\alpha_1 = \varphi$ and $D_1 = \sqrt{5}$, we have the classical Binet's formula for the Fibonacci numbers, where φ represents the Golden Ratio.

Lemma 2.1 (Generalization of Vajda's Identity for k-Fibonacci numbers). *For any positive numbers i, j, k,*

$$F_k(n+i)F_k(n+j) - F_k(n)F_k(n+i+j) = (-1)^n F_k(i)F_k(j).$$

Proof. Multiplying the left hand side by D_k^2 and using Binet's formula (2.1) and the fact that $\alpha_k\bar{\alpha}_k = -1$ yields

$$\begin{aligned}
&D_k^2\left(F_k(n+i)F_k(n+j) - F_k(n)F_k(n+i+j)\right) = (\alpha_k^{n+i} - \bar{\alpha}_k^{n+i})(\alpha_k^{n+j} - \bar{\alpha}_j^{n+j}) \\
&\quad - (\alpha_k^n - \bar{\alpha}_k^n)(\alpha_k^{n+i+j} - \bar{\alpha}_k^{n+i+j}) = \alpha_k^{2n+i+j} - (-1)^n\alpha_k^i\bar{\alpha}_k^j - (-1)^n\bar{\alpha}_k^i\alpha_k^j + \bar{\alpha}_k^{2n+i+j} - \\
&\quad \alpha_k^{2n+i+j} + (-1)^n\alpha_k^{i+j} + (-1)^n\bar{\alpha}_k^{i+j} - \bar{\alpha}_k^{2n+i+j} \\
&\quad = (-1)^n(\alpha_k^i - \bar{\alpha}_k^i)(\alpha_k^j - \bar{\alpha}_k^j) = D_k^2\left((-1)^n F_k(i)F_k(j)\right).
\end{aligned}$$

□

Corollary 2.2. *The following identities hold for any integers $a, b, n \geq 1$:*

$$F_k(a+b) = F_k(a+1)F_k(b) + F_k(a)F_k(b-1) \tag{2.2}$$

$$F_k(a) \leq \frac{1}{k}F_k(a+1) \tag{2.3}$$

$$F_k(a)F_k(b) \leq F_k(a+b-1) \tag{2.4}$$

$$F_k(a+b-1) \leq F_k(a)F_k(b)\left(1+\frac{1}{k^2}\right) \tag{2.5}$$

$$(\textit{D'Ocagne's identity})\ (-1)^a F_k(b-a) = F_k(b)F_k(a+1) - F_k(b+1)F_k(a) \tag{2.6}$$

$$(\textit{Catalan's identity})\ F_k(n)^2 = F_k(n+r)F_k(n-r) + (-1)^{n-r}F_k(r)^2 \tag{2.7}$$

$$(\textit{Simson's identity})\ F_k(n)^2 = F_k(n+1)F_k(n-1) - (-1)^n. \tag{2.8}$$

Moreover, the following equalities hold:

(1) *The equality in* (2.3) *is only attained if* $a = 1$.
(2) *The equality in* (2.4) *is only attained if* $a = 1$ *or* $b = 1$.
(3) *The equality in* (2.5) *is only attained if* $a = b = 2$.

Proof. For (2.2), take $n = 1$, $i = a$ and $j + 1 = b$ in the previous lemma.

For (2.3), we have

$$F_k(a+1) = kF_k(a) + F_k(a-1) \geq kF_k(a)$$

and equality is only attained if $F_k(a-1) = 0$, i.e., if $a = 1$.

For (2.4), substitute a by $a-1$ in identity (2.2). Then

$$F_k(a+b-1) = F_k(a)F_k(b) + F_k(a-1)F_k(b-1) \geq F_k(a)F_k(b).$$

Equality is only attained if $F_k(a-1) = 0$ or $F_k(b-1) = 0$, i.e., if $a = 1$ or $b = 1$.

For (2.5), substitute a by $a-1$ in identity (2.2). Then

$$F_k(a+b-1) = F_k(a)F_k(b) + F_k(a-1)F_k(b-1) \leq F_k(a)F_k(b)\left(1+\frac{1}{k^2}\right).$$

Equality is only attained if $F_k(a-1) = \frac{1}{k}F_k(a)$ and $F_k(b-1) = \frac{1}{k}F_k(b)$, which only happens if $a = b = 2$.

For the D'Ocagne identity (2.6), take $n = a$, $i = b - a$, $j = 1$ in the previous lemma.

For Catalan's identity (2.7), take $n = n - r$, $i = j = r$ in the previous lemma.

Finally, for the Simson identity (2.8), take $r = 1$ in the Catalan identity (2.7). □

Lemma 2.3. *For integers $k \geq 1$ and $N \geq 0$,*

$$\sum_{n=0}^{N} F_k(n)^2 = \frac{1}{k}F_k(N)F_k(N+1).$$

Proof. We will use induction to prove the result. For $N = 0$, the identity is true because $F_k(0) = 0$. Assuming that the result holds for some N, we will

prove it for $N+1$. We begin with the following equation

$$\begin{aligned}\frac{1}{k}F_k(N+1)F_k(N+2) &= \frac{1}{k}F_k(N+1)(kF_k(N+1)+F_k(N))\\ &= F_k(N+1)^2+\frac{1}{k}F_k(N)F_k(N+1)\,.\end{aligned}$$

And, by the induction hypothesis, we have

$$F_k(N+1)^2+\frac{1}{k}F_k(N)F_k(N+1) = F_k(N+1)^2+\sum_{n=0}^{N}F_k(n)^2 = \sum_{n=0}^{N+1}F_k(n)^2\,,$$

which completes the proof. □

Lemma 2.4. *If $k \geq 4$ and $n \geq 1$, then $4F_k(2n-2) \leq F_k(n)^2$.*

Proof. For $n=1$, the inequality becomes $0 = 4F_k(0) \leq F_k(1) = 1$, hence the result holds. Assume that $n \geq 2$. Taking $a = b = n-1$ in equation (2.2), and then multiplying by four, we obtain

$$4F_k(2n-2) = 4F_k(n-1)(F_k(n)+F_k(n-2)). \tag{2.9}$$

If $k \geq 5$, then $4F_k(n-1) \leq 4/5F_k(n)$ and $F_k(n-2) < 1/4F_k(n)$. Combining both inequalities, we get

$$4F_k(n-1)(F_k(n)+F_k(n-2)) < F_k(n)^2.$$

The above inequality and (2.9) prove the lemma for $k \geq 5$. In the case $k=4$, using again (2.9), we have

$$\begin{aligned}4F_4(2n-2) &= 4F_4(n-1)(F_4(n)+F_4(n-2))\\ &= (F_4(n)-F_4(n-2))\,(F_4(n)+F_4(n-2)) = F_4(n)^2-F_4(n-2)^2 \leq F_4(n)^2,\end{aligned}$$

which proves the result. □

Lemma 2.5. *Let $a, b, c \geq 1$. Then*

$$F_2(c) \geq 3F_2(a)F_2(b) \text{ if and only if } c \geq a+b+1 \text{ or } (a,b,c) = (2,2,4), \tag{2.10}$$

and

$$F_k(c) \geq 3F_k(a)F_k(b) \text{ if and only if } c \geq a+b, \text{ for all } k \geq 3. \tag{2.11}$$

Moreover, equality in (2.10) holds when $k=2$ and $(a,b,c) = (2,2,4)$, and equality in (2.11) holds when $k=3$ and $(a,b,c) = (1,1,2)$.

Proof. We first prove (2.10). By identity (2.2), we have that

$$\begin{aligned}F_2(a+b+1) &= F_2(a+1)F_2(b+1)+F_2(a)F_2(b) = (2F_2(a)+F_2(a-1))(2F_2(b)\\ &\quad + F_2(b-1))+F_2(a)F_2(b) \geq (2^2+1)F_2(a)F_2(b) > 3F_2(a)F_2(b)\,.\end{aligned} \tag{2.12}$$

On the other hand,

$$\frac{F_2(a+b)}{F_2(a)F_2(b)} = \frac{F_2(a+1)F_2(b)+F_2(a)F_2(b-1)}{F_2(a)F_2(b)} = \frac{F_2(a+1)}{F_2(a)}+\frac{F_2(b-1)}{F_2(b)}.$$

It is known that successive quotients of Pell numbers $F_2(n+1)/F_2(n)$ form an oscillating sequence converging to α_2, where the sequence of even terms is decreasing and the sequence of odd terms is increasing. As a consequence, the maximum of $F_2(a+1)/F_2(a)$ is $\frac{5}{2}$ and it is attained only at $a=2$, and the maximum of $F_2(b-1)/F_2(b)$ is $\frac{1}{2}$ and it is attained only at $b=2$. Thus,

$$\frac{F_2(a+b)}{F_2(a)F_2(b)} = \frac{F_2(a+1)}{F_2(a)} + \frac{F_2(b-1)}{F_2(b)} \leq \frac{5}{2} + \frac{1}{2} = 3 \tag{2.13}$$

and equality is only attained at $(a,b)=(2,2)$. Combining (2.12) and (2.13) and using the fact that the function $F_2(c)$ is strictly increasing in c, we see that (2.10) holds.

Finally, we prove (2.11). By using again (2.2), if $k \geq 3$

$$\begin{aligned} F_k(a+b) &= F_k(a+1)F_k(b) + F_k(a)F_k(b-1) \\ &= kF_k(a)F_k(b) + F_k(a-1)F_k(b) + F_k(a)F_k(b-1) \geq 3F_k(a)F_k(b), \end{aligned}$$

with equality if and only if $k=3$, $F_k(a-1)=0$ and $F_k(b-1)=0$, i.e., if $a=b=1$. Additionally, for all $k \geq 3$ it follows that

$$F_k(a+b-1) = F_k(a)F_k(b) + F_k(a-1)F_k(b-1) \leq 2F_k(a)F_k(b) < 3F_k(a)F_k(b).$$

By the two previous inequalities and since the function $F_k(c)$ is strictly increasing in c, it follows that (2.11) holds. □

3. Non-minimal Case

Recall that a Markoff m-triple (x,y,z) is a positive integer solution triple of the Markoff m-equation (1.1), where m is a positive integer. Henceforth, we assume that the triple is ordered, i.e. $x \leq y \leq z$. For positive integers a,b,c, we shall denote

$$m_k(a,b,c) = F_k(a)^2 + F_k(b)^2 + F_k(c)^2 - 3F_k(a)F_k(b)F_k(c),$$

so that $(F_k(a), F_k(b), F_k(c))$ is a Markoff m-triple with k-Fibonacci components if and only if $m_k(a,b,c) > 0$. In this section, after deriving conditions on (a,b,c) for which $m_k(a,b,c) \leq 0$, as a straightforward consequence, we prove Theorem 1.1, showing that there exists only one branch of non-minimal Markoff m-triples with k-Fibonacci components. Note that we consider $k \geq 2$, since the case $k=1$ was previously treated in [11].

Lemma 3.1. (1) *For $a \geq 3$, if $c \leq a+b$, then $m_k(a,b,c) \leq 0$, for $k=2$.*
(2) *For $a \geq 1$, if $c < a+b$, then $m_k(a,b,c) \leq 0$, for all $k \geq 3$.*

Proof. We start with (2). We have

$$2F_k(a+1) = 2(kF_k(a) + F_k(a-1)) \leq 2(k+1)F_k(a) \leq 3kF_k(a), \tag{3.1}$$

for $k \geq 2$. Next, from Eqs. (2.2) and (3.1) above, we obtain

$$F_k(a+b) \leq 2F_k(a+1)F_k(b) \leq 3kF_k(a)F_k(b). \tag{3.2}$$

Also, since $c \leq a+b-1$, from (3.2)

$$F_k(c+1)F_k(c) \leq F_k(a+b)F_k(c) \leq 3kF_k(a)F_k(b)F_k(c). \tag{3.3}$$

Now, by Lemma 2.3, assuming a, b, c distinct or $a = b < c - 1$, we have

$$F_k(a)^2 + F_k(b)^2 + F_k(c)^2 \leq \frac{F_k(c+1)F_k(c)}{k}. \tag{3.4}$$

Then, (3.3) and (3.4) yield

$$F_k(a)^2 + F_k(b)^2 + F_k(c)^2 \leq 3F_k(a)F_k(b)F_k(c),$$

which is equivalent to $m_k(a, b, c) \leq 0$.

Observe that in the case $a \leq b = c$, we trivially have $m_k(a, b, c) \leq 0$. Next, we prove the remaining case $a = b = c-1$. As $F_k(c) \leq (k+1)F_k(c-1)$, we have

$$\begin{aligned} 2F_k(c-1)^2 + F_k(c)^2 &\leq 2F_k(c-1)^2 + (k+1)^2 F_k(c-1)^2 \\ &= F_k(c-1)^2 \left(2 + (k+1)^2\right). \end{aligned} \tag{3.5}$$

Since $c \leq a + b - 1 = 2(c-1) - 1$, we may suppose that $c \geq 3$, which leads to

$$2 + (k+1)^2 < 3(k^2+1) \leq 3F_k(c).$$

As a result,

$$F_k(c-1)^2 \left(2 + (k+1)^2\right) < F_k(c-1)^2\, 3\, F_k(c). \tag{3.6}$$

Combining Eqs. (3.5) and (3.6), we obtain

$$2F_k(c-1)^2 + F_k(c)^2 < 3\, F_k(c-1)^2\, F_k(c),$$

which can also be expressed as $m_k(c-1, c-1, c) < 0$.

Finally, we prove (1). The only case to be checked is $c = a + b$ because the proof above is valid if $c \geq a + b + 1$. We aim to prove

$$F_2(a)^2 + F_2(b)^2 + F_2(a+b)^2 \leq 3F_2(a)F_2(b)F_2(a+b).$$

Adding $2F_2(a)F_2(b)$ on both sides,

$$\left(F_2(a) + F_2(b)\right)^2 + F_2(a+b)^2 \leq F_2(a)F_2(b)\left(3F_2(a+b) + 2\right).$$

Since $\left(F_2(a) + F_2(b)\right)^2 \leq 4F_2(b)^2$, it suffices to prove

$$4F_2(b)^2 + F_2(a+b)^2 \leq 3F_2(a)F_2(b)F_2(a+b).$$

Rearranging terms,

$$4F_2(b)^2 \leq F_2(a+b)\left(3F_2(a)F_2(b) - F_2(a+b)\right).$$

Developing $F_2(a+b)$ on the right-hand side, using (2.2),

$$4F_2(b)^2 \leq F_2(a+b)\left(3F_2(a)F_2(b) - F_2(a+1)F_2(b) - F_2(a)F_2(b-1)\right).$$

Using $3F_2(a) - F_2(a+1) = F_2(a-1) + F_2(a-2)$, we obtain

$$4F_2(b)^2 \leq F_2(a+b)\left(F_2(b)(F_2(a-1) + F_2(a-2)) - F_2(a)F_2(b-1)\right),$$

and thus, reordering terms on the right-hand side, we have

$$4F_2(b)^2 \leq F_2(a+b)\left(F_2(b)F_2(a-2) + F_2(b)F_2(a-1) - F_2(a)F_2(b-1)\right).$$

Now, applying D'Ocagne's identity (2.6) to $a-1$ and $b-1$,

$$4F_2(b)^2 \leq F_2(a+b)\left(F_2(b)F_2(a-2) + (-1)^a F_2(b-a)\right). \tag{3.7}$$

To prove the inequality above, we distinguish two cases: a being even and odd. If a is even, since $a \geq 4$, then $F_2(a-2) \geq 2$ and $F_2(a+b) \geq 4F_2(b)$. Consequently,

$$4F_2(b) \leq F_2(a+b)F_2(a-2)$$

and (3.7) holds. If a is odd, since $a \geq 3$, we have $12F_2(b) \leq F_2(a+b)$, and for proving (3.7) it is enough to prove

$$F_2(b) \leq 3F_2(b)F_2(a-2) - 3F_2(b-a).$$

In other words,

$$F_2(b) + 3F_2(b-a) \leq 3F_2(b)F_2(a-2)$$

and this holds because $3F_2(b-a) \leq 3F_2(b-3) \leq \frac{F_2(b)}{4}$ and $F_2(a-2) \geq 1$. □

Lemma 3.2. *The following hold.*

(1) $m_2(1, b, b+1) \leq 0$, *for any* b, *and equality holds only for* $b = 1, 2$.
(2) $m_2(2, b, b+1) < 0$, *for any* $b \geq 2$.

Proof. For (1), it suffices to prove

$$1 + F_2(b)^2 + F_2(b+1)^2 \leq 3F_2(b)F_2(b+1).$$

If $b = 1$, the above inequality holds as an equality. If $b > 1$, by applying Lemma 2.3 to the left-hand side, the above is equivalent to

$$\frac{1}{2}F_2(b+1)F_2(b+2) \leq 3F_2(b)F_2(b+1). \tag{3.8}$$

Equivalently,

$$F_2(b+1)(2F_2(b+1) + F_2(b)) \leq 6F_2(b)F_2(b+1).$$

Dividing by $F_2(b+1) \neq 0$, we obtain $2F_2(b+1) \leq 5F_2(b)$, but this inequality holds because $2F_2(b+1) = 4F_2(b) + 2F_2(b-1)$ and $F_2(b) \geq 2F_2(b-1)$. In this case, equality is only achieved when $b = 2$.

Next, (2) is equivalent to

$$4 + F_2(b)^2 + F_2(b+1)^2 < 6F_2(b)F_2(b+1).$$

If $b = 2$, we can verify the above inequality numerically $(4 + 4 + 25 < 60)$. For $b > 2$, by Lemma 2.3, and Eq. (3.8), we see that the above holds. □

Theorem 3.3 (Theorem 1.1 of the Introduction). *Every non-minimal Markoff* m*-triple with* k*-Fibonacci components is a Markoff* 8*-triple of the form* $(F_2(2), F_2(2n), F_2(2n+2))$, *for* $n \geq 2$.

Proof. First, we start with the case $k \geq 3$. If a Markoff m-triple with k-Fibonacci components $(F_k(a), F_k(b), F_k(c))$ is not minimal then $c < a+b$, by Lemma 2.5. However, by Lemma 3.1 (2), for $k \geq 3$ this restriction implies that $m_k(a, b, c) \leq 0$. Therefore, non-minimal Markoff m-triples with k-Fibonacci components do not exist for $k \geq 3$.

In the case $k = 2$, if a Markoff m-triple with 2-Fibonacci components $(F_2(a), F_2(b), F_2(c))$ is not minimal, then $c \leq a + b$, by Lemma 2.5. This restriction forces $F_2(a)$ to be equal to 1 or 2, because of Lemma 3.1 (1). If

$F_2(a) = 1$, then $a = 1$ and $c \leq b + 1$. In the case $b = c$ it is obvious than $m_2(1, b, b) \leq 0$ and in the case $c = b + 1$, it follows that $m_2(1, b, b + 1) \leq 0$ by Lemma 3.2 (1). Finally, if $F_2(a) = 2 = F_2(2)$, then $a = 2$, and $c \leq 2 + b$. Hence by Lemma 3.2 (2), the triple is of the form $(2, b, b+2)$. Now, we prove that b is an even number. Indeed,

$$\begin{aligned} m_2(2, b, b+2) &= 4 + F_2(b)^2 + F_2(b+2)^2 - 6F_2(b)F_2(b+2) \\ &= 4 + (F_2(b+2) - F_2(b))^2 - 4F_2(b)F_2(b+2) \\ &= 4 + 4F_2(b+1)^2 - 4F_2(b)F_2(b+2) = 4(1 - (-1)^{b+1}) \end{aligned} \tag{3.9}$$

is positive if and only if b is even, where the last equality is a consequence of the Simson identity (2.8). As a result, all the triples of the form $(F_2(2), F_2(2n), F_2(2n+2))$, for $n \geq 1$ are 8-triples and it is straightforward to check that they all lie in a branch of the Markoff 8-tree with minimal triple $(2, 2, 12)$ (See Fig. 1). For $m = 8$, this tree is unique because there are no more minimal triples than $(2, 2, 12)$ as shown in Table 1 of [5]. □

4. Minimal Case

We recall that if (x, y, z) is a minimal Markoff m-triple, i.e. a solution of the Markoff m-equation (1.1), with $z \geq 3xy$, then

$$m = z(z - 3xy) + x^2 + y^2 > 0.$$

Let a, b be any pair of positive integers with $a \leq b$ and let $c = a + b + t$. By Lemma 2.5, if $t \geq 1$ for $k = 2$, or $t \geq 0$ for $k \geq 3$, then $(F_k(a), F_k(b), F_k(c))$ is minimal, therefore $m_k(a, b, c) > 0$. Consequently, there exists an infinite number of minimal Markoff triples with k-Fibonacci components. Clearly, they cannot all correspond to a finite number of values of m, as the number of minimal triples is finite for each m [5]. Hence there are infinitely many values of m that admit minimal Markoff m-triples with k-Fibonacci components. In the rest of the section, we will prove that any $m > 0$ admits at most one minimal Markoff m-triple with k-Fibonacci components, except when $k = 3$, $c = a + b$, a is odd, b is even and $b \geq a + 3$, where $m_3(a, b, a + b)$ admits two such triples.

Lemma 4.1. *Let $1 \leq a \leq b$. Suppose that $k = 2$ and $c = a + b + 1$, or $k \geq 3$ and $c = a + b$. Then*

$$m_k(a, b, c) > L_k \frac{\alpha_k^{2c}}{D_k^2},$$

where $D_k = \alpha_k - \bar{\alpha_k} = \sqrt{k^2 + 4}$ and

$$\begin{aligned} L_2 &= \left(1 - \frac{3}{D_2}\alpha_2^{-1}\right) + 2\left(1 - \frac{3}{D_2}\alpha_2\right)\alpha_2^{-4} - \left(6 + \frac{3}{D_2}\alpha_2 + \frac{9}{D_2}\right)\alpha_2^{-6}, \\ L_3 &= \left(1 - \frac{3}{D_3}\right)(1 + 2\alpha_3^{-2}) - \left(6 + \frac{12}{D_k}\right)\alpha_2^{-4}, \\ L_k &= 1 - \frac{3}{D_k}, \qquad \forall k \geq 4. \end{aligned}$$

Proof. Using Binet's formula (2.1) and taking into account that $\alpha_k\bar{\alpha}_k = -1$, it follows that for any $k \geq 1$

$$F_k(n)^2 = \frac{1}{D_k^2}\left(\alpha_k^{2n} + \alpha_k^{-2n} - 2\cdot(-1)^n\right) > \frac{1}{D_k^2}\left(\alpha_k^{2n} - 2\right).$$

If $k = 2$ and $b = c - 1 - a$, we have

$$\begin{aligned} m_2(a,b,c) &= F_2(c)^2 + F_2(c-1-a)^2 + F_2(a)^2 - 3F_2(c)F_2(c-1-a)F_2(a) \\ &> \frac{1}{D_2^2}\left(\alpha_2^{2c} + \alpha_2^{2c-2-2a} + \alpha_2^{2a} - 6\right) - \frac{3}{D_2^3}(\alpha_2^c - \bar{\alpha}_2^c)(\alpha_2^{c-1-a} - \bar{\alpha}_2^{c-1-a}) \\ &\quad (\alpha_2^a - \bar{\alpha}_2^a). \end{aligned}$$

As $c = a + b + 1 > 1$ and $\alpha_2\bar{\alpha}_2 = -1$, we conclude that

$$\begin{aligned} &(\alpha_2^c - \bar{\alpha}_2^c)(\alpha_2^{c-1-a} - \bar{\alpha}_2^{c-1-a})(\alpha_2^a - \bar{\alpha}_2^a) \leq (\alpha_2^c + \alpha_2^{-c})(\alpha_2^{c-1-a} + \alpha_2^{a-c+1})(\alpha_2^a + \alpha_2^{-a}) \\ &\quad = \alpha_2^{2c-1} + \alpha_2^{2c-1-2a} + \alpha_2^{2a+1} + \alpha_2 + \alpha_2^{-1} + \alpha_2^{-2a-1} + \alpha_2^{2a-2c+1} + \alpha_2^{-2c+1} \\ &\quad < \alpha_2^{2c-1} + \alpha_2^{2c-1-2a} + \alpha_2^{2a+1} + \alpha_2 + 3. \end{aligned}$$

Hence

$$\begin{aligned} &m_2(a,b,c) \\ &\quad > \frac{1}{D_2^2}\left(\alpha_2^{2c} + \alpha_2^{2c-2-2a} + \alpha_2^{2a} - 6\right) - \frac{3}{D_2^3}(\alpha_2^{2c-1} + \alpha_2^{2c-1-2a} + \alpha_2^{2a+1} + \alpha_2 + 3) \\ &\quad = \frac{1}{D_2^2}\alpha_2^{2c}\left[\left(1 - \frac{3}{D_2}\alpha_2^{-1}\right) + \left(1 - \frac{3}{D_2}\alpha_2\right)\left(\alpha_2^{-2-2a} + \alpha_2^{2a-2c}\right)\right. \\ &\qquad \left. - \left(6 + \frac{3}{D_2}\alpha_2 + \frac{9}{D_2}\right)\alpha_2^{-2c}\right]. \end{aligned}$$

As $f(x) = \alpha_2^x$ is a convex function, $c > 1$ and $a \geq 1$, by applying Karamata's inequality [15], we obtain

$$\alpha_2^{-2-2a} + \alpha_2^{2a-2c} \leq \alpha_2^{-2-2} + \alpha_2^{2-2c} = \alpha_2^{-4} + \alpha_2^{2-2c}. \tag{4.1}$$

Since

$$1 - \frac{3}{D_2}\alpha_2 = 1 - \frac{6 + 3\sqrt{8}}{2\sqrt{8}} < 1 - \frac{3}{2} < 0$$

and $c \geq a + b + 1 \geq 3$, we have

$$\begin{aligned} m_2(a,b,c) &> \frac{1}{D_2^2}\alpha_2^{2c}\left[\left(1 - \frac{3}{D_2}\alpha_2^{-1}\right) + \left(1 - \frac{3}{D_2}\alpha_2\right)\left(\alpha_2^{-2-2a} + \alpha_2^{2a-2c}\right)\right. \\ &\qquad \left. - \left(6 + \frac{3}{D_2}\alpha_2 + \frac{9}{D_2}\right)\alpha_2^{-2c}\right] \\ &\geq \frac{1}{D_2^2}\alpha_2^{2c}\left[\left(1 - \frac{3}{D_2}\alpha_2^{-1}\right) + \left(1 - \frac{3}{D_2}\alpha_2\right)\left(\alpha_2^{-4} + \alpha_2^{2-2c}\right)\right. \\ &\qquad \left. - \left(6 + \frac{3}{D_2}\alpha_2 + \frac{9}{D_2}\right)\alpha_2^{-2c}\right] \geq L_2\frac{1}{D_2^2}\alpha_2^{2c}, \end{aligned}$$

as the coefficient of α_2^{-2c} is clearly negative in the previous expression, and therefore its minimum for $c \geq 3$ is attained at $c = 3$.

Analogously, if we assume that $k \geq 3$ and $c = a + b$, we have

$$\begin{aligned}(\alpha_k^c - \bar{\alpha}_k^c)(\alpha_k^{c-a} - \bar{\alpha}_k^{c-a})(\alpha_k^a - \bar{\alpha}_k^a) &\leq (\alpha_k^c + \alpha_k^{-c})(\alpha_k^{c-a} + \alpha_k^{a-c})(\alpha_k^a + \alpha_k^{-a}) \\ &= \alpha_k^{2c} + \alpha_k^{2c-2a} + \alpha_k^{2a} + 2 + \alpha_k^{-2a} + \alpha_k^{2a-2c} + \alpha_k^{-2c} < \alpha_k^{2c} + \alpha_k^{2c-2a} + \alpha_k^{2a} \\ &\quad + 4.\end{aligned}$$

Hence

$$\begin{aligned}m_k(a,b,c) &> \frac{1}{D_k^2}\left(\alpha_k^{2c} + \alpha_k^{2c-2a} + \alpha_k^{2a} - 6\right) - \frac{3}{D_k^3}(\alpha_k^{2c} + \alpha_k^{2c-2a} + \alpha_k^{2a} + 4) \\ &= \frac{1}{D_k^2}\alpha_k^{2c}\left[\left(1 - \frac{3}{D_k}\right)\left(1 + \alpha_k^{-2a} + \alpha_k^{2a-2c}\right) - \left(6 + \frac{12}{D_k}\right)\alpha_k^{-2c}\right].\end{aligned}$$

Now, the factor $1 - \frac{3}{D_k} = 1 - \frac{3}{\sqrt{k^2+4}}$ becomes positive for $k \geq 3$, so this time, we need to apply the opposite Karamata bound [15] (which becomes simply Jensen's inequality in this case)

$$\alpha_k^{-2a} + \alpha_k^{2a-2c} \geq 2\alpha_k^{\frac{-2a+2a-2c}{2}} = 2\alpha_k^{-c},$$

yielding

$$m_k(a,b,c) > \frac{1}{D_k^2}\alpha_k^{2c}\left[\left(1 - \frac{3}{D_k}\right)\left(1 + 2\alpha_k^{-c}\right) - \left(6 + \frac{12}{D_k}\right)\alpha_k^{-2c}\right].$$

Let us consider the polynomial

$$p_k(x) = 2\left(1 - \frac{3}{D_k}\right)x - \left(6 + \frac{12}{D_k}\right)x^2.$$

Then, our bound can be written as

$$m_k(a,b,c) > \frac{1}{D_k^2}\alpha_k^{2c}\left[1 - \frac{3}{D_k} + p_k(\alpha_k^{-c})\right].$$

We know that $c = a + b \geq 2$, so $\alpha_k^{-c} \in (0, \alpha_k^{-2}]$, as $\alpha_k > 1$, and therefore, $\lim_{c\to\infty} \alpha_k^{-c} = 0$. The polynomial $p_k(x)$ is a parabola with a negative leading coefficient, so its minimum in the interval $[0, \alpha_k^{-2}]$ is attained at one of the ends of the interval. A direct computation shows that $p_3(\alpha_3^{-2}) < 0 = p_3(0)$, and hence

$$m_3(a,b,c) > \frac{1}{D_3^2}\alpha_3^{2c}\left[1 - \frac{3}{D_3} + p_3(\alpha_3^{-2})\right] = L_3\frac{1}{D_3^2}\alpha_3^{2c}.$$

On the other hand, for $k \geq 4$, we can prove that $p_k(\alpha_k^{-2}) > 0 = p_k(0)$ as follows. The expression

$$\alpha_k^4 p_k(\alpha_k^{-2}) = 2\alpha_k^2\left(1 - \frac{3}{D_k}\right) - \left(6 + \frac{12}{D_k}\right)$$

is clearly increasing in k, because α_k and D_k are both increasing functions of k. A direct computation shows that for $k = 4$ we have $\alpha_4^4 p_4(\alpha_4^{-2}) > 0$, so

$p_k(\alpha_k^{-2})$ must be positive for all $k \geq 4$. As a consequence,

$$\begin{aligned} m_k(a,b,c) &> \frac{1}{D_k^2}\alpha_k^{2c}\left[1 - \frac{3}{D_k} + p_k(\alpha_k^{-c})\right] \\ &> \frac{1}{D_k^2}\alpha_k^{2c}\left[1 - \frac{3}{D_k} + p_k(0)\right] = \frac{1}{D_k^2}\alpha_k^{2c}\left(1 - \frac{3}{D_k}\right) = L_k\frac{1}{D_k^2}\alpha_k^{2c}. \end{aligned}$$

□

We have the following lower bound for the constant L_k in the lemma above.

Lemma 4.2. *For each $k \geq 2$, the constant L_k satisfies*

$$L_k > \alpha_k^{-2}.$$

Proof. For $k = 2, 3$, a direct computation shows that $\alpha_2^2 L_2 > 1$ and $\alpha_3^2 L_3 > 1$, so $L_k > \alpha_k^{-2}$ for $k = 2, 3$. For $k \geq 4$ we wish to prove that

$$L_k = 1 - \frac{3}{D_k} > \alpha_k^{-2}.$$

Rearranging the equation, this is equivalent to proving that for all $k \geq 4$

$$1 > \frac{3}{D_k} + \alpha_k^{-2} = \frac{3}{\sqrt{k^2+4}} + \frac{4}{(k+\sqrt{k^2+4})^2}.$$

The right-hand side of this expression is decreasing in k and for $k = 4$ a direct computation shows that

$$\frac{3}{D_4} + \alpha_4^{-2} < 1,$$

and hence the inequality holds for all $k \geq 4$. □

Lemma 4.3. *Let $1 \leq a \leq b \leq c$ and $c \geq 3$. Suppose that $a \leq a' \leq c$ and $b \leq b' \leq c$. Then*

$$m_k(a,b,c) \geq m_k(a',b',c)$$

and equality holds if and only if $a = a'$ and $b = b'$. In particular, for an ordered minimal Markoff-Fibonacci m-triple $(F_k(a), F_k(b), F_k(c))$,

$$m_k(1,1,c) \geq m_k(a,b,c) \geq m_k(a, c-a-s, c),$$

where $s = 1$, for $k = 2$ and $s = 0$, for $k \geq 3$.

Proof. The lemma and its proof are entirely analogous to Lemma 4.1 in [11], which addresses the case $k = 1$. In this lemma, the starting point is $a = 2$ because $F_1(2) = F_1(1) = 1$. In our situation, with $k \geq 2$, the case $a = 1$ is also valid since $F_k(2) > F_k(1) = 1$. □

Lemma 4.4. *If $(F_k(a), F_k(b), F_k(c))$ and $(F_k(a'), F_k(b'), F_k(c'))$ are two ordered minimal Markoff-Fibonacci m-triples with $c \geq c'$, then $c = c'$.*

Proof. Assume that $m_k(a,b,c) = m = m_k(a',b',c')$. By applying Lemmas 4.3 and 4.1, it follows that

$$m = m_2(a,b,c) \geq m_2(a, c-a-1, c) > L_2\frac{1}{D_2^2}\alpha_2^{2c}$$

if $k = 2$ and

$$m = m_k(a, b, c) \geq m_k(a, c - a, c) > L_k \frac{1}{D_k^2} \alpha_k^{2c},$$

for any other $k \geq 3$. From Lemma 4.2 we know that $L_k > \alpha_k^{-2}$ for all $k \geq 2$, so

$$m_k(a, b, c) > L_k \frac{1}{D_k^2} \alpha_k^{2c} > \frac{1}{D_k^2} \alpha_k^{2c-2}. \tag{4.2}$$

On the other hand, from Lemma 4.3, we deduce that

$$\begin{aligned} m = m_k(a', b', c') &\leq m_k(1, 1, c') = F_k(c')^2 - 3F_k(c') + 2 \\ &< \frac{1}{D_k^2} \alpha_k^{2c'} + \frac{1}{D_k^2} \bar{\alpha}_k^{2c'} + \frac{2}{D_k^2} (-1)^{c'} - 1 < \frac{1}{D_k^2} \alpha_k^{2c'}. \end{aligned} \tag{4.3}$$

Using Eqs. (4.2) and (4.3) together, we obtain $\alpha_k^{2(c-1)} < D_k^2 m < \alpha_k^{2c'}$. Thus, $c' > c - 1$. As we assumed $c' \leq c$, we conclude that $c' = c$. □

Lemma 4.5. *Let $(F_k(a), F_k(b), F_k(c))$ and $(F_k(a'), F_k(b'), F_k(c))$ be two distinct ordered minimal Markoff-Fibonacci m-triples with the same third element. If $a \leq a'$, then $a < a' \leq b' < b$.*

Proof. Suppose first that $a = a'$. It follows by Lemma 4.3 that the equality $m_k(a, b, c) = m_k(a', b', c') = m_k(a, b', c)$ is only possible if $b = b'$, in which case $(a, b, c) = (a', b', c')$, contradicting the assumption that the two m-triples are distinct. Thus $a < a'$. If $b \leq b'$, then Lemma 4.3 implies $m(a, b, c) < m(a', b', c)$, which is not possible as both are m-triples for the same m. Therefore, it follows that $a < a' \leq b' < b$. □

Lemma 4.6. *Let $(F_k(a), F_k(b), F_k(c))$ and $(F_k(a'), F_k(b'), F_k(c))$ be two ordered minimal Markoff-Fibonacci m-triples. Then $a + b = a' + b'$.*

Proof. By Lemma 4.5 we can assume without loss of generality that $1 \leq a < a' \leq b' < b \leq c$. In particular, $b \geq 3$. Rearranging the equation $m_k(a, b, c) = m_k(a', b', c)$, yields

$$\begin{aligned} &F_k(a)^2 + F_k(b)^2 - F_k(a')^2 - F_k(b')^2 \\ &\quad = 3F_k(c) \left(F_k(a)F_k(b) - F_k(a')F_k(b') \right). \end{aligned} \tag{4.4}$$

Since $b \geq 3$ and $a' \leq b' < b$ we have

$$F_k(b)^2 \geq k^2 F_k(b-1)^2 > 2F_k(b-1)^2 \geq F_k(b')^2 + F_k(a')^2,$$

so the left-hand side of Eq. (4.4) is always positive and, thus, so is the right-hand side. Let us see that this is impossible if $a' + b' > a + b$. Indeed,

$$\begin{aligned} \frac{F_k(a')F_k(b')}{F_k(a)F_k(b)} &= \frac{(\alpha_k^{a'} - \bar{\alpha}_k^{a'})(\alpha_k^{b'} - \bar{\alpha}_k^{b'})}{(\alpha_k^{a} - \bar{\alpha}_k^{a})(\alpha_k^{b} - \bar{\alpha}_k^{b})} \geq \frac{(\alpha_k^{a'} - \alpha_k^{-a'})(\alpha_k^{b'} - \alpha_k^{-b'})}{(\alpha_k^{a} + \alpha_k^{-a})(\alpha_k^{b} + \alpha_k^{-b})} \\ &= \frac{\alpha_k^{a'+b'} - \alpha_k^{b'-a'} - \alpha_k^{a'-b'} + \alpha_k^{-a'-b'}}{\alpha_k^{a+b} + \alpha_k^{b-a} + \alpha_k^{a-b} + \alpha_k^{-a-b}}. \end{aligned}$$

Assume that $a' + b' = a + b + r$ with $r > 0$ and let $s = a + b$. Then $a' + b' = s + r$. Dividing the numerator and denominator by α_k^s yields

$$\frac{\alpha_k^{a'+b'} - \alpha_k^{b'-a'} - \alpha_k^{a'-b'} + \alpha_k^{-a'-b'}}{\alpha_k^{a+b} + \alpha_k^{b-a} + \alpha_k^{a-b} + \alpha_k^{-a-b}} = \frac{\alpha_k^r - \alpha_k^{r-2a'} - \alpha_k^{r-2b'} + \alpha_k^{-2s-r}}{1 + \alpha_k^{-2a} + \alpha_k^{-2b} + \alpha_k^{-2s}}$$
$$= \alpha_k^r \frac{1 - \alpha_k^{-2a'} - \alpha_k^{-2b'} + \alpha_k^{-2s-2r}}{1 + \alpha_k^{-2a} + \alpha_k^{-2b} + \alpha_k^{-2s}}.$$

As $1 \le a < a' \le b' < b$, we have $a \ge 1$, $a' \ge 2$, $b' \ge 2$, $b \ge 3$ and $s = a+b \ge 4$. Thus

$$\alpha_k^r \frac{1 - \alpha_k^{-2a'} - \alpha_k^{-2b'} + \alpha_k^{-2s-2r}}{1 + \alpha_k^{-2a} + \alpha_k^{-2b} + \alpha_k^{-2s}} \ge \alpha_k \frac{1 - 2\alpha_k^{-4}}{1 + \alpha_k^{-2} + \alpha_k^{-6} + \alpha_k^{-8}} \ge 1.92 > 1.$$

Therefore, $F_k(a')F_k(b') > F_k(a)F_k(b)$, which contradicts the positivity of both sides of Eq. (4.4).

Therefore, we must have $a+b \ge a'+b'$. Suppose that $a'+b' = a+b-r$ with $r > 0$ and let $s = a + b$ as before. Following the same logic as in the previous case,

$$\frac{F_k(a')F_k(b')}{F_k(a)F_k(b)} = \frac{(\alpha_k^{a'} - \bar{\alpha}_k^{a'})(\alpha_k^{b'} - \bar{\alpha}_k^{b'})}{(\alpha_k^{a} - \bar{\alpha}_k^{a})(\alpha_k^{b} - \bar{\alpha}_k^{b})} \le \frac{(\alpha_k^{a'} + \alpha_k^{-a'})(\alpha_k^{b'} + \alpha_k^{-b'})}{(\alpha_k^{a} - \alpha_k^{-a})(\alpha_k^{b} - \alpha_k^{-b})}$$
$$= \frac{\alpha_k^{a'+b'} + \alpha_k^{b'-a'} + \alpha_k^{a'-b'} + \alpha_k^{-a'-b'}}{\alpha_k^{a+b} - \alpha_k^{b-a} - \alpha_k^{a-b} + \alpha_k^{-a-b}} = \alpha_k^{-r} \frac{1 + \alpha_k^{-2a'} + \alpha_k^{-2b'} + \alpha_k^{-2s-2r}}{1 - \alpha_k^{-2a} - \alpha_k^{-2b} + \alpha_k^{-2s}}$$
$$\le \alpha_k^{-1} \frac{1 + 2\alpha_k^{-4} + \alpha_k^{-10}}{1 - \alpha_k^{-2} - \alpha_k^{-6}} < 0.53 < \frac{8}{9}.$$

As a result,

$$1 - \frac{F_k(a')F_k(b')}{F_k(a)F_k(b)} > 1 - \frac{8}{9} = \frac{1}{9} \ge \frac{1}{9F_k(a)^2}.$$

Multiplying both sides by $3F_k(a)F_k(b)F_k(c)$, results in

$$3F_k(c)\left(F_k(a)F_k(b) - F_k(a')F_k(b')\right) > \frac{F_k(c)F_k(b)}{3F_k(a)}.$$

Since $(F_k(a), F_k(b), F_k(c))$ is minimal, we have $F_k(c) \ge 3F_k(a)F_k(b)$. Consequently,

$$3F_k(c)\left(F_k(a)F_k(b) - F_k(a')F_k(b')\right) > \frac{F_k(c)F_k(b)}{3F_k(a)} \ge F_k(b)^2$$
$$> F_k(b)^2 - F_k(b')^2 + F_k(a)^2 - F_k(a')^2.$$

This contradicts Eq. (4.4), and thus $a' + b' \ge a + b$ and therefore $a + b = a' + b'$. □

Lemma 4.7. *If a is odd, b is even and $b \ge a + 3$ then*

$$m_3(a, b, a + b) = m_3(a + 1, b - 1, a + b).$$

Proof. Using Simson's identity (2.8) for a odd,

$$F_3(a)^2 - F_3(a+1)^2 = F_3(a)^2 - F_3(a)F_3(a+2) + (-1)^{a+1} =$$
$$= F_3(a)(F_3(a) - F_3(a+2)) + (-1)^{a+1} = -3F_3(a)F_3(a+1) + 1.$$

Using a similar argument for b even, we have

$$F_3(b)^2 - F_3(b-1)^2 = 3F_3(b)F_3(b-1) - 1.$$

Adding both expressions yields

$$F_3(a)^2 + F_3(b)^2 - F_3(a+1)^2 - F_3(b-1)^2 = 3(F_3(b)F_3(b-1) - F_3(a)F_3(a+1)) \quad (4.5)$$

We obtain the following identities by applying Vajda's identity (see Lemma 2.1) and considering that a is odd and b is even:

$$\begin{aligned} &F_3(a)F_3(a+1) \\ &\quad = F_3(b)F_3(b-1) - F_3(a+b)F_3(b-a-1) = (-1)^{b-a-1}F_3(a)F_3(a+1), \\ &\quad F_3(b-a-1) = F_3(a+1)F_3(b-1) - F_3(a)F_3(b) = (-1)^a F_3(1)F_3(b-a-1). \end{aligned}$$

Thus,

$$\begin{aligned} F_3(b)F_3(b-1) - F_3(a)F_3(a+1) &= F_3(a+b)F_3(b-1-a) \\ &= F_3(a+b)(F_3(a+1)F_3(b-1) - F_3(a)F_3(b)). \end{aligned}$$

Substituting back in (4.5) yields

$$\begin{aligned} &F_3(a)^2 + F_3(b)^2 - F_3(a+1)^2 - F_3(b-1)^2 = \\ &\quad 3F_3(a+b)(F_3(a+1)F_3(b-1) - F_3(a)F_3(b)). \end{aligned}$$

Rearranging this equation yields the required result. □

Theorem 4.8 (Theorem 1.2 of the Introduction). *If m admits a minimal Markoff m-triple with k-Fibonacci components then it is unique except for $k = 3$ and all pairs of triples $(F_3(a), F_3(b), F_3(a+b)), (F_3(a+1), F_3(b-1), F_3(a+b))$, for a odd, b even and $b \geq a+3$.*

Proof. Let $(F_k(a), F_k(b), F_k(c))$ and $(F_k(a'), F_k(b'), F_k(c'))$ be a pair of ordered minimal m-triples contradicting the theorem. By Lemma 4.4, it follows that $c = c'$. Moreover, by Lemma 4.5 we can assume without loss of generality that $1 \leq a < a' \leq b' < b \leq c$ and by Lemma 4.6 we must have $a + b = a' + b'$. Taking $n = a$, $i = b' - a$ and $j = b - b' = a' - a$ in Vajda's identity (Lemma 2.1), we transform Eq. (4.4) into

$$\begin{aligned} F_k(a)^2 + F_k(b)^2 - F_k(a')^2 - F_k(b')^2 &= 3F_k(c)\left(F_k(a)F_k(b) - F_k(a')F_k(b')\right) \\ &= (-1)^{a+1}3F_k(c)F_k(b'-a)F_k(b-b')\,. \end{aligned} \quad (4.6)$$

From the proof of Lemma 4.6, the left-hand side of this equality is positive, therefore a is odd, and hence

$$F_k(a)^2 + F_k(b)^2 - F_k(a')^2 - F_k(b')^2 = 3F_k(c)F_k(b'-a)F_k(b-b'). \quad (4.7)$$

In the case $k = 2$, using (2.10) from Lemma 2.5 twice, we obtain that

$$F_2(b) \leq 3F_2(b')F_2(b-b') \leq 9F_2(a)F_2(b'-a)F_2(b-b')\,.$$

Multiplying by $F_2(b)$ and by minimality, $3F_2(a)F_2(b) \leq F_2(c)$, it follows that

$$F_2(b)^2 \leq 9F_2(a)F_2(b)F_2(b'-a)F_2(b-b') \leq 3F_2(c)F_2(b'-a)F_2(b-b')$$

and as a consequence

$$F_2(b)^2 - F_2(b')^2 + F_2(a)^2 - F_2(a')^2 < F_2(b)^2 \leq 3F_2(c)F_2(b'-a)F_2(b-b'),$$

which contradicts Eq. (4.7).

In the case $k \geq 4$, suppose that $c = a + b$. We want to prove

$$F_k(b)^2 - F_k(b')^2 + F_k(a)^2 - F_k(a')^2 > 3F_k(c)F_k(b'-a)F_k(b-b'), \quad (4.8)$$

contradicting (4.7). First, since $F_k(b) \geq kF_k(b-1) \geq 4F_k(b')$ by Eq. (2.3), we have

$$F_k(a')^2 + F_k(b')^2 \leq 2F_k(b')^2 \leq \frac{1}{8}F_k(b)^2 < \frac{F_k(b)^2}{4}. \quad (4.9)$$

Now, using Eq. (2.4) twice, it follows that

$$3F_k(a+b)F_k(b-b')F_k(b'-a) \leq 3F_k(a+b)F_k(b-a-1) \leq 3F_k(2b-2).$$

The inequality above and (4.9) give

$$F_k(a')^2 + F_k(b')^2 + 3F_k(a+b)F_k(b-b')F_k(b'-a) < \frac{F_k(b)^2}{4} + 3F_k(2b-2)$$

and by Lemma 2.4

$$\frac{F_k(b)^2}{4} + 3F_k(2b-2) \leq \frac{F_k(b)^2}{4} + \frac{3}{4}F_k(b)^2 = F_k(b)^2.$$

Due to the two inequalities above, (4.12) holds.

In the case $k = 3$, suppose that $c = a + b$ and $b' \leq b - 2$. We want to prove

$$F_3(b)^2 > F_3(a')^2 + F_3(b')^2 + 3F_3(a+b)F_3(b'-a)F_3(b-b'), \quad (4.10)$$

which contradicts Eq. (4.7). Repeating the above argument,

$$3F_3(a+b)F_3(b'-a)F_3(b-b') \leq 3F_3(2b-2) \leq \frac{3}{4}F_3(b)^2.$$

On the other hand, if $a' \leq b' \leq b-2$, since $F_3(b) \geq 9F_3(b-2)$, we have

$$F_3(a')^2 + F_3(b')^2 \leq 2F_3(b')^2 \leq 2F_3(b-2)^2 \leq \frac{2}{9}F_3(b)^2 < \frac{1}{4}F_3(b)^2.$$

Adding the above two inequalities, (4.10) holds.

In the case $k \geq 3$, we first consider $c \geq a + b + 1$. We will show that

$$F_k(b)^2 - F_k(b')^2 + F_k(a)^2 - F_k(a')^2 < 3F_k(c)F_k(b'-a)F_k(b-b'), \quad (4.11)$$

which contradicts Eq. (4.7). Then, since $F_k(b') > F_k(a)$ it is enough to show that

$$F_k(b)^2 < 3F_k(a+b+1)F_k(b'-a)F_k(b-b'). \quad (4.12)$$

By using Eq. (2.5) twice, we obtain

$$\begin{aligned} 3F_k(a+b+1)F_k(b'-a)F_k(b-b') &\geq 3F_k(a+b+1)\frac{1}{(1+\frac{1}{9})}F_k(b-a-1) \\ &\geq \frac{3}{\left(1+\frac{1}{9}\right)^2}F_k(2b-1) > F_k(2b-1). \end{aligned}$$

On the other hand, applying formula (2.2) to $b-1$ and b, it follows that

$$F_k(2b-1) = F_k(b)^2 + F_k(b-1)^2 > F_k(b)^2.$$

The above two inequalities show that (4.12) holds.

Finally, we study the last case; $k = 3$, $c = a + b$, $b' = b - 1$ and a odd (see Eq. (4.6)). This is precisely addressed in Lemma 4.7, which identifies the minimal pairs of Markoff m-triples with k-Fibonacci components satisfying $m = m_3(a, b, a + b) = m_3(a + 1, b - 1, a + b)$, where b is even. Note that the condition $b \geq a + 3$ in that lemma implies that the triple $(F_3(a+1), F_3(b-1), F_3(a+b))$ is ordered, so $(F_3(a+1), F_3(b-1), F_3(a+b))$ and $(F_3(a), F_3(b), F_3(a + b))$ are distinct. This, however, does not hold if $b = a + 1$. If b were odd, we would have in the last equality of Lemma 4.6

$$\begin{aligned} &F_3(a)^2 + F_3(b)^2 - F_3(a+1)^2 - F_3(b-1)^2 \\ &\quad = 3F_3(a+b)(F_3(a+1)F_3(b-1) - F_3(a)F_3(b)) + 6. \end{aligned}$$

Therefore, if b were odd, $m_3(a, b, a + b) > m_3(a + 1, b - 1, a + b)$. □

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