BACHELOR'S DEGREE IN MATHEMATICAL ENGINEERING AND ARTIFICIAL INTELLIGENCE

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A COMPUTATIONAL ANALYSIS OF ISOMORPHISM CLASSES OF MODULI SPACES OF PARABOLIC VECTOR BUNDLES

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Dedicated to my parents

I hereby declare, under my own responsibility, that the project entitled **A Computational Analysis of Isomorphism Classes of Moduli Spaces of Parabolic Vector Bundles**, submitted at the ICAI School of Engineering – Universidad Pontificia Comillas during the academic year 2024/25, is my own original and unpublished work, and has not been previously submitted for any other purpose.

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A Computational Analysis of Isomorphism Classes of Moduli Spaces of Parabolic Vector Bundles

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Summary

ABSTRACT. This thesis combines computational and mathematical techniques to investigate the structure and symmetries of moduli spaces of stable parabolic vector bundles over smooth complex projective curves with marked points. The main objective is to classify the isomorphism types of these moduli spaces and compute their automorphism groups, with a focus on generic weights and sufficiently large genus.

These moduli spaces vary with respect to a set of parameters called parabolic weights, whose parameter space is divided by a finite collection of walls into stability chambers. Within each chamber, the moduli space remains unchanged, while crossing a wall typically results in a different space. However, a set of basic transformations—pullback, Hecke, tensorization, and dualization—induce isomorphisms between different chambers, grouping moduli spaces into distinct isomorphism classes.

A central contribution of this work is the development of a computational framework centered on two key algorithms. The first is a decision tree that partitions the weight space into stability chambers by recursively splitting polytopes. This procedure relies on enumerating admissible selection vectors that define candidate walls, and on determining which of these walls actually affect the decomposition. The second algorithm builds on the resulting chamber structure to compute isomorphism classes and automorphism groups by applying all basic transformations to chamber representatives, classifying them efficiently using the decision tree.

The computational analysis reveals deep structural patterns and symmetries, motivating a series of conjectures that are subsequently established through rigorous mathematical proofs. We derive a closed formula for the number of geometric walls and present both tight bounds and asymptotic estimates for their growth. For the number of stability chambers, we provide upper and lower bounds based on hyperplane arrangement theory as a function of the number of walls. A key structural result demonstrates that for rank greater than two, dualization does not appear in any automorphism of a generic moduli space.

Keywords: Parabolic bundles, moduli spaces, stability chambers, geometric walls, isomorphism classes, automorphism groups, algebraic geometry, computational algebra, decision trees, polytope decomposition, basic transformations, duality.

Chapter 1: Introduction

This chapter introduces the moduli space $M(r, \alpha, \xi)$ of stable parabolic vector bundles over a smooth projective curve X with marked points D. A parabolic bundle consists of a vector bundle with filtrations and real weights α at the marked points. Varying α changes the stability condition, inducing a wall-and-chamber decomposition of the weight space.

These moduli spaces arise in diverse areas of mathematics and physics, including the theory of differential equations and quantum field theory. Their classification is highly nontrivial and often exceeds the reach of traditional mathematical techniques. To address this challenge, the thesis develops algorithmic methods to analyze stability chambers, isomorphism classes, and automorphism groups. These computational tools are complemented by rigorous mathematical proofs, establishing the correctness and depth of the results.

This work is part of the CIAMOD project at the IIT, where I cotributed as a Student Research Assistant. An initial phase culminated in a poster presented at the 2024 RTGF meeting at ICMAT (CSIC). Now, two papers are in preparation: one on theoretical results and the other on computational approaches.

Chapter 2: Moduli Spaces of Parabolic Bundles and Stability

This chapter introduces the moduli spaces $\mathcal{M}(r, \alpha, \xi)$ of full flag semistable parabolic vector bundles over a smooth projective curve X of genus $g \geq 2$, with marked points $D = \{x_1, \ldots, x_n\}$. Each bundle carries a full flag at each parabolic point and strictly increasing weights $\alpha \in \mathcal{A}_{n,r}$.

Semistability is defined via the parabolic slope and encoded combinatorially through selection vectors, which govern the induced filtrations on subbundles. The weight space $\mathcal{A}_{n,r}$ is divided by a finite set of stability walls $W_{\bar{n},d'}$ into *numerical stability chambers*, where the moduli space remains constant. When $g \geq 1+(r-1)n$, all such walls affect at least one semistable parabolic bundle [AG21], thus separating different moduli spaces.

We invoke an structural simplification from [AG21], which states that shifting all weights at a point by a constant preserves stability, motivating the study of weights in the normalized subspace $\tilde{\mathcal{A}}_{n,r}$ with $\alpha_1(x) = 0$.

Finally, we review the four basic transformations—pullback, dualization, tensor product, and Hecke transformations— forming a group $\mathcal{T}$, which act on the weights α and give rise to isomorphisms between these moduli spaces with fixed (X, D). These transformations preserve semistability and reduce the classification problem to the case of trivial determinant and degree $d = 0$.

Chapter 3: Algorithmic Exploration of Stability Chambers

This chapter develops a computational framework to study the chamber structure and symmetries of moduli spaces $\mathcal{M}(r, \alpha, \xi)$ of parabolic vector bundles. The central object is the space of parabolic weights $\tilde{\mathcal{A}}_{n,r}$, a product of simplices partitioned by hyperplanes $W_{\bar{n},d'}$, which are defined via admissible selection vectors $\bar{n} \in \Omega_{n,r,r'}$.

To address the combinatorial complexity, several algorithms are introduced. First, a Monte Carlo sampling heuristically estimates chamber counts and visualizes configurations for small n and r. Then, a key result identifies geometric walls using intercept bounds, allowing for an enumeration of all geometric walls. A deterministic algorithm replaces the Monte Carlo heuristic with an exact enumeration of chambers via recursive polytope splitting, yielding a decision tree that classifies chambers using exact rational arithmetic.

The final section presents a graph based algorithm that applies basic transformations—pullbacks, Hecke modifications, and dualisation—to detect isomorphisms and compute automorphism groups. Experimental results show that isomorphism classes are far fewer than the total number of chambers, and for $r > 2$, dualisation never appears in any chamber's automorphism group. Several observed patterns motivate conjectures later proved in Chapters 4 and 5.

Chapter 4: Bounds on Geometric Walls and Stability Chambers

This chapter analyzes the number of walls and stability chambers in the space of stability conditions for rank-r parabolic vector bundles over (X, D) with n marked points and degree zero. Each wall is defined by a hyperplane $W_{\bar{n},d'}$ determined by subbundles of subrank r' and selection vectors $\bar{n} \in \Omega_{n,r,r'}$. A key insight (Lemma 4.1.3) shows that walls come in proportional pairs, reducing the count to subranks $r' \leq \lfloor r/2 \rfloor$.

A hyperplane contributes to the decomposition only if its intercept lies in a specific interval $(l_{\bar{n}}, u_{\bar{n}})$ (Lemma 4.1.2). Summing over these intervals leads to a formula (Lemma 4.2.4) for the total number of walls.

For the number of stability chambers, classical bounds from hyperplane arrangement theory apply, giving a linear lower bound and a combinatorial upper bound based on Schläfli's theorem [Sch01]. Though not tight, these bounds offer useful asymptotic estimates. A comparison with actual data reveals that upper bounds significantly overestimate the true number, motivating further refinements.

Overall, the chapter provides both exact formulas and asymptotic behavior, establishing a foundation for understanding the combinatorics behind wall-crossing and chamber structure.

Chapter 5: Duality

For a smooth curve of genus $g \geq \max\{6, 1 + (r-1)n\}$ with generic weights α, the chapter shows that any basic transformation $T = (\sigma, s, L, H) \in \mathcal{T}_\xi$ preserving the moduli space $\mathcal{M}(X, r, \alpha, \xi)$ must satisfy $s = 1$ once $r > 2$. Lemma 5.1.1 first forces the numerical condition $r \mid |H|$ when $\deg \xi = 0$. Lemma 5.1.2 then guarantees, for any automorphism, the existence of a generic weight α' fixed by T inside the same stability chamber. Combining these facts, Lemma 5.1.3 shows that for $r > 2$ the only way to preserve such an α' is to keep $s = 1$, ruling out duality. Thus, for higher rank moduli spaces, dualization is never part of the automorphism group.

References

[AG21] D. Alfaya and T. Gómez. "Automorphism group of the moduli space of parabolic bundles over a curve". In: *Adv. Math.* 393 (2021), p. 108070.

[Sch01] Ludwig Schlafli. "Neue allgemeinen schweizerischen Gesellschaft für die gesamten Naturwissenschaften". In: 38.IV (1901).

Resumen

Resumen. Esta tesis combina técnicas computacionales y matemáticas para investigar la estructura y las simetrías de los espacios de moduli de fibrados vectoriales parabólicos estables sobre curvas proyectivas complejas suaves con puntos marcados. El objetivo principal es clasificar los tipos de isomorfismo de estos espacios de moduli y calcular sus grupos de automorfismos, centrándose en pesos genéricos y en el caso de género suficientemente alto.

Estos espacios de moduli varían con respecto a un conjunto de parametros llamados pesos parabólicos, cuyo espacio de parámetros se divide mediante una colección finita de paredes en cámaras de estabilidad. Dentro de cada cámara, el espacio de moduli permanece invariante, mientras que al cruzar una pared, generalmente se obtiene un espacio distinto. Sin embargo, un conjunto de transformaciones básicas—pullback, Hecke, tensorización y dualización—generan isomorfismos entre diferentes cámaras, agrupando los espacios de moduli en distintas clases de isomorfismo.

Una contribución central de este trabajo es el desarrollo de un marco computacional centrado en dos algoritmos clave. El primero es un árbol de decisión que particiona el espacio de pesos en cámaras de estabilidad mediante la división recursiva de politopos. Este procedimiento se basa en la enumeración de vectores de selección admisibles que definen paredes candidatas y en determinar cuáles de estas afectan realmente a la descomposición. El segundo algoritmo utiliza la estructura de cámaras resultante para calcular clases de isomorfismo y grupos de automorfismos, aplicando todas las transformaciones básicas a representantes de cada cámara, clasificándolos de forma eficiente mediante el árbol de decisión.

El análisis computacional revela patrones estructurales y simetrías profundas, llevando a una serie de conjeturas que posteriormente se demuestran mediante pruebas matemáticas rigurosas. Derivamos una fórmula cerrada para el número de paredes geométricas y presentamos tanto cotas ajustadas como estimaciones asintóticas para su crecimiento. Para el número de cámaras de estabilidad, proporcionamos cotas superior e inferior basadas en la teoría de arreglos de hiperplanos en función del número de paredes. Un resultado estructural clave demuestra que para rangos mayores que dos, la dualización no aparece en ningún automorfismo de un espacio de moduli genérico.

Palabras clave: Fibrados parabólicos, espacios de moduli, cámaras de estabilidad, paredes geométricas, clases de isomorfismo, grupos de automorfismos, geometría algebraica, álgebra computacional, árboles de decisión, descomposición de politopos, transformaciones básicas, dualidad.

Capítulo 1: Introducción

Este capítulo introduce el espacio de moduli $M(r, \alpha, \xi)$ de fibrados vectoriales parabólicos estables sobre una curva proyectiva suave X con puntos marcados D. Un fibrado parabólico consiste en un fibrado vectorial con filtraciones completas y pesos reales α en los puntos marcados. Al variar α cambia la condición de estabilidad, induciendo una descomposición del espacio de pesos en paredes y cámaras.

Estos espacios de moduli aparecen en diversas áreas como matemática y física, incluyendo la teoría de ecuaciones diferenciales y la teoría cuántica de campos. Su clasificación es altamente no trivial y a menudo supera el alcance de técnicas matemáticas tradicionales. Para abordar este desafío, la tesis desarrolla métodos algorítmicos para analizar las cámaras de estabilidad, las clases de isomorfismo y los grupos de automorfismos. Estas herramientas computacionales se complementan con demostraciones matemáticas rigurosas que establecen la corrección y profundidad de los resultados.

Este trabajo forma parte del proyecto CIAMOD en el IIT, donde participé como Alumno Colaborador. La fase inicial culminó en un póster presentado en el encuentro RTGF 2024 en el ICMAT (CSIC). Actualmente, están en preparación dos artículos: uno sobre los resultados teóricos y otro sobre los enfoques computacionales.

Capítulo 2: Espacios de Moduli de Fibrados Parabólicos y Estabilidad

Este capítulo introduce los espacios de moduli $\mathcal{M}(r, \alpha, \xi)$ de fibrados vectoriales parabólicos semiestables con filtraciones completas sobre una curva proyectiva suave X de género $g \geq 2$, con puntos marcados $D = x_1, \ldots, x_n$. Cada fibrado lleva una filtración completa en cada punto parabólico y pesos estrictamente crecientes $\alpha \in \mathcal{A}_{n,r}$.

La semiestabilidad se define mediante la pendiente parabólica y se codifica combinatoriamente a través de vectores de selección, que gobiernan las filtraciones inducidas en los subfibrados. El espacio de pesos $\mathcal{A}_{n,r}$ se divide mediante un conjunto finito de paredes de estabilidad $W_{\bar{n},d'}$ en *cámaras de estabilidad numérica*, donde el espacio de moduli permanece constante. Cuando $g \geq 1 + (r-1)n$, todas estas paredes afectan a al menos un fibrado parabólico semiestable [AG21], separado por tanto espacios de moduli distintos.

Hacemos uso de una simplificación estructural de [AG21], que establece que desplazar todos los pesos en un punto por una constante preserva la estabilidad, lo cual motiva el estudio de los pesos en el subespacio normalizado $\tilde{\mathcal{A}}_{n,r}$ con $\alpha_1(x) = 0$.

Finalmente, repasamos las cuatro transformaciones basicas—pullback, dualización, producto tensorial y transformaciones de Hecke—que forman un grupo $\mathcal{T}$ que actuan sobre los pesos α dando lugar a los isomorfismos entre estos espacios de moduli con (X, D). Estas transformaciones preservan la semiestabilidad y reducen el problema de clasificación a determinante trivial y grado $d = 0$.

Capítulo 3: Exploración Algorítmica de Cámaras de Estabilidad

Este capítulo desarrolla un marco computacional para estudiar la estructura de cámaras y simetrías de los espacios de moduli $\mathcal{M}(r, \alpha, \xi)$ de fibrados vectoriales parabólicos. El objeto central es el espacio de pesos parabólicos $\tilde{\mathcal{A}}n, r$, un producto de símplices particionado por hiperplanos $W\bar{n}, d'$, definidos mediante vectores de selección admisibles $\bar{n} \in \Omega_{n,r,r'}$.

Para abordar la complejidad combinatoria, se introducen múltiples algoritmos. Primero, un muestreo Monte Carlo estima heurísticamente el número de cámaras y visualiza configuraciones para n y r pequeños. Luego, un resultado clave identifica paredes geométricas mediante cotas de término independiente, permitiendo enumerarlas completamente. Un algoritmo determinista reemplaza la heurística de Monte Carlo con una enumeración exacta de cámaras mediante división recursiva de politopos, generando un árbol de decisión que clasifica cámaras usando aritmética racional exacta.

La sección final presenta un algoritmo basado en grafos que aplica transformaciones básicas—pullback, Hecke y dualización—para detectar isomorfismos y calcular grupos de automorfismos. Los resultados experimentales muestran que las clases de isomorfismo son mucho menos numerosas que el total de cámaras, y que para $r > 2$, la dualización nunca aparece en el grupo de automorfismos de ninguna cámara. Varios patrones observados motivan conjeturas que se demuestran más adelante en los Capítulos 4 y 5.

Capítulo 4: Cotas sobre Paredes Geométricas y Cámaras de Estabilidad

Este capítulo analiza el número de paredes y cámaras de estabilidad en el espacio de condiciones de estabilidad para fibrados parabólicos de rango r sobre (X, D) con n puntos marcados y grado cero. Cada pared está definida por un hiperplano $W_{\bar{n},d'}$ determinado por subfibrados de subrango r' y vectores de selección $\bar{n} \in \Omega_{n,r,r'}$. Una observación clave (Lema 4.1.3) muestra que las paredes vienen en pares proporcionales, reduciendo el conteo a subrangos $r' \leq \lfloor r/2 \rfloor$.

Un hiperplano contribuye a la descomposición solo si su término independiente cae en un intervalo específico $(l_{\bar{n}}, u_{\bar{n}})$ (Lema 4.1.2). Al sumar sobre estos intervalos se obtiene una fórmula (Lema 4.2.4) para el número total de paredes.

Para el número de cámaras de estabilidad, se aplican resultados clásicos de teoría de arreglos de hiperplanos para obtener una cota inferior lineal y una cota superior combinatoria basada en el teorema de Schläfli [Sch01]. Aunque no son cotas ajustadas, ofrecen estimaciones asintóticas útiles. Comparando con datos reales observamos que las cotas superiores sobrestiman significativamente el número real, motivando refinamientos adicionales.

En conjunto, el capítulo proporciona tanto fórmulas exactas como comportamiento asintótico, estableciendo una base para comprender la combinatoria detrás del cruce de paredes y la estructura de cámaras.

Capítulo 5: Dualidad

Para una curva suave de género $g \geq \max 6, , 1 + (r-1)n$ con pesos genéricos α, el capítulo demuestra que cualquier transformación básica $T = (\sigma, s, L, H) \in \mathcal{T}_\xi$ que preserve el espacio de moduli $\mathcal{M}(X, r, \alpha, \xi)$ debe satisfacer $s = 1$ cuando $r > 2$. El Lema 5.1.1 impone primero la condición numérica $r \mid |H|$ cuando $\deg \xi = 0$. Luego, el Lema 5.1.2 garantiza, para todo automorfismo, la existencia de un peso genérico α' fijo por T dentro de la misma cámara de estabilidad. Combinando estos hechos, el Lema 5.1.3 muestra que para $r > 2$ la única forma de preservar tal α' es mantener $s = 1$, descartando la dualidad. Por tanto, para espacios de moduli de rango mayor que dos, la dualización nunca forma parte del grupo de automorfismos.

Referencias

[AG21] D. Alfaya and T. Gómez. “Automorphism group of the moduli space of parabolic bundles over a curve”. In: *Adv. Math.* 393 (2021), p. 108070.

[Sch01] Ludwig Schlafli. “Neue allgemeinen schweizerischen Gesellschaft für die gesamten Naturwissenschaften”. In: 38.IV (1901).

Contents

Chapter 1

Introduction

1.1 Context and intuition

Let us begin by unpacking the terminology of the main objects of study in this paper. Roughly speaking, a *moduli space* is a geometric space whose points classify, up to isomorphism, objects of a fixed type; moving inside the space corresponds to smoothly deforming the object one is interested in. A *vector bundle* $(E, E_\bullet)$ of rank r and degree d on a smooth projective curve X is a family of r-dimensional vector spaces parameterised by the points of X. The degree d, also denoted $\deg(E)$, is related to how the bundle twists around the curve, and the rank r indicates the number of dimensions in each fibre of the bundle (See [Gro55] and [Ses82]) for more information on vector bundles).

Endowing a vector bundle with a *parabolic structure* involves choosing a finite set of marked points $D = \{x_1, \dots, x_n\} \subset X$. A parabolic structure on a vector bundle E of rank r consists of assigning, at each point $x_i \in D$, a *full flag filtration* of the fibre $E|_{x_i}$:

$$E|_{x_i} = E_{x_i,1} \supsetneq E_{x_i,2} \supsetneq \cdots \supsetneq E_{x_i,r} \supsetneq E_{x_i,r+1} = 0,$$

along with a corresponding *system of parabolic weights*

$$0 \leq \alpha_1(x_i) < \alpha_2(x_i) < \cdots < \alpha_r(x_i) < 1.$$

These parabolic weights are real numbers that describe how the bundle is allowed to degenerate at the marked points.

We say a vector bundle $(E, E_\bullet)$ is α-stable if it satisfies a certain stability condition with respect to the parabolic weights α (see §2.1). The choice of parabolic weights α determines which parabolic vector bundles are stable. As a result, varying α can change the set of objects classified by the moduli space, and thus alter the moduli space itself. In this work, we treat α as one of the key structural parameters governing the geometry and classification of the moduli spaces under study.

Finally, a *moduli space of stable parabolic vector bundles* is a geometric space that classifies stable parabolic vector bundles of fixed rank r, degree d, determinant ξ and parabolic weights α on a fixed curve X with marked points D and genus g. We will denote this moduli space by $M(r, \alpha, \xi)$ (see [New78] for more details on moduli of parabolic vector bundles).

In this paper, we will study moduli spaces with fixed determinant ξ and a sufficiently high genus $g \geq 1 + (r-1)n$.

Assume that a curve X of high enough genus has been fixed. Throught our analysis, we will consider the following parameters of the moduli space parabolic vector bundles:

- Number of parabolic points n;
- Rank r of the vector bundles;
- Set of automorphisms (symmetries) of the curve X that preserve the marked points D;
- Parabolic weights $0 \leq \alpha_1(x) < \alpha_2(x) < \ldots < \alpha_r(x) < 1$ for each $x \in D$;
- Degree d of the vector bundles.

Studying moduli spaces of parabolic bundles is crucial for understanding the geometry of vector bundles with fields with singularities, and they have become a central object of study in algebraic geometry, representation theory, and mathematical physics. For instance, they arise naturally in the study of isomonodromic deformations of linear differential equations with singularities, an area that connects deeply with Painlevé equations and integrable systems [IIS06] (moduli space of parabolic connections). In physics, moduli spaces of parabolic bundles appear in gauge theory and the geometric Langlands program, where they provide key insights into dualities between quantum field theories [GW22] (moduli space of Higgs bundles). Because of their deep connections across disciplines, the study of moduli spaces of parabolic bundles is an interesting and active area of research.

When studying moduli spaces of parabolic vector bundles, we are often interested in the following questions:

1. Under what conditions do different choices of parameters (e.g., weights) yield exactly the *same* moduli space object?
2. When can two moduli spaces be considered *isomorphic*?
3. How many *distinct* (non-isomorphic) moduli spaces are there when considering a fixed set of parameters?
4. Can we classify the *automorphism* groups of these moduli spaces?

1.2 Motivation

The motivation behind this thesis arises from the mathematical challenges involved in the study of moduli spaces of parabolic bundles over algebraic curves. These moduli spaces encode deep geometric and topological information, and their classification remains a central problem in modern algebraic geometry. Traditional tools, although powerful, quickly reach their limits due to the combinatorial and geometric complexity that escalates even in seemingly simple cases. The spaces involved exhibit intricate structures with walls and chambers—so-called stability chambers—that change according to specific parameters and require detailed analysis to fully understand.

To address this challenge, we adopt a novel perspective that combines ideas from computational geometry and algorithmic techniques, such as Decision Trees, to explore the structure of these moduli spaces. This approach enables us to systematically enumerate and classify the stability chambers, understand the number and nature of isomorphism classes, and determine the automorphism groups associated with each distinct moduli space. The integration of computational tools not only enhances our capacity to manage large parameter spaces but also opens new directions in the investigation of moduli theory.

1.3 Publications

This work is part of a broader research initiative at the IIT – Institute for Research in Technology, within the framework of the CIAMOD project. Over the past two years, I have been involved in this initiative as a Student Research Assistant, contributing to both theoretical and computational developments.

Notably, the first stage of the project culminated in the presentation of a scientific poster at the 2024 Red Temática de Geometría y Física (RTGF) meeting, held at the Institute of Mathematical Sciences (ICMAT) of the Spanish National Research Council (CSIC). This opportunity allowed for valuable interaction with leading researchers in the field and helped refine the objectives of the project.

The next stage of the project is the preparation of two work-in-progress papers, one focusing on the theoretical aspects of the moduli spaces of parabolic vector bundles and the other on the computational methods employed in their analysis, in particular, to the Decision Tree algorithm for partitioning polytopes. These papers will be submitted to peer-reviewed journals, contributing to the academic discourse on this topic.

1.4 Overview of the thesis

This thesis explores the structure and classification of moduli spaces of parabolic vector bundles through the lens of parabolic stability and its combinatorial manifestations, with a particular emphasis on algorithmic enumeration.

We begin in Chapter 1 by laying out the context and motivation for the study of parabolic vector bundles, articulating both the mathematical richness and the algorithmic challenges they pose. The reader is introduced to the conceptual foundation of parabolic stability, including how epsilon shifts affect the parameter space and transformations that affect stability conditions. This introductory framework sets the stage for transitioning smoothly from theoretical constructs to algorithmic procedures.

In Chapter 2, we delve into the definition and construction of moduli spaces of parabolic vector bundles. We introduce the notion of stability chambers—regions of the weight space that preserve stability—and explore their geometric structure. The connection between walls and chambers is established rigorously, providing a necessary backdrop for further computational exploration.

Chapter 3 marks a methodological shift toward algorithmic experimentation. Here, we present a suite of algorithms designed to enumerate stability chambers and identify isomorphisms between moduli spaces. Techniques such as Monte Carlo sampling, enumeration of selection vectors, and symmetry reduction are introduced and carefully analyzed. The emphasis is on both correctness and computational efficiency, showcasing the interplay between mathematical theory and algorithm design.

Building on the computational insights, Chapter 4 establishes upper bounds on the number of geometric walls and stability chambers. These bounds are not merely abstract estimates—they emerge from concrete algorithms and are supported by empirical data. This chapter thus functions as a bridge between theory and experimentation, affirming the robustness of the proposed methods.

Finally, Chapter 5 investigates restrictions on the possible combinations of basic transformations yielding automorphisms of the moduli space. In particular, we show restrictions on the presence of Hecke transfomrations and the dual as part of automorphisms.

1.5 Code Availability

The full source code developed for this thesis, including algorithms for chamber decomposition, moduli space classification, and automorphism group enumeration, is openly available at:

`https://github.com/CIAMOD/stability_chambers.git`

This repository contains scripts, data, and visualization tools used in the computational analysis and in generating the figures and tables throughout this document.

Chapter 2

Moduli Spaces of Parabolic Vector Bundles and Parabolic Stability Chambers

Throughout this work, let X be a smooth complex projective curve of genus $g \geq 2$, and let $D = \{x_1, \ldots, x_n\}$ be a collection of n distinct points on X, referred to as the *parabolic points.*

A *full flag quasi-parabolic vector bundle* of rank r over the pair (X, D) is a pair $(E, E_\bullet)$, where E is a vector bundle of rank r over X, equipped with additional data at each parabolic point $x \in D$: a *full flag* on the fiber $E|_x$, that is, a decreasing sequence of subspaces

$$E|_x = E_{x,1} \supsetneq E_{x,2} \supsetneq \cdots \supsetneq E_{x,r} \supsetneq E_{x,r+1} = 0,$$

where each $E_{x,i}$ is a subspace of dimension $r - i + 1$ inside $E|_x$.

A *parabolic vector bundle* is a quasi-parabolic vector bundle $(E, E_\bullet)$ together with a system of strictly increasing *parabolic weights*

$$0 \leq \alpha_1(x) < \alpha_2(x) < \cdots < \alpha_r(x) < 1$$

associated to each flag at every parabolic point $x \in D$. We will denote the weights as $\alpha = \{\alpha_i(x)\}_{x \in D,\, i=1,\ldots,r}$.

If $F \subset E$ is a subbundle, it inherits a parabolic structure by taking the intersection of $F|_x$ with the flag in $E|_x$ at each parabolic point $x \in D$, that is,

$$F_{x,i} := F|_x \cap E_{x,i}.$$

This filtration defines the *induced parabolic structure* on F.

Definition 2.0.1. *Let $\mathcal{M}(X, D, r, \alpha, \xi)$ denote the moduli space of full flag semistable parabolic vector bundles $(E, E_\bullet)$ of rank r on (X, D) with parabolic system of weights α and such that* $\det(E) \cong \xi$*. When the marked curve (X, D) is clear from the context, we will drop it from the notation and simply write $\mathcal{M}(r, \alpha, \xi) := \mathcal{M}(X, D, r, \alpha, \xi)$.*

Let

$$\mathcal{A}_{n,r} = \{\alpha = (\alpha_i(x)) \,|\, 0 \leq \alpha_1(x) < \ldots < \alpha_r(x) < 1\}$$

denote the set of possible systems of parabolic weights. It is a product of simplexes inside the hypercube $[0, 1]^{nr}$. Given a fixed rank r and determinant ξ, for each $\alpha \in \mathcal{A}_{n,r}$ we have a potentially different moduli space $\mathcal{M}(r, \alpha, \xi)$. The main goal of this work is to understand precisely how many different isomorphism classes of such moduli spaces there are, when α moves over $\mathcal{A}_{n,r}$.

2.1 Parabolic stability

Boden and Yokogawa [BY99] proved that the stability space $\mathcal{A}_{n,r}$ is divided in stability chambers by a set of hyperplanes called stability walls, such that the moduli space $\mathcal{M}(r,\alpha,\xi)$ does not change as long as α remains inside one of these stability chambers. In this section we will review the definition of parabolic stability and the stability walls that partition the space of parabolic weights $\mathcal{A}_{n,r}$ into stability chambers. We will also define the selection vectors that are used to describe the stability walls.

We define the parabolic degree of a parabolic vector bundle as

$$\operatorname{pardeg}_\alpha(E,E_\bullet) := d + \sum_{x\in D}\sum_{i=1}^r \alpha_i(x)$$

and its parabolic slope as the quotient

$$\operatorname{par}-\mu_\alpha(E,E_\bullet) := \frac{\operatorname{pardeg}_\alpha(E,E_\bullet)}{\operatorname{rk}(E)} = \frac{d+\sum_{x\in D}\sum_{i=1}^r \alpha_i(x)}{r}. \qquad (2.1.1)$$

We say that a parabolic vector bundle $(E,E_\bullet)$ is α-stable (respectively α-semistable) if for any proper subbundle $0\neq F\subsetneq E$ we have

$$\operatorname{par}-\mu_\alpha(F,F_\bullet) < \operatorname{par}-\mu_\alpha(E,E_\bullet) \quad (\text{respectively, } \leq)$$

where F is given the induced parabolic structure from $(E,E_\bullet)$. If equality is attained, we say that $(E,E_\bullet)$ is strictly semistable.

Selection vectors encode the choice of the induced filtation of a rank r' subbundle $F\subset E$ by selecting r' components of the flag at each parabolic point.

Definition 2.1.1. *A* selection vector $\bar{n}=(n_i(x))_{i=1,\dots,r;\, x\in D}$ *is a tuple of integers with* $n_i(x)\in\{0,1\}$ *for each* i *and* x*, such that for some integer* r' *with* $1\leq r'<r$*, we have*

$$\sum_{i=1}^r n_i(x) = r' \quad \textit{for all } x\in D.$$

We will denote

$$\Omega_{n,r,r'} = \{\overline{n}=(n_i(x))_{i,x} \mid \sum_{i=1}^r n_i(x)=r' \quad \forall x\in D\}$$

to the space of all possible admissible selection vectors of subrank r' *and rank* r

From [AG21, §2], forn any $F\subset E$ with rank r', $\exists \bar{n}\in\Omega n,r,r'$ such that

$$\operatorname{pardeg}_\alpha(F) = \deg(F) + \sum_{x\in D}\sum_{i=1}^r n_i(x)\alpha_i(x).$$

Then, a parabolic vector bundle $(E,E_\bullet)$ is semistable for the parabolic weights α if and only if

$$\frac{d'+\sum_{x\in D}\sum_{i=1}^r n_i(x)\alpha_i(x)}{r'} \leq \frac{d+\sum_{x\in D}\sum_{i=1}^r \alpha_i(x)}{r} \qquad (2.1.2)$$

for all possible choices of a type $\bar{n}$, a rank r' and a degree d' from a subbundle $F\subset E$.

Given a selection vector $\bar{n} \in \Omega_{n,r,r'}$ and an integer $d' \in \mathbb{Z}$, let $W_{\bar{n},d'}$ be the hyperplane

$$W_{\bar{n},d'} \;:\; r' \sum_{i=1}^{r} \sum_{x\in D} \alpha_i(x) - r \sum_{i=1}^{r} \sum_{x \in D} n_i(x)\alpha_i(x) = rd' - r'd \,. \tag{2.1.3}$$

With a slight abuse of notation, let

$$W_{\bar{n}}(\alpha) = r' \sum_{i=1}^{r} \sum_{x\in D} \alpha_i(x) - r \sum_{i=1}^{r} \sum_{x \in D} n_i(x)\alpha_i(x) \tag{2.1.4}$$

so that

$$W_{\bar{n},d'} \;:\; W_{\bar{n}}(\alpha) = rd' - r'd.$$

Remark 2.1.2. *We call $W_{\bar{n},d'}$ a* numerical stability wall *for the moduli space. If α and β inside $\mathcal{A}_{n,r}$ are not separated by any wall $W_{\bar{n},d'}$, then the moduli spaces $\mathcal{M}(r,\alpha,\xi)$ and $\mathcal{M}(r,\beta,\xi)$ are exactly the same.*

Definition 2.1.3. *A system of weights α is called* generic *in degree d if $\alpha \notin W_{\bar{n},d'}$ for any admissible $\bar{n}$ and any $d' \in \mathbb{Z}$.*

By [BY99], if α is generic, then the moduli space $\mathcal{M}(r,\alpha,\xi)$ is smooth. In this work we will restrict ourselves to the analysis of generic weights. As a consequence of the previous discussion, we observe that the stability space $\mathcal{A}_{n,r}$ is partitioned by the numerical walls $W_{\bar{n},d'}$ into a finite set of polytopes called the *numerical stability chambers* of the moduli space. We say that a stability wall $W_{\bar{n},d'}$ is a *geometrical stability wall* if there exists a parabolic vector bundle $(E,E_\bullet)$ and a subbundle F with type $\bar{n}$ and degree d' such that $\operatorname{par}-\mu_\alpha(F,F_\bullet) = \operatorname{par}-\mu_\alpha(E,E_\bullet)$ for some $\alpha \in W_{\bar{n},d'}$. By [AG21, Theorem 10.6], if $g \geq 1+(r-1)n$, then all stability walls $W_{\bar{n},d'}$ are geometric walls and, therefore, separate different geometrical stability chambers with potentially different isomorphism classes of moduli spaces of parabolic vector bundles.

For the entirety of this work, we will assume that the genus g is high enough so that all stability walls are geometric. This allows us to focus on the geometric properties of the moduli spaces and their classification.

2.2 Epsilon shifts

In this section we will introduce an structural simplification that will allow us to work in a simpler space of parabolic weights, where visualizations in small dimensions are easier to handle, and algorithmic techniques are slightly more efficient.

To begin with, note that there exists a natural translation action on $\mathcal{A}_{n,r}$ which preserves the stability of parabolic vector bundles. Given a vector $\varepsilon = (\varepsilon(x))_{x\in D} \in \mathbb{R}^n$, we define the shift of α by ε, denoted $\alpha[\varepsilon]$, as

$$\alpha[\varepsilon]_i(x) = \alpha_i(x) + \varepsilon(x).$$

Then, for any parabolic vector bundle $(E,E_\bullet)$, we have

$$\operatorname{pardeg}_{\alpha[\varepsilon]}(E,E_\bullet) = \operatorname{pardeg}_\alpha(E,E_\bullet) + r\sum_{x\in D}\varepsilon(x).$$

Hence,

$$\text{par}-\mu_{\alpha[\varepsilon]}(E,E_\bullet)=\text{par}-\mu_\alpha(E,E_\bullet)+\sum_{x\in D}\varepsilon(x).$$

It follows that for any subbundle $F\subset E$, we have

$$\text{par}-\mu_{\alpha[\varepsilon]}(E,E_\bullet)-\text{par}-\mu_{\alpha[\varepsilon]}(F,F_\bullet)=\text{par}-\mu_\alpha(E,E_\bullet)-\text{par}-\mu_\alpha(F,F_\bullet),$$

which shows that $(E,E_\bullet)$ is $\alpha[\varepsilon]$-semistable if and only if it is α-semistable. This provides a canonical identification between $\mathcal{M}(r,\alpha,\xi)$ and $\mathcal{M}(r,\alpha[\varepsilon],\xi)$ for any shift $\varepsilon\in\mathbb{R}^n$ such that $\alpha[\varepsilon]\in\mathcal{A}_{n,r}$. Consequently, it is natural to study our problem within the equivalence class of $\mathcal{A}_{n,r}$ under the shift action. Since shifting is independent at each parabolic point, we may, without loss of generality, assume that $\alpha_1(x)=0$ for all $x\in D$. This leads us to define the subset

$$\tilde{\mathcal{A}}_{n,r}=\{\alpha\in\mathcal{A}_{n,r}\mid\alpha_1(x)=0\,\forall x\in D\}.$$

There is a natural projection $\pi:\mathcal{A}_{n,r}\longrightarrow\tilde{\mathcal{A}}_{n,r}$ given by

$$\pi(\alpha)_i(x)=\alpha_i(x)-\alpha_1(x)\tag{2.2.1}$$

which corresponds to the shift

$$\pi(\alpha)=\alpha[(-\alpha_1(x_1),\ldots,-\alpha_1(x_n))].$$

In particular, this implies that a parabolic vector bundle is α-semistable if and only if it is $\pi(\alpha)$-semistable.

2.3 Basic transformations of quasiparabolic vector bundles and stability weights

In [AG21], it was proven that any isomorphism (or 3-birational transformations) between two moduli spaces of parabolic vector bundles on a marked curve (X,D) can be described as a suitable composition of four kinds of transformations

- Pullback with respect to an automorphism $\sigma:X\longrightarrow X$ such that $\sigma(D)=D$,
- dualization,
- tensoring with a line bundle L on X
- Hecke transformations $\mathcal{H}_H$, where H is an effective divisor on X supported on the parabolic points D (See [AG21, §5] for precise definition).

We call a composition of these four types of transformations a *basic transformation*. In [AG21, §5] and [Alf22, §3], an explicit description of the composition relations between these four basic transformations was provided and explicit presentation of the group $\mathcal{T}$ of basic transformations was given. It was shown that each basic transformation T can be characterized as a tuple $T=(\sigma,s,L,H)$, where

- σ is an automorphism of (X,D),

- $s \in \{1, -1\}$,
- $L \in \text{Pic}(X)$
- $H = \sum_{x \in D} h_x x$ is an effective divisor on X with $0 \leq h_x < r$ for each $x \in D$.

The tuple $T = (\sigma, s, L, H)$ described this way corresponds to the following transformation of quasi-parabolic vector bundles.

$$(E, E_\bullet) = \begin{cases} \sigma^* \left(L \otimes \mathcal{H}_H(E, E_\bullet)\right) & \text{if } s = 1 \\ \sigma^* \left(L \otimes \mathcal{H}_H(E, E_\bullet)\right)^\vee & \text{if } s = -1 \end{cases}, \tag{2.3.1}$$

In [AG21, §5] it is shown that the group of basic transformations $\mathcal{T}$ acts on the equivalence class of $\mathcal{A}_{n,r}$ under the shift action, and in particular, on the subset $\tilde{\mathcal{A}}_{n,r}$. This action is defined as follows:

If $T = (\sigma, s, L, H) \in \mathcal{T}$, then T defines a map $T : \tilde{\mathcal{A}}_{n,r} \longrightarrow \tilde{\mathcal{A}}_{n,r}$ given by

$$T(\alpha) = (\Sigma_\sigma \circ \mathcal{D}^s \circ \mathcal{T}_L \circ \mathcal{H}_H)(\alpha),$$

with

$$\begin{aligned} \Sigma_\sigma(\alpha)_i(x) &= \alpha_i(\sigma^{-1}(x)) \\ \mathcal{D}^s(\alpha)_i(x) &= \begin{cases} \alpha_i(x) & s = 1 \\ \alpha_r(x) - \alpha_{r-i+1}(x) & s = -1 \end{cases} \\ \mathcal{T}_L(\alpha) &= \alpha \\ \mathcal{H}_x(\alpha)_i(y) &= \begin{cases} \alpha_i(y) & y \neq x \\ 1 + \alpha_1(x) - \alpha_2(x) & y = x,\ i = r \\ \alpha_{i+1}(x) - \alpha_2(x) & y = x, i < r \end{cases} \end{aligned} \tag{2.3.2}$$

A direct computation shows that this map preserves $\tilde{\mathcal{A}}_{n,r} \subset \mathcal{A}_{n,r}$, inducing an action on it. It can be proven (see [AG21]) that a quasi-parabolic vector bundle $(E, E_\bullet)$ is α-stable (respectively α-semistable) if and only if $T(E, E_\bullet)$ is $T(\alpha)$-stable (respectively $T(\alpha)$-semistable).

Basic transformations also induce actions on the determinants and degrees of vector bundles. Given $T = (\sigma, s, L, H)$, define

$$T(\xi) = \sigma^* \left(L^r \otimes \xi(-H)\right)^s \tag{2.3.3}$$

$$T(d) = s(\deg(L) + d - |H|) \tag{2.3.4}$$

Computing determinants and degrees of both sides of equation (2.3.1) shows that T sends vector bundles with determinant ξ to vector bundles of determinant $T(\xi)$ and degree d vector bundles to degree $T(d)$ vector bundles. As a consequence, each $T \in \mathcal{T}$ defines an isomorhpism between moduli spaces

$$T : \mathcal{M}(r, \alpha, \xi) \longrightarrow \mathcal{M}(r, T(\alpha), T(\xi))$$

As we will be interested in describing automorphisms and isomorphisms between different moduli spaces of parabolic vector bundles, we will also be interested in the following subgoups of the group of basic transformations $\mathcal{T}$

- For each $\xi \in \text{Pic}(X)$, let $\mathcal{T}_\xi = \{T \in \mathcal{T} \,|\, T(\xi) \cong \xi\}$ be the subgroup of basic transformations preserving a determinant ξ.

- For each $d \in \mathbb{Z}$, let $\mathcal{T}_d = \{T \in \mathcal{T} \,|\, T(d) = d\}$ be the subgroup of transformations preserving the degree d.
- For a given α, let $\mathcal{T}_\alpha = \{T \in \mathcal{T}_0 \mid T(\alpha)$ lies in the same stability chamber as $\alpha\}$ denote the subgroup of transformations that preserve the stability chamber of α in $\tilde{\mathcal{A}}_{n,r}$.

Lemma 2.3.1. *For each α and ξ there exists α' such that $\mathcal{M}(r, \alpha, \xi) \cong \mathcal{M}(r, \alpha', \mathcal{O}_X)$.*

Proof. Let $d = \deg(\xi)$. Suppose that $d = rk + d'$ with $0 \leq d' < r$. Take a parabolic point $x \in D$. Let $L = \mathcal{H}_x^{d'}(\xi)$. Then $\deg(L) = rk$, so there exists L' such that $(L')^r = L$. As a consequence, if we take $T = \mathcal{T}_{(L')^{-1}} \circ \mathcal{H}_x$, then $T(\xi) = \mathcal{O}_X$. Taking $\alpha' = T(\alpha)$, we see that T induces an isomorphism

$$T : \mathcal{M}(r, \alpha, \xi) \longrightarrow \mathcal{M}(r, T(\alpha), T(\xi)) = \mathcal{M}(r, \alpha', \mathcal{O}_X)$$

□

Remark 2.3.2. *A straightforward computation shows that basic transformations $T \in \mathcal{T}$ send numerical stability walls of degree d to numerical stability walls of degree $T(d)$. Since basic transformations are invertible, they act as bijections on $\tilde{\mathcal{A}}_{n,r}$ and any $T \in \mathcal{T}$ induces bijections between the sets of walls and stability chambers in degree d and the sets of walls and stability chambers in degree $T(d)$.*

Remark 2.3.3. *As a consequence of the previous Lemma, in order to study isomorphism classes of moduli spaces of parabolic vector bundles with generic weights it is enough to consider moduli spaces of parabolic vector bundles with trivial determinant. From this point on in this work, we will restrict our study to these moduli spaces and assume $\xi = \mathcal{O}_X$ and therefore, $d = 0$ in all considered moduli spaces.*

Chapter 3

Algorithmic exploration of parabolic chambers and isomorphisms between moduli spaces

This section presents a computational framework for the analysis and classification of stability chambers, walls, isomorphism classes and automorphism groups of moduli spaces of parabolic vector bundles. Given the high-dimensional nature of the weight space, $dim(\mathcal{A}_{n,r}) = n \cdot r$, and the rapid growth in the number of possible wall configurations, the manual exploration of stability regions becomes intractable. The methodology proposed in this work relies on algorithms and computation to generate and visualise data, draw and verify conjectures, and expose underlying symmetries in the moduli problem.

3.1 From Mathematics to Computer Science

At this point, we can bring all the mathematical abstractness of the previous sections into the realm of computer science. Let us state what we have so far:

- A parameter space of parabolic weights $\tilde{\mathcal{A}}_{n,r} = \{\alpha \in [0,1)^{nr} \mid 0 = \alpha_1(x) < \ldots < \alpha_r(x) < 1 \text{ for all } x \in D\}$, where each α is an $n \times r$ matrix of real numbers. The space $\tilde{\mathcal{A}}_{n,r}$ is a product of n simplices of dimension r.
- A set of hyperplanes $W_{\bar{n},d'}$ that partition the space of parabolic weights into stability chambers, where $\bar{n} \in \Omega_{n,r,r'}$ is a selection vector of subrank r' and $d' \in \mathbb{Z}$ is the degree of a subbundle $F \subset E$.
- If $\alpha, \beta \in \tilde{\mathcal{A}}_{n,r}$ belong to the same stability chamber, then the moduli spaces $\mathcal{M}(r, \alpha, \xi)$ and $\mathcal{M}(r, \beta, \xi)$ are identical.
- There is a group of basic transformations $\mathcal{T}$ that acts on the space of parabolic weights $\tilde{\mathcal{A}}_{n,r}$. A single basic transformation $T \in \mathcal{T}$ can be described as a tuple $T = (\sigma, s, L, H)$, where σ is a permutation of the marked points D, s determines whether we take the dual or not, L does not affect the weights and H is a vector of integers h_x that determines the Hecke transformation at each marked point $x \in D$.

- Some transformations $\mathcal{T}_\alpha \subset \mathcal{T}$, preserve the stability chamber of a given $\alpha \in \tilde{\mathcal{A}}_{n,r}$. These are the automorphisms of the moduli space $\mathcal{M}(r, \alpha, \mathcal{O}_X)$. Other transformations connect different stability chambers, grouping them into isomorphism classes.

The goal of this section is to develop algorithms that can efficiently explore the space of parabolic weights $\tilde{\mathcal{A}}_{n,r}$, enumerating all stability walls, chambers, and isomorphism classes, and find the automorphism groups of the moduli spaces of parabolic vector bundles. The algorithms will be based on the mathematical concepts introduced in the previous sections, and will leverage computational techniques to handle the combinatorial complexity of the problem.

3.2 Enumeration of Selection Vectors $\bar{n}$

A central component in the algorithms developed throughout this thesis involves enumerating all possible configurations of admissible selection vectors $\bar{n} \in \Omega_{n,r,r'}$, since they define the hyperplanes $W_{\bar{n},d'}$ that partition the space of parabolic weights $\tilde{\mathcal{A}}_{n,r}$ into stability chambers. Recall that a selection vector is a binary matrix of size $n \times r$ with exactly r' entries equal to 1 in each row.

The enumeration process proceeds by first generating all $\binom{r}{r'}$ ways of choosing r' positions of an r-sized vector. Then, we construct all possible cartesian products of these combinations across n points. There are exactly $\binom{r}{r'}^n$ such matrices, each one corresponding to selection vector $\bar{n} \in \Omega_{n,r,r'}$.

Algorithm 1 Generate Admissible Matrices with Fixed r'

Require: Integers n, r, r'

```
C ← All combinations of r′ elements from {0, 1, ..., r−1}
V ← Cartesian product of C repeated n times
for v ∈ V do
    Initialize matrix n̄ ∈ Z^{n×r} with all zeros
    for j ← 1 to n do
        for all k ∈ v_j do
            n̄[j, k] ← 1
        end for
    end for
    yield n̄
end for
```

This algorithm efficiently generates all admissible selection vectors $\bar{n}$ for a given n, r, and r'. The output is a list of matrices, each representing a unique selection vector configuration. The complexity of this algorithm is $O\left(nr'\binom{r}{r'}^n\right)$, which grows exponentially with n and r. However, for small values of n and r, this approach is feasible and provides a comprehensive enumeration of all admissible configurations.

The function `generate_admissible_matrices_fixed_r_prime` in our implementation encapsulates this enumeration strategy and acts as a backbone for subsequent algorithms that rely on exploring the space of parabolic configurations.

3.3 Monte Carlo sampling of stability chambers

An initial heuristic approach to the enumeration of stability chambers employs Monte Carlo sampling to estimate the number of these regions without explicitely using the hyperplanes $W_{\bar{n},d'}$.

The method starts by generating random systems of weights α uniformly distributed in the space $\tilde{\mathcal{A}}_{n,r}$ and then calculating the invariant $\bar{M}(r,\alpha,d)$ described in [AG21, §10] for each one of them. The $\bar{M}(r,\alpha,d)$ is a vector whose entries are calculated as follows:

$$M(r,\alpha,d,\bar{n}) = \left\lfloor \frac{r'd + r'\sum_{x\in D}\sum_{i=1}^{r}\alpha_i(x) - r\sum_{x\in D}\sum_{i=1}^{r} n_i(x)\alpha_i(x)}{r} \right\rfloor \tag{3.3.1}$$

for all admissible selection vectors $\bar{n} \in \Omega_{n,r} = \bigcup_{r'=1}^{r-1} \Omega_{n,r,r'}$.

This is computed efficiently with tensor operations and batching chunks of selection vectors $\bar{n}$ and randomly sampled α vectors. However, the number of selection vectors $\bar{n}$ grows exponentially with n and r, and the number of random samples needed to obtain a good estimate of the number of stability chambers is in practice orders of magnitude larger than the number of chambers themselves.

This method is inherently limited in precision and scalability, and it only yields lower bounds on the number of stability chambers, since we cannot guarantee that at least one α falls in each chamber. However for very small n and r and a sufficient number of samples it can serve as an empirical validation of more rigorous methods and as a good visualization tool.

Next figure shows the results of this Monte Carlo sampling for $n = 1$ and $r = 3$ and all possible degrees $d \pmod r$. We confirm the previous Remark 2.3.2 that set of stability walls and chambers is independent of the degree $d \pmod r$.

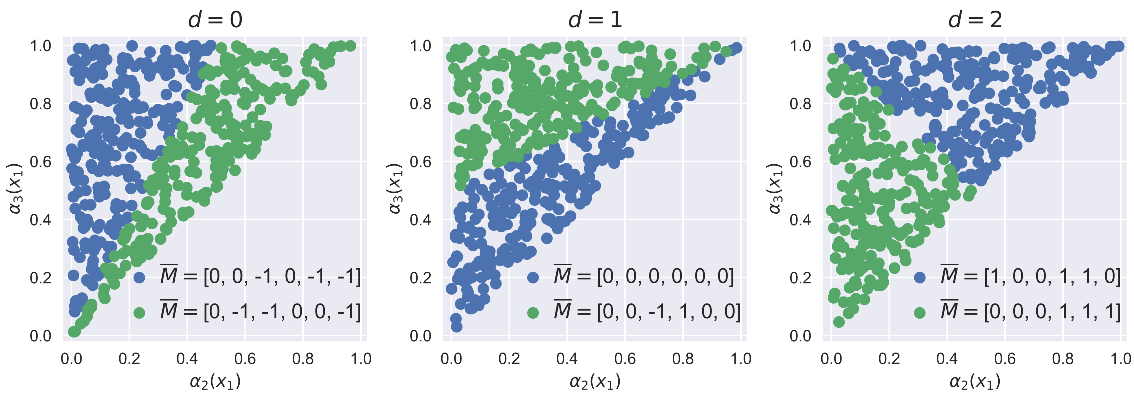


Figure 3.1: Monte Carlo sampling of stability chambers for $n = 1$ and $r = 3$ and all degrees $d \pmod r$. Dots represent 500 uniformly sampled $\alpha \in \tilde{\mathcal{A}}_{1,3}$, projected onto the plane $\alpha_1(x_1) = 0$.

Further illustrative examples of Monte Carlo sampling for small values of n and r can be found in Appendix A. These examples provide additional intuition into how the space $\tilde{\mathcal{A}}_{n,r}$ is partitioned into stability chambers by the hyperplanes $W_{\bar{n},d'}$.

3.4 Geometric walls

The concept of geometric walls is central to understanding the stability chamber decomposition within the space of parabolic weights. These walls correspond to regions where semistability changes, and thus determine the chamber structure of the moduli space $\mathcal{M}(r, \alpha, \xi)$.

A natural question that follows is under what conditions the hyperplane $W_{\bar{n},d'}$ intersects the interior of $\tilde{\mathcal{A}}_{n,r}$, i.e., whether it contributes an actual wall in the chamber decomposition. Since d' could be any integer, there are infinetly many possible hyperplanes, but only actually contribute to the chamber decomposition. Recall the equation for the hyperplane (2.1.3), after letting $d = 0$ the independent term of the hyperplane is given by rd', where d' is the degree of the subbundle F with type $\bar{n}$. We will denote the the independent term rd' as the intercept of the hyperplane. Let us denote $\overline{\tilde{\mathcal{A}}_{n,r}} = \{\alpha \in [0, 1]^{nr} \,|\, 0 \leq \alpha_1(x) \leq \alpha_2(x) \leq \ldots \leq \alpha_r(x) < 1 \text{ for all } x \in D\}$ to the closure of $\tilde{\mathcal{A}}_{n,r}$. We can find limits on the intercept of the hyperplane $W_{\bar{n},d'}$ so that it intersects the interior of $\tilde{\mathcal{A}}_{n,r}$.

Definition 3.4.1. *Let $\bar{n} \in \Omega_{n,r,r'}$ be a selection vector of subrank r' and rank r over (X, D). We define the lower and upper bounds for the intercept of the hyperplane $W_{\bar{n},d'}$ as follows:*

$$l_{\overline{n}} = \min_{\alpha \in \overline{\tilde{\mathcal{A}}_{n,r}}} W_{\bar{n}}(\alpha)$$

and

$$u_{\overline{n}} = \max_{\alpha \in \overline{\tilde{\mathcal{A}}_{n,r}}} W_{\bar{n}}(\alpha)$$

where $W_{\bar{n}}(\alpha)$ is defined in equation (2.1.4).

In Section 3.4, we will show that it suffices to evaluate the expressions on the vertices of $\tilde{\mathcal{A}}_{n,r}$, and we will give explicit formulas to make this calculation tractable and efficient. The lemma states the following:

Lemma 3.4.2. *Let $\bar{n} \in \Omega_{n,r,r'}$ and $d' \in \mathbb{Z}$. The hyperplane $W_{\overline{n},d'}$ intersects the product of simplices $\mathcal{A}_{n,r}$ if and only if*

$$rd' \in (l_{\overline{n}}, u_{\overline{n}}),$$

where

$$l_{\overline{n}} = \sum_{x \in D} \min_{1 \leq j \leq r} \sum_{i=j}^{r} w_{\bar{n},i}(x)$$

and

$$u_{\overline{n}} = \sum_{x \in D} \max_{1 \leq j \leq r} \sum_{i=j}^{r} w_{\bar{n},i}(x),$$

are the lower and upper bounds respectively for the intercept of a hyperplane $W_{\overline{n},d'}$ that intersects the product of simplices and $w_{\bar{n},i}(x) = r' - n_i(x)r$ denotes the i-th component of the normal vector of $W_{\overline{n},d'}$ at the point $x \in D$.

With this result, we can obtain the list of all hyperplanes by enumerating all admissible selection vectors $\bar{n} \in \Omega_{n,r,r'}$ for each subrank $1 \leq r' < r$, computing $u_{\overline{n}}$ and $l_{\overline{n}}$, listing all multiples of r in $(l_{\bar{n}}, u_{\bar{n}})$ and finally discarding any proportional hyperplanes

n\r	2	3	4	5	6
1	0	1	3	11	21
2	1	9	41	215	799
3	4	45	344	3075	21379
4	12	189	2540	39875	515229
5	32	729	17840	491875	11827979
6	80	2673	122384	5871875	264528629
7	192	9477	828416	68421875	–
8	448	32805	5555648	782421875	–

n\r	7	8	9	10	11
1	65	129	307	631	1539
2	3927	15049	65403	254621	–
3	186837	1354856	11062215	–	–
4	8200787	112008812	–	–	–
5	345458281	–	–	–	–

Table 3.1: Number of stability walls with $d = 0$

if any. The output is a list of tuples (w, b), where w is the vector of coefficients of the hyperplane for some selection vector $\bar{n}$ and $b = rd'$ is the intercept of the hyperplane.

These results highlight the intractability of enumerating stability chambers without computational aid for all but the smallest values of n and r. Not only the high dimensionality makes it difficult to visualize the stability chambers, but also the exponential growth of the number of walls as n and r increase generate a combinatorial explosion of possible chamber configurations.

In the process of generating this data, a deep exploration of the hyperplanes and their intercept bounds was conducted, and several structural patterns and symmetries were observed, eventually leading to the exact formula and tight bounds for the total number of geometric walls 4.3.3:

- For each hyperplane $W_{\bar{n},d'}$ found with $\bar{n}' \in \Omega_{n,r,r'}$ there is a proportional hyperplane $W_{\bar{n}',-d}$ for some $\bar{n}' \in \Omega_{n,r,r-r'}$.

  Example 3.4.3 (Proportional hyperplanes for $n = 2$, $r = 3$)**.** *Each hyperplane computed with $r' = 1$ is proportional to one with $r' = 2$ and vice versa. $w_{\bar{n}}$ and $w_{\bar{n}'}$ are flattened for clarity.*

$r' = 1$	$r' = 2$
$w_{\bar{n}}$, rd'	$w_{\bar{n}'}$, rd'
$[-2,\ 1,\ 1,\ -2,\ 1,\ 1],\ 3$	$[2,\ -1,\ -1,\ 2,\ -1,\ -1],\ -3$
$[-2,\ 1,\ 1,\ 1,\ -2,\ 1],\ 0$	$[2,\ -1,\ -1,\ -1,\ 2,\ -1],\ 0$
$[-2,\ 1,\ 1,\ 1,\ 1,\ -2],\ 0$	$[2,\ -1,\ -1,\ -1,\ -1,\ 2],\ 0$
$[1,\ -2,\ 1,\ -2,\ 1,\ 1],\ 0$	$[-1,\ 2,\ -1,\ 2,\ -1,\ -1],\ 0$
$[1,\ -2,\ 1,\ 1,\ -2,\ 1],\ 0$	$[-1,\ 2,\ -1,\ -1,\ 2,\ -1],\ 0$
$[1,\ -2,\ 1,\ 1,\ 1,\ -2],\ 0$	$[-1,\ 2,\ -1,\ -1,\ -1,\ 2],\ 0$
$[1,\ 1,\ -2,\ -2,\ 1,\ 1],\ 0$	$[-1,\ -1,\ 2,\ 2,\ -1,\ -1],\ 0$
$[1,\ 1,\ -2,\ 1,\ -2,\ 1],\ 0$	$[-1,\ -1,\ 2,\ -1,\ 2,\ -1],\ 0$
$[1,\ 1,\ -2,\ 1,\ 1,\ -2],\ -3$	$[-1,\ -1,\ 2,\ -1,\ -1,\ 2],\ 3$

Conjecture 3.4.4. *For each hyperplane $W_{\bar{n},d'}$ with $\bar{n} \in \Omega_{n,r,r'}$, there exists exactly one proportional hyperplane $W_{\bar{n}',-d'}$ for some $\bar{n}' \in \Omega_{n,r,r-r'}$.*

- There are two clear symmetries in the intercept bounds for each selection vector $\bar{n}$. (1) Within each r', $u_{\bar{n}} = -l_{\bar{n}'}$, when $\bar{n}' = (n_{r-i}(x))_{i,x}$. (2) Between r' and $r - r'$ we also have $u_{\bar{n}} = -l_{\bar{n}'}$ but now when $\bar{n}' = (1 - n_i(x))_{i,x}$.

  Example 3.4.5 (Intercept bounds symmetries for $n = 1$, $r = 4$)**.** *We compute the lower and upper bounds for the intercepts associated with each admissible selection vector $\bar{n}$, for subranks $r' \in \{1, 2, 3\}$. Each $\bar{n}$ is a binary vector of length $r = 4$, with exactly r' entries equal to 1.*

  $r' = 1$

$\bar{n}$	$l_{\bar{n}}$	$u_{\bar{n}}$
[1, 0, 0, 0]	0	3
[0, 1, 0, 0]	−1	2
[0, 0, 1, 0]	−2	1
[0, 0, 0, 1]	−3	0

  $r' = 2$

$\bar{n}$	$l_{\bar{n}}$	$u_{\bar{n}}$
[1, 1, 0, 0]	0	4
[1, 0, 1, 0]	0	2
[1, 0, 0, 1]	−2	2
[0, 1, 1, 0]	−2	2
[0, 1, 0, 1]	−2	0
[0, 0, 1, 1]	−4	0

  $r' = 3$

$\bar{n}$	$l_{\bar{n}}$	$u_{\bar{n}}$
[0, 1, 1, 1]	−3	0
[1, 0, 1, 1]	−2	1
[1, 1, 0, 1]	−1	2
[1, 1, 1, 0]	0	3

  Conjecture 3.4.6. *For all n,r,r′ and all selection vector $\bar{n} \in \Omega_{n,r,r'}$, we have that $u_{\bar{n}} = -l_{\bar{n}'}$ when $\bar{n}' = (n_{r-i}(x))_{i,x} \in \Omega_{n,r,r'}$*

  Conjecture 3.4.7. *For all n,r,r′ and all selection vector $\bar{n} \in \Omega_{n,r,r'}$, we have that $u_{\bar{n}} = -l_{\bar{n}'}$ when $\bar{n}' = (1 - n_i(x))_{i,x} \in \Omega_{n,r,r-r'}$*

- A key step on the simplification of the number of walls equation, done in § 4.2 requires knowing how the intercept bound distribute mod r. When exploring this distributions computationally, two properties were observed: (1) all $u_{\bar{n}}$ and $l_{\bar{n}}$ are always multiples of the $gcd(r, r')$. (2) all $u_{\bar{n}}$ and $l_{\bar{n}}$ are distributed uniformly in all $k \propto gcd(r, r')$. We define each count as $L_k = \#\{\bar{n} \in \Omega_{n,r,r'} \,|\, l_{\bar{n}} \equiv k \pmod r\}$ and $U_k = \#\{\bar{n} \in \Omega_{n,r,r'} \,|\, u_{\bar{n}} \equiv k \pmod r\}$. $\bar{L} = (L_k)_{k=0}^{r-1}$ and $\bar{U} = (U_k)_{k=0}^{r-1}$ are the vectors of counts of $l_{\bar{n}}$ and $u_{\bar{n}}$ respectively, for each $k \pmod r$.

  Example 3.4.8 ($u_{\bar{n}}$ and $l_{\bar{n}}$ counts mod r for $n = 1$, $r = 6$)**.**

r′	$\bar{L}$	$\bar{U}$
1	$(1, 1, 1, 1, 1, 1)$	$(1, 1, 1, 1, 1, 1)$
2	$(5, 0, 5, 0, 5, 0)$	$(5, 0, 5, 0, 5, 0)$
3	$(10, 0, 0, 10, 0, 0)$	$(10, 0, 0, 10, 0, 0)$
4	$(5, 0, 5, 0, 5, 0)$	$(5, 0, 5, 0, 5, 0)$
5	$(1, 1, 1, 1, 1, 1)$	$(1, 1, 1, 1, 1, 1)$

  Conjecture 3.4.9. *For fixed n, r and r′, let $U_k = \#\{\bar{n} \in \Omega_{n,r,r'} \,|\, u_{\bar{n}} \equiv k \pmod r\}$ and $L_k = \#\{\bar{n} \in \Omega_{n,r,r'} \,|\, l_{\bar{n}} \equiv k \pmod r\}$, then for all $k \not\propto gcd(r, r')$ we have that $U_k = L_k = 0$ and for each $k \propto gcd(r, r')$ we have*

$$U_k = L_k = \frac{gcd(r, r')\binom{r}{r'}}{r}.$$

All of these observations are proven formally in Chapter 4, and are used to derive tight bounds for the number of geometric walls in the stability space of the moduli space of parabolic vector bundles of rank r and subrank r' over (X, D).

3.5 Decision tree algorithm for enumerating stability chambers

The aim of this section is to describe an exact procedure that replaces the Monte Carlo heuristics of the previous subsection by an *exhaustive* enumeration of stability chambers. The core idea is to cut the ambient simplex $\tilde{\mathcal{A}}_{n,r}$ recursively with the finitely–many geometric walls obtained in §3.4. The recursive structure of this approach gives rise to a binary tree, where each internal node represents a polytope that is split by a wall, and each leaf corresponds to a stability chamber. We can take advantage of the tree structure, and not only count the number of chambers, but construct an efficient classifier for new systems of weights $\alpha \in \tilde{\mathcal{A}}_{n,r}$, which will be crucial for the isomorphism graph algorithm described in §3.6, allowing us to identify when two chambers are isomorphic chambers.

A key point of this method lies in the use of exact arithmetic to perform all operations over rational numbers. This guarantees mathematically correct results, in contrast to floating-point arithmetic which can introduce rounding errors and lead to incorrect chamber counts or invalid polytopes.

The decision tree algorithm is implemented in Python using exact rational arithmetic through the `cddlib` library with GMP backend. The main logic is organized around two classes:

- **`Polytope`**: represents a convex region in halfspace form $Ax \leq b$, where A is an integer matrix and b is an integer column vector. Key methods include:
  - `extreme()`: computes the vertices of the polytope using `cdd.gmp`.
  - `add_halfspace()`: creates two new `Polytope` instances by adding to each one the corresponding halfspace defined by a hyperplane $wx = b$.
- **`TreeNode`**: stores a node in the decision tree, which corresponds to a polytope (chamber candidate). It also stores the list of active hyperplanes (walls), the hyperplane used to split that node, and both its children. Notable methods:
  - `centroid`: computes the center of the current chamber using rational arithmetic (used in §3.6 to find isomorphism classes).
  - `add_child()`: adds a child node created by cutting the current polytope.
  - `classify()`: recursively classifies a point into its corresponding chamber leaf.

Complementing these classes lie several auxiliary functions that facilitate the recursive cutting of the ambient simplex and the traversal of the resulting chamber tree:

- `get_simplex_inequalities(n, r)`: constructs the defining inequalities $Ax \leq b$ of the ambient simplex $\tilde{\mathcal{A}}_{n,r}$, which is a product of n open simplices in dimension r. The coefficients in A and b correspond to the conditions $0 \leq \alpha_1 < \ldots < \alpha_r < 1$ for each parabolic point. The output is a pair (A, b) suitable for initializing a `Polytope` object.

- `cut_polytope_by_hyperplane(polytope, (w, b))`: takes a `Polytope` and a hyperplane defined by $wx = b$, and creates two new polytopes obtained by intersecting with the halfspaces $wx \leq b$ and $wx \geq b$. Both resulting polytopes are updated by computing their vertices via the `extreme()` method.
- `hyperplane_intersects_polytope`: determines whether a given hyperplane (w, b) intersects the interior of a polytope. It evaluates the linear form wx on all vertices of the polytope and returns true if the hyperplane separates them, i.e., if some satisfy $wx < b$ and others $wx > b$.

The decision tree is constructed iteratively: starting from the ambient simplex $\tilde{\mathcal{A}}_{n,r}$, we maintain a queue of nodes (chambers). At each step, we filter out hyperplanes that do not intersect the chamber being consider, and from the remaining hyperplens, we choose a random one to split a chamber into two subchambers. When no such hyperplane exists, the node becomes a leaf, representing a stability chamber.

The runtime of the algorithm is difficult to estimate precisely, as it depends on the intricate structure of the chamber decomposition, the number of hyperplanes involved, and the number of vertices in each polytope. In practice, the execution time of the algorithm transitions from milliseconds for small parameters to several hours as n and r increase only modestly.

Remark 3.5.1. *Despite this, we measured the average depth of the decision tree generated across multiple values of n and r (see Appendix B), which was consistently on the order of $O(\log |C_{n,r}|)$, where $|C_{n,r}|$ denotes the number of stability chambers computed for the given parameters. This observation will be particularly useful in §3.6, where we rely on this efficient classifier to quickly identify the chamber associated with a given system of weights.*

Algorithm 2 Enumerate stability chambers using a decision tree

Let $P_0 \leftarrow \tilde{\mathcal{A}}_{n,r}$ (corner simplexs)
Let $\mathcal{W}$ be the list of all geometric walls (w, b)
Create root node $T_0 \leftarrow$ `TreeNode`(polytope=P_0, candidate_hyperplanes=$\mathcal{W}$)
Initialize queue $Q \leftarrow [T_0]$ and chamber count $C \leftarrow 0$
while Q is not empty **do**
    Pop T from Q
    Discard walls stored in *T.candidate_hyperplanes* not intersecting *T.polytope*
    Let $\mathcal{W}_T$ be the remaining walls in *T.candidate_hyperplanes*
    if $\mathcal{W}_T$ is empty **then**
        $C \leftarrow C + 1$ (chamber found)
        continue
    end if
    Choose a random hyperplane (w, b) from $\mathcal{W}_T$
    Remove (w, b) from $\mathcal{W}_T$
    Split *T.polytope* into $P_1 = \{w \cdot x \leq b\}$ and $P_2 = \{w \cdot x > b\}$
    Compute vertices of P_1 and P_2
    Create children nodes T_1, T_2 from P_1, P_2 and candidate hyperplanes $\mathcal{W}_T$
    Add T_1, T_2 to Q
end while
return T_0 (root node), C (total chambers)

We also provide two auxiliary functions to save and load the resulting decision tree and its statistics in a json file. For memory efficiency we store all hyperplane coefficient vectors w and centroids *flattened and without the redundant* n *dimensions* (Recall that if $\alpha \in \tilde{\mathcal{A}}_{n,r}$, then $\alpha_1(x) = 0$ for all $x \in D$). The json file has the following structure:

```
{
    "n_leaves": N_LEAVES,
    "n_nodes": N_NODES,
    "max_depth": MAX_DEPTH,
    "avg_depth": AVG_DEPTH,
    "tree": [
        {
            "depth": 0,
            "cut_hyperplane": "(w, b)",
            "parent_idx": null,
            "centroid": null
        },
        {
            "depth": 1,
            "cut_hyperplane": "(w', b')",
            "parent_idx": 0,
            "centroid": "(c_1, c_2, ..., c_n·(r-1))"
        },
        ...
    ]
}
```

The following table summarizes the number of stability chambers computed for small values of n and r using this algorithm. The results match the Monte Carlo estimates from §3.4 and provide exact counts.

n\r	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	1	2	4	14	80	1296	76724	$>$ 5315121
2	2	12	640	4748330*	–	–	–	–
3	5	720	$>$ 1984886	–	–	–	–	–
4	24	4868610*	–	–	–	–	–	–
5	409	–	–	–	–	–	–	–
6	31916	–	–	–	–	–	–	–
7	10834621*	–	–	–	–	–	–	–

Table 3.2: Number of stability chambers with $d = 0$

The asterisk (*) indicates that the number was computed before the implementation of exact arithmetic, and thus may not be fully accurate. The greater than sign ($>$) indicates that the number is a lower bound found through Monte Carlo sampling, and the actual number of chambers is likely larger.

3.6 Graph-based enumeration of isomorphism classes and automorphism groups

The purpose of this subsection is to provide an exact algorithm for the enumeration of isomorphism classes as well as the full automorphism group of each class. In [AG21, Theorem 7.23 and Theorem 7.25] it is shown that two moduli spaces $\mathcal{M}(r, \alpha, \xi)$ and $\mathcal{M}(r, \alpha', \xi')$ are isomorphic if and only if there exists a basic transformation T such that $T(\alpha)$ belongs to the same chamber as α' and $T(\xi) \cong \xi'$. The basic transformations are defined in §2.3 and consist of pullback, Hecke and dualisation. As mentioned in Remark 2.3.3, we can restrict our search to the case of degree $d = 0$ without loss of generality, discarding basic transformations that do not preserve the degree.

We can think of the set of all stability chambers as a disconnected directed graph, where each vertex corresponds to a stability chamber and each edge corresponds to a basic transformation that maps one chamber to another. Every connected component of this graph corresponds to an isomorphism class of parabolic vector bundles, and the automorphism group of each class is the set of all basic transformations that map a chamber to itself. We will refer to this graph as the *isomorphism graph* of moduli spaces of parabolic vector bundles.

The algorithm presented in this work employs the decision tree structure from §3.5 to navigate the isomorphism graph of moduli spaces of parabolic bundles. It also relies on an efficient implementation of each of the basic transformations acting on the weight space $\tilde{\mathcal{A}}_{n,r}$, described in Equation (2.3.2). Pullbacks act by permuting the rows of each weight matrix weight matrix α according to the permutation σ. Hecke transformations shift the weights at each point $x \in D$ a specified number of h_x times. Tensorization leaves the weights unchanged, while dualisation reflects them by sending each entry $\alpha_i(x)$ to $\alpha_r(x) - \alpha_{r-i+1}(x)$ for all i and x.

Recall that in the introduction we mentioned the automorphisms (symmetries) of a marked curve (X, D) as a key ingredient in the classification of parabolic vector bundles. Depending on the symmetries of the marked curve, the set of possible basic transformations varies, in particular, the set of possible permutations σ.

We focus on two representative cases of marked curves (X, D): *fully symmetric* and *asymmetric* marked curves. We say a marked curve fully symmetric if $\mathrm{Aut}(X, D) \cong \mathcal{S}_n$ and asymmetric if $\mathrm{Aut}(X, D) = \mathrm{id}$. $\mathcal{S}_n$ is the symmetric group of degree n, which is the group of all permutations of n elements. Any other marked curve with partial symmetry could easily be implemented in our code without any changes to the algorithm, by restricting the set of pullback transformations to those that respect the symmetry of the marked curve.

We know how each transformation affects the degree d of a parabolic vector bundle:

- The pullback transformation does not change the degree, i.e., $d \mapsto d$.
- The Hecke transformation $H_{\mathcal{H}}$ maps $d \mapsto (d - \sum_x h_x) \pmod r$, where h_x is the Hecke operator at point x.
- The dualisation transformation D maps $d \mapsto -d \pmod r$.

To preserve the degree, we need to transformations satisfying:

$$-(d - \textstyle\sum_x h_x) \equiv d \pmod r$$

Since we are considering $d = 0$, we will work with the set of all triples $T_0 = (\sigma, s, H)$, $\mathcal{T}_0 = \{T_0 | \sigma \in \mathcal{S}_n, s \in \{-1, 1\}, H \in \{1, \ldots, r-1\}^n\}$, where $\mathcal{S}_n$ is the symmetric group of degree n and $H = (h_x)_{x \in D}$ is a vector of Hecke operators at each parabolic point meeting the condition $\sum_x h_x \equiv 0 \pmod{r}$.

Lemma 3.6.1. *The size of the set of basic transformations preserving degree $d = 0$ modulo tensorization is:*

$$|\mathcal{T}_0| = n! \cdot 2 \cdot r^{n-1}$$

where $n!$ is the number of permutations of n elements, 2 accounts for the dualisation and identity transformations, and r^{n-1} accounts for the Hecke operators at each parabolic point $x \in D$, restricted to the condition $\sum_x h_x \equiv 0 \pmod{r}$.

The enumeration of isomorphism classes and their automorphism groups proceeds from the classification tree constructed in §3.5, which partitions $\tilde{\mathcal{A}}_{n,r}$ into disjoint stability chambers. Each leaf of the decision tree corresponds to a distinct chamber $\mathcal{C}$ and is uniquely represented by its centroid $\alpha_c \in \mathcal{C}$.

The core idea is to treat each chamber as a representative of a potential isomorphism class and apply all *basic transformations* preserving degree $d = 0$, T_0, to its representative $\alpha_c \in \mathcal{C}$. The transformed weight $T_0(\alpha_c)$ is then classified by querying the decision tree, which identifies the unique chamber $\mathcal{C}'$ it belongs to. If $\mathcal{C}' = \mathcal{C}$, then the transformation is an automorphism of the chamber; if not, the two chambers are identified as isomorphic, and their automorphism groups coincide.

This process is repeated iteratively: once a chamber is determined to be isomorphic to another (via a transformation), it is marked as visited, and its exploration is skipped in future iterations. The algorithm terminates once all chambers have been visited, yielding a list of isomorphism classes, each represented by a tuple (α_c, Aut), where α_c is the centroid of the one of the chambers in the isomorphism class and Aut is the automorphism group of that class, containing all basic transformations acting on α_c that map it back to the same chamber.

Below is the pseudocode for the algorithm that enumerates isomorphism classes and their automorphism groups from the stability chambers obtained in §3.5.

Algorithm 3 Isomorphism classes and automorphism groups from stability chambers

Require: Decision tree $\mathcal{D}$ with chambers $\{\mathcal{C}_i\}$ and representatives $\{\alpha_i\}$
Require: Degree $d = 0$
Require: Set of basic transformations $\mathcal{T}_0$ preserving degree $d = 0$

Initialize visited set $\mathcal{V} \leftarrow \emptyset$
Initialize list of isomorphism classes $\mathcal{I} \leftarrow []$
for all chambers $\mathcal{C}_i$ **do**
  if $\mathcal{C}_i \in \mathcal{V}$ **then**
    continue (skip already visited chamber)
  end if
  Mark $\mathcal{C}_i$ as visited: $\mathcal{V} \leftarrow \mathcal{V} \cup \{\mathcal{C}_i\}$
  Let automorphism group Aut $\leftarrow \{\text{id}\}$
  for all $T \in \mathcal{T}_0$ **do**
    Apply T to α_i to obtain $T(\alpha)$
    Classify α' using $\mathcal{D}$ to find chamber $\mathcal{C}'$
    if $\mathcal{C}' = \mathcal{C}_i$ **then**
      Add T to Aut
    else if $\mathcal{C}' \notin \mathcal{V}$ **then**
      Mark $\mathcal{C}'$ as visited
    end if
  end for
  Append (α_i, Aut) to $\mathcal{I}$
end for
return $\mathcal{I}$: list of isomorphism classes with corresponding automorphism group

The complexity of this output-sensitive algorithm depends on several key parameters: the number of isomorphism classes $|\mathcal{I}|$, the size of the basic transformation group $|\mathcal{T}_0|$, the cost a applying each transformation $T \in \mathcal{T}_0$, and the query time of the decision tree classifier $Q_{\mathcal{D}}$. As discussed in Section 3.5, experimental evidence shows that the average query time of the decision tree is logarithmic in the number of stability chambers; that is, $Q_{\mathcal{D}} = O(\log |\mathcal{C}_{n,r}|)$.

Moreover, it follows directly from the explicit formulas in Equation (2.3.2) that each basic transformation can be implemented with a time complexity of $O(n \cdot r)$, where n is the number of marked points and r the rank. Since each isomorphism class requires checking the action of all transformations in $\mathcal{T}_0$ and classifying the resulting weights via the decision tree, the total complexity of the algorithm is:

$$\Theta\left(nr \cdot |\mathcal{T}_0| \cdot |\mathcal{I}| \cdot \log |\mathcal{C}_{n,r}|\right),$$

where $|\mathcal{C}_{n,r}|$ denotes the total number of stability chambers.

Experimental timings. To complement the theoretical complexity, we measured the execution time of the algorithm under various input configurations. For instance, when running the algorithm for $n = 1$, $r = 8$, the algorithm completes in approximately 40 seconds for both symmetric and asymmetric settings. In contrast, for $n = 6$, $r = 2$, the symmetric case takes around 7 minutes, while the asymmetric case requires only about 20 seconds. These results are consistent with the theoretical expectations, since the number of transformations $|\mathcal{T}_0|$ preserving degree $d = 0$ grows with $n!$.

We summarize the number of isomorphism classes for both fully symmetric and asymmetric cases in Tables 3.3 and 3.4 respectively.

n\r	**2**	**3**	**4**	**5**	**6**	**7**
1	1	1	2	7	40	648
2	1	2	44			
3	2	17				
4	3					
5	8					
6	28					

Table 3.3: Number of isomorphism classes for fully symmetric curves

n\r	**2**	**3**	**4**	**5**	**6**	**7**
1	1	1	2	7	40	648
2	1	2	80			
3	2	42				
4	6					
5	39					
6	1123					

Table 3.4: Number of isomorphism classes for asymmetric curves

Results and conjectures. When looking at the exhaustive lists of automorphism groups and isomorphism classes, we observed the following:

1. If $r > 2$, dualisation is *never* part of an automorphism group.
2. The number of isomorphism classes is significantly less than the number of chambers, which implies that the size of the automorphism groups are generally small.
3. In particular, when considering an asymmetric marked curve, we observe that almost all Hecke transformations induce a non-trivial isomorphism between different moduli spaces.
   (a) In $r = 2$, we find that the number of isomorphism classes is almost equal to the number of chambers divided by the number of Heckes, 2^{n-1}. It is known that in r=2 dualization can be alternatively expressed in terms of tensorization and, therefore, does not affect stability (see [AG21, Lemma 7.23]), thus it does not induce isomorphisms between moduli spaces. This means that (almost) all Hecke transformations are contributing to the isomorphism classes.
   (b) In $r > 2$, we observe that the number of isomorphism classes is almost equal to the number of chambers divided by twice the number of Heckes, $2r^{n-1}$, since now dualisation never contributes to the automorphism group, as stated in the first point.
4. When considering a single parabolic point, the only possible automorphism is the identity, thus the number of isomorphism classes is half the number of chambers with a fixed degree $d = 0$.

In summary, we observed that many types of theoretically possible automorphisms can never happen in a generic weight, and that for $r > 2$ the dualisation transformation is never part of the automorphism group. In Chapter 5 we will formally prove the dualisation conjecture.

Chapter 4

Bounds on the number of geometric walls and stability chambers

In this chapter, we prove the majority of the conjectures formulated in the previous section and use these results to establish both upper and lower bounds on the number of geometric walls and stability chambers in the moduli space of stable parabolic vector bundles of rank r over (X, D) with fixed degree $d = 0$. Beyond these bounds, we also derive several closed formulas that enable a detailed asymptotic analysis of the number of geometric walls as functions of the rank r and the number of marked points n, highlighting the exponential growth of the chamber decomposition in high-dimensional settings.

4.1 Unique stability walls

In this section, we determine the conditions under which the hyperplanes $W_{\bar{n},d'}$ intersect the interior of the space of full flag parabolic weights $\mathcal{A}_{n,r}$. We then prove Conjecture 3.4.4, which asserts that for every hyperplane $W_{\bar{n},d'}$, there exists a proportional hyperplane $W_{\bar{n}',d'}$ for some $\bar{n}' \in \Omega_{n,r,r'}$ such that $W_{\bar{n},d'} \propto W_{\bar{n}',-d'}$. This result enables us to count geometric walls without explicitly checking for proportional hyperplanes algorithmically, thereby allowing the derivation of exact formulas and significantly improving the efficiency of related computations.

Rearranging the expression of $W_{\bar{n},d'}$ from equation (2.1.4) into

$$W_{\bar{n}}(\alpha) = \sum_{x \in D} \sum_{i=1}^{r} (r' - n_i(x)r)\, \alpha_i(x) \tag{4.1.1}$$

and thus define

$$w_{\overline{n}} = (r' - n_i(x)r)_{i=1,\dots,r,x\in D}$$

as the normal vector of the hyperplane $W_{\overline{n},d'}$, with $w_{\overline{n},i}(x) = r' - n_i(x)r$ being the i-th component of the normal vector at the point $x \in D$. This formulation will be more comfortable for several proofs in this section.

Let us start finding the restrictions on the hyperplanes $W_{\bar{n},d'}$ so that they intersect the interior of the space of full flag parabolic weights $\mathcal{A}_{n,r}$. To do that, let us prove the following small lemma that will be convenient for the rest of the section.

Lemma 4.1.1. *Let $\bar{n} \in \Omega_{n,r,r'}$ be a selection vector of subrank r' and rank r over (X, D), then*

$$\sum_{i=1}^{r} w_{\bar{n},i}(x) = 0$$

for all $x \in D$.

Proof. By definition of $\bar{n} \in \Omega_{n,r,r'}$, we have that $\sum_{i=1}^{r} n_i(x) = r'$ for all $x \in D$. Thus, we can write

$$\sum_{i=1}^{r} w_{\bar{n},i}(x) = \sum_{i=1}^{r}(r' - n_i(x)r) = r'r - r\sum_{i=1}^{r} n_i(x) = r'r - rr' = 0.$$

□

Lemma 4.1.2. *Let $\bar{n} \in \Omega_{n,r,r'}$ and $d' \in \mathbb{Z}$. The hyperplane $W_{\overline{n},d'}$ intersects the interior of the product of simplices $\mathcal{A}_{n,r}$ if and only if*

$$rd' \in (l_{\overline{n}}, u_{\overline{n}}),$$

where

$$l_{\overline{n}} = \sum_{x \in D} \min_{1 \leq j \leq r} \sum_{i=j}^{r} w_{\bar{n},i}(x)$$

and

$$u_{\overline{n}} = \sum_{x \in D} \max_{1 \leq j \leq r} \sum_{i=j}^{r} w_{\bar{n},i}(x),$$

are the lower and upper bounds respectively for the intercept of a hyperplane $W_{\overline{n},d'}$ that intersects the product of simplices and $w_{\bar{n},i}(x) = r' - n_i(x)r$ denotes the i-th component of the normal vector of $W_{\overline{n},d'}$ at the point $x \in D$.

Proof. Let $V_{\mathcal{A}_{n,r}}$ be the set of vertices of the product of simplices $\mathcal{A}_{n,r}$. Since $\overline{\mathcal{A}_{n,r}}$ is a convex polytope, the hyperplane $W_{\overline{n},d'}$ intersects the interior of $\mathcal{A}_{n,r}$ if and only if

$$\min_{v \in V_{\mathcal{A}_{n,r}}} W_{\overline{n}}(v) < rd' < \max_{v \in V_{\mathcal{A}_{n,r}}} W_{\overline{n}}(v).$$

Vertices of the product of simplices are binary matrices $v = (v_i(x))$ with $v_i(x) \in \{0, 1\}$ and $v_i(x) < v_{i+1}(x)$ for all $i = 1, \ldots, r-1$ and $x \in D$. Let $k \in \mathbb{Z}$, $1 \leq k \leq r$ and $x \in D$. Then, the vertex v is

$$v_i(x) = \begin{cases} 0 & \text{if } i \leq k \\ 1 & \text{if } i > k \end{cases}$$

Let us define

$$W_{\overline{n},x}(\alpha) = \sum_{i=1}^{r} (r' - n_i(x)r)\, \alpha_i(x).$$

$W_{\overline{n}}(\alpha)$ can be expressed as a sum of n linear functions $W_{\overline{n},x}(\alpha)$, each one depending only on the weights at a single point $x \in D$. Thus, we can write $\min_{v \in V_{\mathcal{A}_{n,r}}} W_{\overline{n}}(v)$ as a sum of the minimum of each function over the set of vertices at a single point $x \in D$,

$$\min_{v \in V_{\mathcal{A}_{n,r}}} W_{\overline{n}}(v) = \min_{v \in V_{\mathcal{A}_{n,r}}} \sum_{x} W_{\overline{n},x}(v) = \sum_{x \in D} \min_{v \in V_{\mathcal{A}_{1,r}}} W_{\overline{n},x}(v)$$

where $V_{\mathcal{A}_{1,r}} = \{(0,\dots,0),(0,\dots,0,1),\dots,(1,\dots,1)\}$ is the set of vertices for a single parabolic point $x \in D$.

Continuing with the proof, we have

$$\sum_x \min_{v\in V_{\mathcal{A}_{1,r}}} W_{\overline{n},x}(v) = \sum_{x\in D} \min\left(W_{\overline{n},x}((0,\dots,0)),\dots,W_{\overline{n},x}((1,\dots,1))\right)$$
$$= \sum_{x\in D} \min\left(0, w_{\bar{n},r}(x), w_{\bar{n},r}(x)+w_{\bar{n},r-1}(x),\dots,\sum_{i=1}^r w_{\bar{n},i}(x)\right)$$
$$= \sum_{x\in D} \min_{1\le j\le r} \sum_{i=j}^r w_{\bar{n},i}(x) = l_{\overline{n}}.$$

Note that $\sum_{i=1}^r w_{\bar{n},i}(x) = 0$ for all $x \in D$, as proved in Lemma 4.1.1, so we can ignore the 0 at the beginning of the minimum.

Same reasoning applies to the maximum of $W_{\overline{n}}(v)$ over the vertices of the product of simplices $\mathcal{A}_{n,r}$,

$$\max_{v\in V_{\mathcal{A}_{n,r}}} W_{\overline{n}}(v) = \sum_{x\in D} \max_{v\in V_{\mathcal{A}_{1,r}}} W_{\overline{n},x}(v) = \sum_{x\in D} \max_{1\le j\le r} \sum_{i=j}^r w_{\bar{n},i}(x) = u_{\overline{n}}.$$

□

Let us now prove that exactly half of the hyperplanes are redundant when considering all $\bar{n} \in \bigcup_{r'} \Omega_{n,r,r'}$ and $d' \in \mathbb{Z}$, i.e., that for each hyperplane $W_{\bar{n},d'}$ with $\bar{n} \in \Omega_{n,r,r'}$ there exists exactly one proportional hyperplane $W_{\bar{n}',-d'}$ such that $\bar{n}' = (1-n_i(x))_{i,x} \in \Omega_{n,r,r-r'}$.

Lemma 4.1.3. *$W_{\bar{n},d'}$ and $W_{\bar{n}',d''}$ are the same hyperplane if and only if $\bar{n} = \bar{n}'$ and $d' = d''$ or $n_i(x)' = 1 - n_i(x)$ for all $i = 1,\dots,n$ and $x \in D$ and $d' = -d''$.*

Proof. If $n_i'(x) = 1 - n_i(x)$, then $\bar{n}'$ is a selection vector of rank $r - r'$. Then

$$W_{\bar{n}}(\alpha) + W_{\bar{n}'}(\alpha) = r' \sum_{i=1}^r \sum_{x\in D} \alpha_i(x) + (r-r') \sum_{i=1}^r \sum_{x\in D} \alpha_i(x)$$
$$- r\sum_{i=1}^r \sum_{x\in D} (n_i(x)+n_i'(x))\alpha_i(x) = r\sum_{i=1}^r \sum_{x\in D} \alpha_i(x) - r\sum_{i=1}^r \sum_{x\in D} \alpha_i(x) = 0.$$

Thus, for all $d' \in \mathbb{Z}$, $W_{\bar{n}}(\alpha) - rd' = -(W_{\bar{n}'}(\alpha) + rd')$.

On the other hand, if $W_{\bar{n},d'}$ and $W_{\bar{n}',d''}$ are the same hyperplane, then, without loss of generality, there exists $\lambda \neq 0$ such that $W_{\bar{n}'} = \lambda W_{\bar{n}}$. Let r'' be the rank of $\bar{n}'$. In virtue of the previous identity, we can also assume without loss of generality, changing $\bar{n}$ or $\bar{n}'$ by their respective complements if necessary that $r', r'' \le r/2$. Then

$$0 = \lambda W_{\bar{n}} - W_{\bar{n}'} = \sum_{i=1}^r \sum_{x\in D} (\lambda r' - r'')\alpha_i(x) - \sum_{i=1}^r \sum_{x\in D} r(\lambda n_i(x) - n_i'(x))\alpha_i(x).$$

Thus, for each i and each $x \in D$ we must have

$$\lambda r' - r'' = r\lambda n_i(x) - r n_i'(x)$$

Now, we have two cases. First, assume that $\lambda r' - r'' = 0$. Then $\lambda = r''/r' \neq 0$ and we have

$$0 = \lambda n_i(x) - n_i'(x)$$

for each i and x. But $n_i(x), n_i'(x) \in \{0, 1\}$, so we must have $n_i(x) = n_i'(x)$ and $1 = \lambda = r''/r'$, so $\bar{n} = \bar{n}'$. Then, it is trivial that in order for the hyperplanes to coincide, we must also have $d' = d''$.

On the other hand, assume that $\lambda r' - r'' \neq 0$. As we assumed that $r', r'' \leq r/2$, then $r' + r'' \leq r$ and, if $\bar{n}'$ is not the complement of $\bar{n}$, then there must exist at least some $i = 1, \ldots, r$ and some $x \in D$ such that $n_i(x) = n_i'(x) = 0$. But then, for that i and x, we would have

$$r\lambda n_i(x) - rn_i'(x) = 0 \neq \lambda r' - r''.$$

Thus, we must have that $\bar{n}'$ is the complementary of $\bar{n}$, so $r'' = r - r'$. As a consequence, $W_{\bar{n}'} = -W_{\bar{n}}$ and, therefore, in this case the hyperplanes coincide if and only if $d'' = -d'$. □

This lemma justifies that exactly one hyperplane can be proportional to another hyperplane, and that the proportional hyperplanes are of the form $W_{\bar{n},d'}$ and $W_{\bar{n}',-d'}$ for some $\bar{n}' \in \Omega_{n,r,r-r'}$ such that $n_i'(x) = 1 - n_i(x)$ for all $i = 1, \ldots, r$ and $x \in D$.

We now have to prove that this hyperplane $W_{\bar{n},d}$ does exists for each $\bar{n} \in \Omega_{n,r,r'}$ and each $d \in I_{\bar{n}} = \{d' \in (l_{\bar{n}}, u_{\bar{n}}) \mid d' \equiv 0 \pmod{r}\}$, where $I_{\bar{n}}$ is the set of possible intercepts for the hyperplane $W_{\bar{n},d'}$.

In the definition of the intercept bounds given above, we see both bounds depend on partial sums of the normal vector $w_{\bar{n}}$ of the hyperplane $W_{\bar{n},d'}$ at each point $x \in D$. From now on, it will be convenient to define the following notation for the partial sums of the normal vector $w_{\bar{n}}$ at each point $x \in D$:

$$S_{j,x}(w_{\bar{n}}) = \sum_{i=j}^{r} w_{\bar{n},i}(x),$$

where $S_{j,x}(w_{\bar{n}})$ is the partial sum starting at index j of the normal vector $w_{\bar{n}}$ at the point $x \in D$.

Lemma 4.1.4. *Let $\mathcal{F} : \Omega_{n,r,r'} \longrightarrow \Omega_{n,r,r-r'}$ be the function $\mathcal{F}(\bar{n}_i(x)) = 1 - \bar{n}_i(x)$, Then*

$$u_{\bar{n}} = -l_{\mathcal{F}(\bar{n})},$$
$$l_{\bar{n}} = -u_{\mathcal{F}(\bar{n})}.$$

Proof. We have

$$S_{j,x}(w_{\mathcal{F}(\bar{n})}) = \sum_{i=j}^{r} w_{\mathcal{F}(\bar{n}),i}(x) = \sum_{i=j}^{r} \left((r - r') - (1 - n_i(x))\, r)\right)$$
$$= \sum_{i=j}^{r} (r - r' - r + n_i(x)r) = -\sum_{i=j}^{r} (r' - n_i(x)r) = -S_{j,x}(w_{\bar{n}}),$$

Thus, we have

$$u_{\bar{n}} = \sum_{x \in D} \max_{1 \leq j \leq r} S_{j,x}(w_{\bar{n}}) = \sum_{x \in D} \max_{1 \leq j \leq r} -S_{j,x}(w_{\mathcal{F}(\bar{n})})$$
$$= -\sum_{x \in D} \min_{1 \leq j \leq r} S_{j,x}(w_{\mathcal{F}(\bar{n})}) = -l_{\mathcal{F}(\bar{n})}.$$

And since $\mathcal{F}(\mathcal{F}(\bar{n})) = \bar{n}$, we also have

$$l_{\bar{n}} = -u_{\mathcal{F}(\bar{n})}$$

□

Combining the previous two lemmas, we have that all $\bar{n} \in \Omega_{n,r,r'}$ can be paired with a unique $\bar{n}' \in \Omega_{n,r,r-r'}$ such that if the set of possible intercepts for $\bar{n}$ is $I_{\bar{n}} = \{d' \in (l_{\bar{n}}, u_{\bar{n}}) \mid d' \equiv 0 \pmod r\}$, then the set of possible intercepts for $\bar{n}'$ is the same set but negated, $I_{\bar{n}'} = \{d' \in (-u_{\bar{n}}, l_{\bar{n}}) \mid d' \equiv 0 \pmod r\}$. Thus, each hyperplane formed by a selection vector $\bar{n} \in \Omega_{n,r,r'}$ and an integer $d' \in I_{\bar{n}}$ is proportional to a hyperplane formed by the selection vector $\bar{n}' \in \Omega_{n,r,r-r'}$ and the integer $-d' \in I_{\bar{n}'}$. This means that exactly half of the hyperplanes $W_{\bar{n},d'}$ are proportional to each other when considering all $r' \in \{1, \ldots, r-1\}$, or similarly, that we only need to consider $r' \in \{1, \ldots, \lfloor r/2 \rfloor\}$ and discard half the hyperplanes with $r' = r/2$ if r is even.

4.2 Explicit formulas for the number of walls

In this section, we begin with a straightforward algorithmic formula for computing the number of geometric walls. We then refine this expression to facilitate further analysis, ultimately leading to tight bounds on the number of geometric walls.

Let us start with a useful lemma on the intercept bounds $l_{\bar{n}}$ and $u_{\bar{n}}$ defined in lemma 4.1.2.

Lemma 4.2.1. *Let $\bar{n} \in \Omega_{n,r,r'}$ be a selection vector of subrank r' and rank r over (X, D), then*

$$u_{\bar{n}} > l_{\bar{n}}$$

Proof. Let $u_{\bar{n}}$ and $l_{\bar{n}}$ be as in lemma 4.1.2, and let $S_{j,x}(w_{\bar{n}}) = \sum_{i=j}^{r} w_{\bar{n},i}(x)$.

We have the following inequality

$$\max_{1 \le j \le r} S_{j,x}(w_{\bar{n}}) \ge \max(S_{1,x}(w_{\bar{n}}), S_{r,x}(w_{\bar{n}})) = \max(0, r' - n_r(x)r)$$

Let's look at each case separately. If $n_r(x) = 0$, then

$$\max_{1 \le j \le r} S_{j,x}(w_{\bar{n}}) \ge \max(0, r') = r' > 0 \ge \min_{1 \le j \le r} S_{j,x}(w_{\bar{n}})$$

If $n_r(x) = 1$, then

$$\max_{1 \le j \le r} S_{j,x}(w_{\bar{n}}) \ge \max(0, r' - r) = 0 > r' - r \ge \min_{1 \le j \le r} S_{j,x}(w_{\bar{n}})$$

Finally, in both cases, we have

$$u_{\bar{n}} = \sum_{x \in D} \max_{1 \le j \le r} S_{j,x}(w_{\bar{n}}) > \sum_{x \in D} \min_{1 \le j \le r} S_{j,x}(w_{\bar{n}}) = l_{\bar{n}}$$

□

Lemma 4.2.2. *The number of different walls in the space of stability conditions of rank r, degree 0 with n parabolic points is*

$$W_{n,r} = \begin{cases} \sum_{r'=1}^{\lfloor r/2 \rfloor} W_{n,r,r'} & \text{if } r \text{ is odd} \\ \sum_{r'=1}^{r/2-1} W_{n,r,r'} + \frac{1}{2} W_{n,r,r/2} & \text{if } r \text{ is even} \end{cases},$$

where

$$W_{n,r,r'} = \sum_{\bar{n} \in \Omega_{n,r,r'}} \left(\left\lceil \frac{u_{\bar{n}}}{r} \right\rceil - \left\lfloor \frac{l_{\bar{n}}}{r} \right\rfloor - 1 \right)$$

Proof. In order to count the number of different walls in the space of stability conditions of rank r, degree 0 with n parabolic points, we will go through all the selection vectors $\bar{n} \in \Omega_{n,r,r'}$, which each produce a single normal vector $w_{\bar{n}}$, and count the number of different intercepts that make the hyperplane $W_{\bar{n},d'}$ intersect the product of simplices $\mathcal{A}_{n,r}$, using the bounds given in Lemma 4.1.2.

However, we need to be careful with the fact that some selection vectors $\bar{n}$ and $\bar{n}'$ can produce the same hyperplane $W_{\bar{n},d'}$, which would lead to double counting. Lemma 4.1.3 tells us that the hyperplanes $W_{\bar{n},d'}$ and $W_{\bar{n}',d''}$ are the same if and only if $\bar{n} = \bar{n}'$ and $d' = d''$ or $\bar{n}'_i(x) = 1 - \bar{n}_i(x)$ for all $i = 1, \ldots, r$ and $x \in D$ and $d' = -d''$.

In the previous lemma, we defined the function $\mathcal{F} : \Omega_{n,r,r'} \longrightarrow \Omega_{n,r,r-r'}$ as the function $\mathcal{F}(\bar{n}_i(x)) = 1 - \bar{n}_i(x)$. Also, we proved that $u_{\bar{n}} = -l_{\mathcal{F}(\bar{n})}$ and $l_{\bar{n}} = -u_{\mathcal{F}(\bar{n})}$. Thus, we can conclude that the set of hyperplanes generated by the selection vectors $\bar{n}$ and $\mathcal{F}(\bar{n})$ are the same.

In the case of r even and $r' = \frac{r}{2}$, the bijection $\mathcal{F}$ is also a permutation, since $\Omega_{n,r,r'} = \Omega_{n,r,r-r'} = \Omega_{n,r,\frac{r}{2}}$. Thus, we can conclude that only half of the hyperplanes generated by the selection vectors $\bar{n} \in \Omega_{n,r,r/2}$ are different.

We define $W_{n,r,r'}$ as the number of pairs $(w_{\bar{n}}, rd')$ for $\bar{n} \in \Omega_{n,r,r'}$ and $rd' \in (l_{\bar{n}}, u_{\bar{n}})$, each one corresponding to a hyperplane $W_{\bar{n},d'}$ that intersects the product of simplices $\mathcal{A}_{n,r}$.

It is a well known fact that the number of multiples of r in an open interval (a, b) with $b > a$ is given by

$$N_r((a,b)) = \left\lceil \frac{b}{r} \right\rceil - \left\lfloor \frac{a}{r} \right\rfloor - 1.$$

By Lemma 4.2.1 we have $u_{\bar{n}} > l_{\bar{n}}$, so we can apply the previous formula to all intervals $(l_{\bar{n}}, u_{\bar{n}})$. Thus, the number of tuples $(w_{\bar{n}}, rd')$ for $\bar{n} \in \Omega_{n,r,r'}$ and $rd' \in (l_{\bar{n}}, u_{\bar{n}})$ is given by

$$W_{n,r,r'} = \sum_{\bar{n} \in \Omega_{n,r,r'}} N_r((l_{\bar{n}}, u_{\bar{n}}))$$

Finally, we can conclude that the number of different walls in the space of stability conditions of rank r, degree 0 with n parabolic points is given by

$$W_{n,r} = \begin{cases} \sum_{r'=1}^{\lfloor r/2 \rfloor} W_{n,r,r'} & \text{if } r \text{ is odd} \\ \sum_{r'=1}^{r/2-1} W_{n,r,r'} + \frac{1}{2} W_{n,r,r/2} & \text{if } r \text{ is even.} \end{cases}$$

□

The next step would be simplifying this expression, getting rid of the floor and ceiling functions, so that further analysis can be done. In order to do so, we will use the following lemma regarding the counts of intercept bounds modulo r, which we mentioned in the previous section, for which the data genarated algorithmically was crucial.

Lemma 4.2.3. *Let $U_{n,r,r',k}$ and $L_{n,r,r',k}$ be respectively,*

$$U_{n,r,r',k} = \#\{\bar{n} \in \Omega_{n,r,r'} \,|\, u_{\bar{n}} \equiv k \pmod{r}\},$$

$$L_{n,r,r',k} = \#\{\bar{n} \in \Omega_{n,r,r'} \,|\, l_{\bar{n}} \equiv k \pmod{r}\}.$$

Then,

$$\begin{cases} U_{n,r,r',k} = L_{n,r,r',k} = \frac{gcd(r,r')}{r}\binom{r}{r'}^n & \text{if } k \equiv 0 \pmod{gcd(r,r')} \\ U_{n,r,r',k} = L_{n,r,r',k} = 0 & \text{otherwise.} \end{cases}$$

Proof. Let us first look at the case $k \not\equiv 0 \pmod{gcd(r,r')}$.

It follows from Lemma 4.1.2 that $u_{\bar{n}}$ and $l_{\bar{n}}$ are sums of terms of the form $r' - n_i(x)r$ for some $i = 1, \ldots, r$ and $x \in D$. Thus, $gcd(r', r - r') = gcd(r', r)$ divide both $u_{\bar{n}}$ and $l_{\bar{n}}$. As a consequence,

$$u(\bar{n}) \equiv l(\bar{n}) \equiv 0 \pmod{gcd(r,r')},$$

i.e., there are no selection vectors $\bar{n}$ such that $u_{\bar{n}} \equiv k \pmod{r}$ or $l_{\bar{n}} \equiv k \pmod{r}$ for $k \not\equiv 0 \pmod{gcd(r,r')}$.

Let's now look at the case $k \equiv 0 \pmod{gcd(r,r')}$. Let $\mathcal{R}_{x_0} : \Omega_{n,r,r'} \longrightarrow \Omega_{n,r,r'}$, with $x_0 \in D$ be the permutation of $\Omega_{n,r,r'}$ such that

$$\left(\mathcal{R}_{x_0}(\bar{n})\right)_i(x) = \begin{cases} \bar{n}_{i-1}(x) & \text{if } x = x_0 \text{ and } i > 1, \\ \bar{n}_r(x) & \text{if } x = x_0 \text{ and } i = 1, \\ \bar{n}_i(x) & \text{otherwise.} \end{cases}$$

Let

$$u_{\bar{n},x} = \max_{1\le j\le r} S_{j,x}(w_{\bar{n}}),\; l_{\bar{n},x} = \min_{1\le j\le r} S_{j,x}(w_{\bar{n}}), \tag{4.2.1}$$

with $S_{x,j}(w_{\bar{n}}) = \sum_{i=j}^{r} w_{\bar{n},i}(x)$ so that $u_{\bar{n}} = \sum_x u_{\bar{n},x}$. Since $\mathcal{R}_{x_0}$ acts only on x_0, only $u_{\bar{n},x_0}$ is affected. We have

$$\begin{aligned} u_{\mathcal{R}_{x_0}(\bar{n}),x_0} &= \max_{1\le j\le r} S_{j,x_0}(w_{\mathcal{R}_{x_0}(\bar{n})}) = \max_{1\le j\le r} \sum_{i=j}^{r} w_{\mathcal{R}_{x_0}(\bar{n}),i}(x_0) \\ &= \max\left(w_{\bar{n},1}(x_0), w_{\bar{n},r}(x_0) + w_{\bar{n},r-1}(x_0), \ldots, \sum_{i=2}^{r} w_{\bar{n},i}(x_0) \right) = \\ &\max\left(w_{\bar{n},1}(x_0), \max_{2\le j\le r} S_{j,x_0}(w_{\bar{n}}) \right). \end{aligned}$$

Now there are two cases. First, if $u_{\bar{n}} > 0$, then $\operatorname{argmax}_{1\le j\le r} S_{j,x_0}(w_{\bar{n}}) = j_0$ for some $j_0 \ge 2$. Then, we have

$$\max_{2\le j\le r} S_{j,x_0}(w_{\bar{n}}) = \max_{1\le j\le r} S_{j,x_0}(w_{\bar{n}})$$

and thus

$$u_{\mathcal{R}_{x_0}(\bar{n}),x_0} = (r' - n_1(x_0)r) + \max_{1\le j\le r} S_{j,x_0}(w_{\bar{n}}) = (r' - n_1(x_0)r) + u_{\bar{n},x_0}.$$

Thus, if $u_{\bar{n}_x,x} \equiv k \pmod r$, then $u_{\mathcal{R}(\bar{n}_x)} \equiv k + r' \pmod r$. Now, looking at the whole $u_{\bar{n}}$ we obtain

$$u_{\mathcal{R}_{x_0}(\bar{n})} = \sum_{x\in D} u_{\mathcal{R}_{x_0}(\bar{n}_x)} = \sum_{\substack{x\neq x_0\\ x\in D}} u_{\bar{n}_x,x} + u_{\mathcal{R}_{x_0}(\bar{n}_{x_0})} = \sum_{\substack{x\neq x_0\\ x\in D}} u_{\bar{n}_x,x} + u_{\bar{n}_{x_0}} + (r' - n_1(x_0)r)$$
$$= \sum_{x\in D} u_{\bar{n}_x,x} + (r' - n_1(x_0)r) = u_{\bar{n}} + (r' - n_1(x_0)r).$$

Therefore, if $u_{\bar{n}} \equiv k \pmod r$, then $u_{\mathcal{R}_{x_0}(\bar{n})} \equiv k + r' \pmod r$.

We know that $u_{\bar{n}}$ is always a multiple of $\gcd(r, r')$. Let us define the set

$$G = \{a \cdot \gcd(r, r') \pmod r \,|\, a \in \mathbb{Z}\}.$$

Then, G is a subgroup of $\mathbb{Z}/r\mathbb{Z}$ of order $r/gcd(r, r')$, and r' is a generator of the group. Let $\Omega_k = \{\bar{n} \in \Omega_{n,r,r'} \,|\, u_{\bar{n}} \equiv k \pmod r\}$, then for any $k, k' \in G$ there exists $p \in \mathbb{Z}$ such that $\mathcal{R}^p_{x_0}(\bar{n})$ induces a bijection between Ω_k and $\Omega_{k'}$.

Thus, there is the same number of elements in Ω_k and $\Omega_{k'}$, and since $U_{n,r,r',k}$ is the number of elements in Ω_k, we have that $U_{n,r,r',k} = U_{r,r',k'}$ for all $k, k' \in G$. This means that all $\bar{n} \in \Omega_{n,r,r'}$ are evenly distributed across all different Ω_k. Same reasoning applies to $L_{n,r,r',k}$. $|\Omega_{n,r,r'}|$ is the number of selection vectors of rank r and subrank r' over (X, D), which is given by $\binom{r}{r'}^n$, for all $r' \neq r/2$. For $r' = r/2$, we have that $\binom{r}{r'}^n$ is even, and thus, the number of selection vectors of rank r and subrank r' over (X, D) is given by $\frac{1}{2}\binom{r}{r'}^n$.

Since there are $r/gcd(r, r')$ different values of $k \pmod r$, we have that

$$\begin{cases} U_{n,r,r',k} = L_{n,r,r',k} = \frac{|\Omega_{n,r,r'}|}{r/gcd(r,r')} = \frac{gcd(r,r')}{r}\binom{r}{r'}^n & \text{if } k \equiv 0 \pmod{gcd(r, r')} \\ U_{n,r,r',k} = L_{n,r,r',k} = 0 & \text{otherwise} \end{cases}$$

□

With this we can find a more refined expression for $W_{n,r,r'}$.

Lemma 4.2.4. *The number of different walls in the space of stability conditions of rank r, degree* 0 *with n parabolic points and subrank r' is*

$$W_{n,r,r'} = \frac{1}{r}\binom{r}{r'}^{n-1}\left(nS_{r,r'} - \binom{r}{r'}gcd(r, r')\right),$$

where $S_{r,r'}$ is defined as

$$S_{r,r'} = \sum_{\bar{n}\in\Omega_{1,r,r'}} (u_{\bar{n}} - l_{\bar{n}}).$$

Proof. We will make use of the following modular arithmetic identities to simplify the expression of $W_{n,r,r'}$ derived in Lemma 4.2.2.

$$\left\lfloor\frac{a}{b}\right\rfloor = \frac{a - (a \bmod b)}{b}, \quad \left\lceil\frac{a}{b}\right\rceil = \left\lfloor\frac{a}{b}\right\rfloor + [a \not\equiv 0 \bmod b],$$

where $[a \not\equiv 0 \bmod b]$ is the Iverson bracket, which is 1 if $a \not\equiv 0 \bmod b$ and 0 otherwise.

We can rewrite the expression for $W_{n,r,r'}$ as follows:

$$\begin{aligned} W_{n,r,r'} &= \sum_{\bar{n}\in\Omega_{n,r,r'}} \left(\frac{u_{\bar{n}} - (u_{\bar{n}} \bmod r)}{r} + [u_{\bar{n}} \not\equiv 0 \bmod r] - \frac{l_{\bar{n}} - (l_{\bar{n}} \bmod r)}{r} - 1 \right) \\ &= \sum_{\bar{n}\in\Omega_{n,r,r'}} \left(\frac{u_{\bar{n}} - (u_{\bar{n}} \bmod r)}{r} - \frac{l_{\bar{n}} - (l_{\bar{n}} \bmod r)}{r} - [u_{\bar{n}} \equiv 0 \bmod r] \right) \\ &= \frac{1}{r} \sum_{\bar{n}\in\Omega_{n,r,r'}} (u_{\bar{n}} - l_{\bar{n}}) - \frac{1}{r} \sum_{\bar{n}\in\Omega_{n,r,r'}} u_{\bar{n}} \bmod r + \frac{1}{r} \sum_{\bar{n}\in\Omega_{n,r,r'}} l_{\bar{n}} \bmod r \\ &\qquad - \sum_{\bar{n}\in\Omega_{n,r,r'}} [u_{\bar{n}} \equiv 0 \bmod r]. \end{aligned}$$

Let us now look at the first term. Since $\Omega_{n,r,r'}$ is a product of n sets $\Omega_{1,r,r'}$, we can represent $\bar{n}$ as $\bar{n} = (\bar{n}_{x_1}, \dots, \bar{n}_{x_n})$, where $\bar{n}_{x_i} = (n_1(x_i), \dots, n_r(x_i)) \in \Omega_{1,r,r'}$ for all $i \in \{1, \dots, n\}$. Also, from the definition of $u_{\bar{n}}$ and $l_{\bar{n}}$, we have that $u_{\bar{n}} = \sum_{x\in D} u_{\bar{n}_x,x}$ and $l_{\bar{n}} = \sum_{x\in D} l_{\bar{n}_x,x}$, with $u_{\bar{n}_x,x}$ and $l_{\bar{n}_x,x}$ as defined in equation (4.2.1). Thus, we can rewrite the first term as follows.

$$\frac{1}{r} \sum_{\bar{n}\in\Omega_{n,r,r'}} (u_{\bar{n}} - l_{\bar{n}}) = \frac{1}{r} \sum_{\bar{n}\in\Omega_{n,r,r'}} \sum_{x\in D} (u_{\bar{n}_x,x} - l_{\bar{n}_x,x}).$$

Notice that $u_{\bar{n}_x,x}$ and $l_{\bar{n}_x}$ only depend on the selection vector $\bar{n}_x \in \Omega_{1,r,r'}$ and not on the choice of point $x \in D$. Thus, we can use the standard notation for $u_{\bar{n}}$ and $l_{\bar{n}}$, with $\bar{n} \in \Omega_{1,r,r'}$, and rewrite the first term as follows.

$$\begin{aligned} \frac{1}{r} \sum_{\bar{n}\in\Omega_{n,r,r'}} \sum_{x\in D} (u_{\bar{n}_x,x} - l_{\bar{n}_x,x}) &= \frac{1}{r} \sum_{\bar{n}_{x_1}\in\Omega_{1,r,r'}} \cdots \sum_{\bar{n}_{x_n}\in\Omega_{1,r,r'}} \sum_{i\in\{0,\dots,n\}} (u_{\bar{n}_{x_i}} - l_{\bar{n}_{x_i}}) \\ &= \frac{1}{r} \left(\sum_{\bar{n}_{x_1}\in\Omega_{1,r,r'}} \cdots \sum_{\bar{n}_{x_n}\in\Omega_{1,r,r'}} (u_{\bar{n}_{x_1}} - l_{\bar{n}_{x_1}}) + \dots \right. \\ &\qquad \left. + \sum_{\bar{n}_{x_1}\in\Omega_{1,r,r'}} \cdots \sum_{\bar{n}_{x_n}\in\Omega_{1,r,r'}} (u_{\bar{n}_{x_n}} - l_{\bar{n}_{x_n}}) \right). \end{aligned}$$

Now, we can reorder each of the n summations, so that we sum over the selection vectors $\bar{n}_{x_i} \in \Omega_{1,r,r'}$ first. Notice the sum

$$\sum_{\bar{n}_{x_i}\in\Omega_{1,r,r'}} (u_{\bar{n}_{x_i}} - l_{\bar{n}_{x_i}})$$

is the same for all $i = 1, \dots, n$, since it only depends on the set $\Omega_{1,r,r'}$. We will denote this sum as $S_{r,r'} = \sum_{\bar{n}\in\Omega_{1,r,r'}} (u_{\bar{n}} - l_{\bar{n}})$, and thus we are left with

$$\frac{1}{r} n S_{r,r'} \sum_{\bar{n}_{x_1}\in\Omega_{n-1,r,r'}} 1 = \frac{1}{r} n S_{r,r'} \binom{r}{r'}^{n-1},$$

where $\binom{r}{r'}$ is the number of selection vectors $\bar{n}_{x_i} \in \Omega_{1,r,r'}$.

Now, let us look at the second term. We have

$$-\frac{1}{r}\sum_{\bar{n}\in\Omega_{n,r,r'}} u_{\bar{n}} \bmod r = -\frac{1}{r}\sum_{0\le j<r} j\cdot U_{n,r,r',j},$$

where $U_{n,r,r',j}$ is the number of selection vectors $\bar{n}\in\Omega_{n,r,r'}$ such that $u_{\bar{n}}\equiv j \pmod r$. Same reasoning applies to the third term, where we have

$$\frac{1}{r}\sum_{\bar{n}\in\Omega_{n,r,r'}} l_{\bar{n}} \bmod r = \frac{1}{r}\sum_{0\le j<r} j\cdot L_{n,r,r',j},$$

where $L_{n,r,r',j}$ is the number of selection vectors $\bar{n}\in\Omega_{n,r,r'}$ such that $l_{\bar{n}}\equiv j \pmod r$. In Lemma 4.2.3, we showed that $U_{n,r,r',j}=L_{n,r,r',j}$ for all n,r,r',j, so both these terms cancel out.

Finally, we have

$$\sum_{\bar{n}\in\Omega_{n,r,r'}} [u_{\bar{n}}\equiv 0 \bmod r] = U_{n,r,r',0} = \frac{\gcd(r,r')}{r}\binom{r}{r'}^{n},$$

where $U_{n,r,r',0}$ is the number of selection vectors $\bar{n}\in\Omega_{n,r,r'}$ such that $u_{\bar{n}}\equiv 0 \pmod r$ as proven in Lemma 4.2.3.

Thus, combining all terms we get the expression for $W_{n,r,r'}$ as follows.

$$W_{n,r,r'} = \frac{1}{r}\binom{r}{r'}^{n-1}\left(nS_{r,r'} - \binom{r}{r'}\gcd(r,r')\right).$$

□

This expression is almost simplified. It only depends on the term $S_{r,r'}$, which is the sum of the range of possible intercepts for all hyperplanes $W_{\bar{n},d'}$, where $\bar{n}\in\Omega_{1,r,r'}$. Attempts to simplify this term further led to a complex expression and recurrence relations. Although $S_{r,r'}$ resists a closed formula, bounding it suffices for an asymptotic analysis of the number of walls in the space of stability conditions.

4.3 Bounds for the number of walls

In this section, we derive bounds on the number of walls in the space of stability conditions for parabolic vector bundles of rank r, degree 0, and n marked points by estimating the term $S_{r,r'}$ defined in Lemma 4.2.4. We also provide an asymptotic analysis of the growth of the number of walls as a function of r and n.

In order to do so, we first need to rewrite the intercept bounds $u_{\bar{n}}$ and $l_{\bar{n}}$ in a more compact expression.

Lemma 4.3.1. *Let $\bar{n}\in\Omega_{1,r,r'}$. Let $u_{\bar{n}}$ and $l_{\bar{n}}$ be the upper and lower bounds of the intercepts of the hyperplanes $W_{\bar{n},d'}$, and let $\mathcal{P}:\Omega_{n,r,r'}\longrightarrow [1,r]^{r'}$ be the natural fuction associating bijectively the set of selection vectors $\bar{n}$ to the ordered set of r' indices $i_1,\ldots,i_{r'}$ such that $\bar{n}_{i_k}(x)=1$ for all $k=1,\ldots,r'$ and $x\in D$. Then, we have*

$$u_{\bar{n}} = \max_{1\le k\le r'} kr - i_k r'$$

$$l_{\bar{n}} = \min_{1\le k\le r'} (k-1)r - (i_k-1)r'$$

Proof. Since we are working with a single parabolic point $x \in D$, we can drop the index x from the notation, so that $w_{\bar{n}}(x) = w_{\bar{n}}$ and $n_i(x) = n_i$ for all $i = 1, \ldots, r$. Recall the definition of $u_{\bar{n}}$ and $l_{\bar{n}}$ from lemma 4.1.2:

$$u_{\bar{n}} = \max_{1 \leq j \leq r} \sum_{i=j}^{r} w_{\bar{n},i}$$

$$l_{\bar{n}} = \min_{1 \leq j \leq r} \sum_{i=j}^{r} w_{\bar{n},i}.$$

and recall $w_{\bar{n},i} = r' - n_i r$. Since $0 < r' < r$ we have the following for all $i = 1, \ldots, r$.

$$\begin{cases} w_{\bar{n},i} < 0 & \text{if } n_i = 1 \\ w_{\bar{n},i} > 0 & \text{if } n_i = 0. \end{cases}$$

Thus, the sequence $\left\{\sum_{i=j}^{r} w_{\bar{n},i}\right\}_j = \{rr' - r\left(\sum_{i=1}^{r} n_i\right), \ldots, r' - rn_r\}$ is 0 for $j = 1$ and decreases until $j = i_1$, since we are removing positive terms from the sums, then it increases at $j = i_1 + 1$ since we remove a negative term from the sum, then it decreases again until $j = i_2$, and so on. Thus, $i_k + 1$ are local maxima, and i_k are local minima, and the maximum and minimum of the sequence are attained at $j = i_k$ and $j = i_k + 1$ for some k, respectively.

$$\begin{aligned} u_{\bar{n}} = \max_{1 \leq k \leq r'} \sum_{i=i_k+1}^{r} w_{\bar{n},i} &= \max_{1 \leq k \leq r'} \sum_{i=i_k+1}^{r} (r' - n_i r) \\ &= \max_{1 \leq k \leq r'} (r - i_k)r' - \left(\sum_{i=i_k+1}^{r} n_i\right) r = \max_{1 \leq k \leq r'} (r - i_k)r' - (r' - k)r \\ &= \max_{1 \leq k \leq r'} rr' - i_k r' - r'r + kr = \max_{1 \leq k \leq r'} kr - i_k r'. \end{aligned}$$

Similarly, we have

$$\begin{aligned} l_{\bar{n}} = \min_{1 \leq k \leq r'} \sum_{i=i_k}^{r} w_{\bar{n},i} &= \min_{1 \leq k \leq r'} \sum_{i=i_k}^{r} (r' - n_i r) \\ &= \min_{1 \leq k \leq r'} (r - i_k + 1)r' - \left(\sum_{i=i_k}^{r} n_i\right) r = \min_{1 \leq k \leq r'} (r - i_k + 1)r' - (r' - k + 1)r \\ &= \min_{1 \leq k \leq r'} rr' - i_k r' + r' - r'r + kr - r = \min_{1 \leq k \leq r'} (k-1)r - (i_k - 1)r'. \end{aligned}$$

□

With this formulation, we can find bounds for the term $S_{r,r'}$.

Lemma 4.3.2. *Let $S_{r,r'} = \sum_{\bar{n} \in \Omega_{1,r,r'}} (u_{\bar{n}} - l_{\bar{n}})$, where $u_{\bar{n}}$ and $l_{\bar{n}}$ are the upper and lower bounds of the intercepts of the hyperplanes $W_{\bar{n},d'}$, respectively. Then, we have*

$$(r - r')\binom{r}{r'} \leq S_{r,r'} \leq r'(r - r')\binom{r}{r'}$$

Proof. To bound this term, we will bound the difference $u_{\bar{n}} - l_{\bar{n}}$ for all $\bar{n} \in \Omega_{1,r,r'}$. Starting with the lower bound, we have

$$
\begin{aligned}
u_{\bar{n}} - l_{\bar{n}} &= \max_{1\le k\le r'} kr - i_k r' - \min_{1\le k\le r'} (k-1)r - (i_k - 1)r' \\
&= \max_{1\le k\le r'} kr - i_k r' + \max_{1\le k\le r'} -(k-1)r + (i_k-1)r' \ge \max_{1\le k\le r'} kr - i_k r' - (k-1)r + (i_k-1)r' \\
&= \max_{1\le k\le r'} r - r' = r - r'.
\end{aligned}
$$

Thus, we have

$$
S_{r,r'} = \sum_{\bar{n}\in\Omega_{1,r,r'}} (u_{\bar{n}} - l_{\bar{n}}) \ge \sum_{\bar{n}\in\Omega_{1,r,r'}} (r - r') = (r - r')\binom{r}{r'}.
$$

Now, let us look at the upper bound. Let's state some inequalities from the previous formulation:

$$
\begin{cases} 1 \le k \le r' \\ k \le i_{k-1} < i_k \le r - (r' - k) \end{cases}
$$

which implies that $k' - k'' \le i_{k'} - i_{k''} \le (r - r') + (k' - k'')$ for all $k' \ge k'' \in [1, r']$. Thus, we have

$$
\begin{aligned}
u_{\bar{n}} - l_{\bar{n}} &= \max_{1\le k\le r'} kr - i_k r' - \min_{1\le k\le r'} (k-1)r - (i_k - 1)r' \\
&= (r - r') + \max_{1\le k\le r'} kr - i_k r' - \min_{1\le k\le r'} kr - i_k r' \\
&= (r - r') + k'r - i_{k'} r' - (k''r - i_{k''} r') = (r - r') + (k' - k'')r - (i_{k'} - i_{k''})r'
\end{aligned}
$$

for some $k', k'' \in [1, r']$.

Let us look at the case $k' \ge k''$. Then, we have

$$
\begin{aligned}
u_{\bar{n}} - l_{\bar{n}} &= (r - r') + (k' - k'')r - (i_{k'} - i_{k''})r' \le (r - r') + (k' - k'')r - (k' - k'')r' \\
&= (r - r') + (r - r')(k' - k'') \le (r - r') + (r - r')(r' - 1) = r'(r - r')
\end{aligned}
$$

And if $k' \le k''$, we can apply the same reasoning, and we have

$$
\begin{aligned}
u_{\bar{n}} - l_{\bar{n}} &= (r - r') - (k'' - k')r + (i_{k''} - i_{k'})r' \\
&\le (r - r') - (k'' - k')r + ((r - r') + (k'' - k'))r' \\
&= (r - r') - (k'' - k')r + (r - r')r' + (k'' - k')r' \\
&= (r - r') - (k'' - k')(r - r') + r'(r - r') \\
&\le (r - r') - (r - r') + r'(r - r') = r'(r - r').
\end{aligned}
$$

Thus, we conclude that

$$
S_{r,r'} = \sum_{\bar{n}\in\Omega_{1,r,r'}} (u_{\bar{n}} - l_{\bar{n}}) \le \sum_{\bar{n}\in\Omega_{1,r,r'}} (r'(r - r')) = r'(r - r')\binom{r}{r'}.
$$

$\square$

Finally, we can plug the bounds for $S_{r,r'}$ into the expression for $W_{n,r,r'}$ to obtain bounds for the number of walls.

Theorem 4.3.3. *The number of different walls in the space of stability conditions of rank r, degree 0 with n parabolic points is bounded by*

$$W_{n,r} = \begin{cases} \sum_{r'=1}^{\lfloor r/2 \rfloor} W_{n,r,r'} & \text{if } r \text{ is odd} \\ \sum_{r'=1}^{r/2-1} W_{n,r,r'} + \frac{1}{2} W_{n,r,r/2} & \text{if } r \text{ is even} \end{cases}$$

where

$$\frac{1}{r}\binom{r}{r'}^n (n(r-r') - gcd(r,r')) \leq W_{n,r,r'} \leq \frac{1}{r}\binom{r}{r'}^n (nr'(r-r') - gcd(r,r')).$$

Proof. The proof follows directly from the previous lemmas. □

To wrap up the section, we will give a precise asymptotic behavior of the number of walls in the space of stability conditions of rank r, degree 0 with n parabolic points.

Theorem 4.3.4. *Let $W_{n,r}$ denote the number of different walls in the space of stability conditions of rank r, degree 0, with n parabolic points. Then, as $r \to \infty$ with n varying, we have*

$$W_{n,r} = O\left(nr \left(\sqrt{\frac{2}{\pi}} \right)^n \cdot \frac{2^{nr}}{r^{n/2}} \right) \quad \text{and} \quad W_{n,r} = \Omega\left(n \left(\sqrt{\frac{2}{\pi}} \right)^n \cdot \frac{2^{nr}}{r^{n/2}} \right).$$

Proof. We analyze the asymptotic behavior of $W_{n,r}$ using the upper bound in Theorem 4.3.3, where the dominant contribution to the sum comes from the term with $r' = \lfloor r/2 \rfloor$.

Let us first consider the case when r is even, say $r = 2k$, and take $r' = k$. Since $\gcd(2k, k) = k$, the lower and upper bounds for $W_{n,r,k}$ are given by

$$W_{n,r,k} \leq \frac{1}{2k}\binom{2k}{k}^n (nk^2 - k) = \frac{1}{2}\left(\frac{nr}{2} - 1\right)\binom{r}{r/2}^n \sim \frac{1}{4} nr \binom{r}{r/2}^n$$

and

$$W_{n,r,k} \geq \frac{1}{2k}\binom{2k}{k}^n (nk - k) = \frac{1}{2}(n-1)\binom{r}{r/2}^n \sim \frac{1}{2} n \binom{r}{r/2}^n.$$

p Let us now look at the case when r is odd, say $r = 2k+1$, and take $r' = k$. Since $\gcd(2k+1, k) = 1$, we have

$$W_{n,r,k} \leq \frac{1}{2k+1}\binom{2k+1}{k}^n (nk(k+1) - 1) \sim \frac{1}{4} nr \binom{r}{\lfloor r/2 \rfloor}^n$$

and

$$W_{n,r,k} \geq \frac{1}{2k+1}\binom{2k+1}{k}^n (n(k+1) - 1) \sim \frac{1}{2}\binom{r}{\lfloor r/2 \rfloor}^n.$$

Indeed, for both cases, the asymptotic behavior of the bounds is the same. Now we can use Stirling's approximation on the central binomial coefficient, which works on both even and odd cases. For large r, we have

$$\binom{r}{\lfloor r/2 \rfloor} \sim \frac{2^r}{\sqrt{\pi r/2}} = \sqrt{\frac{2}{\pi}} \cdot \frac{2^r}{\sqrt{r}},$$

and hence

$$\binom{r}{\lfloor r/2 \rfloor}^n \sim \left(\sqrt{\frac{2}{\pi}}\right)^n \cdot \frac{2^{nr}}{r^{n/2}}.$$

Combining the dominant term with the binomial estimate gives

$$W_{n,r} = O\left(nr\left(\sqrt{\frac{2}{\pi}}\right)^n \frac{2^{nr}}{r^{n/2}}\right),$$

and

$$W_{n,r} = \Omega\left(n\left(\sqrt{\frac{2}{\pi}}\right)^n \frac{2^{nr}}{r^{n/2}}\right)$$

as claimed. □

4.4 Bounds for the number of stability chambers

In this section, we derive preliminary bounds on the number of distinct parabolic chambers in the space of stability conditions for vector bundles of rank r and degree 0 with n marked points. These bounds are obtained using the results established in the previous section.

Lemma 4.4.1. *Let $\mathcal{H}$ be a finite arrangement of hyperplanes in $\mathbb{R}^d$, and let $\mathcal{C}$ be the number of regions (chambers) into which $\mathcal{H}$ divides $\mathbb{R}^d$. Let $P \subset \mathbb{R}^d$ be a convex polytope. Then the number of connected components of $P \setminus \bigcup_{H \in \mathcal{H}} H$ is less than or equal to $\mathcal{C}$.*

Proof. Since P is the intersection of finitely many halfspaces, we can restrict to one halfspace at a time. At each step, a halfspace can remove some chambers (lying entirely outside), and/or intersect some chambers and keep only one of the two portions. In both cases, no new regions are created—only existing ones are possibly truncated or discarded. Therefore, the number of regions in P remains less than or equal to the number in $\mathbb{R}^d$. □

Theorem 4.4.2. *Let $C_{n,r}$ denote the number of distinct parabolic chambers in the space of stability conditions of rank r and degree 0 with n parabolic points. Let $W_{n,r}$ be the number of walls (hyperplanes) in this space. Then:*

$$W_{n,r} + 1 \leq C_{n,r} \leq \sum_{i=0}^{nr} \binom{W_{n,r}}{i}$$

Proof. The lower bound follows from the fact that minimum number of regions a set of m hyperplanes can create inside a convex polytope is $m + 1$, when all hyperplanes are parallel.

The upper bound follows from the classical result in combinatorial geometry given by Schläfli [Sch01], which states that the maximal number of regions formed by n hyperplanes in general position in $\mathbb{R}^d$ is

$$R(n, d) = \sum_{i=0}^{d} \binom{n}{i}.$$

In our context, we consider an arrangement of $W_{n,r}$ hyperplanes in $\mathbb{R}^{nr}$, corresponding to walls in the space of stability conditions.

These hyperplanes partition the entire space $\mathbb{R}^{nr}$ into at most $\sum_{i=0}^{nr} \binom{W_{n,r}}{i}$ regions. However, we are only interested in the regions that lie within a certain convex polytope $\mathcal{A}_{n,r}$, which defines the domain of stability conditions.

By the lemma above, intersecting this global partition with the polytope $\mathcal{A}_{n,r}$ can only reduce (or preserve) the number of chambers, not increase it. Therefore, the number of parabolic chambers $C_{n,r}$ is bounded by:

$$C_{n,r} \leq \sum_{i=0}^{nr} \binom{W_{n,r}}{i}.$$

□

The number of stability chambers increases rapidly with both n and r. Although upper and lower bounds provide insight into the overall growth behavior, the upper bounds are often extremely loose, particularly for larger values of n and r. In such cases, they function more as theoretical ceilings than as practical estimates. In contrast, the lower bounds tend to be much closer to the actual number of chambers for small values of n and r, but they still substantially underestimate the true growth rate as the parameters increase.

Below we show the lower bounds, upper bounds, and known actual values for the number of chambers when $d = 0$, for $n \leq 6$ and $r \leq 6$.

n\r	**1**	**2**	**3**	**4**	**5**	**6**
1	$1 \leq$ 1 ≤ 1	$2 \leq$ 2 ≤ 2	$4 \leq$ 4 < 8	$12 <$ 14 $< 1 \times 10^3$	$22 <$ 80 $< 9 \times 10^4$	$66 <$ 1296 $< 8 \times 10^8$
2	$2 \leq$ 2 < 2	$10 <$ 12 $< 5 \times 10^2$	$42 <$ 640 $< 2 \times 10^8$	–	–	–
3	$5 \leq$ 5 < 16	$46 <$ 720 $< 2 \times 10^9$	–	–	–	–
4	$13 <$ 24 < 3797	–	–	–	–	–
5	$33 <$ 409 $< 2 \times 10^8$	–	–	–	–	–
6	$81 <$ 31916 $< 8 \times 10^{13}$	–	–	–	–	–

Table 4.1: Comparison of lower bound, actual value, and upper bound for the number of stability chambers ($d = 0$).

Chapter 5

Duality

This chapter delves into the impact of duality on the moduli space of parabolic vector bundles. Building on the conjectures derived from the data in §3.6, we examine the interplay between dualization and the structural properties of these spaces.

5.1 Duality breaks automorphisms

While duality can sometimes behave like a symmetry in rank $r = 2$, our findings suggest that this does not extend to higher ranks. In fact, we will show that for $r > 2$, the dual of a parabolic bundle never induces an automorphism of the moduli space. This reflects a deeper asymmetry in the geometry of these spaces as the rank increases.

Lemma 5.1.1. *If $T = (\sigma, s, L, H) \in \mathcal{T}_\xi$ and $\deg(\xi) = 0$, then r divides $|H|$.*

Proof. If $T \in \mathcal{T}_\xi$, then

$$\sigma^*(L^r \otimes \xi(-H))^s \cong \xi$$

Taking degrees on both sides, and taking into account that σ does not change the degree yields

$$s(r\deg(L) + \deg(\xi) - |H|) = \deg(\xi)$$

As $\deg(\xi) = 0$, we have

$$sr\deg(L) - s|H| = 0$$

so $|H| = r\deg(L)$ and, therefore, r divides $|H|$. □

Lemma 5.1.2. *Let $r \geq 2$ and n be positive integers. Let X be a smooth complex projective curve and let $D = \{x_1, \dots, x_n\}$ be a set of n different points on X. Let α be a generic system of weights on (X, D) in the sense of Definition 2.1.3. If $T = (\sigma, s, L, H) \in \mathcal{T}_\xi$ is a basic transformation inducing an automorphism of $\mathcal{M}(X, r, \alpha, \xi)$, then there exists another generic weight $\alpha' \in C_\alpha$ in the same stability chamber as α such that $\alpha' = T(\alpha')$.*

Proof. Let T be a basic transformation inducing an automorphism of $\mathcal{M}(X, r, \alpha, \xi)$. Then, the induced map on the space of parabolic chambers $T : \tilde{SA}_{n,r} \longrightarrow \tilde{SA}_{n,r}$ preserves the parabolic chamber C_α. From [Alf22, Lemma 3.4], we know that the group $\mathcal{T}_\xi$ is finite. Let N be the order of T in $\mathcal{T}_\xi$. Let us consider the orbit of α in $\tilde{SA}_{n,r}$ by the action of the finite subgroup $\langle T \rangle < \mathcal{T}_\xi$ generated by T, i.e., the set of points

$$\{\alpha, T(\alpha), T^2(\alpha), \dots, T^{N-1}(\alpha)\} \subset \tilde{SA}_{n,r}.$$

As T preserves the stability chamber of α and preserves the genericity of the weights, $T^i(\alpha)$ is a generic weight in C_α for each $i = 1, \ldots, |\mathcal{T}_\xi| - 1$. Let

$$\alpha' = \frac{1}{N} \sum_{i=0}^{N-1} T^i(\alpha).$$

From the equations of the action (2.3.2) we observe that T is an affine map. Thus,

$$T(\alpha') = T\left(\frac{1}{N} \sum_{i=0}^{N-1} T^i(\alpha)\right) = \frac{1}{N} \sum_{i=1}^{N} T^i(\alpha).$$

As T has order N, then $T^N(\alpha) = \alpha$, so

$$T(\alpha') = \frac{1}{N} \sum_{i=1}^{N} T^i(\alpha) = \frac{1}{N} \sum_{i=0}^{N-1} T^i(\alpha) = \alpha'.$$

Finally, observe that the parabolic chambers for the moduli space are defined as intersections of open half-spaces. Therefore, they are convex. Since α' is a convex combination of a set of points in the convex chamber C_α, then $\alpha' \in C_\alpha$. □

Lemma 5.1.3. *Let $r > 2$ and n be positive integers. Let X be a smooth complex projective curve of genus $g \geq \max(6, 1 + (r-1)n)$ and let $D = \{x_1, \ldots, x_n\}$ be a set of n different points on X. Let α be a generic system of weights on (X, D) in the sense of Definition 2.1.3. If $T = (\sigma, s, L, H) \in \mathcal{T}_\xi$ is a basic transformation inducing an automorphism of $\mathcal{M}(X, r, \alpha, \xi)$, then $s = 1$.*

Proof. We can reduce the proof to the case where $\deg(\xi) = 0$. To see this, assume first that $d = \deg(\xi)$. Let $x \in D$ be any parabolic point. Let $\alpha' = \mathcal{H}_{dx}(\alpha')$ and $\xi' = \mathcal{H}_{dx}(\xi) = \xi(-dx)$. Then $\mathcal{H}_{dx}$ induces an isomorphism $\mathcal{H}_{dx} : \mathcal{M}(X, r, \alpha, \xi) \xrightarrow{\sim} \mathcal{M}(X, r, \alpha', \xi')$. Let $T' = \mathcal{H}_{dx} \circ T \circ \mathcal{H}_{dx}^{-1}$. Then T' is an automorphism of $\mathcal{M}(X, r, \alpha', \xi')$. By the relations from [AG21, Lemma 5.7], it is straightforward to verify that $T' = (\sigma, s, L', H')$ for some L' and H'. As s does not change, if we prove that $\mathcal{M}(X, r, \alpha', \xi')$ does not admit any automorphisms with $s = 1$, then nor does $\mathcal{M}(X, r, \alpha, \xi)$.

From this point on, we will assume that $d = \deg(\xi) = 0$. Moreover, by Lemma 5.1.2, we can assume without loss of generality that $T(\alpha) = \alpha$. Suppose that $s = -1$ and let $H = \sum_{x \in D} h_x x$. Then, by (2.3.2), $T(\alpha) = \alpha$ implies the following system of linear equations

$$\begin{cases} \alpha_1(\sigma(x)) = 0 = \alpha_{h_x}(x) - \alpha_{h_x}(x) \\ \alpha_2(\sigma(x)) = \alpha_{h_x}(x) - \alpha_{h_x - 1}(x) \\ \vdots \\ \alpha_{h_x}(\sigma(x)) = \alpha_{h_x}(x) - \alpha_1(x) & \forall x \in D \\ \alpha_{h_x+1}(\sigma(x)) = \alpha_{h_x}(x) + 1 - \alpha_r(x) \\ \vdots \\ \alpha_r(\sigma(x)) = \alpha_{h_x}(x) + 1 - \alpha_{h_x+1}(x) \end{cases} \tag{5.1.1}$$

where, by coherence with the periodicity conditions [Sim90], we will take $\alpha_0(x) = \alpha_r(x) - 1$ if $h_x = 0$ for some $x \in D$.

Summing both sides of all the equations from (5.1.1) across $x \in D$ and observing that the permutation does not affect the total sum of the left hand side yields

$$\sum_{x\in D}\sum_{i=1}^{r}\alpha_i(x) = \sum_{x\in D}\left(r\alpha_{h_x}(x) + (r-h_x)\right) - \sum_{x\in D}\sum_{i=1}^{r}\alpha_i(x).$$

Rearranging yields

$$2\sum_{x\in D}\sum_{i=1}^{r}\alpha_i(x) - r\sum_{x\in D}\alpha_{h_x}(x) = |D|r - |H|.$$

Let $D_0 = D\setminus \operatorname{supp}(H) = \{x \in D \mid h_x = 0\}$. Then, by the convention $\alpha_0(x) = \alpha_r(x) - 1$ we have

$$r\sum_{x\in D_0}\alpha_{h_x}(x) = r\sum_{x\in D_0}(\alpha_r(x)-1) = r\sum_{x\in D_0}\alpha_r(x) - r(|D_0|).$$

If we take into account that $\alpha_1(x) = 0$ for all $x \in D$, we can then combine the previous identities into the following.

$$2\sum_{x\in D}\sum_{i=1}^{r}\alpha_i(x) - r\sum_{x\in D_0}(\alpha_1(x)+\alpha_r(x)) - r\sum_{x\in D\setminus D_0}(\alpha_1(x)+\alpha_{h_x}(x)) = |D|r - |H| + r|D_0|. \tag{5.1.2}$$

By Lemma 5.1.1, r divides $|H|$, so the independent term of (5.1.2) is always divisible by r, and the previous expression has almost exactly the form of the equation of a wall with $r' = 2$ as given by (2.1.4), where the weights that have been selected for each x are $\alpha_1(x)$ and $\alpha_{h_x}(x)$ (or $\alpha_r(x)$ if $h_x = 0$). The only problem with this approach is that if $h_x = 1$ for some $x \in D$, then $\alpha_1(x)$ should be selected twice for that point, and this is not possible. However, if this happens, we can show that we can modify the previous expression using the identities from (5.1.1) in order to obtain an alternative equation which is congruent to (5.1.2) modulo r, but in which no weight is selected twice.

For each $x \in D_1$ the equations (5.1.1) simplify into

$$\begin{cases} \alpha_1(\sigma(x)) = \alpha_1(x) = 0 \\ \alpha_2(\sigma(x)) = 1 - \alpha_r(x) \\ \alpha_3(\sigma(x)) = 1 - \alpha_{r-1}(x) \\ \vdots \\ \alpha_r(\sigma(x)) = 1 - \alpha_2(x) \end{cases} \tag{5.1.3}$$

Let us decompose D_1 in a disjoint set of σ-chains and σ-cycles as follows.

$$D_1 = \bigcup_{i\in I_C} C_i \cup \bigcup_{i\in I_R} R_i$$

with $C_i = \{x_{i,1}, \ldots, x_{i,c_i}\}$ is a σ-cycle, i.e., it has the property $\sigma(x_{i,j}) = x_{i,j+1}$ for $j < c_i$ and $\sigma(x_{i,c_i}) = x_{i,1}$ and $R_i = \{y_{i,1}, \ldots, y_{i,r_i}\}$ is a maximal σ-chain with the property $\sigma(y_{i,j}) = y_{i,j+1}$ for $j < r_i$ and $\sigma(x_{i,r_i}) \notin D_1$. Let us further separate the set of cycles $I_C = I_C^o \cup I_C^e$ into the set of cycles of odd degree and the set of cycles of even degree. Analogously, let $I_R = I_R^o \cup I_R^e$ be the split into the set of chains such that r_i is odd or even respectively.

Let C_i with $i \in I_C^o$ be a σ-cycle of odd order contained in D_1. Then, iterating (5.1.3) through the cycle until we reach the starting point (this is done an odd number of times) we obtain that for each $x \in C_i$ we have

$$\begin{cases} \alpha_1(x) = 0 \\ \alpha_2(x) = 1 - \alpha_r(x) \\ \alpha_3(x) = 1 - \alpha_{r-1}(x) \\ \vdots \\ \alpha_r(x) = 1 - \alpha_2(x) \end{cases}$$

In particular, for each $x \in D_1$ belonging to any such cycle of odd degree, we have $\alpha_2(x) + \alpha_r(x) = 1$. Then, for each $i \in I_C^o$

$$\begin{aligned} r\sum_{x\in C_i}(\alpha_1(x) + \alpha_{h_x}(x)) = 0 = r\sum_{x\in C_i}(\alpha_2(x) + \alpha_r(x) - 1) = r\sum_{x\in C_i}(\alpha_2(x) + \alpha_r(x)) - rc_i \\ \equiv r\sum_{x\in C_i}(\alpha_2(x) + \alpha_r(x)) \pmod r \end{aligned} \quad (5.1.4)$$

Let C_i with $i \in I_C^e$ be a σ-cycle with even degree. By (5.1.1) we have for each $i > 1$ we have

$$\begin{aligned} \alpha_i(x_{i,2k}) + \alpha_{r+2-i}(x_{i,2k-1}) &= \alpha_i(\sigma(x_{i,2k-1})) + \alpha_{r+2-i}(x_{i,2k-1}) \\ &= 1 = \alpha_1(x_{i,2k}) + \alpha_1(x_{i,2k-1}) + 1 \end{aligned}$$

for each $k = 1, \dots, c_i/2$. Thus,

$$\begin{aligned} &r\sum_{x\in C_i}(\alpha_1(x) + \alpha_1(x)) \\ &\quad = r\sum_{k=1}^{c_i/2}(\alpha_1(x_{i,2k-1}) + \alpha_2(x_{i,2k-1})) + r\sum_{k=1}^{c_i/2}(\alpha_1(x_{i,2k}) + \alpha_r(x_{i,2k})) - r\frac{c_i}{2} \\ &\quad \equiv r\sum_{k=1}^{c_i/2}(\alpha_1(x_{i,2k-1}) + \alpha_2(x_{i,2k-1})) + r\sum_{k=1}^{c_i/2}(\alpha_1(x_{i,2k}) + \alpha_r(x_{i,2k})) \pmod r \end{aligned} \quad (5.1.5)$$

Analogously, if R_i with $i \in I_C^e$ is a σ-chain with an even number of elements, then

$$\begin{aligned} r\sum_{x\in R_i}(\alpha_1(x)+\alpha_1(x)) = r\sum_{k=1}^{r_i/2}(\alpha_1(y_{i,2k-1})+\alpha_2(y_{i,2k-1}))+r\sum_{k=1}^{r_i/2}(\alpha_1(y_{i,2k})+\alpha_r(y_{i,2k}))-r\frac{r_i}{2} \\ \equiv r\sum_{k=1}^{r_i/2}(\alpha_1(y_{i,2k-1}) + \alpha_2(y_{i,2k-1})) + r\sum_{k=1}^{r_i/2}(\alpha_1(y_{i,2k}) + \alpha_r(y_{i,2k})) \pmod r \end{aligned} \quad (5.1.6)$$

Finally, let $R_i = \{y_{i,1}, \dots, y_{i,r_i}\}$ with $i \in I_R^o$ be a maximal σ-chain with odd degree and let $y = \sigma(y_{i,r_i}) \in D \backslash D_1$. As $r > 2$, there exists $\tau_i \in \{2, \dots, r\}$ such that $r + 2 - i \neq h_y$.

Then

$$r\sum_{x\in R_i}(\alpha_1(x)+\alpha_1(x))+r(\alpha_1(y)+\alpha_{h_y}(y))=r\sum_{k=1}^{(r_i+1)/2}(\alpha_1(x_{i,2k-1})+\alpha_{\tau_i}(x_{i,2k-1}))$$
$$+r\sum_{k=1}^{r_i-/2}(\alpha_1(x_{i,2k})+\alpha_{r+2-\tau_i}(x_{i,2k}))+r(\alpha_{r+2-\tau_i}(y)+\alpha_{h_y}(y))-r\frac{r_i+1}{2}$$
$$\equiv r\sum_{k=1}^{(r_i+1)/2}(\alpha_1(x_{i,2k-1})+\alpha_{\tau_i}(x_{i,2k-1}))$$
$$+r\sum_{k=1}^{r_i-/2}(\alpha_1(x_{i,2k})+\alpha_{r+2-\tau_i}(x_{i,2k}))+r(\alpha_{r+2-\tau_i}(y)+\alpha_{h_y}(y)). \quad (5.1.7)$$

Taking these into account, we can build the following selection vector with $r'=2$. Let $D_R=\{\sigma(y_{i,r_i})\,|\,i\in I_R^o\}$. Then, let $\bar{n}=\{n_1(x),\dots,n_r(x)\}$ be the selection vector with $r'=2$ selected elements for each $x\in D$ with the following properties

$$\begin{array}{ll}
n_1(x)=1 & \forall x\in D\backslash(D_R\cup\bigcup_{i\in I_C^o}C_i)\\
n_{h_x}(x)=1 & \forall x\in D\backslash D_1\\
n_2(x)=1 & \forall x\in C_i\forall i\in I_C^o\\
n_r(x)=1 & \forall x\in C_i\forall i\in I_C^o\\
n_2(x_{i,2k-1})=1 & \forall i\in I_C^e,\ \forall k=1,\dots,c_i/2\\
n_r(x_{i,2k})=1 & \forall i\in I_C^e,\ \forall k=1,\dots,c_i/2\\
n_2(y_{i,2k-1})=1 & \forall i\in I_R^e,\ \forall k=1,\dots,r_i/2\\
n_r(y_{i,2k})=1 & \forall i\in I_R^e,\ \forall k=1,\dots,r_i/2\\
n_{\tau_i}(y_{i,2k-1})=1 & \forall i\in I_R^o,\ \forall k=1,\dots,(r_i+1)/2\\
n_{r+2-\tau_i}(y_{i,2k})=1 & \forall i\in I_R^o,\ \forall k=1,\dots,(r_i-1)/2\\
n_{r+2-\tau_i}(\sigma(y_{i,r_i})=1 & \forall i\in I_R^e\\
n_k(x)=0 & \text{otherwise.}
\end{array}$$

Combining equations (5.1.2), (5.1.4), (5.1.5), (5.1.6) and (5.1.7) yields

$$2\sum_{x\in D}\sum_{i=1}^{r}\alpha_i(x)-r\sum_{x\in D}n_i(x)\alpha_i(x)\equiv 0 \pmod{r}.$$

Thus, there exists $d'\in\mathbb{Z}$ such that $W_{\overline{n}}(\alpha)=rd'$ and, therefore, $\alpha\in W_{\overline{n},d'}$, so α is non-generic. □

Chapter 6

Conclusion and Future Work

This bachelor's thesis has explored the structure, classification, and symmetries of moduli spaces of stable parabolic vector bundles over complex projective curves with marked points. By integrating algorithmic methods with rigorous mathematical theory, it developed a comprehensive computational framework to analyze stability chambers, identify geometric walls, and detect isomorphisms under a group of basic transformations. The framework yielded significant structural insights, including asymptotic formulas, bounds, and automorphism properties. This concluding chapter summarizes the main contributions and outlines promising directions for future research.

6.1 Conclusion

The central goal of this work was to classify moduli spaces $\mathcal{M}(r, \alpha, \xi)$ of parabolic vector bundles up to isomorphism and to compute their automorphism groups. To achieve this, we first formalized the wall-and-chamber decomposition of the weight space $\tilde{\mathcal{A}}_{n,r}$, using admissible selection vectors to define the hyperplanes $W_{\bar{n},d'}$ which determine transitions in stability.

We introduced several algorithmic tools: a Monte Carlo-based estimator for small ranks and points, and an exact recursive decomposition algorithm to enumerate chambers using rational arithmetic and polytope partitioning. The latter enabled the construction of a decision tree that classifies whether two weights lie in the same chamber. By applying basic transformations to each chamber representative, we traversed a graph-like structure encoding all isomorphism classes and automorphism groups. This process revealed striking structural phenomena, such as the absence of dualization in automorphisms for $r > 2$.

Mathematical analysis complemented the computational work. We derived formulas and asymptotic estimates for the number of geometric walls and established tight bounds based on hyperplane arrangement theory. In rank $r \geq 3$, we proved that the dualization transformation cannot fix a generic chamber, demonstrating its role as a nontrivial outer symmetry. This result, among others, illustrates the power of combining computational heuristics with structural theorems to discover and verify complex geometric properties.

6.2 Future Work

Several promising directions emerge from this work. First, the decision tree algorithm developed for stability chamber classification can be abstracted and generalized into a standalone software library for polytope decomposition with hyperplane arrangements. Such a library would find broad application beyond the specific setting of moduli spaces, including in combinatorics, optimization, and computational geometry. The key advantage lies in its ability to recursively partition high-dimensional simplicial domains via exact rational computation, a technique underexplored in general-purpose libraries.

A related technical challenge involves optimizing the incremental computation of convex hulls and intersections of polytopes. Existing tools like `cddlib` rely on the double description method [FP01], but the lack of Python bindings for incremental operations limits flexibility and performance. Developing efficient bindings or alternative libraries would improve scalability, especially in high-dimensional configurations typical in parabolic bundle moduli.

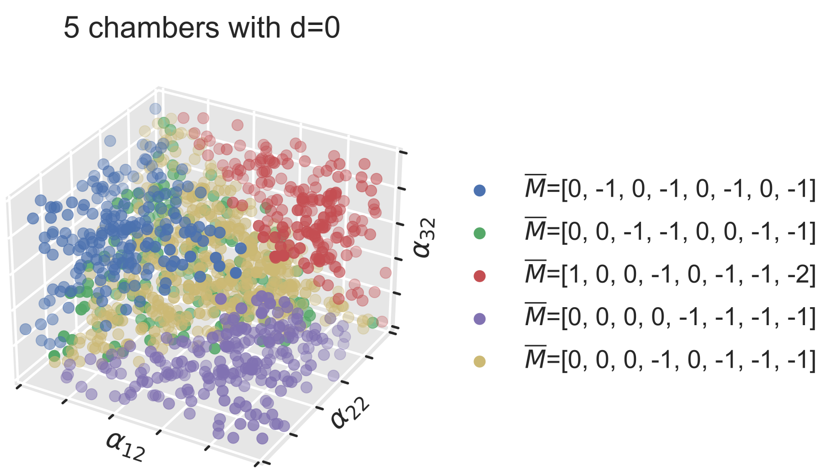
Another direction concerns the improvement of the bounds on the number of stability chambers. While asymptotic upper bounds via Schläfli-type estimates are available, they tend to significantly overcount. A more refined analysis using the intersection poset of the hyperplane arrangement and its associated characteristic polynomial [Zas75] could lead to tighter results and a deeper understanding of the chamber combinatorics. In particular, exact formulas for the number of regions in special arrangements may become feasible through careful enumeration of poset elements and Möbius function calculations.

Finally, a frontier worth exploring is the application of machine learning and artificial intelligence to uncover hidden patterns in the chamber structure and transformation symmetries. For instance, one could train models to predict whether a given weight $\alpha \in \tilde{\mathcal{A}}_{n,r}$ lies in the same chamber as another, or whether two weights correspond to isomorphic moduli spaces. Another goal could be detecting which basic transformations a given moduli space admits as automorphisms. These tasks are highly nontrivial due to the high dimensionality and combinatorial nature of the input: weights are $n \times r$ matrices, and both n and r vary. Constructing unbiased, representative datasets for training such models would itself be a major challenge. Moreover, designing architectures invariant under increases in both dimensions requires novel techniques—possibly drawing on ideas from graph learning or permutation-equivariant networks. Despite the complexity, the potential to automate the discovery of new symmetries or conjectures makes this a compelling direction.

In conclusion, the framework developed here opens multiple avenues for theoretical refinement, computational optimization, and exploratory AI-based investigation. The combination of geometric, algebraic, and algorithmic insights forms a solid foundation for the continued study of moduli spaces and their rich internal structure.

Appendix A

More examples of Monte Carlo sampling of stability chambers



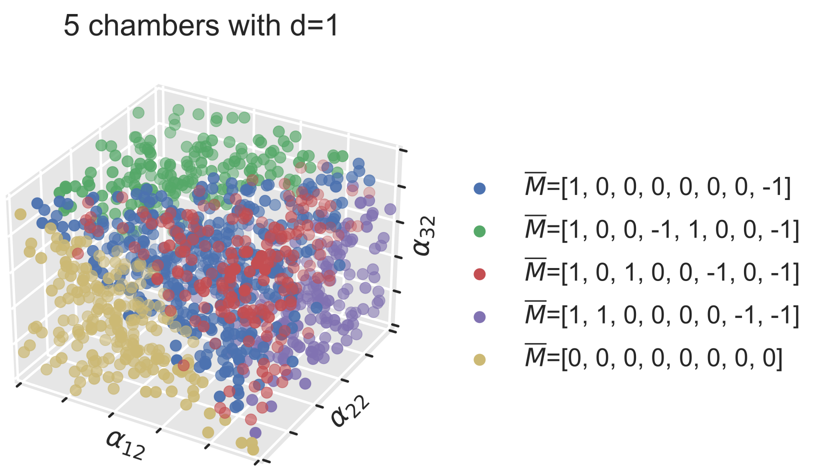


Figure A.1: Monte Carlo sampling of stability chambers for $n = 3$ and $r = 2$ and all degrees $d \pmod r$. Dots represent 1200 uniformly sampled $\alpha \in \tilde{\mathcal{A}}_{1,3}$, projected onto the plane $\alpha_1(x_1) = 0$, $\alpha_1(x_2) = 0$, $\alpha_1(x_3) = 0$.

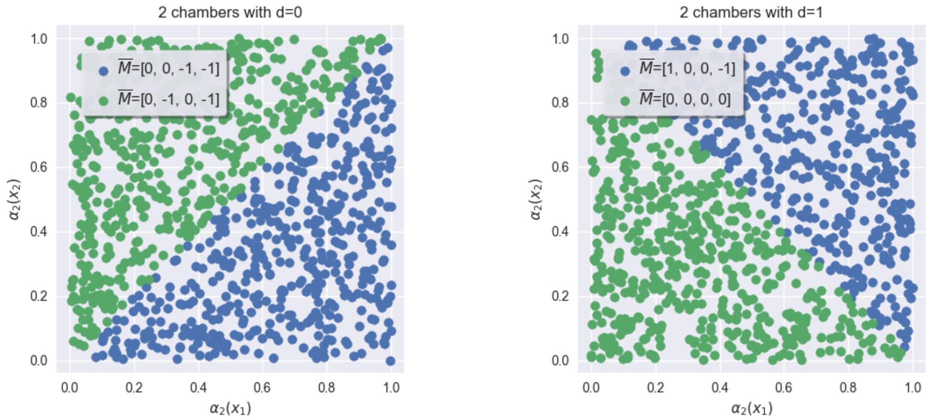


Figure A.2: Monte Carlo sampling of stability chambers for $n = 2$ and $r = 2$ and all degrees $d \pmod r$. Dots represent 1000 uniformly sampled $\alpha \in \tilde{\mathcal{A}}_{2,2}$, projected onto the plane $\alpha_1(x_1) = 0$, $\alpha_1(x_2) = 0$.

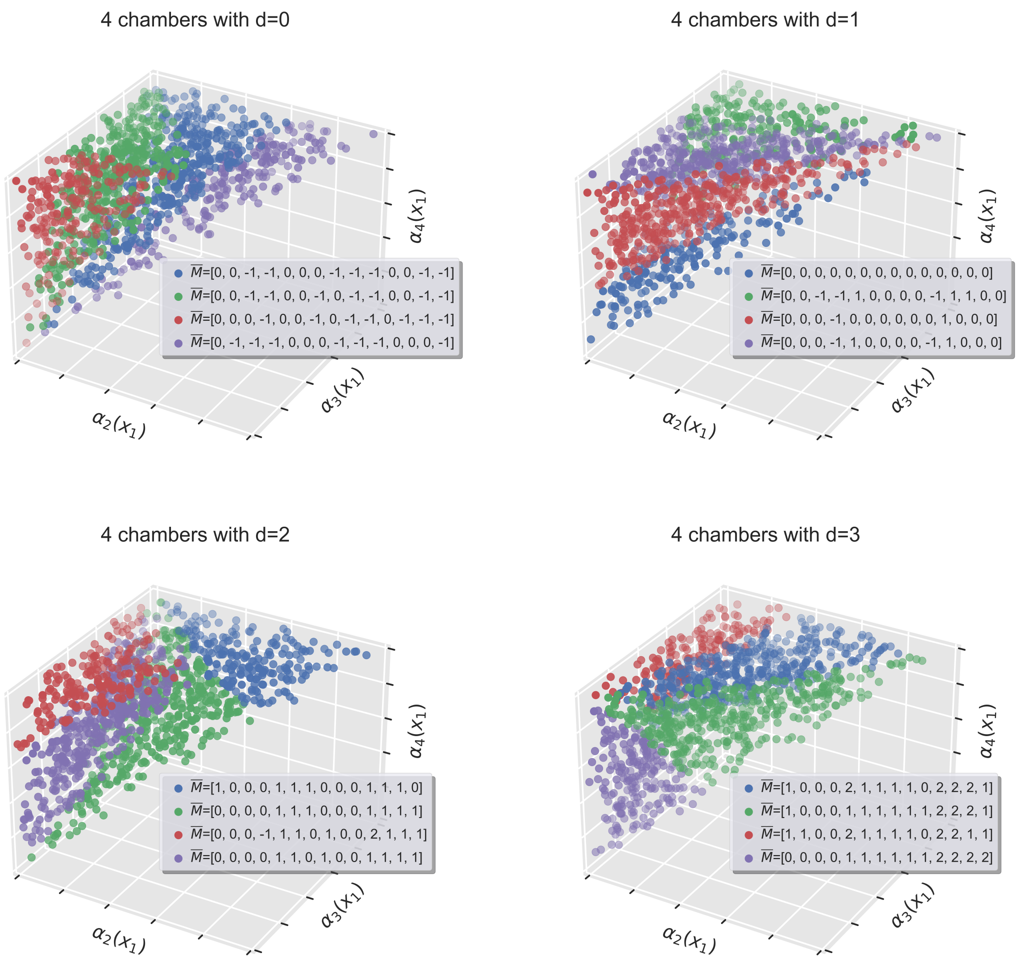


Figure A.3: Monte Carlo sampling of stability chambers for $n = 1$ and $r = 4$ and all degrees $d \pmod r$. Dots represent 1200 uniformly sampled $\alpha \in \tilde{\mathcal{A}}_{1,4}$, projected onto the plane $\alpha_1(x) = 0$.

Appendix B

Examples of Decision Tree JSON files

The structure of the JSON files for the decision trees is as follows:

- **n_leaves**: The number of leaves in the decision tree. Each leaf corresponds to a unique stability chamber.
- **n_nodes**: The number of nodes in the decision tree, including both internal nodes and leaves.
- **max_depth**: The maximum depth of the decision tree, which indicates the longest path from the root to a leaf.
- **avg_depth**: The average depth of the decision tree, calculated as the sum of depths of all leaves divided by the number of leaves.
- **tree**: A list of nodes in the decision tree, where each node is represented by a dictionary containing:
  - **depth**: The depth of the node in the tree.
  - **cut_hyperplane**: The hyperplane that defines the cut at this node, represented as a tuple (w, b), where w is the normal vector and b is the offset.
  - **parent_idx**: The index of the parent node in the list. The root node has a parent index of `null`.
  - **centroid**: The centroid of the region defined by this node. Only computed for leaves.

More examples of decision trees for different values of n and r can be found in the folder `data/trees` of our code repository (see Section 1.5).

B.1 Decision tree for $n = 2$, $r = 2$


```
{
    "n_leaves": 2,
    "n_nodes": 3,
    "max_depth": 1,
    "avg_depth": 0.67,
    "tree": [
```

```
        {
            "depth": 0,
            "cut_hyperplane": "([1, -1], 0)",
            "parent_idx": null,
            "centroid": null
        },
        {
            "depth": 1,
            "cut_hyperplane": null,
            "parent_idx": 0,
            "centroid": "[Fraction(2, 3), Fraction(1, 3)]"
        },
        {
            "depth": 1,
            "cut_hyperplane": null,
            "parent_idx": 0,
            "centroid": "[Fraction(1, 3), Fraction(2, 3)]"
        }
    ]
}
```


Listing B.1: Decision tree structure for $n = 2$, $r = 2$

B.2 Truncated decision tree for $n = 1$, $r = 8$


```
{
    "n_leaves": 76724,
    "n_nodes": 153447,
    "max_depth": 40,
    "avg_depth": 23.34,
    "tree": [
        {
            "depth": 0,
            "cut_hyperplane": "([1, -7, 1, 1, 1, 1, 1], 0)",
            "parent_idx": null,
            "centroid": null
        },
        {
            "depth": 1,
            "cut_hyperplane": "([3, 3, -5, -5, 3, 3, -5], 0)",
            "parent_idx": 0,
            "centroid": null
        },
        {
            "depth": 2,
            "cut_hyperplane": "([-4, -4, -4, 4, 4, 4, 4], 8)",
            "parent_idx": 1,
            "centroid": null
        },
        {
```

```
            "depth": 3,
            "cut_hyperplane": "([4, -4, -4, -4, 4, 4, 4], 8)",
            "parent_idx": 2,
            "centroid": null
        },
        ...
    ]
}
```


Listing B.2: Truncated decision tree structure for $n = 1$, $r = 8$

B.3 Truncated decision tree for $n = 5$, $r = 2$


```
{
    "n_leaves": 409,
    "n_nodes": 817,
    "max_depth": 18,
    "avg_depth": 11.64,
    "tree": [
        {
            "depth": 0,
            "cut_hyperplane": "([1, -1, -1, -1, 1], -2)",
            "parent_idx": null,
            "centroid": null
        },
        {
            "depth": 1,
            "cut_hyperplane": "([1, 1, 1, -1, 1], 2)",
            "parent_idx": 0,
            "centroid": null
        },
        {
            "depth": 2,
            "cut_hyperplane": "([1, 1, -1, -1, 1], 2)",
            "parent_idx": 1,
            "centroid": null
        },
        {
            "depth": 3,
            "cut_hyperplane": null,
            "parent_idx": 2,
            "centroid": "[Fraction(5, 6), Fraction(5, 6), Fraction(1, 6), Fraction(1, 6), Fraction(5, 6)]"
        },
        ...
    ]
}
```


Listing B.3: Truncated decision tree structure for $n = 5$, $r = 2$

Appendix C

Example of isomorphism classes and automorphisms for $n = 2$, $r = 3$ for a fully symmetric curve X

The structure of the json file `morph_info_n2_r3.json` is as follows:

- **n**: The number of parabolic points, $n = 3$.
- **r**: The rank of the vector bundles, $r = 3$.
- **n_isomorphism_classes**: The number of isomorphism classes of parabolic vector bundles.
- **isomorphism_classes**: A list of isomorphism classes, each represented by a dictionary containing:
  - **alpha_representative**: Parabolic weight $\alpha \in \tilde{\mathcal{A}}_{n,r}$ representative of the isomorphism class. Given in matrix form, with values being rational numbers.
  - **n_chambers**: The number of distinct moduli spaces $\mathcal{M}(r, \alpha, \xi)$ in the isomorphism class.
  - **n_automorphisms**: The number of automorphisms of the moduli space $\mathcal{M}(X, r, \alpha, \xi)$.
  - **automorphisms**: A list of automorphisms of the moduli space $\mathcal{M}(r, \alpha, \xi)$, each represented by a dictionary containing:
    - **sigma**: Tuple of integers representing the permutation of the parabolic points.
    - **s**: The integer s representing whether to apply dualization ($s = -1$) or not ($s = 1$).
    - **H**: A tuple of integers representing the parabolic divisor H at each point.

More examples of isomorphism classes and automorphisms for different values of n and r can be found in the folder `data/morphs` of our code repository (see Section 1.5).

```
{
    "n": 2,
    "r": 3,
    "n_isomorphism_classes": 2,
    "isomorphism_classes": [
        {
            "alpha_representative": [
                "[0 Fraction(1, 5) Fraction(9, 10)]",
                "[0 Fraction(1, 5) Fraction(9, 10)]"
            ],
            "n_chambers": 5,
            "n_automorphisms": 2,
            "automorphisms": [
                "((0, 1), 1, (0, 0))",
                "((1, 0), 1, (0, 0))"
            ]
        },
        {
            "alpha_representative": [
                "[0 Fraction(1, 2) Fraction(7, 10)]",
                "[0 Fraction(1, 2) Fraction(7, 10)]"
            ],
            "n_chambers": 5,
            "n_automorphisms": 2,
            "automorphisms": [
                "((0, 1), 1, (0, 0))",
                "((1, 0), 1, (0, 0))"
            ]
        }
    ]
}
```


Listing C.1: Isomorphism classes and automorphisms for $n = 2$, $r = 3$ and a fully symmetric curve X

List of Figures

List of Tables

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