Complex oscillations in a thermosyphon viscoelastic model

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Abstract. Thermoyphons, in the engineering literature, is a device composed of a closed loop containing a fluid who motion is driven by several actions such as gravity and natural convection. In this work we consider a viscoelastic fluid described by the Maxwell constitutive equation. Their dynamics are governing for a coupled differential nonlinear systems, and in several previous work we show chaos in the fluid. This work is, a generalization of Proposition 2 in [11], and also in some sense, a generalization of some previous results on standard (Newtonian) fluids obtained by A. Rodrguez-Bernal and E.S. Van Vleck [21,22], when we consider a viscoelastic fluid.

Keywords: Thermosyphon, Viscoelastic fluid, Asymptotic behaviour.

1 Introduction

Instabilities and chaos in fluids subject to temperature gradients have been the subject of intense work for its applications in engineering and in atmospheric sciences. The interest on this system comes both from engineering and as a *toy* model of natural convection.

In the engineering literature a thermosyphon is a closed loop containing a fluid whose motion is driven by the effect of several actions such as gravity and natural convection.The flow inside the loop is driven by an energetic balance between thermal energy and mechanical energy.

Here, we consider a thermosyphon model in which the confined fluid is viscoelastic. This has some *a-priori* interesting peculiarities that could affect the dynamics with respect to the case of a Newtonian fluid. On the one hand, the dynamics has memory so its behavior depends on the whole past history and, on the second hand, at small perturbations the fluid behaves like an elastic solid and a characteristic resonance frequency could, eventually, be relevant (consider for instance the behavior of jelly or toothpaste).

The simplest approach to viscoelasticity comes from the so-called Maxwell model Morrison [18].

Viscoelastic behavior is common in polymeric and biological suspensions and, consequently, our results may provide useful information on the dynamics of this sort of systems inside a thermosyphon.

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In a thermosyphon the equations of motion can be greatly simplified because of the quasi-one-dimensional geometry of the loop. Thus, we assume that the section of the loop is constant and small compared with the dimensions of the physical device, so that the arc length co-ordinate along the loop (x) gives the position in the circuit. The velocity of the fluid is assumed to be independent of the position in the circuit, i.e. it is assumed to be a scalar quantity depending only on time. This approximations comes from the fact that the fluid is assumed to be incompressible.

On the contrary temperature is assumed to depend both on time and position along the loop.

The derivation of the thermosyphon equations of motion is similar to that in Ref. keller [16] and are obtained in Yasappan and Jiménez-Casas et al. [14]. Finally, after adimensionalizing the variables (to reduce the number of free parameters) we get the followingo ODE/PDE system (see Yasappan and Jiménez-Casas et al. [14] and Bravo-Gutierrez and Castro et al. [1]):

$$\begin{cases} \varepsilon \dfrac{d^2 v}{dt^2} + \dfrac{dv}{dt} + G(v)v = \oint Tf, \\ \dfrac{\partial T}{\partial t} + v \dfrac{\partial T}{\partial x} \qquad = l(v)(T_a - T) + c\frac{\partial^2 T}{\partial x^2} \end{cases} \tag{1}$$

with $v(0) = v_0$, $\frac{dv}{dt}(0) = w_0$ and $T(0, x) = T_0(x)$.

Here $v(t)$ is the velocity, $T(t, x)$ is the distributions of the temperature of the viscoelastic fluid into the loop, $G(v)$, is the friction law at the inner wall of the loop, the function f is the geometry of the loop and the distribution of gravitational forces. In this cases $l(v)(T_a - T)$ is the Newton's linear cooling law as in Jiménez-Casas and Rodríguez-Bernal [5–7], Yasappan, Jiménez-Casas et al. [14], Morrison [18] , Rodríguez-Bernal and Van Vleck [21] or Welander [24], where l represents the heat transfer law across the loop wall, and is a positive quantity depending on the velocity, and T_a is the (given) ambient temperature distribution.

In this case we consider also in the transport equation for the temperature (beside of the Newton's linear cooling law) a term of the diffusion given by $c\frac{\partial^2 T}{\partial x^2}$, where c is a positive constant which denoting the thermal diffusivity.

ε in Eq. (1) is the viscoelastic parameter, wich is the dimensionless version of the viscoelastic time. Roughly speaking, it gives the time scale in which the transition from elastic to fluid-like occurs in the fluid.

We assume that $G(v)$ is positive and bounded away from zero. This function has been usually taken to be $G(v) = G_0$, a positive constant for the linear friction case [16], or $G(v) = |v|$ for the quadratic law [4,17], or even a rather general function given by $G(v) = g(Re)|v|$,where Re is a Reynolds-like number that is assumed to be large [22,23] and proportional to $|v|$. The functions G, f, l and T_a incorporate relevant physical constants of the model, such as the cross sectional area, D, the length of the loop, L, the Prandtl, Rayleigh, or Reynolds numbers, etc see [23]. Usually G, l are given continuous functions, such that $G(v) \geq G_0 > 0$, and $l(v) \geq l_0 > 0$, for G_0 and l_0 positive constants.

Finally, for physical consistency, it is important to note that all functions considered must be 1-periodic with respect to the spatial variable, and $\oint =$

$\int_0^1 dx$ denotes integration along the closed path of the circuit. Note that $\oint f = 0$.

The contribution in this paper (section 3) is to prove that, under suitable conditions, any solution either converges to the rest state or the oscillations of velocity around $v = 0$ must be large enough. This result is a generalization of the asymptotic behaviour proposed in [11] with $c = 0$ i.e. without diffusion of temperature, when we consider a the thermal diffusivity $c \neq 0$, i.e with a term of the diffusion given by $c\frac{\partial^2 T}{\partial x^2}$.

Moreover, this result given by Proposition 2 generalizes, in some sense, the some previous results proposed on standard (Newtonian) fluid in Rodríguez-Bernal and Van Vleck [21,22] when we consider a thermosyphon model with a one-component **viscoelastic** fluid.

2 Previous results about well posedness, global attractor and inertial manifold

2.1 Well posedness and global attractor

First, we note that in this paper we consider the case in which all periodic functions in Eq. (1) have zero average, i.e. we work in $\mathcal{Y} = I\!R^2 \times \dot{H}^1_{per}(0,1)$, where

$$\dot{L}^2_{per}(0,1) = \{u \in L^2_{loc}(I\!R),\, u(x+1) = u(x) a.e., \oint u = 0\},$$

$$\dot{H}^m_{per}(0,1) = H^m_{loc}(I\!R) \cap \dot{L}^2_{per}(0,1).$$

In effect, we observe that, if we integrate the equation for the temperature along the loop, taking into account the periodicity of T, that is, $\oint(\partial T/\partial x) = \oint(\partial^2 T/\partial x^2) = 0$, we have that $\frac{d}{dt}(\oint T) = l(v)\oint(T_a - T)$. Therefore, $\oint T \to \oint T_a$, exponentially as $t \to \infty$, for every $\oint T_0$.

Moreover, if we consider $\theta = T - \oint T$, then from the second equation of system (1), we obtain that θ verifies the same equation of T replacing T_a, T_0, respectively by $T_a - \oint T_a, T_0 - \oint T_0$ with zero average.

Finally, since $\oint f = 0$, we have that $\oint Tf = \oint \theta f$, and the equation for v reads

$$\varepsilon\frac{d^2v}{dt^2} + \frac{dv}{dt} + G(v)v = \oint \theta . f, \quad v(0) = v_0, \frac{dv}{dt}(0) = w_0. \tag{2}$$

Thus, hereafter we consider the system Eq. (1) with $\oint T_0 = \oint T_a = 0$ and $\oint T(t) = 0$ for every $t \geq 0$.

Also, if $c > 0$, the operator $-c(\partial^2/\partial x^2)$, together with periodic boundary conditions, is an unbounded, self adjoint operator with compact resolvent in $L^2_{per}(0,1)$ that is positive when restricted to the space of zero average functions $\dot{L}^2_{per}(0,1)$. Hence, the equation for the temperature T in (1) is of parabolic type for $c > 0$.

Therefore, we can apply the result about sectorial operator of Henry [3] to prove the existence of solutions of system (1). Moreover, if we consider some aditionally hypothesis (H) to add for the friction G using in the technique Lemma 5 in Yasappan and Jiménez-Casas [14], which are satisfied for all friction functions G consider in the previous works,i.e., the thermosyphon models where G is constant or linear or quadratic law, and also for $G(s) \equiv A|s|^n$, as $s \to \infty$. Then, we have the next result.

Proposition 1. *We suppose that $H(r) = rG(r)$ is locally Lipschitz, $f, l \in \dot{L}^2_{per}(0,1)$ $l(v) \geq l_0 > 0$ and $T_a \in \dot{H}^1_{per}(0,1)$. Then, given $(w_0, v_0, T_0) \in \mathcal{Y} = I\!R^2 \times \dot{H}^1_{per}(0,1)$, there exists a unique solution of (1) satisfying*

$$(w, v, T) \in C([0,\infty], \mathcal{Y}) \text{ and } (\dot{w}, w, \frac{\partial T}{\partial t}) \in C([0,\infty), I\!R^2 \times \dot{L}^2_{per}(0,1)),$$

where $w = \dot{v} = \frac{dv}{dt}$ and $\dot{w} = \frac{d^2v}{dt^2}$. In particular, (1) defines a nonlinear semigroup, $S(t)$ in $\mathcal{Y}$, with $S(t)(w_0, v_0, T_0) = (w(t), v(t), T(t))$.

Moreover, from (H) (see [14]) Eq. (1) has a global compact and connected attractor, $\mathcal{A}$, in $\mathcal{Y}$. Also if $T_a \in \times \dot{H}^m_{per}(0,1)$ with $m \geq 1$, the global attractor $\mathcal{A} \subset I\!R^2 \times \dot{H}^{m+2}_{per}(0,1)$ and is compact in this space.

Proof. This result has been proved in Theorem 3, Theorem 5 and Corollary 11 from Yasappan and Jiménez-Casas at al.[14].

□

2.2 Inertial manifold

In this section we assume also that $G^*(r) = rG(r)$ is locally Lipschitz satisfying (H) (see [14]), and $f, T_a \in \dot{L}^2_{per}$ are given by following Fourier expansions

$$T_a(x) = \sum_{k \in Z\!\!Z^*} b_k e^{2\pi kix}; \quad f(x) = \sum_{k \in Z\!\!Z^*} c_k e^{2\pi kix}; \text{ where } Z\!\!Z^* = Z\!\!Z - \{0\},$$

while $T_0 \in \dot{H}^2_{per}$ is given by $T_0(x) = \sum_{k \in Z\!\!Z^*} a_{k0} e^{2\pi kix}$.

Finally assume that $T(t,x) \in \dot{H}^2_{per}$ is given by

$$T(t,x) = \sum_{k \in Z\!\!Z^*} a_k(t) e^{2\pi kix} \text{ where } Z\!\!Z^* = Z\!\!Z - \{0\}.$$

We note that $\bar{a_k} = -a_k$ since all functions consider are real and also $a_0 = 0$ since they have zero average.

Now we observe the dynamics of each Fourier mode and from Eq. (1), we get the following system for the new unknowns, v and the coefficients $a_k(t)$.

$$\left\{ \begin{array}{c} \varepsilon \frac{d^2v}{dt^2} + \frac{dv}{dt} + G(v)v = \sum_{k \in Z\!\!Z^*} a_k(t) c_{-k} \\ \dot{a}_k(t) + \left[2\pi kiv(t) + 4c\pi^2 k^2 + l(v(t)) \right] a_k(t) = l(v(t)) b_k \end{array} \right. \tag{3}$$

Assume that the given ambient temperature $T_a \in \dot{H}^m_{per}$, are given by

$$T_a(x) = \sum_{k \in K} b_k e^{2\pi kix}, \text{ and } b_k \neq 0 \text{ for every } k \in K \subset Z\!\!Z,$$

with $0 \neq K$, since $\oint T_a = 0$. We denote by V_m the clousure of the subspace of $\dot{H}^m_{per}$ generated by $\{e^{2\pi kix}, k \in K\}$.

Then we have from Theorem 13 in Yasappan and Jiménez-Casas et al.[14] the set $\mathcal{M} = I\!R^2 \times V_m$ is an **inertial manifold** for the flow of $S(t)(w_0, v_0, T_0) = (w(t), v(t), T(t))$ in the space $\mathcal{Y} = I\!R^2 \times \dot{H}^m_{per}(0,1)$.

By this, the dynamics of the flow is given by the flow in $\mathcal{M}$ associated to the given ambient temperature T_a. This is

$$\left\{ \begin{array}{c} \varepsilon \frac{d^2 v}{dt^2} + \frac{dv}{dt} + G(v)v = \sum_{k \in K} a_k(t) c_{-k} \\ \dot{a}_k(t) + \left[2\pi kiv(t) + 4c\pi^2 k^2 + l(v(t))\right] a_k(t) = l(v(t)) b_k, \; k \in K \end{array} \right. \tag{4}$$

3 Complex oscillations

In this section we consider the linear friction law [16] where $G(v) = G_0$, and $l(v) = l_0$ with G_0 and l_0 positive constants, and we consider the diffusion temperature in the transport equation for the temperature, as we have commented in the previous section 1.This is, we consider the thermal diffusivity $c \geq 0$.

The aim in this section is to prove the Proposition 2, for this linear friction case [16] when we consider the diffusion temperature. This way, we generalize the result of thermosyphon model without diffusion temperature ($c = 0$)[11] and also, in some sense, the result of thermosyphon models for Newtoniann fluids of Rodríguez-Bernal and Van Vleck [21,22].

We note proving the Proposition 2, we get to generalize for this general case with thermal diffusivity different to zero $c > 0$, the results which show the complex oscillations, this is: under suitable conditions,for large time the velocity reaches the equilibrium - null velocity -, or takes a value to make its integral diverge, which means that either it remains with a constant value without changing its sign or it will alternate an infinite number of times so the oscillations around zero become large enough to make the integral diverge.

3.1 Reduced subsystem for the relevant temperature coefficients

First of all, we note that hereafter, we consider de functions T_a(ambient temperature) and f(the function associated to the geometry of the loop), are given by the following Fourier expansions

$$T_a(x) = \sum_{k \in K} b_k e^{2\pi kix}, \quad f(x) = \sum_{k \in J} c_k e^{2\pi kix}, \tag{5}$$

where

$K = \{k \in Z\!\!\!Z^* / b_k \neq 0\}, J = \{k \in Z\!\!\!Z^* / c_k \neq 0\}$ with $Z\!\!\!Z^* = Z\!\!\!Z - \{0\}$.

Next, from the equations Eq.(3), regarding the right hand side of the first equation, we can observe that the velocity of the fluid is independent of the coefficients for temperature $a_k(t)$ for every $k \in Z\!\!\!Z^* - (K \cap J)$.

That is, the **relevant coefficients** for the velocity are only $a_k(t)$ with k belonging to the set $K \cap J$. This important result about the asymptotic behaviour has been proved in Propositions 14 and 15 from Yasappan and Jiménez-Casas at al.[14].

We also note that $0 \notin K \cap J$ and since $K = -K$ and $J = -J$ then the set $K \cap J$ has an even number of elements, which we denote by $2n_0$, where n_0 is the number of the positive elements of $K \cap J$, $(K \cap J)_+$.

Moreover the equations for a_{-k} are conjugates of the equations for a_k and therefore we have $\sum_{k \in K \cap J} a_{k(t)} c_{-k} = 2Re(\sum_{k \in (K \cap J)_+} a_k(t) c_{-k})$.

Thus,

$$\oint Tf = 2Re(\sum_{k \in (K \cap J)_+} a_k(t) c_{-k}). \tag{6}$$

with $\bar{a_k} = -a_k$ since all functions consider are real and also $a_0 = 0$ since they have zero average. From Eq. (1), we get the following system for the new unknowns, v and the relevant coefficients $a_k(t)$.

$$\begin{cases} \varepsilon \frac{d^2 v}{dt^2} + \frac{dv}{dt} + G(v)v = \sum_{k \in (K \cap J)_+} a_k(t) c_{-k} \\ \dot{a}_k(t) + \left[2\pi k i v(t) + 4c\pi^2 k^2 + l(v(t))\right] a_k(t) = l(v(t)) b_k, k \in (K \cap J)_+ \end{cases} \tag{7}$$

3.2 Result about complex oscillations

In order to prove the Proposition 2, we consider the following Lemma which have been proved in [11].

Lemma 1. *We consider the linear equation given by*

$$\varepsilon \frac{d^2 v}{dt^2} + \frac{dv}{dt} + G_0 v = I(t), \tag{8}$$

then there exist $v_p(t)$ particular solution of (8) such that

$$limsup_{t \to \infty} |v_p(t)| \leq limsup_{t \to \infty} |I(t)| \tag{9}$$

and

$$liminf_{t \to \infty} |v_p(t)| \geq liminf_{t \to \infty} |I(t)| \tag{10}$$

Proof. See Lemma 1 pag 250 in [11]

□

Proposition 2. *We consider the linear friction case, i.e.* $G(v) = G_0$, *and* $l(v) = l_0$ *with* G_0 *and* l_0 *positive constants and with the thermal diffusivity* $c \geq 0$.

i) if $K \cap J = \emptyset$, *then the global attractor for system Eq. (1) in* $\mathbb{R}^2 \times \dot{H}^1_{per}(0,1)$ *is reduced to a point* $\{(0, 0, \theta_\infty)\}$. *This is, for every* $(w_0, v_0, T_0) \in \mathbb{R}^2 \times \dot{H}^1_{per}(0,1)$, *one has that the associated solution verifies* $(w(t), v(t), T(t)) \to$

$(0,0,\theta_\infty)$ *in* $I\!R^2 \times \dot{H}^1_{per}(0,1)$ *as* $t \to \infty$*; where* $\theta_\infty(x)$ *is the unique solution in* $\dot{H}^2_{per}(0,1)$ *of the equation*

$$-c\frac{\partial^2 \theta_\infty}{\partial x^2} + l_0\theta_\infty = l_0 T_a. \qquad (11)$$

Moreover, we note that if $c = 0$*, one gets* $T(t) \to T_a$*.*

ii) We assume that

$$I_0 = Re(\sum_{k\in(K\cap J)_+} \frac{b_k c_{-k}}{4c\pi^2k^2 + l_0}) = 0, \qquad (12)$$

with $K \cap J$ *finite set, and that a solution of Eq.* (3) *satisfies* $\int_0^\infty |v(s)|ds < \infty$*. Then the system reaches the rest stationary solution, this is:*

$$\begin{cases} v(t) \to 0, \ \text{and } w(t) \to 0, \ \text{as } t \to \infty \\ a_k(t) \to l_0 \frac{b_k}{4c\pi^2k^2+l_0}, \ \text{as } t \to \infty \end{cases}$$

Therefore, he also have in this situation the global attractor for the system Eq. (1) *in* $I\!R^2 \times \dot{H}^1_{per(0,1)}$ *is reduced to a point* $\{(0,0,\theta_\infty)\}$*, with* θ_∞ *satisfying the equation (11).*

iii) Conversely, if $I_0 = Re(\sum_{k\in(K\cap J)_+} \frac{b_k c_{-k}}{4c\pi^2k^2 + l_0}) \neq 0$*, then for every solution* $\int_0^\infty |v(s)|ds = \infty$*, and* $v(t)$ *does not converge to zero.*

Proof. We cover several steps.

Step 1: First, we study the behaviour for large time of the coefficients $a_k(t)$.

From the second equation in (3), for t_0 enough large, we known that for every $t > t_0$, we have

$$a_k(t) = a_k(t_0)e^{-\int_{t_0}^t 2\pi kvi + 4c\pi^2k^2 + l_0} + l_0 b_k \int_{t_0}^t e^{-\int_r^t 2\pi kvi + 4c\pi^2k^2 + l_0} dr. \qquad (13)$$

Now, we note that

$$a_k(t) = a_k(t_0)e^{-\int_{t_0}^t 2\pi kvi + 4c\pi^2k^2 + l_0} + l_0 b_k(I_1(t) + (I_2(t)) \qquad (14)$$

where

$$I_1(t) = \int_{t_0}^t e^{-\int_r^t 4c\pi^2k^2 + l_0} dr = \frac{(1 - e^{-(4c\pi^2k^2 + l_0)(t - t_0)})}{4c\pi^2k^2 + l_0} \qquad (15)$$

and

$$I_2(t) = \int_{t_0}^t e^{-\int_r^t 4c\pi^2k^2 + l_0} \left(e^{-\int_r^t 2\pi kvi} - 1\right) dr \qquad (16)$$

Next, taking limits when $t \to \infty$, in (14), we have that:

i)

$$a_k(t_0)e^{-\int_{t_0}^t 2\pi kvi+4c\pi^2k^2+l_0} \to 0 \text{ as } t \to \infty$$

since $\left|e^{-\int_{t_0}^t 2\pi kvi}\right| = 1$ and c, l_0 are positive constants. And from (15), we also have

ii)

$$I_1(t) \to \frac{1}{4c\pi^2k^2 + l_0} \text{ as } t \to \infty.$$

Therefore, if we assume that

$$I_2(t) \to 0 \text{ as } t \to \infty, \tag{17}$$

then taking limits when $t \to \infty$, in (14), we obtain that:

$$a_k(t) \to l_0 \frac{b_k}{4c\pi^2k^2 + l_0}, \text{ as } t \to \infty. \tag{18}$$

Thus, to conclude this step, this is to get (18), it is enough to prove (17).

In order to prove (17), we consider a solution of Eq.(3) satisfying $\int_0^\infty |v(s)ds < \infty$, and then we will prove that for every $\eta > 0$ there exists t_0 such that $I_2(t) \le \eta$ for every $t \ge t_0$.

In effect, if $\int_0^\infty |v(s)|ds < \infty$, then for all δ there exits $t_0 > 0$ such that for every $t_0 \le r \le t$ we have $|\int_r^t v| \le \delta$. Then, for any $\eta > 0$ we can take t_0 large enough such that

$$|e^{-\int_r^t 2\pi ikv} - 1| \le (4c\pi^2k^2 + l_0)\eta \text{ for all } t_0 \le r \le t. \tag{19}$$

Hence, from (16) we get

$$I_2(t) = \int_{t_0}^t e^{-\int_r^t 4c\pi^2k^2+l_0}\Big(e^{-\int_r^t 2\pi kvi} - 1\Big)dr \le \eta\Big(1 - e^{-(4c\pi^2k^2+l_0)(t-t_0)}\Big) \le \eta,$$

since $\left|1 - e^{-(4c\pi^2k^2+l_0)(t-t_0)}\right| \le 1$, and we get (17). Thus, we conclude (18)

Moreover, with $K \cap J$ a finite set, we also have:

$$\begin{cases} a_k(t) \to l_0 \frac{b_k}{4c\pi^2k^2+l_0} \\ I(t) = 2Re(\sum_{k\in(K\cap J)_+} a_k(t)c_{-k}) \to 2l_0I_0 \end{cases} \tag{20}$$

with $I_0 = Re(\sum_{k\in(K\cap J)_+} \frac{b_kc_{-k}}{4c\pi^2k^2 + l_0})$.

Step 2: We study now the asymptotic behaviour for the temperature $T(t, x)$.

In this step, we will prove that

$$T(t,x)=\sum_k a_k(t)e^{2\pi kix}\to\theta_\infty=l_0\sum_k\frac{b_k}{4c\pi^2k^2+l_0}e^{2\pi kix}\ \text{in}\ \dot{H}^1_{per}(0,1),\quad (21)$$

and we also note

$$-c\frac{\partial^2\theta_\infty}{\partial x^2}+l_0\theta_\infty=l_0\sum_k b_k e^{2\pi kix}=l_0T_a(x).$$

In effect, first from $T(t,x)=\sum_k a_k(t)e^{2\pi kix}\in\dot{H}^1_{per}(0,1)$ for every $t\geq t_0\geq 0$ we have that $\sum_k k^2|a_k(t)|^2<\infty$ for every $t\geq t_0\geq 0$, and using $T_a(x)=\sum_k b_k e^{2\pi kix}\in\dot{L}^2_{per}(0,1)$ we also have that $\sum_k|b_k|^2<\infty$, and then

$$\sum_{k=m+1}^{\infty}k^2|a_k(t)|^2\to 0,\ \text{for every}\ t\geq t_0\geq 0\ \text{and}\ \sum_{k=m+1}^{\infty}|b_k|^2\to 0\ \text{as}\ m\to\infty.$$

Next, we will prove that

$$\sum_{k=m+1}^{\infty}k^2|a_k(t)|^2\to 0\ \text{as}\ m\to\infty,\ \text{uniformly for}\ t\ \text{large}.\quad (22)$$

From (13), taking into account that $|e^{-\int_r^t 2\pi ikv}|=1$ together with (15), we get

$$|a_k(t)|\leq|a_k(t_0)|e^{-(4\mu\pi^2k^2+l_0)(t-t_0)}+\frac{l_0|b_k|}{4c\pi^2k^2+l_0}(1-e^{(-4c\pi^2k^2+l_0)(t-t_0)}).$$

Therefore, using now $e^{-(4c\pi^2k^2(t-t_0)+l_0}\leq 1$ and $(1-e^{-(4c\pi^2k^2+l_0)(t-t_0)})\leq 1$, we get

$$|a_k(t)|\leq|a_k(t_0)|+\frac{l_0|b_k|}{4c\pi^2k^2+l_0}.$$

Thus, we obtain that

$$\sum_{k=m+1}^{\infty}k^2|a_k(t)|^2\leq C\Big(\sum_{k=m+1}^{\infty}k^2|a_k(t_0)|^2+l_0^2\sum_{k=m+1}^{\infty}\frac{k^2|b_k|^2}{(4c\pi^2k^2+l_0)^2}\Big),$$

and since $\frac{k^2}{(4c\pi^2k^2+l_0)^2}\leq\frac{1}{16c^2\pi^4k^2}\leq\frac{1}{16c^2\pi^4}$, we get

$$\sum_{k=m+1}^{\infty}k^2|a_k(t)|^2\leq C\Big(\sum_{k=m+1}^{\infty}k^2|a_k(t_0)|^2+\frac{l_0^2}{16c^2\pi^4}\sum_{k=m+1}^{\infty}|b_k|^2\Big),$$

with $C>0$ independent of k,m and t, and we conclude (22).

Finally, we note that

$$\|(T(t,x)-\theta_\infty)_x\|^2_{\dot{L}^2_{per}(0,1)}\leq 4\pi^2\sum_k k^2|a_k(t)-\frac{l_0b_k}{4c\pi^2k^2+l_0}|^2\leq$$

$$\leq 4\pi^2 \sum_{k=1}^{m} k^2 |a_k(t) - \frac{l_0 b_k}{4c\pi^2 k^2 + l_0}|^2 +$$

$$+4\pi^2 \sum_{k=m+1}^{\infty} k^2 |a_k(t) - \frac{l_0 b_k}{4c\pi^2 k^2 + l_0}|^2 = 4\pi^2 S_m(t) + 4\pi^2 R_{m+1}(t),$$

where

$$R_{m+1}(t) = \sum_{k=m+1}^{\infty} k^2 |a_k(t) - \frac{l_0 b_k}{4c\pi^2 k^2 + l_0}|^2 \leq C \sum_{k=m+1}^{\infty} k^2 |a_k(t)|^2 +$$

$$+\frac{C}{16c^2\pi^4} \sum_{k=m+1}^{\infty} |b_k|^2 \to 0$$

as $m \to \infty$ uniformly for t large thanks to (22).

Then, for every $\eta > 0$ there exists $m_0(\eta) > 0$ such that $4\pi^2 R_{m_0+1}(t) < \frac{\eta}{2}$. Therefore, using again (20), we obtain $t_0(\eta) > 0$ enough large, such that we also have $4\pi^2 S_{m_0}(t) < \frac{\eta}{2}$ for every $t \geq t_0$, where

$$S_{m_0}(t) = \sum_{k=1}^{m_0} k^2 |a_k(t) - \frac{l_0 b_k}{4c\pi^2 k^2 + l_0}|^2.$$

This is, we get

$$\|(T(t,x) - \theta_\infty)_x\|^2_{\dot{L}^2_{per}(0,1)} \to 0 \text{ as } t \to \infty.$$

Analogously, we also prove that

$$\|T(t,x) - \theta_\infty\|^2_{\dot{L}^2_{per}(0,1)} \to 0 \text{ as } t \to \infty,$$

and we get that $T(t,x) \to \theta_\infty$ in $\dot{H}^1_{per}(0,1)$.

To conclude, we study now when the velocity $v(t)$ and the acceleration $w(t)$ go to zero.

Step 3: We study now the asymptotic behaviour for the velocity $v(t)$.

From (6) we can reading the equation for v, the first equation of system Eq. (3), as

$$\varepsilon \frac{d^2 v}{dt^2} + \frac{dv}{dt} + G(v)v = I(t).$$

we consider now, $G(v) = G_0 > 0$ and then we note that:

I) First, we consider $v_p(t)$ the particular solution of the above equation given by Lemma 1 and we denoted by $v_H(t)$ the solution of linear homogeneous equation given by:

$$\varepsilon \frac{d^2 v}{dt^2} + \frac{dv}{dt} + G_0 v = 0$$

such that $v(t) = v_p(t) + v_H(t)$. We now that since $v_H(t) \to 0$ as $t \to \infty$, we have that: $v(t) - v_p(t) \to 0$ as $t \to \infty$.

II) Second,using (20) for every $\delta > 0$ there exists t_0 such that $|I(t) - I_0^*| \le \delta$ for ever $t \ge t_0$, with $I_0^* = 2l_0 I_0$ and using Lemma 1, we conclude that

$$limsup_{t\to\infty}|v(t)| \le \frac{I_0^* + \delta}{G_0} \text{ and } liminf_{t\to\infty}|v(t)| \ge \frac{I_0^* - \delta}{G_0} \tag{23}$$

for every $\delta > 0$.

Step 4: Next, we study the asymptotic behaviour for the acceleration $w(t)$, and using the above steps to conclude the proof.

First, we will prove that if $v(t) \to 0$ then $w(t) \to 0$ as $t \to \infty$ exponentially.
In effect, if $v(t) \to 0$, for every $\delta > 0$ there exists t_0 such that $|G_0 v| \le \delta$ and $\varepsilon \frac{d|w|}{dt} + |w| \le \delta$ for every $t \ge t_0$, this is

$$|w(t)| \le |w(t_0)|e^{-\frac{1}{\varepsilon}(t-t_0)} + \delta[1 - e^{-\frac{1}{\varepsilon}(t-t_0)}] \le \delta \tag{24}$$

i.e $w(t) \to 0$ as $t \to \infty$ exponentially.

i)-ii) From (23), in particular when $K \cap J = \emptyset$ or $I_0 = 0$,(i.e $I_0^* = 2l_0 I_0 = 0$) we get $v(t) \to 0$ and from (24), $w(t) \to 0$ as $t \to \infty$. Then, taking into account that (21) we conclude.

iii) Finally, we also note that from (23)

$$liminf_{t\to\infty}|v(t)| \ge \frac{I_0^* - \delta}{G_0}$$

for every $\delta > 0$ and in the case of $I_0 \ne 0$,(i.e $I_0^* = 2l_0 I_0 \ne 0$) taking enough small δ we get $liminf_{t\to\infty}|v(t)| > 0$, which implies that $\int_0^\infty |v(s)|ds = \infty$. This result is in contradiction with the initial condition $\int_0^\infty |v(s)|ds < \infty$, what implies that it is not a valid hypothesis, and we conclude.

□

3.3 Concluding remarks

Recalling that functions associated to circuit geometry, f, and to prescribed ambient temperature, T_a, are given by $f(x) = \sum_{k\in J} c_k e^{2\pi kix}$ and $T_a(x) = \sum_{k\in K} b_k e^{2\pi kix}$, respectively.

In Yasappan and Jiménez-Casas et. al [14], using the operator abstract theory, it is proved that if $K \cap J = \emptyset$, then the global attractor for system Eq. (1) in $I\!R^2 \times \dot{H}^1_{per}(0,1)$ is reduced to a point for every $c \ge 0$.

In this sense the Proposition 2 offers the possibility to obtain the same asymptotic behaviour for the dynamics, i.e., the attractor is also reduced to a point taking functions f and T_a without this condition, that is with $K \cap J \ne \emptyset$, its enough that the set $(K \cap J) \ne \emptyset$, but

$Re(\sum_{k\in(K\cap J)_+} \frac{b_k c_{-k}}{4c\pi^2k^2+l_0}) = 0$, when we consider the linear friction case $G = G_0$, and $l = l_0$ in the Newton calling law and also considering diffusion temperature i.e. ($l_0(T_a - T) + c\frac{\partial^2 T}{\partial x^2}$).

We note, the result about the inertial manifold (Yasappan and Jiménez-Casas et. al [14]) reduces the asymptotic behaviour of the initial system Eq. (1) to the dynamics of the reduced explicit system Eq. (4) with $k \in K \cap J$.

We observe also that from the analysis above, it is possible to design the geometry of circuit, f, and/or ambient temperature, T_a, so that the resulting system has an arbitrary number of equations of the form $N = 4n_0 + 1$ where n_0 is the number of elements of $(K \cap J)_+$ and we consider the real an imaginary parts of relevant coefficients for the temperature $a_k(t)$ and solute concentration $d_k(t)$ with $k \in (K \cap J)_+$.

Note that it may be the case that K and J are infinite sets, but their intersection is finite. Also, for a circular circuit we have $f(x) \sim asen(x) + bcos(x)$, i.e. $J = \{\pm 1\}$ and then $K \cap J$ is either $\{\pm 1\}$ or the empty set.

Finally, we also note if the ambient temperature T_a is constant in this case for every f geometry of the loop, we have the set $K \cap J = \emptyset$, this is, the global attractor for system Eq. (1) in $I\!R^2 \times \dot{H}^1_{per}(0,1)$ is reduced to a point for every $c \geq 0$.

Recently, we have considered a thermosyphon model containing a viscoelastic fluid and we have shown chaos in some closed-loop thermosyphon model with one-component viscoelastic fluid not only in this model [14], also in other kind of transfer law ([12], Jiménez-Casas and Castro [8], Yasappan and Jiménez-Casas et al. [13],[9]), and even in some cases with a viscoelastic binary fluid (Yasappan and Jiménez-Casas et al. [15]) and Jiménez-Casas and Castro [8]).

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