

# Asymptotic Behaviour of a Thermosyphon model with binary fluids

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**Abstract.** The aim of this work is the generalization for a thermosyphon model where one can be considered a binary fluid, of the theoretical results about the asymptotic behaviour of the thermosyphon model for a fluid with one-component from ([9]).

## INTRODUCTION

The concern of this study is to analyze the dynamics of a fluid transporting an inner solute along an arbitrary close geometry and subject to gravity and convection forces (thermosyphon). In the present work will be considered the thermosyphon model with a binary fluid - fluid with a solute substance -, and we study the evolution of the velocity  $v$ , the temperature of the fluid,  $T$ , and the solute concentration,  $S$ .

In this case the Soret's effect is considered, which consists in a solute flux generation due to a temperature gradient in the fluid (see [1],[7],[12]). The physical reason for this effect seems to rest on the system auto-organization in a manner to get the most efficient heat transport (minimum entropy production). In general, the heaviest specie for the one with the strongest interaction is accumulated in the coldest system region. The Soret's effect is mathematically represented as a first approximation by a coefficient ratio product and the temperature gradient, this coefficient is denoted by Soret's coefficient.

The evolution of the above quantities is given by the following coupled ODE/PDE system (see [6], [4], [8], [3], [11] for further details) with periodic boundary conditions for  $T, S$

$$\begin{cases} \varepsilon \frac{dv}{dt} + G(v)v = \oint [T - S] f(x) dx, v(0) = v_0 \\ \frac{\partial T}{\partial t} + v \frac{\partial T}{\partial x} = H(v) (T_a - T), T(0, x) = T_0(x) \\ \frac{\partial S}{\partial t} + v \frac{\partial S}{\partial x} = c \frac{\partial^2 S}{\partial x^2} - b \frac{\partial^2 T}{\partial x^2}, S(0, x) = S_0(x). \end{cases} \quad (1)$$

We assume that the section of the loop is constant and small compared with the dimension of the physical circuit, so that the arc length co-ordinate along the loop  $x$  gives the position in the circuit. The velocity is assumed to be independent of the position in the circuit.

The function  $f$  describes the geometry of the loop and the distribution of gravitational forces,  $\oint = \int_0^1 dx$  denotes the path integration along the closed path of the circuit. We assume that  $G(v)$ , which specifies the frictional law at the inner wall of the loop, and  $H$ , associated to the heat flux across the loop wall, are positive and bounded away from zero, with  $G(v) \geq G_0 > 0$ ,  $H(v) \geq H_0 > 0$ . We consider  $G, f$  and  $H$  are given functions together with the constants,  $b, c > 0$ , and the ambient temperature,  $T_a(x)$ . The constant  $b$  is proportional to the Soret's coefficient ([1]) and  $c$  is the solute diffusion coefficient.

We note that  $\oint f = 0$  and all functions depending on  $x$  are 1-periodic respect to  $x$  and have zero average (see [6], [4]). We denote by  $\dot{L}_{per}^2$  the functions with zero average on  $L_{per}^2(0, 1)$  as the Sobolev's space  $\dot{H}_{per}^2$ .

The result about the well-posedness and asymptotic behaviour for the system (1) it can be found in ([6], [4]). In this previous works the existence and uniqueness of solution is proved for every time with initial data in suitable framework (see also [5]). In this previous works have been studied the asymptotic behaviour of the system (1) when time goes to infinity.

In this sense has been proved the existence of a inertial manifold associated to the functions  $f$  (loop-geometry) and  $T_a$  (ambient temperature), using the the abstract operators theory ([2], [10]).

We denote by  $\dot{L}_{per}^2$  the functions of  $L_{per}^2(0, 1)$  with zero average as the Sobolev's space  $\dot{H}_{per}^2$ . We assume that  $G^*(r) = rG(r)$  is locally Lipschitz,  $H \in C^1$ ,  $T_a \in \dot{H}_{per}^2$  and  $f \in \dot{L}_{per}^2$  are given by following Fourier expansions

$$T_a(x) = \sum_{k \in Z^*} b_k e^{2\pi k i x}; \quad f(x) = \sum_{k \in Z^*} c_k e^{2\pi k i x}; \quad \text{where } Z^* = Z - \{0\}, \quad (2)$$

while  $T_0 \in \dot{H}_{per}^2$  and  $S_0 \in \dot{L}_{per}^2$  are given by  $T_0(x) = \sum_{k \in Z^*} a_{k0} e^{2\pi k i x}$ ,  $S_0(x) = \sum_{k \in Z^*} d_{k0} e^{2\pi k i x}$ .

Finally assume that  $T(t, x) \in \dot{H}_{per}^2$  and  $S(t, x) \in \dot{L}_{per}^2$  are given by

$$T(t, x) = \sum_{k \in Z^*} a_k(t) e^{2\pi k i x} \quad \text{and} \quad S(t, x) = \sum_{k \in Z^*} d_k(t) e^{2\pi k i x} \quad Z^* = Z - \{0\} \quad (3)$$

We note that  $\bar{a}_k = -a_k$  ( $\bar{d}_k = -d_k$ ) since all functions consider are real and also  $a_0 = d_0 = 0$  since they have zero average.

Now we observe the dynamics of each Fourier mode and from (1), we get the following system for the new unknowns,  $v$  and the coefficients  $a_k(t)$  and  $d_k(t)$ .

$$\begin{cases} \varepsilon \frac{dv}{dt} + G(v)v = \sum_{k \in Z^*} (a_k(t) - d_k(t))c_{-k} \\ \dot{a}_k(t) + [2\pi k i v(t) + H(v(t))] a_k(t) = H(v(t))b_k \\ \dot{d}_k(t) + [2\pi k i v(t) + 4c\pi^2 k^2] d_k(t) = 4b\pi^2 k^2 a_k(t) \end{cases} \quad (4)$$

We consider the functions  $T_a$  and  $f$  are given by following Fourier expansions

$$T_a(x) = \sum_{k \in K} b_k e^{2\pi k i x}; \quad f(x) = \sum_{k \in J} c_k e^{2\pi k i x}; \quad \text{where } J = \{k \in Z^* / c_k \neq 0\}, K = \{k \in Z^* / b_k \neq 0\} \quad \text{with } Z^* = Z - \{0\}, \quad (5)$$

$$\text{and } \oint (T - S)f = \sum_{k \in (K \cap J)} (a_k(t) - d_k(t))c_{-k}.$$

We prove working with the equations (4) the velocity of the fluid is independent of the coefficients for temperature  $a_k(t)$  and the salinity  $d_k(t)$  for every  $k \in Z^* - (K \cap J)$ . That is, the relevant coefficients for the evolution of the velocity are only  $a_k(t)$  and  $d_k(t)$  with  $k$  belonging to the set  $K \cap J$ . This important result about the asymptotic behaviour has been proved in ([6], [4]) using the inertial manifold theory ([2], [10]).

## ASYMPTOTIC BEHAVIOUR FOR SMALL OF PARAMETER $\varepsilon$

In this section the fluid velocity behaviour is studied under the assumption of small values of the parameter  $\varepsilon$ , which is related to the ratio between the effects of the fluid acceleration due to viscous forces and the acceleration due to the effect of temperature in the fluid density (origin of the Soret effect). Considering both effects of the same order to enable one to consider the Soret effect (relevant effect in the thermosyphon problem), there results a direct association between the parameter  $\varepsilon$  and the fluid inertia acceleration; therefore small values of the parameter means small fluid acceleration, which involves a behaviour close to the stationary regime (result to be proved theoretically in [6]).

We shall consider the case where the functions  $H$  and  $G$  depend on  $\varepsilon$ , i.e.

$$G = H = H_\varepsilon(v) = \frac{H_0(v)}{\varepsilon}, \quad (6)$$

and we shall study the system behaviour for small values of  $\varepsilon$ .

The aim is to show, Proposition 2, that for sufficient small values of  $\varepsilon$  every solution behaves as a stationary solution with null velocity.

Now, the reduced system behaviour is studied, equivalent to whole system in the asymptotic behaviour in the large time, as it was shown above.

**Lemma** If we assume that  $\int_0^\infty |v(s)|ds < \infty$ , then for every  $\eta > 0$  there exists  $t_0$  such that

$$\int_{t_0}^t H(r)e^{-\int_r^t H} (e^{-\int_r^t 2\pi ikv} - 1)dr \leq \eta \text{ with } t \geq t_0 \quad (7)$$

**Proof:** If  $\int_0^\infty |v(s)|ds < \infty$ , it means that  $\forall \delta, \exists t_0(\delta) > 0 / \int_r^t |v| \leq \delta$  if  $t_0 \leq r \leq t < \infty$ . Thus,  $\forall \eta$  there exists  $t_0 > 0$  large enough such that

$$|e^{-\int_r^t 2\pi ikv} - 1| \leq 4\pi|k| \int_r^t |v| \leq \eta \quad \forall t \geq t_0 \quad (8)$$

and therefore  $\int_{t_0}^t H(r)e^{-\int_r^t H} (e^{-\int_r^t 2\pi ikv} - 1)dr \leq \eta \int_{t_0}^t H(r)e^{-\int_r^t H} dr = \eta(1 - e^{-\int_{t_0}^t H}) \leq \eta, \forall t \geq t_0$ , since  $H$  is strictly bounded away from zero, and we get (7).  $\square$

**Proposition 1 i)** If  $I = \sum_{k \in K \cap J} b_k c_{-k} = \oint T_a f = 0$  with  $K \cap J$  a finite set, and  $\int_0^\infty |v(s)|ds < \infty$ , then the system goes to the rest stationary solution, that is to say:

$$(v(t), a_k(t), d_k(t)) \rightarrow (0, b_k, \frac{b}{c} b_k), i.e. \quad (v(t), T(t), S(t)) \rightarrow (0, T_a, \frac{b}{c} T_a)$$

ii) If  $I = \sum_{k \in K \cap J} b_k c_{-k} \neq 0$  then for every solution,  $\int_0^\infty |v(s)|ds = \infty$  and  $v(t)$  does not converge to zero.

**Proof:**(see [6])

However, this last result can be refined showing the coefficients tendency to their equilibrium values as a function of the parameter under study,  $\varepsilon$ , as following:

**Proposition 2** If  $I = \sum_{k \in K \cap J} b_k c_{-k} = 0$  with  $K \cap J$  a finite set, and  $\int_0^\infty |v(s)|ds < \infty$ , then

$$\limsup_{t \rightarrow \infty} |a_k(t) - b_k| \leq \varepsilon 4\pi |k| |b_k| \limsup_{t \rightarrow \infty} \frac{|v(t)|}{H_0(v(t))} \quad (9)$$

$$\limsup_{t \rightarrow \infty} |d_k(t) - \frac{b}{c} b_k| \leq \varepsilon (4c\pi^2 k^2) 4\pi |k| |b_k| \limsup_{t \rightarrow \infty} \frac{|v(t)|}{H_0(v(t))} \quad (10)$$

and

$$\begin{aligned} \limsup_{t \rightarrow \infty} |v(t)| &\leq \varepsilon 4\pi \sum_{K \cap J} |k| |b_k| |c_{-k}| \limsup_{t \rightarrow \infty} \frac{1}{H_0(v(t))} + \\ &+ \varepsilon 16c\pi^3 \sum_{K \cap J} |k|^3 |b_k| |c_{-k}| \limsup_{t \rightarrow \infty} \frac{1}{H_0(v(t))} \end{aligned} \quad (11)$$

**Proof:** To study  $a_k(t)$  we defined  $\widehat{a}_k = b_k \int_{t_0}^t H_\varepsilon(r) e^{-\int_r^t H_\varepsilon} dr = b_k (1 - e^{-\int_{t_0}^t H_\varepsilon})$  that converges to  $b_k$  for  $t \rightarrow \infty$  ( $\lim_{t \rightarrow \infty} \widehat{a}_k = b_k$ ), and defining  $\tau_k(t) = a_k(t) - \widehat{a}_k$ ,  $\tau_k$  satisfies,

$$\tau_k(t) = a_k(t_0) e^{-\int_{t_0}^t (2\pi kvi + H_\varepsilon)} + b_k \int_{t_0}^t e^{-\int_r^t H_\varepsilon} (e^{-\int_r^t 2\pi kvi} - 1) H_\varepsilon(r) dr$$

taking modulus and applying that  $|e^{-\int_{t_0}^t 2\pi kvi}| = 1$ , together with the above lemma we have that  $\limsup_{t \rightarrow \infty} |\tau_k(t)| = 0$  and moreover, we get

$$|\tau_k(t)| \leq |a_k(t_0)| e^{-\int_{t_0}^t H_\varepsilon} + |b_k| \frac{\int_{t_0}^t H_\varepsilon(r) e^{\int_{t_0}^r H_\varepsilon} |e^{-\int_r^t 2\pi kvi} - 1| dr}{e^{\int_{t_0}^t H_\varepsilon}} \text{ with } |e^{-\int_r^t 2\pi kvi} - 1| \leq 4\pi |k| \int_r^t |v|,$$

and applying the L'Hopital Lemma (see [9]) together with (6), we have

$$\limsup_{t \rightarrow \infty} |\tau_k(t)| \leq \varepsilon 4\pi |k| |b_k| \limsup_{t \rightarrow \infty} \frac{|v(t)|}{H_0(v(t))} (1 - e^{-\int_{t_0}^t H_\varepsilon})$$

and we obtain (9). For  $d_k(t)$  we define  $\widehat{d}_k = (1 - e^{-4c\pi^2 k^2(t-t_0)})\frac{b}{c}b_k$  that converges to  $\frac{b}{c}b_k$  for  $t \rightarrow \infty$  ( $\lim_{t \rightarrow \infty} \widehat{d}_k = \frac{b}{c}b_k$ ), and defining  $\sigma_k(t) = d_k(t) - \widehat{d}_k$ , thus  $\sigma_k$  satisfies,

$$\sigma_k(t) = d_k(t_0)e^{-\int_{t_0}^t (2\pi k v_i + 4c\pi^2 k^2)} + \int_{t_0}^t (-4c\pi^2 k^2)e^{-4c\pi^2 k^2(t-r)}\frac{b}{c}(a_k(r)e^{-\int_r^t 2\pi k v_i} - b_k)dr.$$

Now, from the above Lemma, and considering (9), we get (10). Next, reading the equation for  $v$  as

$$\varepsilon \frac{dv}{dt} + G(v)v = (I_1(t) - I) + (I_2(t) - \frac{b}{c}I) + (1 + \frac{b}{c})I \quad \text{with } G = H_\varepsilon(v) = \frac{H_0(v)}{\varepsilon}$$

where  $I_1(t) = \sum_{k \in K \cap J} a_k(t)c_{-k}$  and  $I_2(t) = \sum_{k \in K \cap J} d_k(t)c_{-k}$ , we get

$$v(t) = v_{t_0}e^{-\int_{t_0}^t G(v)} + \frac{1}{\varepsilon} \int_{t_0}^t (I_1(t) - I)e^{-\int_r^t G(v)}dr + \frac{1}{\varepsilon} \int_{t_0}^t Ie^{-\int_r^t G(v)}dr + \frac{1}{\varepsilon} \int_{t_0}^t (I_2(t) - \frac{b}{c}I)e^{-\int_r^t G(v)}dr + \frac{b}{c\varepsilon} \int_{t_0}^t Ie^{-\int_r^t G(v)}dr.$$

Finally, we have that for every  $\delta > 0$  there exists  $t_0$  such that  $|I_1(t) - I| \leq \varepsilon M + \delta$  and  $|I_2(t) - \frac{b}{c}I| \leq \varepsilon N + \delta$ , for  $t \geq t_0$  where  $M = 4\pi \sum_{k \in K \cap J} |k||b_k||c_{-k}|$  and  $N = 16c\pi^2 \sum_{k \in K \cap J} |k|^3|b_k||c_{-k}|$  and using again L'Hopital's Lemma ( see [9]) for the function  $F(t) = \int_{t_0}^t \frac{1}{\varepsilon} e^{-\int_r^t G} dr$ , together with (6), we have  $\limsup_{t \rightarrow \infty} F(t) = \limsup_{t \rightarrow \infty} \frac{\frac{1}{\varepsilon} \int_{t_0}^t e^{\int_0^t G} dr}{e^{\int_0^t G}} \leq \limsup_{t \rightarrow \infty} \frac{1}{\varepsilon G} = \limsup_{t \rightarrow \infty} \frac{1}{H_0}$ , and from (9) together with (10), we get (for  $I = 0$ ) the estimates on the limit of  $|v(t)|$  and we conclude.  $\square$

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