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Option pricing in a stochastic delay volatility model

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This work introduces a new stochastic volatility model with delay parameters in the volatility process, extending the Barndorff–Nielsen and Shephard model. It establishes an analytical expression for the log price characteristic function, which can be applied to price European options. Empirical analysis on S&P500 European call options shows that adding delay parameters reduces mean squared error. This is the first instance of providing an analytical formula for the log price characteristic function in a stochastic volatility model with multiple delay parameters. We also provide a Monte Carlo scheme that can be used to simulate the model.

KEYWORDS

Barndorff–Nielsen and Shephard model, closed formula, option pricing, stochastic delay differential equations

MSC CLASSIFICATION

91G20, 91G30, 34K50

1 | INTRODUCTION

In this paper, we present a new model for pricing European options, which is based on the Barndorff–Nielsen and Shephard stochastic volatility model [1, 2]. The variance in the Barndorff–Nielsen and Shephard model follows a non-Gaussian Ornstein–Uhlenbeck process. A Lévy subordinator generates this Ornstein–Uhlenbeck process; this means that the Ornstein–Uhlenbeck process is nonnegative. In that way, it can be used to model the variance. Nicolato and Venardos [3] derived the log price's characteristic function, and used it to price European options. In the context of option pricing, this model has different variations. For example, SenGupta [4] presented a model that includes two correlated Lévy processes. The author showed that this model can replicate the implied volatility surface more accurately. Bannör and Scherer [5] proposed a variant of the Barndorff–Nielsen and Shephard model with two background-driving Lévy processes, one for positive jumps and the other for negative jumps. The authors constructed the characteristic function for this model and used it to price Forex options, showing that the proposed model did a better job at replicating the volatility surface. Another variation is a time-changed model in which the stochastic business time follows an integrated non-Gaussian Ornstein–Uhlenbeck process [6]. Muhle-Karbe et al. [7] gave a multivariate version of the Barndorff–Nielsen and Shephard model, which can be used to price options on several assets. Other model modifications try to account for the presence of long memory in the volatility. For instance, Barndorff-Nielsen and Stelzer [8] used the superposition of the Ornstein–Uhlenbeck processes to model long-range effects. Salmon and SenGupta [9] used a fractional Ornstein–Uhlenbeck process to get the long-memory property. This paper introduces a new version of the Barndorff–Nielsen and Shephard model in which the variance follows a stochastic delay differential equation with Lévy noise. We would like to see if the presence of delay parameters allows us to replicate European option prices more accurately. Apart from option pricing, the Barndorff–Nielsen and Shephard model is also used in other areas. For example, it can be used to model commodities and energy [10–12]. High-frequency and geophysical data can also be modeled using Barndorff–Nielsen and Shephard models [13, 14]. This means that the proposed model could have other applications besides option pricing.

Stochastic delay differential equations have been applied to finance previously. Arriojas et al. [15] proposed a delay version of the geometric Brownian motion and used it for pricing European options. The price of European options under this model can be approximated using finite difference methods [16]. An extension of this model that includes jumps generated by a Poisson process is given by Kim et al. [17]. Lee et al. [18] constructed a delay geometric Brownian motion in which the volatility is stochastic. Apart from the delay geometric Brownian motion, other models based on stochastic delay differential equations have been used for pricing options. In the literature, several models exist in which the volatility follows a stochastic delay differential equation. Kazmerchuk et al. [19] presented a continuous time version of the GARCH model and derived a formula for pricing European options. Swishchuk and Xu [20] and Swishchuk and Vadori [21] proposed several stochastic volatility models with delay for pricing variance swaps. Tambue et al. [22] proposed a delay nonlinear model for pricing corporate liabilities. In the case of bond pricing, Flore and Nappo [23] constructed a delayed version of the Cox–Ingersoll–Ross model, while Coffie [24] presented a delay Ait-Sahalia short rate model with jumps. More recently, Gómez-Valle and Martínez-Rodríguez [25] used a delay geometric Brownian motion to price commodity futures. Stochastic delay differential equations have applications outside of the world of finance, biology [26], physics [27], epidemiology [28], among others.

This paper proposes a delay version of the Barndorff–Nielsen and Shephard model with $N \in \mathbb{N}$ delay parameters. Under this model, we prove that it is possible to get an analytical formula for the characteristic function of the log price. So, we can price European options and compute the likelihood function using Fourier methods. As far as we know, this is the first time someone has derived the log price's characteristic function in a stochastic volatility model with several delay parameters.

Volatility clustering and significant autocorrelation of absolute returns can be empirically observed in financial markets, implying the existence of past dependence in the volatility structure of financial returns [29, p. 12]. Because of that, the proposed model includes several delay parameters in the volatility process that will capture this behavior. Apart from that, the implied volatility of short-maturity options tends to be higher than the implied volatility of long-maturity options [30, p. 42]. Introducing delays in volatility allows different behaviors for short-maturity options and long-maturity options.

The delay parameters included in the model represent the volatility's dependence on its past values. Discrete time series models usually introduce delays when modeling the volatility (see GARCH model [31] and autoregressive stochastic volatility model [32]). One can consider the proposed model as a continuous time counterpart of those discrete time series models.

The Black–Scholes–Merton model [33] was one of the first option pricing models, and it has become the standard in the industry. One of the most significant flaws of the Black–Scholes–Merton model is the assumption of constant volatility. Market option prices show that the volatility can not be constant [30]. Several models have been introduced to mimic volatility dynamics, for example, local volatility models [34], stochastic volatility models [35], and more recently, local stochastic volatility models and rough volatility models [36]. We aim to examine if the proposed model can replicate market option prices accurately. We hope that the flexibility given by the delay parameters allows the model to capture the differences between options with short- and long-term maturities.

In addition, we will give the analytical solution to a delay differential equation. Due to the increasing interest in stochastic delay models, delay differential equations have started to appear in finance. For example, Flore and Nappo [23] proved the existence of a delay Riccati equation, which appears when pricing a zero coupon bond in a delayed version of the Cox–Ingersoll–Ross model [37]. Delay differential equations have been used in other areas of science like population dynamics [38], physics [39], or biology [40], among others. Another area of interest is the study of the limiting behavior on delay differential equations; see [41, 42]. Regarding the delayed Ornstein–Uhlenbeck process with Brownian noise, Küchler and Mensch [43] studied its limiting distribution.

Other differential equations recently used in finance are fractional differential equations and elliptic differential equations. For example, fractional differential equations appear on option pricing involving Lévy processes [44, Section 5] and in models that include fractional Brownian motions [45]. At the same time, elliptic differential equations are used in the context of infinite time horizon stochastic control problems [46, Section 4.4]. Recent advancements in fractional and elliptic differential equations can be found in the work of Liu and Zhao [47] and Liu and Liu [48].

The structure of this paper is as follows. In Section 2, we introduce the model and prove the existence of a strong solution. We derive the characteristic function in Section 3. Section 4 performs numerical experiments with real data. In Appendix A, we propose a simulation method for the variance process.

2 | THE MODEL

Let $\left(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{t\geq 0}\right)$ be a filtered probability space. In this space, we define two independent and adapted processes: the first one is a Brownian motion $W = (W_t)_{t\geq 0}$ and the second one is a Lévy subordinator $Z = (Z_t)_{t\geq 0}$. We assume that the stock price $S = (S_t)_{t\geq 0}$ can be written as

$$S_t = e^{X_t} \text{ for } t \geq 0,$$

where $X = (X_t)_{t\geq 0}$ represents the log price. The log price X also depends on a stochastic variance process $V = (V_t)_{t\geq 0}$. Both of them satisfy the following system of stochastic differential equations

$$dX_t = \left(r - \frac{1}{2}V_{t^-} - \kappa(\rho)\right)dt + \sqrt{V_{t^-}}dW_t + \rho dZ_t \text{ with } X_0 \in \mathbb{R}, \tag{1}$$

$$dV_t = \left(a + bV_{t^-} + \sum_{j=1}^{N} c_j V_{(t-\tau_j)^-}\right)dt + dZ_t \tag{2}$$

where $N \in \mathbb{N} \cup \{0\}$, $c_1, c_2, \dots, c_N \geq 0$, $\tau_1, \tau_2, \dots, \tau_N > 0$, $a \geq 0$, $b < 0$, $\rho \leq 0$, $r \geq 0$ is the interest rate and κ is the cumulant generating function of Z_1, that is,

$$\kappa(\theta) = \ln E\left[e^{\theta Z_1}\right] \text{ for } \theta \in \mathbb{R}.$$

We assume that

$$0 < \tau_1 < \tau_2 < \dots < \tau_N, \tag{3}$$

and

$$V_t = \phi(t) \text{ for } t \in [-\tau_N, 0], \tag{4}$$

where $\phi : [-\tau_N, 0] \to (0, \infty)$ is a bounded deterministic left-continuous function. Under these assumptions, it is possible to show that the differential systems (1) and (2) have a strong solution. We proceed as in the work of Flore and Nappo [23, Theorem 2.2] and Arriojas et al. [15, Theorem 1] to prove the existence of the strong solution of Equation (2).

Proposition 1. *The strong solution of the stochastic differential Equation (2) with initial value (4) is given by*

$$V_t = V_0 e^{bt} + a\int_0^t e^{-b(s-t)}ds + \int_0^t \left(e^{-b(s-t)} \sum_{j=1}^{N} c_j V_{(s-\tau_j)^-}\right)ds + \int_0^t e^{-b(s-t)}dZ_s, \tag{5}$$

for all $t \geq 0$. In addition, the process V has càdlàg paths almost surely.

Proof. First, we will show that Equation (5) is well-defined, and then we will show that the process V defined in Equation (5) satisfies the differential Equation (2).

Because the process Z is a subordinator, the trajectory of Z has locally finite variation almost surely [49, p. 471]. Because of this, the stochastic integral that appears in (5) is well-defined [50, p. 105] and its trajectories are càdlàg almost surely [51, p. 209]. We still need to verify if the second Lebesgue integral in (5) is well-defined.

When $t \in [0, \tau_1]$, Equation (5) satisfies

$$V_t = V_0 e^{bt} + a\int_0^t e^{-b(s-t)}ds + \int_0^t \left(e^{-b(s-t)} \sum_{j=1}^{N} c_j \phi(s-\tau_j)\right)ds + \int_0^t e^{-b(s-t)}dZ_s. \tag{6}$$

Since ϕ is left-continuous and bounded, the second Lebesgue integral that appears in equation 6 is well-defined and the process $(V_t)_{t\in[0,\tau_1]}$ has càdlàg paths almost surely. For $t \in [\tau_1, \tau_2]$, Equation (5) gives us

$$
\begin{aligned}
V_t = V_0 e^{bt} + a\int_0^t e^{-b(s-t)}ds + \int_0^t \left(e^{-b(s-t)} \sum_{j=2}^{N} c_j \phi(s-\tau_j) \right) ds \\
+ c_1 \int_0^{\tau_1} e^{-b(s-t)} \phi(s-\tau_1)ds + c_1 \int_{\tau_1}^t e^{-b(s-t)} V_{(s-\tau_1)^-} ds \\
+ \int_0^t e^{-b(s-t)} dZ_s.
\end{aligned} \tag{7}
$$

Since $(V_t)_{t\in[0,\tau_1]}$ has càdlàg paths and ϕ is left-continuous, the second, third, and fourth Lebesgue integrals that appear in Equation (7) are well-defined, and the process $(V_t)_{t\in[0,\tau_2]}$ has càdlàg paths almost surely. Using induction, one can show that Equation (5) is well-defined for $t \in [0, \tau_N]$ and that the process $(V_t)_{t\in[0,\tau_N]}$ has càdlàg paths almost surely.

The general case for $t \in [0, k\tau_N]$ where $k \in \mathbb{N}$ can also be proved by induction. Assume that Equation (5) is well-defined for $t \in [0, k\tau_N]$ and the process $(V_t)_{t\in[0,k\tau_N]}$ has càdlàg paths almost surely, then if $t \in [k\tau_N, k\tau_N + \tau_1]$, Equation (5) can be written as

$$
V_t = V_0 e^{bt} + a\int_0^t e^{-b(s-t)}ds + \int_0^t \left(e^{-b(s-t)} \sum_{j=1}^{N} c_j V_{(s-\tau_j)^-} \right) ds + \int_0^t e^{-b(s-t)} dZ_s. \tag{8}
$$

By the induction step, we know that $(V_t)_{t\in[0,k\tau_N]}$ has càdlàg paths almost surely, and hence, the second Lebesgue integral of Equation (8) is well-defined. Because of this, the process $(V_t)_{t\in[0,k\tau_N+\tau_1]}$ has càdlàg paths almost surely. In the case, $t \in [k\tau_N + \tau_1, k\tau_N + \tau_2]$, Equation (5) satisfies

$$
\begin{aligned}
V_t = V_0 e^{bt} + a\int_0^t e^{-b(s-t)}ds + \int_0^t \left(e^{-b(s-t)} \sum_{j=2}^{N} c_j V_{(s-\tau_j)^-} \right) ds \\
+ c_1 \int_0^{k\tau_N+\tau_1} e^{-b(s-t)} V_{(s-\tau_1)^-} ds + c_1 \int_{k\tau_N+\tau_1}^t e^{-b(s-t)} V_{(s-\tau_1)^-} ds \\
+ \int_0^t e^{-b(s-t)} dZ_s.
\end{aligned} \tag{9}
$$

Observe that the second, third, and fourth Lebesgue integrals in (9) are well-defined by the previous step since $(V_t)_{t\in[0,k\tau_N+\tau_1]}$ has càdlàg paths almost surely. Proceeding in a similar manner, one can show that Equation (5) is well-defined for $t \in [0, (k+1)\tau_N]$ and the process $(V_t)_{t\in[0,(k+1)\tau_N]}$ defined as in Equation (5) has càdlàg paths almost surely, completing the induction procedure.

We have shown that Equation (5) defines a process with càdlàg paths. Now we will show that the process $V = (V_t)_{t\geq 0}$ defined in Equation (5) satisfies the differential Equation (2). If we define the process $Y = (Y_t)_{t\geq 0}$ as $Y_t = e^{-bt}V_t$ for $t \geq 0$, then from Equation (5), we have that

$$
dY_s = \left(ae^{-bs} + \sum_{j=1}^{N} c_j e^{-bs} V_{(s-\tau_j)^-} \right) ds + e^{-bs} dZ_s \text{ for } s \geq 0.
$$

Applying integration by parts formula to $e^{bt}Y_t$ [52, Corollary 2.6.2, Theorem 2.6.28], we obtain

$$
d\left(Y_s e^{bs}\right) = \left(a + bY_{s^-} e^{bs} + \sum_{j=1}^{N} c_j V_{(s-\tau_j)^-} \right) ds + dZ_s \text{ for } s \geq 0.
$$

Since $Y_t e^{bt} = V_t$ for all $t \geq 0$, we have just obtained the desired result. □

Since Z is a subordinator and from Equation (5), we have that the process V is nonnegative as long as $a \geq 0$ and $c_1, \dots, c_n \geq 0$. Due to this fact, the process X satisfies

$$X_t = X_0 + rt - \kappa(\rho)t - \int_0^t \frac{1}{2} V_{s^-} ds + \int_0^t \sqrt{V_{s^-}} dW_s + \rho Z_s \text{ for } t \geq 0. \tag{10}$$

From Proposition 1 and the nonnegative property of V, the Lebesgue and Itô integrals that appear in Equation (10) are well-defined.

If we define the jump $\Delta Z_s = Z_s - Z_{s^-}$, the subordinator Z can be written as

$$Z_t = b_Z t + \sum_{0<s\leq t} \Delta Z_s = b_Z t + \int_0^t \int_0^\infty x J(dx, dt), \tag{11}$$

where $b_Z \geq 0$ and J is the jump measure of the Lévy process Z [53, Corollary 3.1]. From properties of subordinators, the process Z has Lévy triplet $(\gamma_Z, 0, \nu)$ where

$$\gamma_Z = b_Z + \int_0^1 x\nu(dx), \int_0^\infty (1 \wedge x)\nu(dx) < \infty,$$

and the cumulant generating function of Z_1 has the following formula

$$\kappa(\theta) = \ln E_{\mathbb{Q}} \left[e^{\theta Z_1}\right] = b_Z \theta + \int_0^\infty \left(e^{\theta x} - 1\right) \nu(dx) \tag{12}$$

for all $\theta \in \mathbb{R}$ [53, Corollary 3.1]. In general, the function κ is not finite for every value of θ, because of that we define the value

$$\hat{\kappa} = \sup\{\theta \in \mathbb{R} : \kappa(\theta) < \infty\}.$$

In this way, the value $\kappa(\theta)$ is finite if $\theta < \hat{\kappa}$. Because Z is a nonnegative process, we have that $\hat{\kappa} > 0$.

Remark 1. Depending on the subordinator Z, we can have different formulas for the cumulant function κ.

1. If Z is a compound Poisson process with parameter $\lambda > 0$ whose jumps are exponentially distributed with parameter $\eta > 0$, the function κ is
$$\kappa(\theta) = \frac{-\lambda\eta}{\theta - \eta} - \lambda \text{ for } \theta \in \mathbb{R}.$$
In this case, $\hat{\kappa} = \eta$ [54, p. 52].
2. If the subordinator Z is a gamma process with parameters $a_g, b_g > 0$, the cumulant function is
$$\kappa(\theta) = -a_g \ln\left(1 - \frac{\theta}{b_g}\right) \text{ for } \theta \in \mathbb{R}.$$
For this cumulant function, we have $\hat{\kappa} = b_g$ [54, p. 52].
3. In this case, the subordinator Z is an inverse Gaussian process with parameters $a_I, b_I > 0$; the function κ is defined as
$$\kappa(\theta) = a_I \left(b_I - \sqrt{b_I^2 - 2\theta}\right) \text{ for } \theta \in \mathbb{R}.$$
For this subordinator, we have $\hat{\kappa} = \frac{b_I^2}{2}$ [54, p. 53].

From Equation (11), we have that the process X and the jump process ΔX satisfy

$$dX_t = \left(r - \kappa(\rho) - \frac{1}{2} V_{t^-} + \rho b_Z\right) dt + \sqrt{V_{t^-}} dW_t + \int_0^\infty \rho x J(dx, dt)$$
$$\Delta X_t = X_t - X_{t^-} = \rho \Delta Z_t = \int_0^\infty \rho x J(dx, dt) \text{ for } t \geq 0.$$

The discounted stock price $\tilde{S} = \left(\tilde{S}_t\right)_{t\geq 0}$ is defined as

$$\tilde{S}_t = e^{-rt}S_t = e^{-rt+X_t} \text{ for } t \geq 0.$$

We can show that discounted stock price is a martingale.

Proposition 2. *The discounted stock price $\tilde{S}$ is a martingale in the filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{t\geq 0}\right)$.*

Proof. By application of the Itô formula for semimartingales [52, Theorem 2.7.33], we obtain

$$\begin{aligned}\tilde{S}_t = S_0 &+ \int_0^t -r\tilde{S}_{s^-}ds + \int_0^t \tilde{S}_{s^-}\left(r - \kappa(\rho) - \frac{1}{2}V_{s^-} + \rho b_Z\right) ds \\ &+ \int_0^t \tilde{S}_{s^-}\sqrt{V_{s^-}}dW_s + \int_0^t \tilde{S}_{s^-}\int_0^\infty \rho x J(dx, ds) \\ &+ \int_0^t \frac{1}{2}\tilde{S}_{s^-}V_{s^-}ds + \sum_{0\leq s\leq t}\left(\tilde{S}_s - \tilde{S}_{s^-} - \tilde{S}_{s^-}\rho\Delta Z_s\right).\end{aligned}$$

From the fact that

$$\tilde{S}_s = e^{-rs+X_{s^-}+\Delta X_s} = \tilde{S}_{s^-}e^{\rho\Delta Z_s},$$

we arrive to the formula

$$\begin{aligned}\tilde{S}_t &= S_0 + \int_0^t \tilde{S}_{s^-}\left(-\kappa(\rho) + \rho b_Z\right) ds + \int_0^t \tilde{S}_{s^-}\sqrt{V_{s^-}}dW_s \\ &\quad + \int_0^t \tilde{S}_{s^-}\int_0^\infty \rho x J(dx, ds) + \int_0^t \tilde{S}_{s^-}\int_0^\infty \left(e^{\rho x} - 1 - \rho x\right) J(dx, ds) \\ &= S_0 + \int_0^t \tilde{S}_{s^-}\left(-\kappa(\rho) + \rho b_Z\right) ds + \int_0^t \tilde{S}_{s^-}\sqrt{V_{s^-}}dW_s \\ &\quad + \int_0^t \tilde{S}_{s^-}\int_0^\infty \left(e^{\rho x} - 1\right)\nu(dx)ds + \int_0^t \tilde{S}_{s^-}\int_0^\infty \left(e^{\rho x} - 1\right)\tilde{J}(dx, ds) \\ &= S_0 + \int_0^t \tilde{S}_{s^-}\sqrt{V_{s^-}}dW_s + \int_0^t \tilde{S}_{s^-}\int_0^\infty \left(e^{\rho x} - 1\right)\tilde{J}(dx, ds), \end{aligned} \tag{13}$$

where $\tilde{J}(dx, dt) = J(dx, dt) - \nu(dx)dt$ is the compensated jump measure of the Lévy process Z and in the last equality, we have used Equation (12). From Equation (13), we know that the discounted stock price is a local martingale [51, Theorem 4.2.12]. Since the process $\tilde{S}$ is a nonnegative local martingale, to prove the martingale property, we only need to show that

$$E\left[\tilde{S}_t\right] = S_0$$

[55, Theorem 7.24]. By independence of V and W, we have that

$$\int_0^t \sqrt{V_{s^-}}dW_s\bigg|\mathcal{F}_t^Z \sim N\left(0, \int_0^t V_{s^-}^2 ds\right), \tag{14}$$

where $\mathcal{F}^Z = \left(\mathcal{F}_t^Z\right)_{t\geq 0}$ is the filtration generated by the process Z. By application of the tower property and the result in (14), we obtain

$$\begin{aligned}E[\tilde{S}_t] &= E\left[E\left[\tilde{S}_t\middle|\mathcal{F}_t^Z\right]\right] \\ &= E\left[S_0 e^{-\int_0^t \frac{1}{2}V_{s^-}ds - \kappa(\rho)t + \rho Z_t}E\left[e^{\int_0^t \sqrt{V_{s^-}dW_s}}\middle|\mathcal{F}_t^Z\right]\right] \\ &= S_0 E\left[S_0 e^{-\kappa(\rho)t+\rho Z_t}\right] = S_0,\end{aligned}$$

where the last equality comes from Theorem 13.50 in Pascucci [49]. □

3 | CHARACTERISTIC FUNCTION

In this section, we would like to compute the conditional characteristic function of X_T for a fixed $T > 0$, which is defined as

$$\Phi_{t,T}(u) = E\left[e^{iuX_T}\middle|\mathcal{F}_t\right] \tag{15}$$

for $u \in \mathbb{R}$ and $t \in [0, T]$. Using the technique proposed by Flore and Nappo [23], we obtain the next result that will allow us to express the characteristic function in terms of an affine function of X and V.

Proposition 3. *The conditional characteristic function defined in (15) can be expressed as*

$$\Phi_{t,T}(u) = e^{iuX_t + A(T-t) + B(T-t)V_t + \sum_{j=1}^{N}\int_{t-\tau_j}^{t\wedge(T-\tau_j)}(c_j B(T-s-\tau_j)V_{s^-})ds}, \tag{16}$$

where $A, B : [0,\infty) \to \mathbb{C}$ satisfies the system of delay differential equations

$$B'(\ell) = -\frac{1}{2}(iu + u^2) + bB(\ell) + \sum_{j=1}^{N} c_j B(\ell - \tau_j)\mathbb{1}_{[\tau_j,T]}(\ell) \tag{17}$$

$$A'(\ell) = iu(r - \kappa(\rho)) + aB(\ell) + \kappa(iu\rho + B(\ell)) \tag{18}$$

with initial values $A(0) = 0$ and $B(\ell) = 0$ for all $\ell \in [-\tau_N, 0]$. In particular, when $t = 0$, we have

$$\Phi_{0,T}(u) = e^{iuX_0 + A(T) + B(T)V_0 + \sum_{j=1}^{N}\int_{-\tau_j}^{0\wedge(T-\tau_j)}(c_j B(T-s-\tau_j)\phi(s))ds}. \tag{19}$$

Proof. To save space in this proof, we will use $\Phi_{t,T}$ instead of $\Phi_{t,T}(u)$. Before starting the computations for the characteristic function, we define the processes

$$\gamma_{t,j} = \int_{t-\tau_j}^{t} \Gamma_j(s)V_{s^-}\mathbb{1}_{[-\tau_j,T-\tau_j]}(s)ds$$

for $t \geq 0$ and $j = 1, 2, \ldots, N$, where Γ_j is a measurable deterministic function that will be defined later. Observe that $\gamma_{T,j} = 0$ for all $j = 1, \ldots, N$, and hence, we can write the characteristic function as

$$\Phi_{t,T} = E\left[e^{iuX_T + \sum_{j=1}^{N}\gamma_{T,j}}\middle|\mathcal{F}_t\right].$$

A more convenient way of expressing the processes $\gamma_{\cdot,j}$ is

$$\begin{aligned}\gamma_{t,j} &= \int_{-\tau_j}^{t} \Gamma_j(s)V_{s^-}\mathbb{1}_{[-\tau_j,T-\tau_j]}(s)ds - \int_{0}^{t}\Gamma_j(s-\tau_j)V_{(s-\tau_j)^-}ds \\ &= \gamma_{0,j} + \int_0^t \left(\Gamma_j(s)V_{s^-}\mathbb{1}_{[0,T-\tau_j]}(s) - \Gamma_j(s-\tau_j)V_{(s-\tau_j)^-}\right)ds.\end{aligned}$$

Let us assume that the conditional characteristic function $\Phi_{\cdot,T}$ satisfies the expression

$$\Phi_{t,T} = e^{iuX_t + C(t) + D(t)V_t + \sum_{j=1}^{N}\gamma_{t,j}},$$

where $C, D : [0,\infty) \to \mathbb{C}$ are two continuous deterministic functions with $C(T) = 0$ and $D(t) = 0$ for all $t \in [T, T+\tau_N]$. By application of the Itô formula for semimartingales [52, Theorem 2.7.33], we obtain

$$\begin{aligned}\Phi_{t,T} &= \Phi_{0,T} + \int_0^t \Phi_{s^-,T}\left(C'(s) + D'(s)V_{s^-}\right) ds \\ &+ \int_0^t iu\Phi_{s^-,T}\left(r - \frac{1}{2}V_{s^-} - \kappa(\rho) + \rho b_Z\right) ds + \int_0^t iu\Phi_{s^-,T}\sqrt{V_{s^-}}dW_s \\ &+ \int_0^t \int_0^\infty iu\Phi_{s^-,T}\rho x J(dx, ds) \\ &+ \int_0^t D(s)\Phi_{s^-,T}\left(a + bV_{s^-} + \sum_{j=1}^N c_j V_{(s-\tau_j)^-} + b_Z\right) ds \\ &+ \int_0^t \int_0^\infty D(s)\Phi_{s^-,T} x J(dx, ds) \\ &+ \sum_{j=1}^N \int_0^t \Phi_{s^-,T}\left(\Gamma_j(s)V_{s^-}\mathbb{1}_{[-\tau_j, T-\tau_j]}(s) - \Gamma(s-\tau_j)V_{(s-\tau_j)^-}\right) ds \\ &- \int_0^t \frac{1}{2}\Phi_{s^-,T}u^2 V_{s^-} ds \\ &+ \sum_{s\le t}\left(\Phi_{s,T} - \Phi_{s^-,T} - \left(iu\rho\Phi_{s^-,T}\Delta Z_s + D(s)\Phi_{s^-,T}\Delta Z_s\right)\right),\end{aligned} \tag{20}$$

where

$$\Phi_{t^-,T} = e^{iuX_{t^-} + C(t) + D(t)V_{t^-} + \sum_{j=1}^N \gamma_{t,j}}.$$

We define the deterministic functions Γ_j in such a way that all delay terms that appear in Equation (20) disappear. Because of that, Γ_j is defined as

$$\Gamma_j(t) = \begin{cases} c_j D(t+\tau_j) & \text{if } t \in [-\tau_j, T-\tau_j], \\ 0 & \text{if } t \in (T-\tau_j, T]. \end{cases} \tag{21}$$

From the fact that

$$\Phi_{t,T} = \Phi_{t^-,T}e^{(i\rho u + D(t))\Delta Z_t},$$

and from Equation (21), the result in Equation (20) can be written as

$$\begin{aligned}\Phi_{t,T} &= \Phi_{0,T} + \int_0^t \Phi_{s^-,T}\left(C'(s) + D'(s)V_{s^-}\right) ds \\ &+ \int_0^t iu\Phi_{s^-,T}\left(r - \frac{1}{2}V_{s^-} - \kappa(\rho) + \rho b_Z\right) ds \\ &- \int_0^t \frac{1}{2}\Phi_{s^-,T}u^2 V_{s^-} ds \\ &+ \int_0^t D(s)\Phi_{s^-,T}\left(a + bV_{s^-} + b_Z\right) ds \\ &+ \sum_{j=1}^N \int_0^t \Phi_{s^-,T}\left(c_j D(s+\tau_j)V_{s^-}\mathbb{1}_{[-\tau_j, T-\tau_j]}(s)\right) ds \\ &+ \int_0^t iu\Phi_{s^-,T}\sqrt{V_{s^-}}dW_s + \int_0^t \int_0^\infty iu\Phi_{s^-,T}\rho x J(dx, ds) \\ &+ \int_0^t \int_0^\infty D(s)\Phi_{s^-,T} x J(dx, ds) \\ &+ \int_0^t \Phi_{s^-,T}\int_0^\infty \left(e^{(iu\rho + D(t))x} - 1 - (iu\rho x + D(s)x)\right) J(dx, ds)\end{aligned}$$

$$
\begin{aligned}
&= \Phi_{0,T} \\
&+ \int_0^t iu\Phi_{s^-,T}\sqrt{V_{s^-}}dW_s + \int_0^t\int_0^\infty \Phi_{s^-,T}\left(e^{(i\rho u + D(s))x} - 1\right)\tilde{J}(dx, ds) \\
&+ \int_0^t \Phi_{s^-,T}V_{s^-} \\
&\times \left(D'(s) + bD(s) - \frac{1}{2}(iu + u^2) + \sum_{j=1}^{n} c_j D(s+\tau_j)\mathbb{1}_{[0,T-\tau_j]}(s)\right) ds \\
&+ \int_0^t \Phi_{s^-,T} \\
&\times \Big(C'(s) + iu(r - \kappa(\rho)) + aD(s) \\
&+ b_Z(iu\rho + D(s)) + \int_0^\infty \left(e^{(i\rho u + D(s))x} - 1\right)\nu(dx)\Big) ds,
\end{aligned}
$$

where $\tilde{J}(dx, dt) = J(dx, dt) - \nu(dx)dt$ is the compensated jump measure of the Lévy process Z. From Equation (15), we know that $\left(\Phi_{t,T}\right)_{t\in[0,T]}$ is a martingale, and hence, we can obtain the following system of differential equations

$$
-D'(t) = -\frac{1}{2}(iu + u^2) + bD(t) + \sum_{j=1}^{N} c_j D(t+\tau_j)\mathbb{1}_{[0,T-\tau_j]}(t), \tag{22}
$$

$$
-C'(t) = iur - iu\kappa(\rho) + aD(t) + \kappa(iu\rho + D(t)), \tag{23}
$$

where we have used the definition of κ, Equation (12). If we define the variable $\ell = T - t$, the functions $A(\ell) = C(t)$ and $B(\ell) = D(t)$, then systems (22) and (23) can be written as (17) and (18). □

Equation (17) is a linear delay differential equation. This type of equation has been studied in the literature before. By continuity of the initial condition $B(\ell) = 0$ for all $\ell \in [-\tau_N, 0]$ and linearity of Equation (17), the method of steps could be used to prove the existence of a unique solution [56, Section 3.1]. However, we will prove in Proposition 4 that Equation (17) has an analytical solution on the interval $[0, T]$ for a fixed $T > 0$. To solve this equation, we will proceed as in [57] and use the Laplace transform. For a suitable function $f : [0, \infty) \to \mathbb{C}$, the Laplace transform is defined as

$$
\mathcal{L}[f](s) = \int_0^\infty f(u)e^{-su}du \text{ for } s \in \mathbb{C}. \tag{24}
$$

Proposition 4. *The solution of the delay differential Equation (17), with initial value $B(\ell) = 0$ for all $\ell \in [-\tau_N, 0]$, is*

$$
B(\ell) = -\frac{iu + u^2}{2}\sum_{n=0}^{\left\lfloor \frac{\ell}{\tau_1}\right\rfloor}\sum_{|\alpha|=n}\frac{c^\alpha}{\alpha!}D_{n,\alpha}(\ell), \tag{25}
$$

where

$$
D_{n,\alpha}(\ell) = \frac{(-1)^n n!}{b^{n+1}}\left[e^{b(\ell - \langle\alpha,\tau\rangle)}\sum_{r=0}^{n}(-1)^{-r}\frac{1}{r!}b^r(\ell - \langle\alpha,\tau\rangle)^r - 1\right]H(\ell - \langle\alpha,\tau\rangle), \tag{26}
$$

with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$, $|\alpha| = \sum_{i=1}^N \alpha_i$, $c^\alpha = \prod_{i=1}^N c_i^{\alpha_i}$, $\alpha! = \prod_{i=1}^N \alpha_i!$, $\langle\alpha,\tau\rangle = \sum_{i=1}^N \alpha_i\tau_i$, $\alpha_i \geq 0$ for every $i = 1, \dots, N$. and H is the Heaviside function.

Proof. Observe that Equation (17) can be written as

$$
B'(\ell) = -\frac{1}{2}(iu + u^2) + bB(\ell) + \sum_{j=1}^{N} c_j B(\ell - \tau_j)H(\ell - \tau_j) \tag{27}
$$

for $\ell \in [0, T]$, where H is the Heaviside function. For brevity, we define

$$L(s) = \mathcal{L}[B](s).$$

By properties of the Laplace transform, we have

$$\mathcal{L}[B'](s) = sL(s) - B(0^-) = sL(s), \tag{28}$$

$$\mathcal{L}[1](s) = \frac{1}{s}, \tag{29}$$

$$\mathcal{L}\left[B(. - \tau_j)H(. - \tau_j)\right](s) = e^{-\tau_j s}L(s), \tag{30}$$

from some $s \in \mathbb{C}$ [58, p. 15, p. 227, p. 18]. Applying the Laplace transform to both sides of Equation (27), we obtain

$$sL(s) = -\frac{0.5(iu + u^2)}{s} + bL(s) + \sum_{j=1}^{N} c_j e^{-\tau_j s}L(s) \tag{31}$$

for some $s \in \mathbb{C}$. Rearranging the terms in (31), we have

$$sL(s) = -\frac{0.5(iu + u^2)}{s - b - \sum_{j=1}^{N} c_j e^{-\tau_j s}}. \tag{32}$$

From property (28) and the geometric series, Equation (32) can be expressed as

$$\mathcal{L}[B'](s) = -\frac{0.5(iu + u^2)}{s - b}\frac{1}{1 - \frac{\sum_{j=1}^{N} c_j e^{-\tau_j s}}{s-b}} = -\frac{0.5(iu + u^2)}{s - b}\sum_{n=0}^{\infty}\frac{\left(\sum_{j=1}^{N} c_j e^{-\tau_j s}\right)^n}{(s-b)^n}, \tag{33}$$

where in the last equality we have assumed that $\left|\frac{\sum_{j=1}^{N} c_j e^{-\tau_j s}}{s-b}\right| < 1$. From the multinomial theorem, we obtain

$$\left(\sum_{j=1}^{N} c_j e^{-\tau_j s}\right)^n = \sum_{|\alpha|=n}\frac{n!}{\alpha!}c^{\alpha}e^{-\langle\alpha,\tau\rangle s}, \tag{34}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$, $|\alpha| = \sum_{j=1}^{N}\alpha_j$, $\alpha! = \prod_{j=1}^{N}\alpha_j!$, $c^{\alpha} = \prod_{j=1}^{N} c_j^{\alpha_j}$ and $\langle\alpha,\tau\rangle = \sum_{j=1}^{N}\alpha_j\tau_j$ with $\alpha_j \geq 0$ for all $j = 1, \dots, N$. Substituting Equation (34) into Equation (33), we get

$$\mathcal{L}[B'](s) = -\frac{iu + u^2}{2}\sum_{n=0}^{\infty}\sum_{|\alpha|=n}\frac{n!}{\alpha!}\frac{c^{\alpha}e^{-\langle\alpha,\tau\rangle s}}{(s-b)^{n+1}}. \tag{35}$$

Observe that by the inverse of the Laplace transform, we have

$$\mathcal{L}^{-1}\left[\frac{n!e^{-\langle\alpha,\tau\rangle s}}{(s-b)^{n+1}}\right](\ell) = (\ell - \langle\alpha,\tau\rangle)^n e^{b(\ell - \langle\alpha,\tau\rangle)}H(\ell - \langle\alpha,\tau\rangle)$$

for $\ell > 0$. Applying the inverse of the Laplace transform on both sides of Equation (35), we arrive at

$$B'(\ell) = -\frac{iu + u^2}{2}\sum_{n=0}^{\infty}\sum_{|\alpha|=n}\frac{c^{\alpha}}{\alpha!}(\ell - \langle\alpha,\tau\rangle)^n e^{b(\ell - \langle\alpha,\tau\rangle)}H(\ell - \langle\alpha,\tau\rangle). \tag{36}$$

Integrating both sides of Equation (36) gives

$$B(\ell) = -\frac{iu+u^2}{2}\sum_{n=0}^{\infty}\sum_{|\alpha|=n}\frac{c^\alpha}{\alpha!}\int_{\langle\alpha,\tau\rangle}^{\ell\vee\langle\alpha,\tau\rangle}(v-\langle\alpha,\tau\rangle)^n e^{b(v-\langle\alpha,\tau\rangle)}dv. \tag{37}$$

Observe that the terms in the first summation of Equation (37) disappear when $n > \left\lfloor \frac{\ell}{\tau_1}\right\rfloor$ because τ_1 is the smallest delay parameter. So we can write

$$B(\ell) = -\frac{iu+u^2}{2}\sum_{n=0}^{\left\lfloor \frac{\ell}{\tau_1}\right\rfloor}\sum_{|\alpha|=n}\frac{c^\alpha}{\alpha!}\int_{\langle\alpha,\tau\rangle}^{\ell\vee\langle\alpha,\tau\rangle}(v-\langle\alpha,\tau\rangle)^n e^{b(v-\langle\alpha,\tau\rangle)}dv. \tag{38}$$

If $\ell > \langle\alpha,\tau\rangle$, the integral of Equation (38) can be written as

$$\begin{aligned}
&\int_{\langle\alpha,\tau\rangle}^{\ell}(v-\langle\alpha,\tau\rangle)^n e^{b(v-\langle\alpha,\tau\rangle)}dv \\
&= \frac{1}{b^{n+1}}\int_0^{b(\ell-\langle\alpha,\tau\rangle)} h^n e^h dh \\
&= \frac{n!}{b^{n+1}}\left[(-1)^n e^h \sum_{r=0}^{n}(-1)^{-r}\frac{1}{r!}h^r\right]\Bigg|_0^{b(v-\langle\alpha,\tau\rangle)} \\
&= \frac{(-1)^n n!}{b^{n+1}}\left[e^{b(\ell-\langle\alpha,\tau\rangle)}\sum_{r=0}^{n}(-1)^r\frac{1}{r!}b^r(\ell-\langle\alpha,\tau\rangle)^r - 1\right].
\end{aligned}$$

Finally, we get to the desired result by substituting the previous result into Equation (37),

$$\begin{aligned}
B(\ell) = &-\frac{iu+u^2}{2}\sum_{n=0}^{\left\lfloor \frac{\ell}{\tau_1}\right\rfloor}\sum_{|\alpha|=n}\frac{c^\alpha}{\alpha!} \\
&\times \frac{(-1)^n n!}{b^{n+1}}\left[e^{b(\ell-\langle\alpha,\tau\rangle)}\sum_{r=0}^{n}(-1)^{-r}\frac{1}{r!}b^r(\ell-\langle\alpha,\tau\rangle)^r - 1\right] H(\ell-\langle\alpha,\tau\rangle).
\end{aligned}$$

□

Remark 2. From Proposition 4, we can compute the deterministic function B for different values of N.

1. In the case $N = 0$, the function B can be expressed as

$$B(\ell) = -\frac{iu+u^2}{2b}\left(e^{b\ell}-1\right).$$

2. When $N = 1$, Equation (25) satisfies

$$B(\ell) = -\frac{iu+u^2}{2}\sum_{n=0}^{\left\lfloor \frac{\ell}{\tau_1}\right\rfloor}\frac{c_1^n(-1)^n}{b^{n+1}}\left[e^{b(\ell-n\tau_1)}\sum_{r=0}^{n}(-1)^{-r}\frac{1}{r!}b^r(\ell-n\tau_1)^r - 1\right].$$

3. If $N = 2$, Equation (25) can be written as

$$B(\ell) = -\frac{iu+u^2}{2}\sum_{n=0}^{\left\lfloor \frac{\ell}{\tau_1}\right\rfloor}\sum_{\alpha_1=0}^{n}\frac{c_1^{\alpha_1}c_2^{n-\alpha_1}}{\alpha_1!(n-\alpha_1)!}$$
$$\times\frac{(-1)^n n!}{b^{n+1}}\left[e^{b(\ell-\alpha_1\tau_1-(n-\alpha_1)\tau_2)}\sum_{r=0}^{n}(-1)^{-r}\frac{1}{r!}b^r(\ell-\alpha_1\tau_1-(n-\alpha_1)\tau_2)^r-1\right]$$
$$\times H(\ell-\alpha_1\tau_1-(n-\alpha_1)\tau_2).$$

We have proved in Proposition 4 that Equation (17) has an analytical solution on the interval $[0,T]$. Notice that this analytical solution is only valid for the initial condition $B(\ell)=0$ for all $\ell\in[-\tau_N,0]$ and when $b\neq 0$. It would be possible to obtain a more general formula, but this is outside the context of the presented application. In Proposition 5, we will prove that the analytical solution is a complex number with a nonpositive real part under the conditions $c_1, c_2, \dots, c_n > 0$, and $b<0$. These conditions make sense from a financial point of view since they allow us to obtain a positive variance process with mean-reverting property.

Notice that the deterministic function B obtained in Proposition 4 is continuous, and hence, the first integral that appears in the following equation

$$A(\ell) = iur\ell - iu\kappa(\rho)\ell + a\int_0^\ell B(s)ds + \int_0^\ell \kappa(iu\rho + B(s))ds \tag{39}$$

is well-defined. For the second Lebesgue integral in (39) to be finite, we need to verify that

$$\Re(B(s)) < \hat{\kappa} \tag{40}$$

for all $s\in[0,\ell]$. If $\Re(B(s))\leq 0$, then condition (40) will be satisfied since $\hat{\kappa}>0$. Since $c_1,\dots,c_N\geq 0$, we have that $\Re(B(\ell))\leq 0$ if and only if the functions $D_{n,\alpha}$ in (26) are nonnegative.

Proposition 5. *The functions $D_{n,\alpha}$ defined in Proposition 4 are nonnegative for all $n\in\mathbb{N}\cup\{0\}$ and all $\alpha=(\alpha_1,\dots,\alpha_n)$ with $\alpha_1,\dots,\alpha_n\geq 0$ and $\sum_{i=1}^n \alpha_i = n$.*

Proof. First, notice that

$$\frac{(-1)^n}{b^{n+1}} < 0 \tag{41}$$

for every $n\in\mathbb{N}\cup\{0\}$ because $b<0$. If $\ell\geq\langle\alpha,\tau\rangle$, we have that

$$\begin{aligned}\left|e^{b(\ell-\langle\alpha,\tau\rangle)}\sum_{r=0}^{n}(-1)^{-r}\frac{1}{r!}b^r(\ell-\langle\alpha,\tau\rangle)^r\right| &\leq e^{b(\ell-\langle\alpha,\tau\rangle)}\sum_{r=0}^{n}\frac{|b|^r}{r!}(\ell-\langle\alpha,\tau\rangle)^r\\ &< e^{b(\ell-\langle\alpha,\tau\rangle)}\sum_{r=0}^{\infty}\frac{|b|^r}{r!}(\ell-\langle\alpha,\tau\rangle)^r\\ &= e^{b(\ell-\langle\alpha,\tau\rangle)}e^{|b|(\ell-\langle\alpha,\tau\rangle)} = 1.\end{aligned} \tag{42}$$

Using inequality (42), we have that

$$e^{b(\ell-\langle\alpha,\tau\rangle)}\sum_{r=0}^{n}(-1)^{-r}\frac{1}{r!}b^r(\ell-\langle\alpha,\tau\rangle)^r - 1 < 0, \text{ when } \ell\geq\langle\alpha,\tau\rangle. \tag{43}$$

From (41) and (43), we arrive at the fact that (26) is nonnegative □

We have shown that the system of delay differential Equations (17) and (18) has an analytical expression on the time interval $[0,T]$ and proved that the characteristic function can be computed analytically. However, the analytical results obtained for the system of differential Equations (17) and (18) are only valid under the strong conditions imposed on the parameters and initial conditions. For other parameters and initial conditions values, the system in (17) and (18) could not have a solution.

4 | NUMERICAL EXPERIMENTS AND EMPIRICAL APPLICATIONS

In this section, we perform numerical simulations to analyze the proposed model, and we use the characteristic function in (15) to calibrate the model against real market options.

In Figures 1 and 2, we plot one realization of the variance process V for different values of N and when the background-driving Lévy process Z is a gamma process or an inverse Gaussian process. Since the process Z has

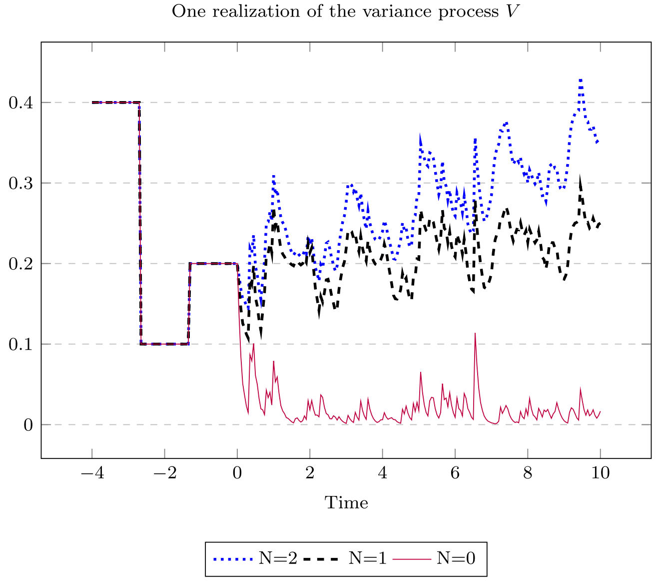

FIGURE 1 One realization of the variance process V when the background driving Lévy process is a gamma process, for different values of N and with the following values for the parameters: $a = 0$, $b = -10$, $c_1 = 10$, $c_2 = 1$, $\tau_1 = 2$, $\tau_2 = 4$, $a_g = 5$, $b_g = 10$, and $V_0 = 0.2$. [Colour figure can be viewed at wileyonlinelibrary.com]

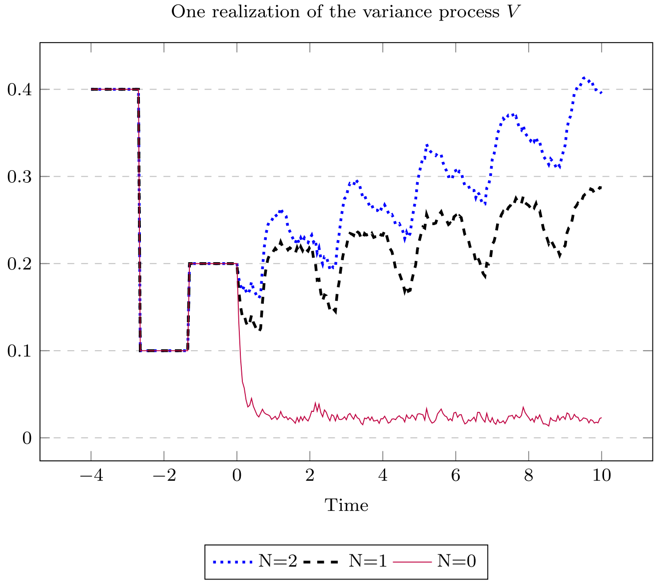

FIGURE 2 One realization of the variance process V when the background driving Lévy process is an inverse Gaussian process, for different values of N and with the following values for the parameters: $a = 0$, $b = -10$, $c_1 = 10$, $c_2 = 1$, $\tau_1 = 2$, $\tau_2 = 4$, $a_I = 5$, $b_I = 10$, and $V_0 = 0.2$. [Colour figure can be viewed at wileyonlinelibrary.com]

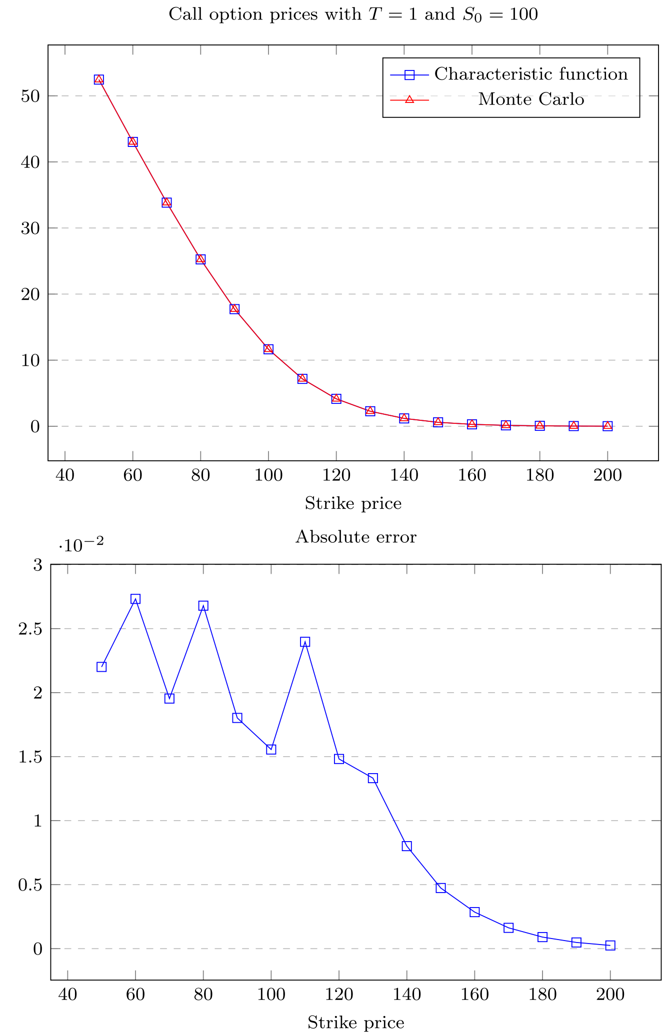

FIGURE 3 Call option prices and absolute value of the error, when the model has the following parameters: $r = 0.05$, $a = 0$, $b = -10$, $c_1 = 0.2$, $c_2 = 0.3$, $\tau_1 = 0.25$, $\tau_2 = 0.5$, $\rho = -0.7$, $V_0 = 0.2$; the background-driving Lévy process is a gamma process with parameters $a_g = 5$ and $b_g = 20$ and $\phi(t) = 0.2$ for all $t \in [-\tau_2, 0]$. The call prices are given by the characteristic function and by the Monte Carlo approximation. The error is the difference between the prices given by the characteristic function and those given by the Monte Carlo approximation. [Colour figure can be viewed at wileyonlinelibrary.com]

locally finite variation paths, we can use Equation (5) to simulate the process V. We can approximate process V using Riemann–Stieltjes type approximations (see Appendix A). We simulate the process V for different values of N. When $N = 0$, the process V does not have any delay parameter; when $N = 1$, the process has one delay parameter τ_1; and when $N = 2$, the process has two delay parameters τ_2 and τ_1. These realizations have been generated using the following initial value function

$$\phi(t) = \begin{cases} 0.4 & \text{if } t \in \left[-4, -\frac{8}{3}\right] \\ 0.1 & \text{if } t \in \left(-\frac{8}{3}, -\frac{4}{3}\right] \\ 0.2 & \text{if } t \in \left(-\frac{4}{3}, 0\right]. \end{cases} \tag{44}$$

We observe that delay parameters allow us to generate trajectories with periodicities. However, the periodic property is lost when the variance process has no delay parameter. Evidence in the literature shows that the volatility can behave periodically [59–61].

We compare the call option prices given by the characteristic function in (15) with the call prices provided by Monte Carlo. In that way, we can verify that the pricing method proposed in Section 3 is well-implemented. The Monte Carlo approximation is based on the Romano and Touzi formula [62], and we used 200,000 realizations and 1000 time steps.

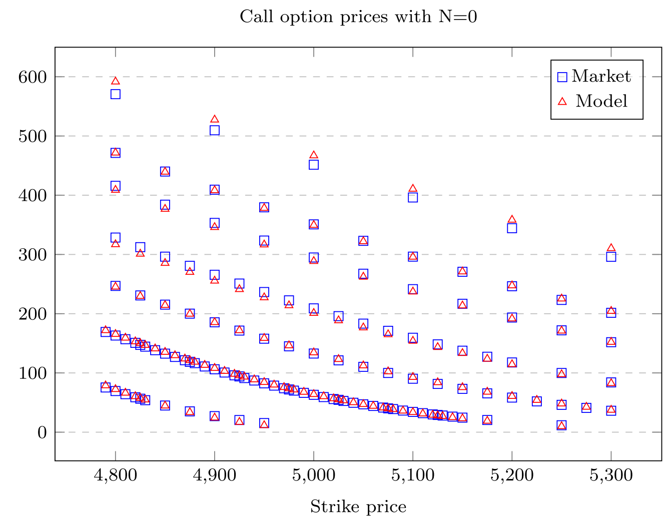

FIGURE 4 Call option prices from the S&P500 index taken on the date 2024/01/11 and call option prices given by the calibrated model with $N = 0$. [Colour figure can be viewed at wileyonlinelibrary.com]

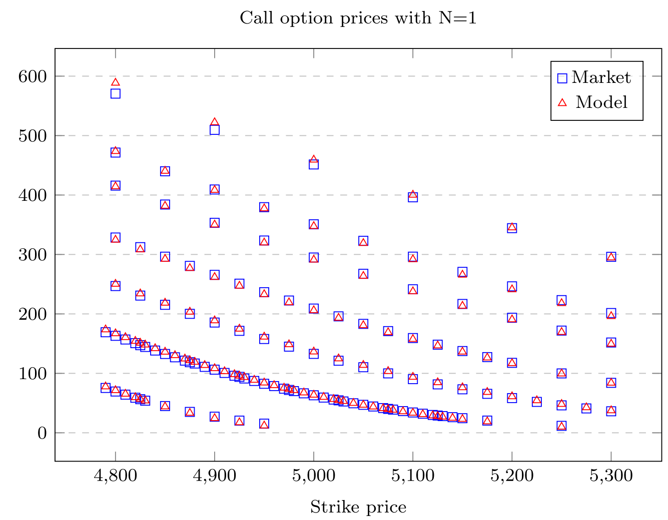

FIGURE 5 Call option prices from the S&P500 index taken on the date 2024/01/11 and call option prices given by the calibrated model with $N = 1$. [Colour figure can be viewed at wileyonlinelibrary.com]

The option prices given by the characteristic function are computed using the method proposed by Carr and Madan [63]. The results obtained are shown in Figure 3. Notice that the results provided by the characteristic function are similar to the ones given by the Monte Carlo approximation.

Remark 3. The Lebesgue integrals that appear in Equations (19) and (39) are approximated by Riemann sums in the following way

$$\int_0^T B(t)dt \approx \sum_{i=1}^{M_0} B(t_{i-1})(t_i - t_{i-1}),$$

$$\int_0^T \kappa\left(iu\rho + B(t)\right)dt \approx \sum_{i=1}^{M_0} \kappa\left(iu\rho + B(t_{i-1})\right)(t_i - t_{i-1}),$$

$$\int_{-\tau_j}^0 \mathbb{1}_{[-\tau_j, T-\tau_j]}(s)B(T - s - \tau_j)\phi(s)ds \approx \sum_{k=1}^{N_0} \mathbb{1}_{[-\tau_j, T-\tau_j]}(s_{k-1})B(T - s_{k-1} - \tau_j)\phi(s_{k-1})(s_k - s_{k-1}),$$

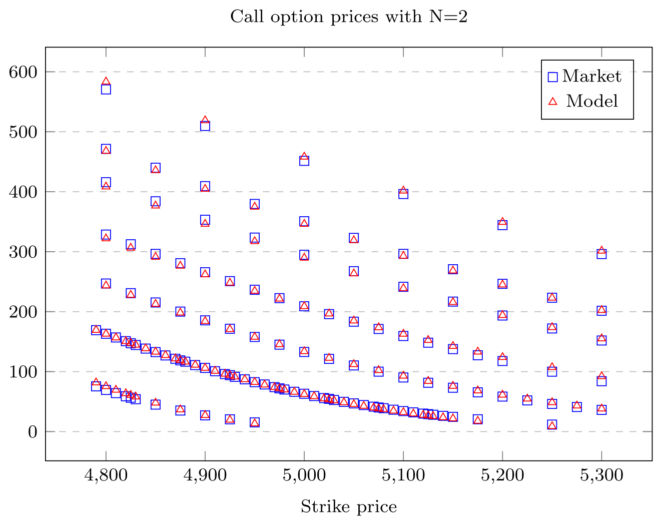

FIGURE 6 Call option prices from the S&P500 index taken on the date 2024/01/11 and call option prices given by the calibrated model with $N = 2$. [Colour figure can be viewed at wileyonlinelibrary.com]

where $M_0 \in \mathbb{N}$, $t_i = i\frac{T}{M_0}$ for $i = 0, 1, \dots, M_0$, $N_0 = \left\lfloor \frac{\tau_N M_0}{T} \right\rfloor$, and $s_k = -\tau_N + \tau_N \frac{j}{N_0}$ for $k = 0, 1, \dots, N_0$. In the experiments shown in Figure 3, we use $M_0 = 1000$, while in the experiments shown in Figures 4–6, we use $M_0 = 200$.

Finally, we calibrate the model against real market data and show that the inclusion of delays improves the accuracy of the model. We use near out of the money European call options on the S&P500 index to test the proposed model. The data are from Yahoo Finance (https://finance.yahoo.com/quote/%5ESPX/options/). We take 125 call options on the date 2024/01/11 with the following expiration dates $T = 0.1177$, $T = 0.3477$, $T = 0.5968$, $T = 0.8459$, $T = 1.1909$, $T = 1.4401$, and $T = 1.9383$. The strikes vary from 4790 to 5300, and the underlying has a value of 4783.45. To calibrate the parameters of our model, we minimize the mean squared error

$$MSE = \frac{1}{125} \sum_{i=1}^{125} (Call^i_{Market} - Call^i_{Model})^2. \tag{45}$$

The $Call^i_{Market}$ is computed using the average value between ask and bid prices. We would like the mid prices to be representative of the true option prices; because of that, we only take market options that satisfy

$$\frac{\text{Ask price} - \text{Bid price}}{\text{Ask price}} < 0.1.$$

We calibrate our model for $N = 0$, $N = 1$, and $N = 2$ when the background-driving Lévy process is a gamma process. When $N = 1$, we use the following initial value function

$$\phi(t) = \begin{cases} a_\phi & \text{if } t \in \left[-\tau_1, -\frac{\tau_1}{2}\right] \\ b_\phi & \text{if } t \in \left(-\frac{\tau_1}{2}, 0\right], \end{cases} \tag{46}$$

where $a_\phi, b_\phi > 0$. In the case $N = 2$, the initial value function is

$$\phi(t) = \begin{cases} a_\phi & \text{if} t \in \left[-\tau_2, -\frac{2\tau_2}{3}\right] \\ b_\phi & \text{if} t \in \left(-\frac{2\tau_2}{3}, -\frac{\tau_2}{3}\right] \\ c_\phi & \text{if} t \in \left(-\frac{\tau_2}{3}, 0\right], \end{cases} \tag{47}$$

Number of delay parameters	Mean squared error
$N = 0$	26.8509
$N = 1$	11.7504
$N = 2$	12.5112

TABLE 1 Mean squared error.

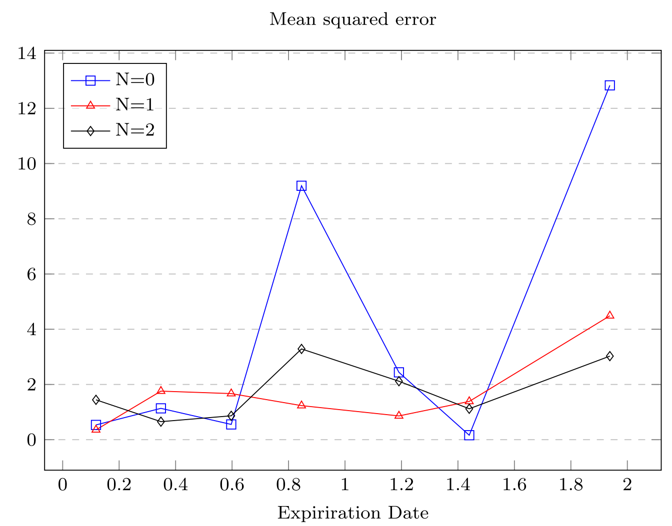

FIGURE 7 Mean squared error obtained for each expiration date when comparing S&P500 call option prices with call option prices obtained by the calibrated model. [Colour figure can be viewed at wileyonlinelibrary.com]

where $a_\phi, b_\phi, c_\phi > 0$. For the interest rate, we use the 3-month USD LIBOR rate on 2024/01/11, $r = 0.055756$, extracted online (https://www.marketwatch.com/investing/interestrate/liborusd3m?countrycode%3Dmr). We must impose some conditions on the delay parameters τ_1 and τ_2 to calibrate the parameters. We impose that $\tau_i \leq 2$ for $i = 1, 2$, so the maximum value of the delay parameters is near the maximum expiry date. We also require that $\tau_i \geq 0.05$ for $i = 1, 2$. If the delay parameters are too small, we encounter problems when computing Equation (25). The last condition we impose is $\tau_2 - \tau_1 \geq 0.05$, since we need $\tau_2 > \tau_1$. The calibration procedures were done in Python, and in particular, we used the sequential least squares optimization procedure in SciPy [64].

In Figures 4–6, we plot the market option prices and the option prices given by our model. In Table 1, we give the mean squared error obtained for the different values of N. Observe that the presence of delay parameters reduces the mean squared error significantly. When $N = 1$, the error is reduced by 56.23% concerning the base model, while when $N = 2$, the error is reduced by 53.40%. However, no improvement is obtained when comparing the two delay parameters model with the one delay parameter model. Figure 7 and Table 2 show the mean squared error obtained for each expiration date. Delay parameters make the error more stable through expiration dates. The presence of delay parameters reduces the error in long-term maturity options. In Table 3, we show the calibrated parameters of the proposed model. The values of the parameter a are almost zero when $N = 1$ and $N = 2$. Hence, we could eliminate the parameter a from the model. Notice that the absolute value of the parameter b increases when we increase the number of delay parameters. When $N = 2$, the values for delay parameters τ_1 and τ_2 are near each other. This could explain why the model with one delay parameter and the model with two delay parameters obtain a similar mean squared error. When $N = 1$, the value of the delay parameter τ_1 is near 2, the maximum value permitted.

The main drawback of this model is that every time we add a delay parameter, we need to add a summation in Equation (25). This makes the pricing procedure more complex when the number of delay parameters is high. Adding the delay parameter τ_j also adds the parameter c_j. Because of that, calibration is very tedious when having several delay parameters. In Table 4, we show the time it took us to calibrate the model for various delay parameters. We can observe that increasing the number of delay parameters increases the time needed for calibration.

TABLE 2 Mean squared error obtained for each expiration date.

Expiration date	$N = 0$	$N = 1$	$N = 2$
$T = 0.1177$	0.5345	0.356	1.444
$T = 0.3477$	1.1349	1.7584	0.6527
$T = 0.5969$	0.5525	1.6694	0.8649
$T = 0.846$	9.1962	1.2336	3.2869
$T = 1.191$	2.4389	0.8607	2.1171
$T = 1.4401$	0.1621	1.3855	1.1194
$T = 1.9384$	12.8318	4.4868	3.0263

TABLE 3 Calibrated parameters.

Parameter	$N = 0$	$N = 1$	$N = 2$
a	0.1262	-6×10^{-13}	-9×10^{-17}
b	−44.3850	−58.3168	−180.6986
ρ	−0.9051	−2.1411	−3.0623
c_1	–	2.6333	45.7331
c_2	–	–	4×10^{-15}
a_g	4.7513	1.5165	7.5323
b_g	25.4040	27.6887	87.6807
V_0	0.00098	–	–
τ_1	–	1.9895	0.8873
τ_2	–	–	0.9373
a_ϕ	–	0.1052	0.0055
b_ϕ	–	1×10^{-13}	0.0133
c_ϕ	–	–	0.0287

TABLE 4 Calibration time of the model when it has a different number of delay parameters.

Number of delay parameters	Calibration time (s)
$N = 0$	559.9894
$N = 1$	1780.1121
$N = 2$	7390.9387

Note: These results were obtained on a PC with Intel Core i5-1335U CPU and 16 GB of RAM.

5 | CONCLUSION

This paper introduces a novel stochastic volatility model that incorporates delay parameters within the volatility process. This represents a significant extension of the traditional Barndorff–Nielsen and Shephard model. We proved that obtaining an analytical formula for the characteristic function of the log price is possible. This is the first time someone has given an analytical formula for the characteristic function of the log price on a stochastic volatility model with several delay parameters in the variance process. In addition, we provided a Monte Carlo method that can be used to price options as well. As shown in Figure 5, the results obtained by the characteristic function and the Monte Carlo method are similar. Furthermore, we priced European call options for the S&P500 index and showed that delay parameters reduce the mean squared error. The results showed that when $N = 1$, the error was reduced by 56.23% concerning the base model, while when $N = 2$, the error was reduced by 53.40%; see Table 1. The delay parameters made the error more stable through expiration dates and reduced the error in long-term maturity options; see Table 2. The presented results are of importance in the area of finance for two main reasons. The first reason is that we can price options in a reasonable time using the Carr and Madan [63] formula; hence, the model can be used in the industry. The second reason is that the proposed model can be considered a continuous time counterpart for well-known discrete time series models like the GARCH model [31] and the autoregressive stochastic volatility model [32]; this opens a new area of research.

Implementing the proposed model also introduces additional complexity in calibration and computational demands. Because of that, we think that it would be possible to use a neural network to decrease the calibration time [65]. Another option to decrease the computation time of the calibration might be to use numerical approximations to solve the system of differential Equations (17) and (18). Numerical approximation might be faster than the analytical solution in Equation (25), since numerical approximations do not need to calculate the summations that appear in (25). For example, we can use the package in R deSolve [66] to solve numerically the system of delay differential Equations (17) and (18).

The model defined in (1) and (2) is already on a risk-neutral probability measure, as shown in Proposition 2. As it happened with the usual Barndorff–Nielsen and Shephard model [3], it would be possible to show that the proposed model is incomplete. This implies the existence of several risk-neutral measures. For example, Arai and Imai [67] studied the price of options in the Barndorff–Nielsen and Shephard model under the equivalent minimal martingale measure. For future work, we would like to study the price of options under different martingale measures.

Several versions of the Barndorff–Nielsen and Shephard model exist in the literature. For example, Salmon and SenGupta [9] and Tong [68] constructed a fractional variation of the model, Bannör and Scherer [5] and SenGupta [4] built a Barndorff–Nielsen and Shephard models with two jumps, and Muhle-Karbe et al. [7] proposed a multivariate version of the Barndorff–Nielsen and Shephard model. The presented model can be combined with all previous models, generating a plethora of new models. In addition, Barndorff–Nielsen and Shephard models can be used to model the dynamics of commodities and energy prices [11, 12, 69]. The introduced model exhibits periodic trajectories in volatility, aligning with observed seasonality in commodities and energy prices [59]. Consequently, the delayed Barndorff–Nielsen and Shephard model presents practical implications in the valuation of commodities and energy derivatives. The effect of delay parameters on the trajectories of process V could be studied in future works to understand the relation between delay parameters and seasonality behavior. The effect of the time delay on the dynamics of delay differential equations has been studied before by Huang and Liu [70].

AUTHOR CONTRIBUTIONS

Álvaro Guinea Juliá: Conceptualization; formal analysis; writing—original draft; writing—review and editing; supervision. **Raquel Caro-Carretero:** Supervision; writing—review and editing.

CONFLICT OF INTEREST STATEMENT

The authors declare no potential conflict of interests.

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APPENDIX A: APPROXIMATION OF THE VARIANCE PROCESS

In this section, we propose an approximation of the process V, defined in Equation (5). This approximation is based on the fact that the process Z has locally finite variation trajectories almost surely.

If we define a sequence of partitions $0 = t_0^{(m)} < t_1^{(m)} < \ldots < t_m^{(m)} = T > 0$ of the time interval $[0, T]$ such that $\max_{\{1\le k\le m\}}\left\{ t_k^{(m)} - t_{k-1}^{(m)} \right\} \to 0$ as $m \to \infty$, then we have that

$$\lim_{m\to\infty} \sum_{k=1}^{m} e^{-bt_{k-1}^{(m)}} \mathbb{1}_{(-\infty,t]}\left(t_{k-1}^{(m)}\right)\left(Z_{t_k^{(m)}} - Z_{t_{k-1}^{(m)}}\right) = \int_0^T e^{-bs}\mathbb{1}_{(-\infty,t]}(s)dZ_s = \int_0^t e^{-bs}dZ_s \text{ for } t \in [0,T] \tag{A1}$$

almost surely [71, Proposition 5.19]. The limit defined in (A1) is well-defined since Z has locally finite variation càdlàg paths, and $\left(\mathbb{1}_{(-\infty,t]}(s)e^{-bs}\right)_{s\in[0,T]}$ is a bounded left-continuous function on $[0, T]$.

Based on the result on (A1), we will create an approximation that converges almost surely. This approximation can be used to simulate the process V.

We will construct the approximation in the presence of a unique delay parameter. However, this approximation can be generalized easily for N delay parameters.

Instead of approximating the process V, we will approximate the process $Y = (Y_t)_{t\geq 0}$, which is defined as

$$Y_t = e^{-bt}V_t \text{ for } t \geq 0,$$

and then we will recover V_t by multiplying by e^{bt}.

When there is only a one delay parameter τ_1, Y_T can be written as

$$Y_T = Y_t + \int_t^T ae^{-bs}ds + \int_t^T c_1 e^{-b\tau_1} Y_{(s-\tau_1)^-} ds + \int_t^T e^{-bs} dZ_s. \tag{A2}$$

Let us define $M = \left\lfloor \frac{T}{\tau_1} \right\rfloor$, and the following sequences of partitions

$$(k-1)\tau_1 = t_0^{(m_k)} < t_1^{(m_k)} < \ldots < t_{m_k}^{(m_k)} = k\tau_1 \wedge T$$

such that $\max_{\{1\leq j\leq m_k\}} \left\{ t_j^{(m_k)} - t_{j-1}^{(m_k)} \right\} \to 0$ as $m_k \to \infty$ for $k = 1, \ldots, M+1$. For every $t \in [(k-1)\tau_1, k\tau_1 \wedge T]$ with $k \in \{1, \ldots, M+1\}$, we define the approximation of Y_t recursively as

$$\begin{aligned} Y_t^{m_k} = Y_{\tau_1(k-1)}^{m_{k-1}} &+ \sum_{j=1}^{m_k} ae^{-bt_{j-1}^{(m_k)}} \mathbb{1}_{(-\infty,t]}\left(t_{j-1}^{(m_k)}\right)\left(t_j^{(m_k)} - t_{j-1}^{(m_k)}\right) \\ &+ \sum_{j=1}^{m_k} c_1 e^{-b\tau_1} Y^{m_{k-1}}_{(t_{j-1}^{(m_k)}-\tau_1)} \mathbb{1}_{(-\infty,t]}\left(t_{j-1}^{(m_k)}\right)\left(t_j^{(m_k)} - t_{j-1}^{(m_k)}\right) \\ &+ \sum_{j=1}^{m_k} e^{-bt_{j-1}^{(m_k)}} \mathbb{1}_{(-\infty,t]}\left(t_{j-1}^{(m_k)}\right)\left(Z_{t_j^{(m_k)}} - Z_{t_{j-1}^{(m_k)}}\right), \end{aligned} \tag{A3}$$

with the following initial condition

$$\begin{aligned} Y_t^{m_1} = Y_0 &+ \sum_{j=1}^{m_1} ae^{-bt_{j-1}^{(m_1)}} \mathbb{1}_{(-\infty,t]}\left(t_{j-1}^{(m_1)}\right)\left(t_j^{(m_1)} - t_{j-1}^{(m_1)}\right) \\ &+ \sum_{j=1}^{m_1} c_1 e^{-bt_{j-1}^{(m_1)}} \phi(t_{j-1}^{(m_1)} - \tau_1) \mathbb{1}_{(-\infty,t]}\left(t_{j-1}^{(m_1)}\right)\left(t_j^{(m_1)} - t_{j-1}^{(m_1)}\right) \\ &+ \sum_{j=1}^{m_1} e^{-bt_{j-1}^{(m_1)}} \mathbb{1}_{(-\infty,t]}\left(t_{j-1}^{(m_1)}\right)\left(Z_{t_j^{(m_1)}} - Z_{t_{j-1}^{(m_1)}}\right) \text{ when } t \in [0, \tau_1]. \end{aligned} \tag{A4}$$

It is possible to show that the recursive relation defined in Equations (A3) and (A4) converges to the process Y.

Proposition 6. *The recursive relation defined in Equations (A3) and (A4) satisfy*

$$lim_{m_k,\ldots,m_1\to\infty} Y_t^{m_k} = Y_t \textit{ almost surely} \tag{A5}$$

for all $t \in [(k-1)\tau_1, k\tau_1 \wedge T]$ *with* $k \in \{1, \ldots, M+1\}$.

Proof. When $t \in [0, \tau_1]$, we have

$$\begin{aligned}\lim_{m_1\to\infty} Y_t^{m_1} &= Y_0 + \int_0^t ae^{-bs}ds \\ &+ \int_0^t c_1 e^{-bs}\phi(s-\tau_1)ds \\ &+ \int_0^t e^{-bs}dZ_s = Y_t \text{ almost surely,}\end{aligned} \tag{A6}$$

by application of the result in (A1) and the Riemann–Lebesgue theorem [72, Theorem 3.21]. In the case $t \in [\tau_1, 2\tau_1]$, the limit when $m_1 \to \infty$ of the approximation $Y_t^{m_2}$ satisfies

$$\begin{aligned}\lim_{m_1\to\infty} Y_t^{m_2} &= \lim_{m_1\to\infty} Y_{\tau_1}^{m_1} + \sum_{j=1}^{m_2} ae^{-bt_{j-1}^{(m_2)}} \mathbb{1}_{(-\infty,t]}\left(t_{j-1}^{(m_2)}\right)\left(t_j^{(m_2)} - t_{j-1}^{(m_2)}\right) \\ &+ \sum_{j=1}^{m_2} c_1 e^{-b\tau_1} Y^{m_1}_{(t_{j-1}^{(m_2)}-\tau_1)} \mathbb{1}_{(-\infty,t]}\left(t_{j-1}^{(m_2)}\right)\left(t_j^{(m_2)} - t_{j-1}^{(m_2)}\right) \\ &+ \sum_{j=1}^{m_2} e^{-bt_{j-1}^{(m_2)}} \mathbb{1}_{(-\infty,t]}\left(t_{j-1}^{(m_2)}\right)\left(Z_{t_j^{(m_2)}} - Z_{t_{j-1}^{(m_2)}}\right) \\ &= Y_{\tau_1} + \sum_{j=1}^{m_2} ae^{-bt_{j-1}^{(m_2)}} \mathbb{1}_{(-\infty,t]}\left(t_{j-1}^{(m_2)}\right)\left(t_j^{(m_2)} - t_{j-1}^{(m_2)}\right) \\ &+ \sum_{j=1}^{m_2} c_1 e^{-b\tau_1} Y_{(t_{j-1}^{(m_2)}-\tau_1)} \mathbb{1}_{(-\infty,t]}\left(t_{j-1}^{(m_2)}\right)\left(t_j^{(m_2)} - t_{j-1}^{(m_2)}\right) \\ &+ \sum_{j=1}^{m_2} e^{-bt_{j-1}^{(m_2)}} \mathbb{1}_{(-\infty,t]}\left(t_{j-1}^{(m_2)}\right)\left(Z_{t_j^{(m_2)}} - Z_{t_{j-1}^{(m_2)}},\right),\end{aligned} \tag{A7}$$

where in the last equality, we have used the result in (A6). Taking the limit when $m_2 \to \infty$ on both sides of Equation (A7), we obtain

$$\begin{aligned}\lim_{m_2\to\infty}\lim_{m_1\to\infty} Y_t^{m_2} &= Y_{\tau_1} + \int_{\tau_1}^t ae^{-bs}ds \\ &+ \int_{\tau_1}^t c_1 e^{-b\tau_1} Y_{(s-\tau_1)}ds \\ &+ \int_{\tau_1}^t e^{-bs}dZ_s = Y_t \text{ almost surely,}\end{aligned}$$

where we have applied (A1) and the Riemann–Lebesgue theorem [72, Theorem 3.21].

The general case will be proved by induction. Let us assume that the limit (A5) is satisfied for all $t \in [(k-1)\tau_1, k\tau_1]$; we would like to prove the result in (A5) when $t \in [k\tau_1, (k+1)\tau_1 \wedge T]$. By induction assumption, the approximation $Y_t^{m_{k+1}}$ satisfies

$$\begin{aligned}&lim_{m_k,\dots,m_1\to\infty} Y_t^{m_{k+1}} \\ &= Y_{\tau_1 k} + \sum_{j=1}^{m_{k+1}} ae^{-bt_{j-1}^{(m_{k+1})}} \mathbb{1}_{(-\infty,t]}\left(t_{j-1}^{(m_{k+1})}\right)\left(t_j^{(m_{k+1})} - t_{j-1}^{(m_{k+1})}\right) \\ &+ e^{bt}\sum_{j=1}^{m_{k+1}} c_1 e^{-b\tau_1} Y_{(t_{j-1}^{(m_{k+1})}-\tau_1)} \mathbb{1}_{(-\infty,t]}\left(t_{j-1}^{(m_{k+1})}\right)\left(t_j^{(m_{k+1})} - t_{j-1}^{(m_{k+1})}\right) \\ &+ e^{bt}\sum_{j=1}^{m_{k+1}} e^{-bt_{j-1}^{(m_{k+1})}} \mathbb{1}_{(-\infty,t]}\left(t_{j-1}^{(m_{k+1})}\right)\left(Z_{t_j^{(m_{k+1})}} - Z_{t_{j-1}^{(m_{k+1})}}\right) \text{ almost surely.}\end{aligned}$$

Taking the limit when $m_{k+1} \to \infty$ on both sides of the previous equation, we finally arrive at

$$\begin{aligned} lim_{m_{k+1}, m_k, \ldots, m_1 \to \infty} Y_t^{m_{k+1}} = Y_{\tau_1 k} &+ \int_{k\tau_1}^{t} ae^{-bs} ds \\ &+ \int_{k\tau_1}^{t} c_1 e^{-b\tau_1} Y_{(s-\tau_1)} ds \\ &+ \int_{k\tau_1}^{t} e^{-bs} dZ_s = Y_t \text{ almost surely. } \square \end{aligned}$$

We have shown that the approximation defined in (A3) and (A4) can be used to approximate the process Y on the interval $[0, T]$; to that end, we only need to simulate the increments of Z. Using Equations (A3) and (A4), the approximation of Y_T satisfies

$$\begin{aligned} Y_T^{m_{M+1}} = Y_0 &+ \sum_{i=1}^{M+1} \sum_{j=1}^{m_i} ae^{-bt_{j-1}^{(m_i)}} \left(t_j^{(m_i)} - t_{j-1}^{(m_i)} \right) \\ &+ \sum_{i=1}^{M+1} \sum_{j=1}^{m_i} c_1 e^{-b\tau_1} Y^{m_{i-1}}_{(t_{j-1}^{(m_i)} - \tau_1)} \left(t_j^{(m_i)} - t_{j-1}^{(m_i)} \right) \\ &+ \sum_{i=1}^{M+1} \sum_{j=1}^{m_i} e^{-bt_{j-1}^{(m_i)}} \left(Z_{t_j^{(m_i)}} - Z_{t_{j-1}^{(m_i)}} \right), \end{aligned} \tag{A8}$$

with

$$Y^{m_{1-1}}_{(t_{j-1}^{(m_1)} - \tau_1)} = \phi(t_{j-1}^{(m_1)} - \tau_1) e^{-b\left(t_{j-1}^{m_1} - \tau_1\right)}.$$

Using Equation (A8), we construct the following simulation scheme. For a partition $0 = t_0 < t_1 < \ldots < t_H = T$ of the interval $[0, T]$ with $t_i = i\Delta$ for $i = 0, 1, \ldots, H$, $\Delta = T/H$, and $\tau_1 = H_{\tau_1}\Delta$ where H_{τ_1} is an integer, approximation (A8) can be rewritten as

$$Y_T^H = Y_0 + \sum_{j=1}^{H} ae^{-bt_{j-1}} \Delta + \sum_{j=1}^{H} c_1 e^{-bH_{\tau_1}\Delta} Y^H_{t_{j-1} - H_{\tau_1}\Delta} \Delta + \sum_{j=1}^{H} e^{-bt_{j-1}} \left(Z_{t_j} - Z_{t_{j-1}} \right), \tag{A9}$$

with

$$Y^H_{t_{i-1} - H_{\tau_1}\Delta} = \phi\left(t_{i-1} - H_{\tau_1}\Delta\right) e^{-b\left(t_{i-1} - H_{\tau_1}\Delta\right)} \text{ when } \left(t_{i-1} - H_{\tau_1}\Delta\right) \leq 0.$$

Approximation (A9) can be expressed recursively as

$$Y_{t_i}^H = Y_{t_{i-1}}^H + \left(ae^{-bt_{i-1}} + c_1 e^{-bH_{\tau_1}\Delta} Y^H_{t_{i-1} - H_{\tau_1}\Delta} \right) \Delta + e^{-bt_{i-1}} \left(Z_{t_i} - Z_{t_{i-1}} \right). \tag{A10}$$

From Proposition (6), we know that for large values of H, Equation (A9) is a good approximation of Y_T and hence $V_T^H = e^{bT} Y_T^H$ is a good a approximation of V_T.

Finally, if we define $V_{t_i}^H = e^{bt_i} Y_{t_i}^H$, then from Equation (A10), we have that

$$V_{t_i}^H = V_{t_{i-1}}^H e^{b\Delta} + \left(ae^{b\Delta} + c_1 e^{b\Delta} V^H_{t_{i-1} - H_{\tau_1}\Delta} \right) \Delta + e^{b\Delta} \left(Z_{t_i} - Z_{t_{i-1}} \right) \tag{A11}$$

with

$$V^H_{t_{i-1} - H_{\tau_1}\Delta} = \phi\left(t_{i-1} - H_{\tau_1}\Delta\right) \text{ when } t_{i-1} - H_{\tau_1}\Delta \leq 0.$$

We use Equation (A11) to simulate the process V in the numerical experiments of Section 4. We use Algorithm A1 and Remark 4 to generate one realization that approximates the process V.

Algorithm 1: Simulation of the process V.

Input: $N_{\tau_1}, H, \Delta, \phi, a, b, c,$ parameters of the process Z ;
Output: $\left\{v_{i\Delta}^H\right\}_{i=-N_{\tau_1}}^H$;
for $i = 1, 2, \ldots, N_{\tau_1} + 1$ **do**
  Initialize: $v_{(i-1)\Delta - N_{\tau_1}\Delta}^H = \phi\left((i-1)\Delta - N_{\tau_1}\Delta\right)$;
end
for $i = 1, 2, \ldots, H$ **do**
  Draw a sample Δz_i from the random variable $Z_{i\Delta} - Z_{(i-1)\Delta}$;
  $v_{i\Delta}^H = v_{(i-1)\Delta}^H e^{b\Delta} + \left(ae^{b\Delta} + c_1 e^{b\Delta} v_{(i-1)\Delta - H_{\tau_1}\Delta}^H\right)\Delta + e^{b\Delta}\Delta z_i$;
end
return $\left\{v_{i\Delta}^H\right\}_{i=-N_{\tau_1}}^H$;

Remark 4. The increments $Z_{t_i} - Z_{t_{i-1}}$ that appear in (A11) follow a different distribution depending on the type of Lévy process that Z satisfies.

1. If the subordinator Z is a gamma process with parameters $a_g, b_g > 0$, then the increment $Z_{t_i} - Z_{t_{i-1}}$ satisfies a gamma distribution with parameters $a_g\Delta$, b_g [54, p. 52].
2. In the case the subordinator Z is an inverse Gaussian process with parameters $a_I, b_I > 0$, then the increment $Z_{t_i} - Z_{t_{i-1}}$ satisfies an inverse Gaussian distribution with parameters $a_I\Delta$, b_I [54, p. 53].