Debt Maturity and Growth Options

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Abstract

Firms anticipate bad times by issuing long-term debt with procyclical rollover activity, yet the 2008 crisis is associated with an excess of short-term debt. Both patterns arise in a model of declining cash flows and an upside event— a growth option, whereby expired debt is refinanced with short- or long-term bonds. Larger upside events induce a higher fraction of equilibria in which outstanding short-term debt falls in bad times, namely, a procyclical rollover policy and a higher time to default. For sufficiently large upside options, this pattern is the only model equilibrium, where firms *engineer* a later default via longer maturity.

The amounts borrowed from the BoE must be repaid by 2022; as a result Ms MacMahon [head of financial institutions group at Citi] said, "we are definitively seeing some banks starting to ramp up their regular funding activity to repay the BoE." UK banks' debt sales near decade high as regulations tighten and cheap funding ends. (*Financial Times*, London, May 14, 2018).

1   Introduction

Debt maturity and leverage are crucial for any firm. The former is less constrained than leverage—defined by supply and demand factors such as financial regulation, credit ratings, or binding covenants. A firm's debt-maturity structure matters because, along with endogenous default, it provides an extra layer of financial flexibility. In an influential study, Graham and Harvey (2001) provide a clear example of this interaction: Firms, which are forward-looking, anticipate the cost of refinancing in bad times, which leads to the issuance of long-term debt and procyclical refinancing activity (Chen et al., 2013; Mian and Santos, 2018; Xu, 2018). In the 2008 credit crunch, by contrast, an excess of short-term debt led to severe rollover and liquidity problems (Brunnermeier, 2009; Krishnamurthy, 2010). Although both patterns arise in a model of dynamic debt maturity (He and Milbradt, 2016), the mechanisms behind them are largely unexplored.

In this paper, we show how the tradeoff between two firm fundamentals—the bond-recovery sensitivity to the default time and a growth option—explains these rich debt-maturity patterns. Specifically, in the model, the dynamic tradeoff between short- and long-term debt is given by the tradeoff between these two firm fundamentals. A positive tradeoff, namely, a poor growth option, induces an equilibrium rollover policy in which outstanding short-term (long-term) debt increases (falls) in bad times, leading to a shorter time to default; by contrast, a negative tradeoff, that is, a large growth option, implies the opposite—a procyclical rollover policy and a higher time to default. Moreover, we derive several novel predictions. For instance, the issuance/rollover policy

of new short-term debt is (independent of this tradeoff and is) procyclical—more short-term debt is issued in good than in bad times. Although multiple equilibria exist, they are unique given the issuance policy, and hence this policy, namely, the change in debt maturity (rather than maturity itself), is more revealing about a firm's debt-maturity strategy. Very short-term debt or a large principal of debt reduces the fraction of the equilibria with procyclical rollover activity (and a higher time to default), because in both scenarios, the amount of debt is too large to roll over. Lastly, equity is less sensitive to the maturity structure of debt in good times, which fits with a less cyclical refinancing pattern of investment-grade firms.

To understand corporate debt maturity means studying debt markets more deeply. In contrast to a fixed-maturity mortgage, a firm is permanently indebted and never repays all debt. In addition, covenants limit further indebteness/leverage. Corporate debt-maturity choice is a dynamic refinancing problem. Short-term debt is riskier than long-term debt because it leads more often to refinancing, an endogenous earlier default (Leland and Toft, 1996; He and Xiong, 2012). Further, any debt problem involves borrowers and lenders and is solved in equilibrium. Also, the rollover choice today depends on both the rollover choice in the future and the default policy, and must be consistent with future decisions.

In this paper, in a model that builds on He and Milbradt (2016), we study how a firm manages its debt maturity, namely, the tradeoff between short- and long-term debt. In the model, the volume of expiring debt depends on its maturity structure, and is rolled over by issuing short- or long-term bonds that have the same seniority, principal (normalized to 1), and coupon (equal to the risk-free rate of a flat yield curve), but a distinct maturity rate. Operational income is declining, contains no volatility risk, and is subject to an upside event—a growth option. (Declining operational income recognizes that credit risk is linked to poor cash flows.) As in Leland and Toft, equityholders absorb the firms' net cash flow: operational income minus the debt service—coupons and rollover costs. The latter per expired bond equals 1 minus the price of a newly

issued bond, such that book leverage is constant (the principal of outstanding debt is always 1). The firm chooses both the default time and the rollover policy to maximize equity value. Equity depends not only on cash flows and the default time, but also on the two bond prices and the fractions of outstanding and new issues of short-term debt.

No volatility implies the optimal default policy T^* is straightforward: Default occurs at the instant T^* at which the declining net cash flow, upside-event adjusted, hits zero. Any other default policy $(T \neq T^*)$ is not credible. Therefore, any later default time $(T > T^*)$ is not credible, and we assume that at time T^*, lenders do not extend credit beyond T^*, and equityholders default. Equityholders do not commit to default at time T^*; rather, lenders cut financing. Value-matching and smooth-pasting conditions hold at the time of default, T^*. With dynamic debt maturity, the default policy T^* also depends on the fraction of outstanding short-term debt: Not only does more of this fraction lead to an earlier default, the default time is also more sensitive to low than to large fractions of outstanding short-term debt.

We study this model based on the integral form of the equity price. This strategy, compared to the use of the partial differential equation, simplifies the joint analysis of both problems: the optimal default time and the equilibrium rollover policy. To understand our contribution, we first review the main results in He and Milbradt.

They show two types of equilibria exist based on the fraction of short-term debt in new debt issues: corner (interior) policies in which this fraction is either zero or 1 along the entire equilibrium path (between zero and 1 at the time of default). These rollover policies interact with the default policy, and although multiple equilibria exist, they are unique going backwards from the default time. They emphasize a corner-shortening equilibrium in which all newly issued debt is short term, leading to a shortening time to default. Notably, this shortening path, which has adverse welfare costs, emerges when bond recovery is higher if default occurs earlier. However, they do not address key questions such as the interior equilibria, in which the rollover policy is not explicitly found, or what type of equilibrium (e.g., shortening over lengthening) prevails.

3

In this paper, we provide many insights over He and Milbradt: Overall, we solve the model analytically and provide a fuller characterization of equilibria, from which we derive new economic insights. Specifically, we (1) provide closed-form solutions for the equilibrium paths of the fraction of short-term debt in new issues and in outstanding debt, (2) show equilibria are threefold based on the path of outstanding debt, and (3) address the key question of when a type of path prevails, and show the tradeoff that dictates this prevalence. Last but not least, we relate the rollover activity implied by these equilibrium paths to the refinancing patterns of creditworthy firms.

First, we solve for all equilibrium paths and describe the interior equilibria (IE) in detail. Any IE remains interior; that is, the fraction of new issues of short-term debt is strictly between zero and 1 along the entire path. Hence, only pure corner or interior equilibria exist. Further, this fraction of new issues of short-term debt steadily decays with time in the run-up to default. Yet despite this fall in short-term debt issuance, the fraction of short-term debt in *outstanding* debt may be hump shaped or increasing. We also extend the insight that a corner-shortening equilibrium emerges in the face of declining cash flows so that bond recovery is higher if default occurs earlier.

We show the rollover policy on the path toward default is increasing in the rollover policy at the instant of default. This link implies that as the sensitivity of bond recovery to the default time rises, more short-term debt is issued not only at the instant of default, but also along the entire equilibrium path. This result holds for any IE and leads to an earlier default. Moreover, because this sensitivity, which is weakly negative, decreases with time (i.e., becomes less negative), the choice of an earlier default time (linked to larger fractions of outstanding short-term debt) also implies the issuance of more short-term debt along the entire equilibrium path. If this sensitivity is sufficiently negative, the equilibrium becomes a corner-shortening path. In addition, the higher this bond-recovery convexity to the default time is, the fewer IE paths exist.

Second, besides the described corner/IE based on the issuance policy of new debt, equilibria are also sorted and threefold based on the path of outstanding debt. An

equilibrium exists in which (the fraction of) outstanding short-term debt rises along the entire path to default, and in which the time to default is shorter—the shortening equilibrium. A lengthening equilibrium also exists in which outstanding short-term debt falls, as well as a hump-shaped equilibrium that is lengthening closer to default. These equilibria (linked to large, small, and intermediate fractions of outstanding short-term debt at the time of default) raise two bigger questions: "When is an equilibrium of one type and not another (e.g., lengthening vs. shortening or hump-shaped)?" and "When does one type of equilibrium prevail over the others?"

Third, although the *issuance* of new short-term debt is linked to the sensitivity of bond recovery to an earlier default time (lenders anticipate a deteriorating recovery at default), lengthening equilibria depend on the tradeoff between this sensitivity and the upside-event expected payoff (so equityowners do not pass up a good reward). Thus, (i) an equilibrium path is lengthening if this tradeoff is mainly negative. When multiple equilibria exist away from the default boundary, although the shortening path implies a higher bond-recovery sensitivity, borrowers and lenders bet longer for the upside option in a lengthening path, which is riskier for both. And (ii) the larger the upside-event expected payoff, the larger the fraction of lengthening paths. For sufficiently large upside options, this pattern is the only model equilibrium.

That a debt-maturity path trades off a higher bond-recovery sensitivity to the default time versus a higher good-upside-option likelihood extends He and Milbradt, for whom "the maturity choice trades off rollover losses today versus higher rollover frequencies tomorrow," as well as Graham and Harvey, who report firms issue long-term debt to bypass refinancing in bad times. For these firms, "financial flexibility, credit ratings, and matching debt maturity and assets life are important," which ultimately leads to less credit risk. As in the latter work, the mechanism our reveals is an equilibrium outcome, namely, a dynamic risk-management policy that depends on the assets in place. Likewise, foreign-denominated debt is a natural hedge of foreign revenues (Géczy et al., 1997). In our setting, not missing the growth option is key.

This paper helps us understand the incentives associated with the endogenous choice of debt maturity in a dynamic setting that includes endogenous default, risky cash flows, and debt-equity conflict. Specifically, issuing short-term debt lowers current rollover losses but requires refinancing more often, leading to an earlier default. The basic debt-equity agency conflict in which equityholders set their endogenous default decision, as well as how to absorb rollover losses, at the expense of debt holders is also present here. Lenders incorporate this endogeneity in the pricing of new debt.

Endogenous debt refinancing along with endogenous default provides additional flexibility to any firm. Chen et al. (2013) show aggregate corporate debt maturity has a clear procyclical pattern: The average debt maturity is longer in economic expansions than in recessions. In fact, Xu (2018) shows that speculative-grade firms manage the maturity profile of debt to avoid having to borrow in bad times via early refinancing along with maturity lengthening, whereas investment-grade firms are more sensitive to the term spread. Mian and Santos (2018) report firms also refinance their syndicated loans in normal times. These future bad times, which lead to more long-term debt, could be exogenous if liquidity becomes scarce or, simply, if the value of the firm drops. In brief, adding to early work on static debt maturity (e.g., Barclay and Smith, 1995; Guedes and Opler, 1996; Baker et al., 2003; Johnson, 2003), these new papers find procyclical refinancing activity, which is more relevant to our dynamic work.

In the model, in any IE, the fraction of new issues of short-term debt is larger than the fraction of new issues of long-term debt in good times (i.e., when the firm's cash flow is large). Further, this fraction of new issues of short-term debt steadily decays at the same time as the operational income. It follows that the issuance of new short-term debt is procyclical; namely, more short-term debt is always issued in good than in bad times (although this issuance policy does not necessarily lead to a lower fraction of outstanding short-term debt—a lengthening path to default). Also, although multiple equilibria exist, they are unique given the rollover policy, and hence the change in outstanding debt and not debt maturity or outstanding debt itself fully determines an

6

equilibrium path.

A lengthening path in which the fraction of outstanding short-term debt is falling and hence average maturity is rising implies a procyclical refinancing activity. In a setting of declining operational income, a path is lengthening if the tradeoff between the upside-event expected payoff and the bond-recovery sensitivity to the default time is positive. This tradeoff is intuitive: better to wait longer if a good reward is expected. It follows that although the slope of the default time to the outstanding short-term debt is negative, and although shortening and lengthening paths coexist, firms with a good upside option *engineer* a later default via a large fraction of lengthening equilibria.

Thus, we find a rationale in which, in most of the equilibria, more long-term debt is newly issued, outstanding short-term debt falls in bad times, the rollover policy is procyclical, and the time to default rises, complementing the evidence of similar procyclical refinancing patterns for speculative-grade firms (Mian and Santos, 2018; Xu, 2018) or the debt-overhang problem (Diamond and He, 2014). For example, Guedes and Opler (1996) find speculative-grade firms typically borrow in the middle of the maturity spectrum, and do not issue short-term debt. Moreover, issuing the short-term debt far away from default/in good times, when equity is only slightly sensitive to the maturity structure of debt, conforms with a less cyclical refinancing pattern of investment-grade firms (Xu, 2018).

For example, consider an oil-production firm. Oil is a stored commodity that hardly depreciates. In the US shale oil and gas finance industry, Reserve-Based Lending facilities have a five-year tenor with a bullet maturity (Azar, 2017). Our analysis suggests both borrowers and lenders expect the oil price to hike in this extended five-year period.

Naturally, a third factor of a firm's debt-maturity equilibrium policy (besides the bond recovery and the upside option) is the maturity spread between short- and long-term bonds. Very short-term debt not only accelerates default and depresses the issuance of new short-term debt, but also reduces the fraction of lengthening equilibria. The reason is that a large fraction of outstanding very short-term debt, which quickly

expires, is costly to roll over and cannot be easily unwound, but the firm defaults. That the default time is more sensitive to small than to large fractions of outstanding short-term debt implies that in the presence of very short-term debt, a shortening equilibrium leads to a much earlier default than a lengthening path. It follows that outstanding overnight (e.g., repo) debt, poor cash flows, and steadily declining asset values (or collateral) lead to a quicker default in equilibrium.

During the 2007–2008 credit-crunch crisis, in which the secured repo market was a key financing tool in the entire (including the shadow) banking system, banks reduced the maturity of their debt while the collateral was sliding (Brunnermeier, 2009; Krishnamurthy, 2010). This pattern fits with a shortening equilibrium. Further, a supply of short-term funding exists, such as money-market funds restrained to invest at the front end of the yield curve (e.g., in commercial paper, Gorton et al., 2015).

In sum, although the default and rollover policies are highly interconnected, the path of outstanding debt captures the interaction between the two. The implications of the model inputs in this path of outstanding debt sum up as follows: Very short-term debt or a large principal of debt reduces the fraction of lengthening equilibria. Although short-term debt issuance depends on the bond-recovery sensitivity to the default time, the upside-event intensity or reward increases the lengthening equilibria. Although a faster declining rate of operational income speeds up default, the operational-income level rises only slightly. It follows that shortening equilibria depend not only on a nonzero bond-recovery sensitivity to the default time and a larger fraction of outstanding short-term debt in place, but also on very short-term debt or large book leverage (a poor upside event), namely, too much debt to roll over (waiting longer is pointless).

The existence of shortening equilibria, as well as where to place these paths, depends critically on whether or not default occurs when operational income is negative, which determines the definition of bond recovery. If operational income is negative at the instant of default, shortening (lengthening) paths are unambiguously tied to larger (smaller) fractions of outstanding short-term debt at the instant of default. However,

a discontinuity in the bond-recovery value exists for lower values of the upward-event expected payoff. In this case, some operational income is positive at the time of default, implying the bond recovery becomes less sensitive to this time and the opposite result happens; that is, large fractions of outstanding short-term at the time of default are linked to paths with procyclical rollover activity. If, indeed, operational income is only positive at default, all paths are (corner or interior but) lengthening.

In welfare terms, a shortening equilibrium is not desirable in general; equityholders maximize equity and not the entire firm value. This result is easily illustrated if the bond-recovery value is a fraction of the enterprise value (i.e., lenders are less efficient running the firm post default). A later default time, such as a lengthening path implies, always yields an entire firm value larger than earlier default (because of this loss of efficiency). As pointed out, very short-term debt reduces the fraction of lengthening equilibria, especially in the case of a large amount of short-term debt in place.

In Leland and Toft's model, in which debt maturity is constant, as well as in He and Milbradt and our paper that relax this assumption, rollover losses hit shareholders' deep pockets in Leland's tradition. In Brunnermeier and Oehmke (2013), by contrast, rollover losses are also absorbed by equityholders promising a sufficiently high new face value to new lenders. That is, old lenders are the most diluted in the event of default, because new bondholders are promised a new larger face value (that no covenant in place prevents) at the time of refinancing. We abstract from these frictions that may affect the supply/demand of debt maturity, including liquidity risk (Diammond, 1991), asymmetric information (Flannery, 1994), excess of leverage and different-seniority lenders (Brunnermeier and Oehmke, 2013), market timing (Greenwood et al., 2010), or debt financing of new projects (Diamond and He, 2014).

Our paper relates to the novel area that studies credit risk and endogenous debt dynamics, especially He and Milbradt.[1] We extend their work by providing a fuller

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[1] Dynamic debt is important too in studies of sovereign credit risk, in which risk-averse households smooth consumption shocks by managing debt maturity and leverage (e.g., Arellano and Rama-narayanan, 2012; Lorenzoni and Werning, 2014; or Aguiar et al., 2018).

description of equilibria and especially linking the debt-maturity structure and refinancing activity to the firm's fundamentals and empirical evidence. We largely avoid assuming a specific functional form for the declining operational income. Early work in debt maturity examines the repricing of short-term debt (Diammond, 1991; Flannery, 1994), linked to refinancing costs. He and Xiong (2012) point out that the riskiness of short-term debt arises from funding risk. Huang et al., (2017) study multi-period debt contracts. Friewald et al. (2018) relate equity returns to debt refinancing. Volatile cash flows or dynamic leverage (DeMarzo and He, 2017) are two extensions left for future research.

The paper is structured as follows. Section 2 presents a model of optimal default in a static setting. Section 3 moves to study optimal default with dynamic debt. Section 4 describes the rollover-choice problem, and section 5 solves this dynamic problem. Section 6 provides some extensions. Section 7 concludes. Proofs are in the Appendix.

2   Static Debt

Because the optimal default decision is similar in a static or dynamic debt setting, we focus first on a model with static debt. We define equity value as follows:

$$F\left(t;T\right) = \int_t^T I\left(u;T\right) \times du, \ t \leq T, \tag{1}$$

in which $t \geq 0$ is time, T is the default policy, and I is the (discounted) payout rate/net cash flow to equityholders. I is deterministic and, like risky debt, depends on the default policy T. This setting is transparent, and it is convenient to understand the optimal default decision here.

$F\left(t;t\right) = 0$ is the value-matching condition, and equity value is worthless at default. $F\left(t;t\right) = 0$ implies $F\left(t;T\right) \geq 0$ by switching to the default policy (i.e., $t = T$). Non-negative equity prices, the value-matching condition, and endogenous default, which is

10

shown next, are three standard properties of any model of equity prices (Leland, 1994).

2.1 The optimal default policy

Equityholders are not committed to any default policy, and may default at any time. The only credible policy, denoted by $T = T^*$, is the optimal default policy. The lenders, that is, bondholders, make sure this decision is consistent by not extending credit beyond T^*. The smooth-pasting condition holds at time $t = T^*$. Formally, this is as follows.

The first-order derivative of the value of equity with respect to T is given by

$$F_T(t;T) = I(T;T) + \int_t^T I_T(u;T) \times du. \tag{2}$$

In particular, for $t = T$, the integral vanishes and

$$F_T(T;T) = I(T;T),$$

where we denote $F(T;T) = F(t;T|t = T)$ and $I(T;T) = I(t;T|t = T)$.

If $T = T^*$ is an interior optimal default policy,

$$F_T(T^*;T^*) = I(T^*;T^*) = 0. \tag{3}$$

This zero net cash flow, a necessary optimality condition, determines the optimal default policy, T^*. Because $F_t(T;T) = -I(T;T)$, it follows that

$$F_t(T^*;T^*) = 0, \tag{4}$$

which implies smooth-pasting is a first-order optimality condition.

11

We define the optimal default time as the following (deterministic) stopping time:

$$\mathcal{T} = \{\inf_{u \geq 0} : I(u; u) = 0\}, \ 0 \leq \mathcal{T} < \infty.$$

If T^* is unique (e.g., in a model of declining cash flows), $T^* = \mathcal{T}$.

A sufficient condition for T^* to be optimal We assume $I(0; 0) > 0$ and

$$\frac{dI(T; T)}{dT} \leq -\mu < 0, \ 0 \leq T,$$

which imply a unique T^* exists, where $I(T^*; T^*) = 0$. To prove $T = T^*$ is the optimal default policy, we show any other default policy $T < T^*$ $(T > T^*)$ is not credible; better to delay (accelerate) default with respect to such a policy $T \neq T^*$. Then lenders do not refinance any new bond beyond the only credible policy T^*, which pushes equityholders to optimal default at time T^*, subject to $0 \leq T \leq T^*$.

Remark. If the expired debt at time T^* cannot be refinanced, equityholders will suffer bigger losses by no defaulting than the expected loss if they default (the latter equals zero because T^* is a credible default policy, equation (3)). For example, a simple covenant, associated with already-poor cash flows, constrains the maturity of all bonds to less than T^*. Otherwise, a default policy in equilibrium may not exist, which shows that equityholders find it profitable to postpone default by rolling over debt. This outcome is not an equityholder's commitment to default at time T^*, but rather the consequence of a strategic action of lenders (and helps us simplify the rollover problem after T^*). Likewise, in the case of many lenders, as in a syndicated bank loan, they can fail to coordinate in refinancing this debt (Bolton and Scharfstein, 1996).

Proposition 1 *Let $T = T^*$ be such that $I(T^*; T^*) = 0$. Then $F_T(T^*; T^*) = 0$ and the smooth-pasting property holds, that is, $F_t(T^*; T^*) = 0$. In addition, assume that $I(0; 0) > 0$ and $\frac{dI(T; T)}{dT} \leq -\mu < 0$, $0 \leq T$. Then T^* exists, $T \neq T^*$ is not a credible default policy, and $T = T^*$ is the optimal default policy (subject to $0 \leq T \leq T^*$).*

12

Further, a weaker condition than $\frac{dI(T;T)}{dT} \leq -\mu<0$, for $T \geq 0$, is that the following equation $I(T,T) = 0$ has a unique solution (denoted by $T = T^*$) and $\frac{dI(T;T)}{dT}\Big|_{T=T^*} < 0$, subject to $I(0;0) > 0$.

Proof. See Appendix A.

From Proposition 1, $I(T;T) > 0$ if $T < T^*$ (and $I(T;T) < 0$ if $T > T^*$). It follows that delaying (accelerating) default slightly from T to $T + dT$ if $T < T^*$ (to $T - dT$ if $T > T^*$) always dominates any default policy in which $T \neq T^*$, where $dT > 0$.[2]

2.1.1 Example: a model of declining cash flows

Below, where debt maturity is dynamic, we define the firm's payout rate in equation (1) akin to

$$I(u;T) = A(u) - m \times e^{-R\times(T-u)} \times l(T).$$

$A(u)$ is operational income, $m > 0$ is the amount of expiring debt, $e^{-R\times(T-u)}$ is the present value of risky debt, and $1 \geq l(T) > 0$ is the debt loss given default. It follows that

$$I(T;T) = A(T) - m \times l(T).$$

We assume that $A'(T) < 0$, i.e., operational income declines, and $l'(T) \geq 0$, i.e., debt losses at default are also (weakly) increasing with time. It follows that

$$\frac{dI(T;T)}{dT} = A'(T) - m \times l'(T) < 0. \tag{5}$$

Then if a T^* exists such that $I(T^*;T^*) = 0$, T^* is the optimal default policy. For example, $A'(T) \leq -\mu < 0$ implies T^* does exist, where $A(T^*) = m \times l(T^*)$.

Remark. At time $t = T^*$, delaying default from T^* to a sufficiently large $S > T^*$,

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[2]We do not consider a second-order condition, because $F(t;T)$ is defined only for $0 \leq t \leq T$ and the optimal T policy is binding in this support.

which immediately yields a positive net cash flow $I\left(T^*;S\right) \times dt$ (if $R > 0$),

$$
\begin{aligned}
I\left(T^*;S\right) &= A\left(T^*\right) - m \times e^{-R \times (S-T^*)} \times l\left(S\right) \approx A\left(T^*\right) \ \text{(e.g., if } S \to \infty) \\
&> A\left(T^*\right) - m \times l\left(T^*\right) = I\left(T^*;T^*\right) = 0,
\end{aligned}
$$

is not a credible policy, because at time $t = S$, the cash flow is negative,

$$
I\left(S;S\right) = A\left(S\right) - m \times l\left(S\right) < I\left(T^*;T^*\right) = 0.
$$

In effect, this noncredible default policy would increase the value of debt, that is, lower rollover losses by delaying default sufficiently further away from T^*. (If $R = 0$, because $l'\left(S\right) \geq 0$, directly $I\left(T^*;S\right) = A\left(T^*\right) - m \times l\left(S\right) \leq I\left(T^*;T^*\right) = 0$).

3 Dynamic Debt Maturity: Optimal Default

We now study optimal default with dynamic debt maturity. Consider the payout rate I depends on a new variable, $\phi_t \in [0, 1]$, the fraction of outstanding short-term debt ($1 - \phi_t$ is the fraction of outstanding long-term debt). We assume a default-policy function, $S\left(\phi\right)$. Because operational income is declining, $\{(t, \phi) : t \geq S\left(\phi\right)\}$ is the defaulting region; that is, $t < S\left(\phi_t\right)$ implies no defaulting at time t. For any path ϕ_u, $0 \leq u$, the default time is a (deterministic) stopping time defined as follows:

$$
T = \{\inf_{u \geq 0} : u = S\left(\phi_u\right)\}, \ 0 \leq T < \infty.
$$

The payout rate and the equity value are given by $I\left(t, \phi_t; f_t, T\right)$ and

$$
F\left(t, \phi_t; T\right) = \int_t^T e^{-R \times (u-t)} \times I\left(u, \phi_u; f_u, T\right) \times du, \ t \leq T. \tag{6}
$$

Equity value, F, depends on two variables, t and ϕ_t, as well as the default-policy

14

function $S(\phi)$ (instead of two variables in the static case, t and T^*). I also depends on a new variable, the fraction of newly issued short-term bonds, f_t. We wait until the next section to describe this dynamic setting in detail.

3.1 The optimal default policy, $T^*(\phi)$

Similar to the previous static setting,

$$F_T(T, \phi_T; T) = I(T, \phi_T; f_T, T). \tag{7}$$

We assume I does not depend on f_T at the default time (i.e., short- and long-term bonds are equal at default, same principal and same seniority). If $S(\phi) = T^*(\phi)$ is an interior optimal default-policy function,

$$I(T, \phi; f, T)|_{T=T^*(\phi)} = 0, \tag{8}$$

and a zero payout rate also determines the optimal default policy, $T^*(\phi)$. It also follows the two smooth-pasting conditions are given by (see Appendix A)

$$F_t(T, \phi; T)|_{T=T^*(\phi)} = 0 \quad \text{and} \quad F_\phi(T, \phi; T)|_{T=T^*(\phi)} = 0. \tag{9}$$

The optimal default time is given by the following stopping time:

$$\mathcal{T} = \{\inf_{u \geq 0} : I(u, \phi_u; f_u, u) = 0\}, \ 0 \leq \mathcal{T} < \infty.$$

In a model of declining cash flows, $T^*(\phi)$ is a monotone function ($T_\phi^*(\phi) < 0$), and $\mathcal{T} = T^*(\phi_{\mathcal{T}})$. For instance, a constant $\phi_u = \phi_0$, for $u \geq 0$, implies the previous static-debt solution, where the default time and default policy are the same number, $\mathcal{T} = T^*(\phi_0)$.

The following assumption implies the zero-payout-rate condition (and hence smooth-

pasting conditions) is sufficient for optimal default. Akin to the proof leading to Proposition 1, assume $I\left(0, \phi; f, 0\right) > 0$ and

$$\frac{dI\left(T, \phi; f, T\right)}{dT} \leq -\mu < 0, \ 0 \leq T, \tag{10}$$

which implies a function $T^{*}\left(\phi\right) > 0$ exists such that the payout rate becomes zero; that is,

$$I\left(T^{*}\left(\phi\right), \phi; f, T^{*}\left(\phi\right)\right) = 0,$$

as in equation (8). Like the static-debt case, equation (10) is implied by positive but declining operational income. It follows $T^{*}\left(\phi\right)$ is the optimal default policy (subject to $0 \leq S\left(\phi\right) \leq T^{*}\left(\phi\right)$). We also assume that no issuance policy pulls the firm away from the default boundary (which is equation (27)).

We collect all information of the equity value at this optimal default time, $T^{*}\left(\phi\right)$.

Lemma 2 *Let $\phi \in [0, 1]$ and $T^{*}\left(\phi\right)$ be such that $I\left(T^{*}\left(\phi\right), \phi; f, T^{*}\left(\phi\right)\right) = 0$. Then*

$$
\begin{aligned}
\textit{value-matching condition} \quad &: \quad F\left(T^{*}\left(\phi\right), \phi; T^{*}\left(\phi\right)\right) = 0, \\
\textit{first-order optimality condition} \quad &: \quad F_{T}\left(T^{*}\left(\phi\right), \phi; T^{*}\left(\phi\right)\right) = I\left(T^{*}\left(\phi\right), \phi; f, T^{*}\left(\phi\right)\right) = 0, \\
\textit{implicit optimal default policy} \quad &: \quad I\left(T^{*}\left(\phi\right), \phi; f, T^{*}\left(\phi\right)\right) = 0, \\
\textit{smooth-pasting conditions} \quad &: \\
F_{t}\left(T^{*}\left(\phi\right), \phi; T^{*}\left(\phi\right)\right) &= -I\left(T^{*}\left(\phi\right), \phi; f, T^{*}\left(\phi\right)\right) \times \left(1 - \left.\frac{d\mathcal{T}}{dt}\right|_{t=T^{*}\left(\phi\right)}\right) = 0, \\
F_{\phi}\left(T^{*}\left(\phi\right), \phi; T^{*}\left(\phi\right)\right) &= I\left(T^{*}\left(\phi\right), \phi; f, T^{*}\left(\phi\right)\right) \times \left.\frac{d\mathcal{T}}{d\phi_{t}}\right|_{t=T^{*}\left(\phi\right)} = 0,
\end{aligned}
$$

Assume $I\left(0, \phi; f, 0\right) > 0$ and $\frac{dI\left(T, \phi; f, T\right)}{dT} \leq -\mu < 0, \ 0 \leq T$, independent of f. Then $S\left(\phi\right) = T^{*}\left(\phi\right)$ exists and is the optimal default time, subject to $0 \leq S\left(\phi\right) \leq T^{*}\left(\phi\right)$.

Proof. See Appendix A. ∎

16

4 The Rollover-Choice Problem

We follow He and Milbradt's (HM) notation and layout. Consider two noncallable bonds that have different maturity, which is exponentially distributed. d_S^{-1} (d_L^{-1}) is the expected maturity and d_S (d_L) is the maturity rate of short- (long-term) bonds, respectively, $d_S > d_L \geq 0$. Let $f_t \in [0,1]$ be the fraction of short-term bonds that are newly issued by the firm at time t, the variable of rollover choice.

f_t controls the dynamic of ϕ_t, the fraction of outstanding short-term bonds,

$$\phi_t' = -\phi_t \times d_S + m(\phi_t) \times f_t. \tag{11}$$

$m(\phi_t) > 0$ (if $d_L > 0$) is the total number of expiring outstanding bonds,

$$m(\phi_t) = \phi_t \times d_S + (1 - \phi_t) \times d_L, \tag{12}$$

like a sinking-fund provision (a la Leland). A large fraction of newly issued short-term bonds allows equityholders to shorten the debt's maturity structure of the firm (i.e., $\phi_t' > 0$ if $f_t > \frac{\phi_t \times d_S}{m(\phi_t)}$), and a smaller fraction extends it. $m(\phi_t)$ dictates the firm refinancing activity.

The two bonds pay the same continuous coupon (c) and have the same principal (normalized to 1) and seniority. The only difference between them is the maturity rate. We assume a flat yield curve, where r is the riskless rate. The coupon equals this rate, $c = r$. If a bond randomly expires, the price equals the principal of 1. All these features produce a more tractable setting. For instance, the total coupon to be paid does not depend on the level of outstanding short-term debt (i.e., $c = \phi_t \times c + (1 - \phi_t) \times c$).

4.1 Net cash flows and the value of equity and debt

We assume a firm in which the operational income is deterministically declining, that is, $y'(t) \leq -\mu < 0$. Although this zero-volatility premise is due to tractability, it recog-

nizes that credit risk is linked to poor/negative cash flows. The same firm, however, is subject to an upside event, with intensity $\zeta \geq 0$, in which the assets in place mature and the value of these assets jumps to $X \geq 1$. This event is like a "growth option"—a successful investment/project, a discovery, or a new technology that boosts the value of the firm. Then the company pays back the debt, cancels all operations, and is sold by $X - 1$ (e.g., the firm goes public or is purchased by outside investors).

Accordingly, the (instantaneous and expected) net cash flow is as follows:

$$I(t, \phi_t; f_t, T) = A(t) + m(\phi_t) \times (G(\tau; f_t, T) - 1), \tag{13}$$

$\tau = T - t$ is the time to default, $A(t)$ is net operational income, G is the value of newly issued debt, and 1 is the principal of the expired debt; any coupon payment and jump adjustment is included in A. Specifically,

$$A(t) = y(t) - c + \zeta \times (X - 1),$$

where $c = r$ is the coupon payment and $\zeta \times (X - 1)$ is the firm's expected payoff due to the upside option. To simplify the notation, the optimal default time is denoted by T (i.e., $T = \mathcal{T}$ and $T = T^*(\phi_T)$).

$1 - G$ is the rollover cost of debt. The value of debt depends on quantity and prices (f and $D_{\{S,L\}}$, respectively) of the two bonds, and is given by

$$G(\tau; f_t, T) = f_t \times D_S(\tau; T) + (1 - f_t) \times D_L(\tau; T), \tag{14}$$

where D_S and D_L are the prices of short- and long-term debt, respectively. It follows that book leverage is constant; the promised principal of outstanding debt is always 1.

Assuming both bonds have the same priority at default,

$$G(0; f_T, T) = D_S(0; T) = D_L(0; T) = 1 - l(T) \geq 0,$$

18

which is the bond-recovery value. The term $l\left(T\right)$ is the bond loss given default. We assume the following: $0 < l\left(T\right) \leq 1$, the definition of credit risk; $l'\left(T\right) \geq 0$, the later the default, the larger the losses; and $l''\left(T\right) \leq 0$, losses are concave, which is equivalent to the bond-recovery value being a convex function. For instance, we can define $l\left(T\right)$ as follows: If bondholders are less efficient running the firm post-default, $0 < a \leq 1$,

$$1 - l\left(T\right) = a \times \int_0^{T_a - T} e^{-R \times u} \times \left(y\left(T + u\right) + \zeta X\right) \times du, \qquad (15)$$

where $T_a \geq T$ is the optimal abandoning time. Based on this definition of $l\left(T\right)$, we show that declining cash flows (along with $y''\left(t\right) \geq 0$) imply the previous three properties of $l\left(T\right)$ (see Appendix B).

Having explained the inputs of the model, we price the two corporate securities, equity and debt. Given the default time T, debt is priced by discounting expected cash flows:

$$D_i\left(\tau; T\right) = 1 - e^{-(R + d_i) \times \tau} \times l\left(T\right), \quad i = \{S, L\}, \qquad (16)$$

where $R \geq 0$ equals the riskless rate plus the intensity (i.e., $R = r + \zeta$). $d_S > d_L \geq 0$ implies $D_S > D_L$; namely, short-term debt is more expensive than long-term debt.

Bond prices, D_i, equal the principal minus the discounted probability of default multiplied by the loss given default. The default event is due to a random surviving event (between t and T), which depends on two standard Poisson processes, and $e^{-\zeta \times \tau} \times e^{-d_i \times \tau}$ gives the bond-surviving probability (i.e., nonupside event and nonexpire, respectively). Without default risk, these two bonds are two par-coupon bonds with random maturity and a flat yield curve. For any expired bond, the rollover loss depends on $l\left(T\right)$ (i.e., from equation (14), $1 - D_i\left(\tau; T\right) = e^{-(R + d_i) \times \tau} \times l\left(T\right), i = \{S, L\}$).

The net cash flow, I, reduces to

$$I\left(t, \phi_t; f_t, T\right) = A\left(t\right) - m\left(\phi_t\right) \times \left(f_t \times e^{-(R + d_S) \times \tau} + \left(1 - f_t\right) \times e^{-(R + d_L) \times \tau}\right) \times l\left(T\right), \quad (17)$$

and at the time of default,

$$I(t, \phi_t; f_t, T)|_{t=T} = A(T) - m(\phi_T) \times l(T).$$

The value of equity is given by

$$F(t, \phi_t; T) = \int_t^T e^{-R \times (u-t)} \times \Big(y(u) - c + \zeta \times (X - 1) \tag{18}$$
$$- m(\phi_u) \times \big(f_u \times e^{-(R+d_S) \times (T-u)} + (1 - f_u) \times e^{-(R+d_L) \times (T-u)} \big) \times l(T) \Big) \times du.$$

Equity depends on bond prices through the loss given default, $l(T)$. See Appendix B.

4.2   The optimal default boundary, $\phi^*(T)$

The optimal default policy, $T^*(\phi)$, is implicit in the zero net cash-flow condition,

$$A(T^*(\phi)) = m(\phi) \times l(T^*(\phi)).$$

The optimal default time is given by the first time that net operational income is equal (from above) to the rollover cost, $A(T)$ and $m(\phi) \times l(T)$, respectively. Net operational income is positive at the time of default (i.e., $A(T) > 0$ if $l(T) > 0$).

Larger losses (l) imply larger net operational income (A) at default, and thus an earlier default (lower $T^*(\phi)$) because $A' < 0$, as in a Leland-type setting. It follows that

$$\phi^*(T) = \frac{1}{d_S - d_L} \times \left(\frac{A(T)}{l(T)} - d_L \right), \tag{19}$$

which is the optimal default boundary.[3] $A' \leq -\mu < 0$ and $l' \geq 0$ imply $\frac{dI(T,\phi;f,T)}{dT} < 0$, from which follows the optimality of the zero net cash flow at default ($I(t, \phi_t; f_t, t)|_{t=T^*(\phi)} = 0$). Because $d_S > d_L$, both assumptions $A'(T) < 0$ and $l'(T) \geq 0$ also imply $T^*_\phi < 0$,

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[3]In a Leland-diffusion setting, instead of the analytical default boundary $\phi^*(T)$, we obtain the equity's gamma from the equity boundary conditions at the default point (Ibáñez, 2017).

namely,

$$\phi_T^*(T) = \frac{1}{d_S - d_L} \times \frac{A'l - Al'}{l^2} < 0.$$

If $A'' \geq 0$ (i.e., $A' = y'$ and $A'' = y''$), the default boundary and default time are convex functions ($\phi_{TT}^*(T) > 0$ and $T_{\phi\phi}^* > 0$, Appendix B). This implies that at larger fractions of outstanding short-term debt, the default time $T^*(\phi)$ is less sensitive to this fraction. Hence, all equilibria are more alike in this region. Conversely, very short-term debt (i.e., a large maturity-rate spread, $d_S - d_L$) lowers $\phi^*(T)$ and accelerates default, but $T^*(\phi)$ becomes more sensitive to lower fractions of outstanding very short-term debt. From equation (19), $\phi^*(T) = 0$ ($\phi^*(T) = 1$) does not depend on d_S (d_L).

4.3 Equilibrium

We assume a Markov perfect equilibrium, which depends on (1) the maximization of the value of equity subject to the optimal default policy and (2) the two bond prices being consistent with the default policy. This equilibrium is as follows:

(1) For any initial state $(0, \phi_0)$, the issuance strategy of equityholders $f_t \in [0, 1]$, $0 \leq t \leq T$, maximizes equity value given by equation (18).

Then the optimal issuance strategy, the dynamics of the fraction of outstanding debt, and the optimal default boundary (i.e., f_t, ϕ_t' in equation (11), and $\phi^*(T)$ in equation (19), respectively) determine the default time T (i.e., $\phi^*(T) = \phi_T$).

(2) Given the optimal default time T, bond prices (i.e., D_S and D_L) are given by equation (16).

5   The Equilibrium Path of Debt Maturity

In this section, we provide the main results of the paper (proofs are in Appendix C). (1) We solve in closed form the equilibrium paths of the fractions of outstanding and new issues (ϕ_t and f_t) of short-term debt, (2) provide a fuller description of equilibria

based on the path of the fraction of outstanding debt (ϕ_t), and (3) show the tradeoff that determines which type of equilibrium path is more abundant.

5.1 Equityholder's objective function

Before optimal default (i.e., $t < T^*(\phi_t)$), the equityholder's instantaneous net gain depends on time (or operational income, $y(t)$), outstanding debt, and rollover choice (t, ϕ_t, and f_t, respectively), and is given by

$$(I(t, \phi_t; f_t, T) + F_\phi(t, \phi_t; T) \times \phi'_t) \times dt. \tag{20}$$

This objective function is equivalent to

$$IC_t \times f_t = \left(-\left(e^{-(R+d_S)\times\tau} - e^{-(R+d_L)\times\tau}\right) \times l(T) + F_\phi(t, \phi_t; T)\right) \times f_t, \tag{21}$$

in which IC_t is the term multiplying f_t (and $m(\phi_t) > 0$ is factorized out). IC_t is an incentive-compatibility condition (borrowing HM's name). The rollover decision is linear in f_t, the newly issued fraction of short-term bonds. This linear form implies a corner solution except for a zero linear term, namely, $IC_t = 0$. A zero (nonzero) IC_t condition determines an interior (corner) policy, f_t, in equilibrium.

For instance, if $IC_t = 0$,

$$F_\phi(t, \phi_t; T) = \left(e^{-(R+d_S)\times\tau} - e^{-(R+d_L)\times\tau}\right) \times l(T) < 0.$$

and the price difference between short- and long-term debt equals the change in equity value with regard to outstanding short-term debt, which is an equilibrium condition. More short-term debt reduces the value of equity, that is, $F_\phi(t, \phi_t; T) < 0$. Far away from default (i.e., if $\tau \to \infty$), this sensitivity $F_\phi(t, \phi_t; T)$ becomes small. This latter observation implies that in good times, when cash flows are large, the specific (interior) equilibrium path and hence whether multiple equilibria exist is less relevant.

Notably, HM show equilibria are unique going backwards from the default time T. To solve these unique equilibria, we use the following approach. We solve the optimal default boundary and bond prices in advance, $\phi^*(T)$ and $D_{\{S,L\}}$, respectively, which depend on the loss given default $l(T)$, an input of the model. Next, we study the rollover-choice policy, f_t, $t \leq T$. We solve f_t backward in time, given $\phi^*(T)$ and $D_{\{S,L\}}$, implying f_t is consistent and is the equilibrium policy.

We focus first on nonconstrained rollover policies, $f_t^{nc} \in \mathcal{R}$, for which any equilibrium is defined by $IC_t = 0$, $t \leq T$. For constrained policies, $0 \leq f_t \leq 1$, two types of equilibria exist: corner and interior. The interior equilibrium (IE) is the same nonconstrained one, that is, $0 \leq f_t = f_t^{nc} \leq 1$, $t \leq T$.[4]

Summary of main results   Akin to HM's results, we show the following in a simple way:

- f_T^{nc} and f_T, the rollover policies at the time of default, are given in closed-form solution (HM's Lemma 1 and Proposition 1).

- $f_t \in \{0, 1\}$ if $f_T^{nc} \notin (0, 1)$. A corner solution is determined at default; namely, $f_T^{nc} \notin (0, 1)$ is a sufficient condition for corner equilibria (as in HM's Proposition 2).

- Our analytical expression for f_t implies any interior backwards path that starts from the boundary is unique (as in HM's Proposition 5).

In addition, based on the new ϕ_t and f_t's analytical solutions (see our Proposition 3), we show the following for the IE (i.e., $0 < f_T^{nc} < 1$), $t \leq T$:

(1) Corollary 4: $0 < f_t^{nc} < 1$; any IE remains interior along the whole path, which implies only pure (i.e., either corner or interior) equilibria exist. $\frac{df_t^{nc}}{dt} < 0$; fewer and fewer new issues of short-term debt are made in the run-up to default for any IE. $\frac{df_t^{nc}}{df_T^{nc}} > 0$; the larger f_T^{nc}, the more short-term debt is issued along the entire IE path.

(2) Proposition 5: In addition to corner-lengthening and -shortening ($f_t \in \{0, 1\}$) and interior ($f_t \in (0, 1)$) equilibria, we have lengthening, in which outstanding short-

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[4]Nonconstrained policies allow the issuance of short positions, e.g., buying short-term debt, which may reverse a default situation. As HM argue, changing the maturity profile in practice is not easy.

term debt falls (i.e., $\phi'_t < 0$), shortening, in which outstanding short-term debt rises ($\phi'_t > 0$), and hump-shaped (but lengthening closer to default, otherwise) equilibria.

(3) The shorter the maturity-rate spread or the lower the principal of debt, the larger the fraction on lengthening equilibria. Assume linear cash flows: the larger the upside-option expected payoff, ζX, the larger the fraction of lengthening equilibria.

(4) Proposition 6: Although multiple equilibria exist for a same ϕ_t, interior equilibria are unique given the issuance policy f_t (i.e., independent of ϕ_t).

5.2 Nonconstrained equilibria in the vicinity of default

We study first the vicinity of default, in which we easily prove the constrained equilibrium rollover policy. Because of smooth-pasting (see equations (49) and (50) in Appendix A), IC_t is equivalent to

$$IC_t \sim - \left(e^{-(R+d_S) \times \tau} - e^{-(R+d_L) \times \tau} \right) + \int_t^T \frac{e^{-R \times (u-t)}}{l(T)} \times \frac{d}{d\phi_t} I(u, \phi_u; f_u, T) \times du, \quad (22)$$

in which $l(T) > 0$ is factorized out. Then, from a first-order Taylor approximation with regard to τ (where T is fixed and $t = T - \tau$),

$$IC_t \approx IC_T + IC'_T \times \tau = IC'_T \times \tau,$$

in which $IC_T = 0$ and

$$IC'_T \sim (d_S - d_L) + \frac{\frac{d}{d\phi_t} I(u, \phi_u; f_u, T) \Big|_{u=t,t=T}}{l(T)}.$$

For $t \to T$, $IC_t = 0$ implies $IC'_T = 0$.

Computing $\frac{d}{d\phi_t} I(u, \phi_u; f_u, T) \Big|_{u=t,t=T}$ (see Appendix C),

$$IC'_T \sim m(\phi_T) \times (d_S - d_L) \times (f_T - f_T^{nc}) \times T^*_\phi, \quad (23)$$

24

where (we advance that)

$$f_T^{nc} = \frac{\frac{l'(T)}{l(T)} - (R + d_L)}{d_S - d_L} \tag{24}$$

denotes the nonconstrained rollover policy at the default time. It follows that

$$IC_T' = 0 \iff f_T = f_T^{nc}, \tag{25}$$

which is the unique solution of a nonconstrained equilibrium.

The constrained equilibria, $0 \le f_T \le 1$

For an interior equilibrium, $0 \le f_T = f_T^{nc} \le 1$. For a corner equilibrium,

(i) if $f_T^{nc} > 1$, $f_T = 1$ (implies that $IC_T' > 0$ and) maximizes $IC_T' \times f_T$ and is a shortening equilibrium, and

(ii) if $f_T^{nc} < 0$, $f_T = 0$ (implies that $IC_T' < 0$ and) maximizes $IC_T' \times f_T$ and is a lengthening equilibrium.

Equivalently, and more compactly (for both interior and corner equilibria),

$$f_T = \min \left\{ \max \left\{ 0, f_T^{nc} \right\}, 1 \right\}, \tag{26}$$

that is, the constrained optimal rollover policy in the vicinity of default (which is the unique equilibrium)—HM's Proposition 1.

Remark. From equation (24), HM derive a key insight: the issuance of new short-term debt at the instant of default depends on the bond-recovery sensitivity (i.e., $f_T^{nc} > 0$ implies $l'(T) > 0$). Below, we show this result extends along the entire equilibrium path: as the sensitivity of bond recovery to the default time increases, the more short-term debt is issued not only at the instant of default, f_T^{nc}, but also along the entire equilibrium path (Corollary 4).

The default-time slope Short-term borrowing (i.e., $f_T > 0$) implies $l'(T) > 0$. From the default-boundary slope (see equation (19)),

$$\phi_T^*(T) = \frac{A'(T) - m(\phi_T) l'(T)}{(d_S - d_L) \times l(T)} < 0,$$

it follows that $f_T > 0$ also requires a flatter default time. That is,

$$\frac{(d_S - d_L) \times l(T)}{A'(T)} < \frac{(d_S - d_L) \times l(T)}{A'(T) - m(\phi_T) l'(T)} = T_\phi^* < 0,$$

where the first unequality follows if $l'(T) > 0$. Otherwise, default is sufficiently delayed by issuing long-term debt. Moreover, as in HM, we assume

$$\phi_T^*(T) < \phi_T' = -\phi_T d_S + m(\phi_T) f_T, \ 0 \le f_T \le 1, \tag{27}$$

which implies no issuance policy pulls the firm away from the default boundary. Given that $\phi_T = \phi^*(T)$, the previous equation simplifies to $\phi_T^*(T) + \phi^*(T) d_S < 0$.

5.3 Nonconstrained equilibria

We find studying the rollover problem using a second integral form for the price of equity is easier. Equity equals the value of assets minus the value of the debt, in which the former is given by (discounting expected) operating revenues until default and the asset's recovery value at default:

$$F(t, \phi_t; T) = \int_t^T e^{-R \times (u-t)} \times (y(u) + \zeta X) \times du + e^{-R \times \tau} \times (1 - l(T)) \tag{28}$$
$$- \left(\phi_t \times \left(1 - e^{-(R+d_S) \times \tau} \times l(T)\right) + (1 - \phi_t) \times \left(1 - e^{-(R+d_L) \times \tau} \times l(T)\right) \right).$$

Both boundary conditions, value matching and smooth pasting, hold (Appendix C).

We analyze the problem backwards in time, from a fixed default time T. The IC_t

condition implies (Appendix C)

$$IC_t \sim -(d_S - d_L) \times \left(\phi_T - f_T^{nc} + \phi_t \times e^{-d_S \times \tau} \left(f_T^{nc} - 1 \right) + (1 - \phi_t) \times e^{-d_L \times \tau} f_T^{nc} \right) \times e^{-R \times \tau},$$ (29)

where f_T^{nc} is given in equation (24). If $\tau \to \infty$ (i.e., $t \to -\infty$), $IC_t \to 0$ (if $\phi_t \in [0,1]$), and, from equation (21), $F_\phi(t, \phi_t; T) \to 0$. As emphasized above, equity depends little on the maturity structure of debt far away from default, in good times.

Two results follow from the last equation for a nonconstrained equilibrium:

Proposition 3 *If $IC_t = 0$, $t \leq T$, the equilibrium paths of the (nonconstrained) fraction of outstanding and new issues, ϕ_t^{nc} and f_t^{nc}, of short-term debt are given by*

$$\phi_t^{nc} = \frac{f_T^{nc} + (\phi_T^{nc} - f_T^{nc}) \times e^{d_L \times \tau}}{f_T^{nc} + (1 - f_T^{nc}) \times e^{-(d_S - d_L) \times \tau}},$$ (30)

$$f_t^{nc} = \frac{f_T^{nc}}{f_T^{nc} + (1 - f_T^{nc}) \times e^{-(d_S - d_L) \times \tau}}.$$ (31)

Proof. See Appendix C.

Note ϕ_t^{nc} is near exponential if $\phi_T^{nc} \neq f_T^{nc}$ (and $d_L > 0$), and the denominator vanishes if $\tau > 0$ and $f_T^{nc} = \frac{-e^{-(d_S - d_L) \times \tau}}{1 - e^{-(d_S - d_L) \times \tau}} < 0$. Three salient insights follow from equation (31).

Corollary 4 *If $0 < f_T^{nc} < 1$,*

(i) $0 < f_t^{nc} < 1$, $t \leq T$; that is, any interior equilibrium at the default boundary remains interior along the entire path (even if $\phi_t \notin [0,1]$).

(ii) $\frac{df_t^{nc}}{d\tau} \sim f_T^{nc} \times (1 - f_T^{nc}) \times e^{-(d_S - d_L) \times \tau} > 0$, $t \leq T$; that is, less and less short-term debt is newly issued in the run-up to default, where $\lim_{\tau \to \infty} f_t^{nc} = 1_{\{f_T^{nc} > 0\}}$.

And (iii) $\frac{df_t^{nc}}{df_T^{nc}} \sim e^{-(d_S - d_L) \times \tau} > 0$, $t < T$; that is, the issuance of short-term debt on the path toward default is increasing in the issuance of short-term debt at the instant of default (for a default time T fixed, which corresponds to comparing different models).

Proof. The proof immediately follows from equation (31) and $0 < f_T^{nc} < 1$.[5]

Next, we focus on constrained equilibria (i.e., $0 \leq f_t \leq 1$), studying separately the two types of corner equilibria and the interior equilibrium.

Corner equilibria (i) if $f_T^{nc} > 1$, $f_t = 1$ is a corner-shortening equilibrium (i.e., $IC_t > 0$ and $IC_t \times f_t$ is maximized for $f_t = 1$, where $IC_t \times f_t = IC_t$), $t \leq T$; and

(ii) if $f_T^{nc} < 0$, $f_t = 0$ is a corner-lengthening equilibrium (i.e., $IC_t < 0$ and $IC_t \times f_t$ is maximized for $f_t = 0$, where $IC_t \times f_t = 0$), $t \leq T$.

That is, corner equilibria are determined at the default time (i.e., if $f_T^{nc} \notin (0, 1)$), which is HM's Proposition 2. We just assume $\phi_t \in [0, 1]$. See Appendix C.

Interior equilibrium For $f_T^{nc} \in [0, 1]$, any interior equilibrium path of short-term debt is given by

$$\phi_t = \phi_t^{nc} \quad \text{and} \quad f_t = f_t^{nc}, \, t \leq T, \tag{32}$$

in which ϕ_t^{nc} and f_t^{nc} are the nonconstrained counterparts (equations (30) and (31)).

More compactly, for both interior and corner equilibria,

$$f_t = \frac{f_T}{f_T + (1 - f_T) \times e^{-(d_S - d_L) \times \tau}} \quad \text{and} \quad \phi_t = \frac{f_T + (\phi_T - f_T) \times e^{d_L \times \tau}}{f_T + (1 - f_T) \times e^{-(d_S - d_L) \times \tau}}, \, t \leq T, \tag{33}$$

in which $f_T = \min\left\{\max\left\{0, f_T^{nc}\right\}, 1\right\}$. As stated in Corollary 4, any interior equilibrium remains interior along the entire equilibrium path in the run-up to default.

The fraction of interior paths Note that

$$\frac{df_T^{nc}}{dT} = \frac{1}{d_S - d_L} \times \left(\frac{l''(T)}{l(T)} - \left(\frac{l'(T)}{l(T)}\right)^2\right) \leq 0, \tag{34}$$

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[5]For instance, if $\tau \to 0$, $\phi_t^{nc} = \phi_T^{nc}$. If $f_T^{nc} \to 0$, $\phi_t^{nc} = \phi_T^{nc} \times e^{d_S \times \tau}$; and if $f_T^{nc} \to 1$, $\phi_t^{nc} - 1 = (\phi_T^{nc} - 1) \times e^{d_L \times \tau}$. If $\phi_T^{nc} = f_T^{nc}$, $f_t^{nc} = \phi_t^{nc}$, for $t \leq T$, and $\phi_t^{nc} = 1_{\{f_T^{nc} > 0\}}$ if $\tau \to \infty$.

In addition, if $\tau \to 0$, $f_t^{nc} = f_T^{nc}$; and if $\tau \to \infty$, $f_t^{nc} = 1_{\{f_T^{nc} > 0\}}$. If $f_T^{nc} \to 0$, $f_t^{nc} = 0$; and if $f_T^{nc} \to 1$, $f_t^{nc} = 1$. That is, $f_t^{nc} \in \{0, 1\}$, $t \leq T$, are two specific examples of corner equilibria.

Define the default times T_0 and T_1 by $\phi^*(T_0) = 0$ and $\phi^*(T_1) = 1$, where $\phi_T^*(T) < 0$ implies $T_0 > T_1$. If $f_{T_0}^{nc} > 0$, that is,

$$\frac{l'(T_0)}{l(T_0)} > r + \zeta + d_L,$$

no corner-lengthening equilibrium exists. Conversely, if $f_{T_1}^{nc} < 1$, that is,

$$\frac{l'(T_1)}{l(T_1)} < r + \zeta + d_S,$$

no corner-shortening equilibrium exists (e.g., if $l'(T_1)$ is sufficiently small). These two results, however, require that $l(T)$ be a continuous function, which may fail if default occurs at both positive and negative values of $y(T)$ and is explained below.

Now, define $T_a > T_b$ by $f_{T_a}^{nc} = 0$ and $f_{T_b}^{nc} = 1$, and assume they exist. The fraction of interior equilibria is given by $\phi^*(T_b) - \phi^*(T_a) > 0$. Assume the higher the distance $T_a - T_b$, the higher $\phi^*(T_b) - \phi^*(T_a)$. The slope of the function f_T^{nc} between T_a and T_b is given by $-(T_a - T_b)^{-1}$. The more negative this slope, the lower $T_a - T_b$, which implies a lower fraction of interior paths. From equation (34), a higher convexity $-l''(T) \geq 0$ leads to a higher derivative $\frac{-df_T^{nc}}{dT}$. Hence, besides the sensitivity $l'(T)$, a higher bond-recovery convexity to the default time also leads to less interior (more corner) paths.

5.4 Equilibria based on the path of outstanding debt

Next, we show that in addition to corner/interior equilibria, which are defined in terms of the rollover policy at the time of default, equilibria are also sorted and threefold based on the path of outstanding debt. This result fully explains the interior paths.

Corner-shortening (-lengthening) equilibria are defined by $f_t = 1$ ($f_t = 0$), $t \leq T$. We provide all equilibria defined by $\phi_t' > 0$ ($\phi_t' < 0$), $t \leq T$. An equilibrium exists in which outstanding short-term debt increases, and in which the time to default is

shorter—a shortening equilibrium. A lengthening equilibrium exists as well, in which outstanding short-term debt falls, and a hump-shaped equilibrium exists that is lengthening closer to default.

Proposition 5 *Assume nonperpetual long-term debt, $d_L > 0$. If $0 < \phi_T + f_T < 2$, for $t \leq T$,*

if $\phi_T \geq f_T$, $\phi'_t < 0$, a lengthening equilibrium;

if $\phi_T < \frac{d_L \times f_T}{d_L \times f_T + d_S \times (1 - f_T)}$, $\phi'_t > 0$, a shortening equilibrium; and

if $\frac{d_L \times f_T}{d_L \times f_T + d_S \times (1 - f_T)} < \phi_T < f_T$, $\phi'_T > 0$, but ϕ_t is hump-shaped (where $\phi'_T = 0$ if $\phi_T = \frac{d_L \times f_T}{d_L \times f_T + d_S \times (1 - f_T)}$).

And if $\phi_T = f_T = 0$ ($\phi_T = f_T = 1$), $\phi_t = 0$ ($\phi_t = 1$) follows from equation (11).

Proof. See the Appendix C.

Because of optimal default (i.e., $\phi_T = \phi^*(T)$), two results follow: First, an equilibrium is lengthening if the fraction of short-term debt in outstanding debt is larger than or equal to the fraction of short-term debt in new issues at the time of default, that is, if $\phi^*(T) \geq f_T$ (i.e., if $A(T) \geq l'(T) - (r + \zeta) \times l(T)$) Equivalently, an equilibrium is lengthening if the upside-option expected payoff, ζX, holds that

$$\zeta X \geq l'(T) + (r + \zeta) \times (1 - l(T)) - y(T), \tag{35}$$

which implies $\zeta X \geq l'(T)$ if $y(T) \leq 0$ (because $1 - l(T) \geq 0$). Equation (35) fails to hold when $l'(T)$ is sufficiently large—an earlier T because $l''(T) \leq 0$. Namely, a nonpositive tradeoff between ζX and $l'(T)$ is a sufficient condition for a nonlengthening path, whereas a positive tradeoff is necessary for a lengthening path.

Second, the fraction of lengthening equilibria equals the fraction of outstanding short-term debt such that $\phi^*(T) = f_T^{nc}$ (i.e., $\max\{0, \min\{\phi^*(T), 1\}\}$ if this fraction $\phi^*(T) \notin [0, 1]$). That is, this fraction is given by the intersection point between two

functions, the optimal default boundary and the equilibrium rollover policy at default,

$$\phi^* (T) = f_T^{nc}.$$

However, this second result about the fraction of lengthening equilibria (as well as where to place these paths) depends critically on whether default occurs when operational income is negative, which shapes the definition of bond recovery. This link is because at the time of default, if bondholders are less efficient running the firm post default, the operational income $y(T)$ is adjusted to $a_y y(T)$ as follows:

$$a_y \geq 1 \text{ if } y(T) < 0 \ \text{ but } \ 0 \leq a_y \leq 1 \text{ if } y(T) > 0,$$

and the growth option is also reduced to $a_X X$, $0 < a_X \leq 1$.

If operational income is negative at the instant of default, $y(T) < 0$, shortening (lengthening) paths are unambiguously linked to larger (smaller) fractions of outstanding short-term debt at the instant of default. By contrast, if $y(T) \geq 0$, the bond recovery becomes less sensitive to this default time and the opposite result happens: large fractions of outstanding short-term debt at default are linked to lengthening paths. If $y(T) \geq 0$, the default penalty in the positive operational income (i.e., $(1 - a_y) \times y(T)$) is reduced by delaying default (i.e., $y(T)$ is closer to zero) through lengthening paths. We show all these results in the case of linear cash flows, and stress the sign of $y(T)$ depends on the upside-event expected payoff, ζX.

Moreover, $\phi^* (T) = f_T^{nc}$ implies the intersection point, which is denoted by $T(\zeta X)$, solves

$$\zeta X = l'(T(\zeta X)) + (r + \zeta) \times (1 - l(T(\zeta X))) - y(T(\zeta X)), \tag{36}$$

which does not depend on the two bonds' maturity rates, d_S and d_L. Although the spread, $d_S - d_L$, does not change $T(\zeta X)$, it lowers the $\phi^* (T)$ and f_T^{nc} functions and

hence lowers $\phi^* (T (\zeta X))$, the fraction of lengthening equilibria. That is,

$$\frac{dT (\zeta X)}{d [d_S - d_L]} = 0 \quad \text{and} \quad \frac{d\phi^* (T (\zeta X))}{d [d_S - d_L]} < 0,$$

which directly follow from equations (36) and (19). Very short-term debt (i.e., a large maturity-rate spread, $d_S - d_L$) implies larger rollover costs and fewer lengthening paths.

Only f_T^{nc}, the new issuance of short-term debt, depends on $l' (T)$, where

$$\frac{df_T^{nc}}{d [l' (T)]} > 0.$$

The larger the sensitivity $l' (T)$, the lower the fraction of lengthening equilibria $\phi^* (T (\zeta X))$. Note that $l' (T)$ is a function and not a parameter. Next, assuming linear cash flows, we show

$$\frac{d\phi^* (T (\zeta X))}{d [\zeta X]} > 0.$$

Namely, the larger ζX, the larger the fraction of lengthening paths (if $y (T) < 0$).

In sum, although the *issuance* of short-term debt is linked to whether a later default time implies a lower bond recovery $l' (T) > 0$ (lenders anticipate a deteriorating recovery at default), lengthening equilibria depend on the tradeoff between this sensitivity and the upside-event expected payoff ζX (borrowers do not pass up a good reward). If this tradeoff is zero or positive, the associated equilibrium path is nonlengthening; a lengthening path implies a later default compared to constant outstanding debt. The examples below are fully consistent with the tradeoff between ζX and $l' (T)$.

5.5 The fraction of lengthening equilibria

We derive additional properties of this fraction by assuming operational income is linear, declining at a constant rate, $y' (t) = -\mu < 0$ and $y'' (t) = 0$. Then

$$1 - l (T) = \int_0^{T_a - T} e^{-R \times u} \times (a_y y (T) - \mu \times u + a_X \zeta X) \times du, \tag{37}$$

where $0 < a_x \leq 1$ and $a_y > 0$. It follows that

$$l'(T) = a_y \mu \times \frac{1 - e^{-R \times (T_a - T)}}{R} > 0,$$

where $T_a - T = \frac{a_y y(T) + a_X \zeta X}{\mu}$. The following results depend critically on the two bond-recovery sensitivity parameters, a_X and especially a_y, and hence on the sign of $y(T)$.

Where are equilibria placed? First, the slope of the difference between $\phi^*(T)$ and f_T^{nc} evaluated at the crossing point is given by

$$\left. \frac{d[\phi^*(T) - f_T^{nc}]}{dT} \right|_{T=T(\zeta X)} \sim \mu \times (a_y - 1), \tag{38}$$

which is a key result to understand where equilibria are placed. If $a_y \neq 1$, and a_y does not depend on y, a unique intersection point exists. $a_y > 1$ implies shortening (lengthening) equilibria are associated with larger (smaller) fractions of short-term debt at the instant of default. $a_y < 1$ implies the opposite. That is, the fraction of lengthening equilibria is given by $\phi^*(T)$ and $1 - \phi^*(T)$, respectively.

However, assume $a_y \geq 1$ ($a_y \leq 1$) depends on negative operational income at default, $y(T) \leq 0$ (on positive, $y(T) \geq 0$). Further, $y(T) \leq 0$ ($y(T) \geq 0$) is linked to larger (smaller) ζX, and we can have a setting with an intermediate ζX—and both positive and negative operational income at default, which implies the bond-recovery function is discontinuous and the previous results mix.

What type of equilibria exist? Second,

$$\frac{\phi^*(T) - f_T^{nc}}{l(T)^{-1}} = (1 - a_X) \underbrace{\zeta X}_{>0} - \left(\underbrace{y(T)}_{\leqslant 0} + \mu \times \underbrace{\frac{1 - e^{-R \times (T_a - T)}}{R}}_{>0} \right) \times (a_y - 1). \tag{39}$$

We present three special cases: (i) If $a_y = a_X < 1$, $\phi^*(T) > f_T^{nc}$ and only a lengthening equilibrium exists if $y(T) \geq 0$ (which is a sufficient condition). (ii) If $a_y = a_X = 1$,

33

$\phi^*(T)$ and f_T^{nc} overlap (i.e., $\phi^*(T) = f_T^{nc}$ for all T independent of the sign of $y(T)$) and only a specific type of path exists, namely, the nonexponential in equation (31). In this scenario, all paths are both interior (i.e., both short- and long-term debt are issued) and lengthening. And (iii) if $a_y > 1$ and $y(T) \leq 0$, the tradeoff can be both positive or negative (i.e., lengthening and nonlengthening paths, respectively, coexist).

In general, if $a_y > 1$, a_y increases the convexity $-l''(T) > 0$, but the effect on the sensitivity $l'(T)$ is less clear and depends on T. That is, a higher a_y changes f_T^{nc} so that $-l''(T)$ and $\frac{-df_T^{nc}}{dT}$ increase, but the effect on $\phi^*(T(\zeta X))$ is less straightforward.[6] A steeper f_T^{nc} implies more corner equilibria. A higher a_X increases $l'(T)$.

The growth option and lengthening equilibria Third, for simplicity, assume $a_X = 1$. If $a_y > 0$,

$$\frac{d\phi^*(T(\zeta X))}{d[\zeta X]} > 0, \tag{40}$$

and the larger the upside-option expected payoff, the more valuable waiting is by means of lengthening paths (if $a_y > 1$, because $\phi^*(T(\zeta X)) = 1$ if $a_y \leq 1$). For X sufficiently large, all equilibria are lengthening (and the opposite holds for $X \geq 1$ small enough).

In sum, either $a_y \geq 1$ or $a_y \leq 1$ depends on the sign of $y(T)$. As ζX falls, the lengthening-equilibria fraction falls, but at the same time, $y(T)$ increases and a discontinuity in the bond-recovery value may exist. So it is convenient to separate the two cases of negative and positive values for $y(T)$. The following examples show all scenarios exist, with both lengthening and shortening equilibria but also without one of the two. In practice, negative operational income ($y(t) \leq 0$) is more relevant.[7]

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[6]For example, if $T \to T_a$, $l(T) \approx 1$, $l'(T) \approx 0$, and $-l''(T) \approx a_Y^2 \mu > -y'(T) = \mu$ (if $a_Y > 1$), which implies $l'(T)$ grows faster than $y(T)$ if T falls, in the right-hand-side of equation (35), explaining the tradeoff between ζX and $l'(T)$.

[7]The fraction of shortening equilibria is given by $1 - \phi^*(T)$, where T holds that

$$\phi^*(T) = \frac{d_L \times f_T}{d_L \times f_T + d_S \times (1 - f_T)},$$

and a larger $\phi^*(T)$ also implies fewer shortening paths. A lower maturity spread and a larger short-

5.5.1 Numerical examples

Equipped with ϕ_t and f_t's formulae, we compute several interior paths (as a function of $y = y(t)$). We use the same parameters as HM's Figures 1 and 4.

In our Figure 1, we show the default boundary, $\phi^*(T)$, and three interior equilibrium paths of the fraction of outstanding short-term debt, ϕ_t, that is, a lengthening path that ends at $\phi_T = 0.50$, a hump-shaped path that ends at $\phi_T = 0.58$ (which replicates HM's Figure 4, right panel), and a shortening path that defaults at $\phi_T = 0.64$. We also show the following equilibrium regions at the default boundary: corner lengthening, $0 \leq \phi_T \leq 0.46$; corner shortening, $0.65 \leq \phi_T \leq 1$; and interior, $0.46 < \phi_T < 0.65$. In addition, we have lengthening, $0 < \phi_T \leq 0.56$, shortening, $0.63 < \phi_T < 1$, and hump-shaped, $0.56 < \phi_T < 0.63$.

*** to include Figure 1 ***

In Figure 2, we provide the rollover policy at default, f_T, and the three equilibrium paths of the fraction of new issues of short-term debt, f_t. Figure 2 illustrates Proposition 4; namely, interior policies at default remain interior and the issuance policy increases with both an earlier default time or a larger time to default. In addition, the three interior issuance-policy paths never intersect, implying a unique equilibrium exists, given the rollover policy (see Proposition 6 below).

*** to include Figure 2 ***

The example in Figures 1 and 2, in which $a_y = 3$ and $a_X = 0.95$, implies the existence of more corner-lengthening/-shortening equilibria than interior equilibria. The function $f_T \in [0, 1]$, the rollover choice at default, is very steep (see Figure 2).

term debt issuance $(d_S/d_L$ and f_T, respectively) imply a larger ratio:

$$\frac{d_L \times f_T}{d_L \times f_T + d_S \times (1 - f_T)} = \frac{1}{1 + \frac{d_S}{d_L} \times \frac{1 - f_T}{f_T}}.$$

Using similar reasonings as above, we can study the fractions of shortening/hump-shaped equilibria.

A lower (higher) payoff compared to $X = 13$, for example, $X \leq 7$ ($X \geq 16$), implies fewer or zero corner-lengthening (-shortening) equilibria exist, as follows from equation (40). That is, a lower (higher) payoff ζX implies $\phi^*(T)$ and f_T intersect closer to $\phi^*(T) = 0$ ($\phi^*(T) = 1$), implying from Proposition 5 fewer corner-lengthening (-shortening) paths.

In Figure 3, we increase the number of nonlengthening equilibria by either decreasing the upside payoff (from $X = 13$ to $X = 10$) or by assuming shorter short-term debt (increasing $dS = 5$ to $dS = 11$). We show both the optimal default boundary and the equilibrium rollover policy at default, $\phi^*(T)$ and f_T, and stress the intersection point. Compared to Figure 1, the fraction of lengthening equilibria is reduced from 56% by half, approximately. Note how close some operational income is to being positive at the instant of default.

*** **to include Figure 3** ***

In Figure 4, however, we assume a different recovery, $0 < 0.8 = a_y = a_X < 1$. Compared to Figures 1 and 2, in which $a_y = 3$, $a_y < 1$ implies a flatter f_T^{nc} and fewer shortening paths. We show both ϕ_t and f_t's paths. All equilibria are (either corner or interior but) lengthening, that is, $\phi_T > f_T$ ($\phi_T \in [0, 1]$) at the default time. Although $a_y < 1$, $y(T) < 0$ because $\zeta = 0.35$ and $X = 13$ are as in the other figures (if $a_y < 1$, $y(T) \geq 0$ is a sufficient but not necessary condition for lengthening paths).

*** **to include Figure 4** ***

Perpetual-debt example. For nonexpiring debt, $d_L = 0$,

$$\phi^*(T) = \frac{A(T)}{d_S \times l(T)} \quad \text{and} \quad f_T^{nc} = \frac{-R + l'(T)/l(T)}{d_S},$$

and any equilibrium path simplifies to

$$f_t = \frac{f_T}{f_T + (1 - f_T) \times e^{-d_S \times \tau}} \quad \text{and} \quad \phi_t = \frac{\phi_T}{f_T + (1 - f_T) \times e^{-d_S \times \tau}}, \quad t \leq T. \qquad (41)$$

36

Assume $\phi_T > 0$. If $f_T = 1$, $\phi_t = \phi_T$, which is a static corner-shortening equilibrium; if $f_T < 1$, $\phi'_t < 0$, all equilibrium paths reduce short-term debt in the run-up to default. If $f_T > 0$, ϕ_t is not exponential. These three features are linked to perpetual debt.

One can wonder whether a Leland-static equilibrium, which is defined by a constant level $\phi_t = \phi_T$, $t \leq T$, exists. The latter constraint implies $e^{-d_S \times \tau} \times (1 - f_T) + f_T = 1$; hence, $f_T = 1$. A static equilibrium is any corner-shortening equilibrium (in case one exists), which is given by $f_T^{nc} \geq 1$, implying, because $f_T^{nc} = \frac{1}{d_S} \times \left(-R + \frac{l'(T)}{l(T)} \right)$,

$$\phi^* (T) \leq 1 : \quad \frac{l'(T)}{l(T)} \geq R + d_S.$$

A Leland-static equilibrium is linked to perpetual debt and large loads of outstanding short-term debt at the time of default (besides the all long-term debt case, $\phi^* (T) = 0$).

6   Extensions: More Outstanding Face Value

Consider the bond principal (instead of $L = 1$) is $L > 0$. $B(T) = L - l(T)$ is the bond recovery given default, and $l(T)$ is the loss given default. It can be shown that the optimal default boundary changes to

$$\phi^* (T) = \frac{1}{d_S - d_L} \times \left(\frac{y(T) + \zeta X - (r + \zeta) \times L}{l(T)} - d_L \right),$$

because the coupon and the intensity depend on the bond principal (L), but the rollover policy at the time of default is the same:

$$f_T^{nc} = \frac{\frac{l'(T)}{l(T)} - (R + d_L)}{d_S - d_L}.$$

Likewise, the intersection point $\phi^* (T) = f_T^{nc}$ implies

$$y(T(\zeta X)) + \zeta X = l'(T(\zeta X)) + (r + \zeta) \times \underbrace{(L - l(T(\zeta X)))}_{= B(T(\zeta X))}.$$

The bond recovery $B(T)$ does not depend on L but on the default time T (i.e., on operational income, $y(T)$). Therefore, the last equation is independent of L (or $d_S - d_L$, as we show above). It follows that

$$\frac{dT(\zeta X)}{dL} = 0 \quad \text{and} \quad \frac{d\phi^*(T(\zeta X))}{dL} = \frac{-(r+\zeta) \times l(T(\zeta X)) - A(T(\zeta X))}{(d_S - d_L) \times l(T(\zeta X))^2} < 0, \quad (42)$$

because $l(T) = L - B(T)$ implies $\frac{dl(T)}{dL} = 1$ (and $A(T(\zeta X)) > 0$). The larger the debt principal L, the lower the fraction of lengthening equilibria. This result complements Proposition 4 in HM; that is, the fraction of corner-shortening paths increases with L.

6.1   Recovering a unique equilibrium

Although equilibria are not necessarily unique (ϕ_t is consistent with many ϕ_T, $t < T$; see Figure 1), we can recover a unique default time T from the rollover policy, f_t. For corner equilibria, this uniqueness is clear. As HM show, corner paths (same type) do not cross between each other, and, as we show, the interior equilibrium remains interior. Hence, $f_t \in \{0, 1\}$ is only associated with corner paths. Given the threesome, that is, $f_t \in \{0, 1\}$, ϕ_t, and t (or $y(t)$), we have a unique corner equilibrium.

For interior paths, we have a unique equilibrium given only f_t and t. Intuitively, the rollover policies do not cross between each other, implying interior equilibria are unique based on the rollover policy. It follows that the issuance policy of new debt—namely, the change in outstanding debt (and not debt maturity itself)—determines a specific equilibrium path. From f_t and ϕ_t's analytical solutions, we have that

$$\phi_t = \frac{f_u + (\phi_u - f_u) \times e^{d_L \times (u-t)}}{f_u + (1 - f_u) \times e^{-(d_S - d_L) \times (u-t)}}, \; u \in \mathcal{R}, \quad (43)$$

$$f_t = \frac{f_u}{f_u + (1 - f_u) \times e^{-(d_S - d_L) \times (u-t)}}, \; u \in \mathcal{R}. \quad (44)$$

From the former equation, ϕ_t depends on both ϕ_u and f_u (besides $u - t$). From the latter, f_t depends only on f_u. Formally,

Proposition 6 *If* $0 < f_T^{nc} < 1$,

$$\frac{df_t}{df_u} \sim e^{-(d_S - d_L) \times (u-t)} > 0,$$

Proof. The proof directly follows from the equation (44). ∎

As f_t increases so does f_u for any $u \in \mathcal{R}$ (i.e., along the entire equilibrium path), which implies interior issuance-policy paths never cross. In addition, for a given equilibrium path of the rollover policy, the default time T can also be characterized as the intersection point between two curves f_T and f_u, where f_T (f_u) denotes the issuance policy at the time of default for all equilibria (along the entire interior-equilibrium path, $u \leq T$). Both results are consistent with Figures 2 and 4.

6.2   Welfare analysis

We complete the paper by looking to welfare. From equation (28), we see the total firm value $V(t, \phi_t; T)$ is given by

$$
\begin{aligned}
&F(t, \phi_t; T) + \phi_t \times \left(1 - e^{-(R+d_S) \times \tau} \times l(T)\right) + (1 - \phi_t) \times \left(1 - e^{-(R+d_L) \times \tau} \times l(T)\right) \\
&= \int_t^T e^{-R \times (u-t)} \times (y(u) + \zeta X) \times du + e^{-R \times \tau} \times (1 - l(T)).
\end{aligned}
\tag{45}
$$

If we assume the bond-recovery value is given by equation (15), $0 \leq a \leq 1$,

$$
V(t, \phi_t; T) = \int_t^T e^{-R \times (u-t)} \times (y(u) + \zeta X) \times du + a \times \int_T^{T_a} e^{-R \times (u-t)} \times (y(u) + \zeta X) \times du.
\tag{46}
$$

Importantly, the abandoning time (T_a) is independent of the default time (T), and it holds that

$$y(T_a) + \zeta X = 0.$$

For the states (t, ϕ_1) and (t, ϕ_2), consider the equilibrium default times T_1 and T_2, respectively, where $T_2 > T_1$. If $\phi_1 = \phi_2$, the two states are the same, which is an

example of the multiplicity of equilibria. Clearly,

$$V\left(t, \phi_1; T_2\right) - V\left(t, \phi_2; T_1\right) \tag{47}$$

$$= (1-a) \times \int_{T_1}^{T_2} e^{-R \times (u-t)} \times \left(y\left(u\right) + \zeta X\right) \times du > 0 \quad (\text{if } 0 \le a < 1),$$

because the integrals cancel between 0 and T_1 and between T_2 and T_a, $0 < T_1 < T_2 < T_a$.

The latter inequality implies the later T_2 always welfare dominates the earlier T_1 (e.g., a lengthening and a shortening, respectively) equilibrium. Only by changing $l'\left(T\right)$, the bond loss given default, can we reverse this result. HM provide some examples in which a shortening path is welfare improving close to default. Yet all equilibria are welfare equal if $a = 1$.

7 Concluding Remarks

We study how a firm manages the maturity of its debt in a model of declining cash flows subject to an upside event (i.e., growth option), where expired debt is refinanced with short- or long-term bonds. Shortening, hump-shaped, and lengthening equilibrium paths in time to default coexist. In the latter, outstanding short-term debt falls in bad times, which leads to a procyclical rollover policy and a higher time to default.

Although the *issuance* of short-term debt is linked to whether an earlier default time leads to a larger bond recovery (bondholders anticipate a deteriorating recovery at default, He and Milbradt, 2016), lengthening equilibria depend on the tradeoff between this sensitivity and the upside-event expected payoff (so equityholders do not pass up a good reward). The larger this upside event, the higher the fraction of lengthening paths; for sufficiently large upside options, this pattern is the only model equilibrium. It follows that firms with a good upside option engineer a later default via longer maturity.

The rationale of issuing long-term debt and increasing the fraction of outstanding

long-term debt because of a larger upside option is a sensible outcome. As Graham and Harvey (2001) or Xu (2018) show, speculative-grade firms manage the maturity profile of debt to avoid having to borrow in bad times, by early refinancing along with maturity lengthening. It follows that growth options and a higher fraction of lengthening equilibria fit with the evidence of procyclical refinancing activity of speculative-grade firms. Moreover, issuing short-term debt in good times, where equity is only somewhat sensitive to the maturity structure of debt, conforms with a less cyclical refinancing pattern of investment-grade firms. For these high credit-rating firms, we expect debt-maturity choices to also be more heterogenous and depend on the term premium.

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8 Appendix A: Optimal Default Conditions

Proof of Proposition 1. We make the following two assumptions on the payout-rate function, $I(t;T)$. First, $I(0;0) > 0$, and second,

$$\frac{dI(T;T)}{dT} \leq -\mu < 0, \; 0 \leq T,$$

which imply a unique point $T^* > 0$ exists such that the payout rate becomes zero; that is, $I(T^*, T^*) = 0$, because $I(0,0) > 0$.

Remark. $I(T^*, T^*) = 0$ and $\frac{dI(T;T)}{dT} < 0$ are analogous to necessary and sufficient conditions, respectively, to show that T^* is the optimal default time. In addition, we assume lenders do not extend credit beyond T^*. Consequently, equityholders optimize with regard to T in the interval $0 \leq T \leq T^*$.

The proof is as follows. Let $dT > 0$. If $T < T^*$,

$$F(T;T+dT) - \underbrace{F(T;T)}_{=0} \approx \underbrace{F(T;T)}_{=0} + \underbrace{I(T;T)}_{=F_T(T;T)} \times dT + \mathrm{O}\left(dT^2\right)$$

$$= I(T;T) \times dT + \mathrm{O}\left(dT^2\right),$$

which follows from a first-order Taylor expansion in F's second variable. If $I(T;T) > I(T^*;T^*) = 0$, $F(T;T+dT) > F(T;T)$, which implies it is optimal to (delay) default beyond T, that is, $dT > 0$.

If $T > T^*$,

$$F\left(T - dT; T\right) - \underbrace{F\left(T - dT; T - dT\right)}_{=0} \approx \underbrace{F\left(T; T\right)}_{=0}\underbrace{-I\left(T; T\right)}_{=F_t(T;T)} \times \left(-dT\right) + \mathrm{O}\left(dT^2\right)$$
$$= I\left(T; T\right) \times dT + \mathrm{O}\left(dT^2\right),$$

which follows from a first-order Taylor expansion in F's first variable. If $I\left(T; T\right) < I\left(T^*; T^*\right) = 0$, $F\left(T - dT; T\right) < F\left(T - dT; T - dT\right)$. It is optimal to (accelerate) default before T, that is, $-dT < 0$.

From both scenarios, it follows that $T \neq T^*$ is not a credible default policy, but $T = T^*$. Then lenders do not extend credit beyond T^*, and equityholders optimize with regard to T in the interval $0 \leq T \leq T^*$. It follows that $T = T^*$ is optimal. Finally, $F_T\left(T^*; T^*\right) = 0$ and $F_t\left(T^*; T^*\right) = 0$ follow from $I\left(T^*, T^*\right) = 0$ and section 2.1.

Further, instead of assuming the function $\frac{dI(T;T)}{dT} \leq -\mu < 0$, $0 \leq T$, we have a weaker condition: The equation $I\left(T, T\right) = 0$ has a unique solution, which is denoted by $T = T^*$, and $\left.\frac{dI(T;T)}{dT}\right|_{T=T^*} < 0$. The proof is exactly as above, because $I\left(T, T\right) > 0$ if $T < T^*$ (and $I\left(T, T\right) < 0$ if $T > T^*$). $\blacksquare$

Proof of Lemma 2. The partial derivatives F_t and F_ϕ are given by

$$F_t\left(t, \phi_t; T\right) = \left.-I\left(u, \phi_u; f_u, T\right)\right|_{u=t} + e^{-R\times(T-t)} \times \left.I\left(u, \phi_u; f_u, T\right)\right|_{u=T} \times \frac{dT}{dt} \quad (48)$$
$$+ \int_t^T \frac{d}{dt}\left[e^{-R\times(u-t)} \times I\left(u, \phi_u; f_u, T\right)\right] du,$$

$$F_\phi\left(t, \phi_t; T\right) = e^{-R\times(T-t)} \times \left.I\left(u, \phi_u; f_u, T\right)\right|_{u=T} \times \frac{dT}{d\phi_t} \quad (49)$$
$$+ \int_t^T e^{-R\times(u-t)} \times \frac{d}{d\phi_t}I\left(u, \phi_u; f_u, T\right) du.$$

Because the two integrals vanish for $t = T$, from equation (8), the smooth-pasting conditions are given by

$$\left.F_t\left(T, \phi; T\right)\right|_{T=T^*(\phi)} = 0 \quad \text{and} \quad \left.F_\phi\left(T, \phi; T\right)\right|_{T=T^*(\phi)} = 0. \quad (50)$$

The other properties in Lemma 2 follow directly from the exposition in section 3 and Proposition 1. ∎

9 Appendix B: The Value of Equity and Debt

Given the default time T, a bond price is given by discounting expected cash flows at the interest rate r. The two exponential distributions imply the probability of surviving between t and T is given by $e^{(\zeta+\delta)\times(T-t)}$. For $t < u \le T$, the probability of the bond being cancelled and repaid at u is given by $e^{(\zeta+\delta)\times(u-t)} \times (\zeta+\delta) \times du$. Consequently,

$$
\begin{aligned}
& D_i\left(T - t; T\right) \\
={} & \int_t^T e^{-(r+\zeta+\delta_i)\times(u-t)} \times (c + (\zeta+\delta_i) \times 1) \times du + e^{(r+\zeta+\delta_i)\times(T-t)} \times (1 - l\left(T\right)) \\
={} & -e^{-(r+\zeta+\delta_i)\times u}\big|_{u=0}^{u=T-t} + e^{(r+\zeta+\delta_i)\times(T-t)} \times (1 - l\left(T\right)) \\
={} & 1 - e^{(r+\zeta+\delta_i)\times(T-t)} \times l\left(T\right).
\end{aligned}
$$

The coupon is $c = r$ and $i = \{S, L\}$. Similarly, equity value is given by

$$
F\left(t, \phi_t; T\right) = \int_t^T e^{-(r+\zeta)\times(u-t)} \times (y\left(u\right) - c + \zeta \times (X - 1)
$$

$$
+\, m\left(\phi_u\right) \times (f_u \times D_S\left(T - u; T\right) + (1 - f_u) \times D_L\left(T - u; T\right) - 1)) \times du.
$$

$$
= \int_t^T e^{-(r+\zeta)\times(u-t)} \times (y\left(u\right) - r + \zeta \times (X - 1)
$$

$$
-\, m\left(\phi_u\right) \times \left(f_u \times e^{-(r+\zeta+d_S)\times(T-u)} + (1 - f_u) \times e^{-(r+\zeta+d_L)\times(T-u)}\right) \times l\left(T\right)) \times du,
$$

as in equations (16) to (18).

9.1 The Loss Given Default, l.

We assume operational income is declining, $y'\left(t\right) \le -\mu < 0$, $t \ge 0$. We also assume $r + \zeta \ge 0$ and $y''\left(t\right) \ge 0$, $t \ge 0$.

The default time, $T = T^* (\phi_T)$, implies $A (T)$ holds that

$$y (T) - r + \zeta \times (X - 1) = m (\phi_T) \times l (T).$$

If bondholders are less efficient running the firm post-default, $0 < a \leq 1$,

$$1 - l (T) = a \times \int_0^{T_a - T} e^{-(r+\zeta) \times u} \times (y (T + u) + \zeta X) \times du.$$

Likewise, the abandon time, T_a, is also given by a zero net cash-flow condition,

$$y (T_a) + \zeta X = 0, \text{ and } \frac{dT_a}{dT} = 0,$$

where $T_a > T$ if $y (T) + \zeta X > 0$ (and $T_a = T$ if $y (T) + \zeta X \leq 0$). Hence, $l (T) \leq 1$.

Further,

$$
\begin{aligned}
l' (T) &= a \times \left(e^{-(r+\zeta) \times (T_a - T)} \times (y (T_a) + \zeta X) - \int_0^{T_a - T} e^{-(r+\zeta) \times u} \times y' (T + u) \times du \right) \\
&= -a \times \int_0^{T_a - T} e^{-(r+\zeta) \times u} \times y' (T + u) \times du, \text{ and} \\
l'' (T) &= -a \times \left(-e^{-(r+\zeta) \times (T_a - T)} \times y' (T_a) + \int_0^{T_a - T} e^{-(r+\zeta) \times u} \times y'' (T + u) \times du \right),
\end{aligned}
$$

and hence

$$
\begin{aligned}
l' (T) &> 0 \text{ if } y' (T + u) < 0, u \geq 0, \\
l'' (T) &< 0 \text{ if } y' (T_a) < 0 \text{ and } y'' (T + u) \geq 0, u \geq 0.
\end{aligned}
$$

Next we prove $l (T) > 0$. We show that if $A (\widehat{T}) \leq 0$, $l (\widehat{T}) > 0$. Hence, $\widehat{T} < T^* (\phi)$, $\phi \in [0, 1]$, where T^* is the solution of the default boundary equation,

$$A (T^* (\phi)) = m (\phi) \times l (T^* (\phi)) > 0, \text{ if } d_L > 0,$$

46

because $A' < 0$ and $l' \geq 0$. If $T_a = T$, $l(T) = 1$, so assume $T_a > \widehat{T}$.

$A\left(\widehat{T}\right) \leq 0$ implies $y\left(\widehat{T}\right) + \zeta X \leq r + \zeta$. If $r + \zeta = 0$, $T_a = \widehat{T}$ and $l\left(\widehat{T}\right) = 1$, so assume $r + \zeta > 0$. Then

$$
\begin{aligned}
1 - l\left(\widehat{T}\right) &= a \times (r + \zeta) \times \int_0^{T_a - \widehat{T}} e^{-(r+\zeta)\times u} \times \frac{y\left(\widehat{T} + u\right) + \zeta X}{r + \zeta} \times du \qquad (51) \\
&< a \times (r + \zeta) \times \int_0^{T_a - \widehat{T}} e^{-(r+\zeta)\times u} \times du \\
&= -a \times e^{-(r+\zeta)\times u}\Big|_0^{T_a - \widehat{T}} = a \times \left(1 - e^{-(r+\zeta)\times\left(T_a - \widehat{T}\right)}\right).
\end{aligned}
$$

The inequality follows from declining cash flow, $\frac{y\left(\widehat{T}+u\right)+\zeta X}{r+\zeta} < 1$ if $u > 0$. It follows that

$$
l\left(\widehat{T}\right) > 1 - a \times \left(1 - e^{-(r+\zeta)\times\left(T_a - \widehat{T}\right)}\right) \geq 0.
$$

In addition,

$$
y(T) + \zeta X = \underbrace{r + \zeta}_{\geq 0} + \underbrace{m(\phi_T) \times l(T)}_{>0} > 0
$$

implies $T_a > T$ and $0 < l(T) < 1$ (i.e., $m(\phi_T) = 0$ only if $\phi_T = 0$ and $d_L = 0$).

∎

If $A'' \geq 0$, the default boundary is a convex function, $\phi^*_{TT}(T) > 0$. Note that $A'' = y''$. Convexity follows from $(Al''l + (A'l - Al')\,2) < 0$ and

$$
\begin{aligned}
(d_S - d_L) \times \phi^*_{TT}(T) &= \frac{(A''l + A'l' - A'l' - Al'')\,l^2 - (A'l - Al')\,2l}{l^4} \\
&= \frac{A''}{l} - \frac{Al''l + (A'l - Al')\,2}{l^3} > 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
T^*_{\phi\phi}(\phi) &= \frac{d}{d\phi}\left[T^*_\phi(\phi)\right] = \frac{d}{d\phi}\left[1/\phi^*_T(T^*(\phi))\right] \\
&= \frac{-1}{\phi^*_{TT}(T)} \times \underbrace{T^*_\phi(\phi)}_{<0} > 0 \quad \text{if } \phi^*_{TT}(T) > 0.
\end{aligned}
$$

10   Appendix C: Equilibrium Rollover Policies

Derivation of equation (23).   Note that

$$
\frac{(d_S - d_L)^{-1}}{l(T)} \times \frac{d}{d\phi_t} [I(u, \phi_u; f_u, T)]
$$

$$
= -\frac{m(\phi_u)}{d_S - d_L} \times \left(e^{-(R+d_S)\times(T-u)} - e^{-(R+d_L)\times(T-u)}\right) \times \frac{df_u}{d\phi_t}
$$

$$
- \left(f_u \times e^{-(R+d_S)\times(T-u)} + (1 - f_u) \times e^{-(R+d_L)\times(T-u)}\right) \times \frac{d\phi_u}{d\phi_t}
$$

$$
- m(\phi_u) \times \left(f_u \times e^{-(R+d_S)\times(T-u)} \times (f_T^{nc} - 1)\right.
$$

$$
\left. + (1 - f_u) \times e^{-(R+d_L)\times(T-u)} \times f_T^{nc}\right) \times T_\phi^* \times \frac{d\phi_T}{d\phi_t},
$$

where

$$
f_T^{nc} = \frac{\frac{l'(T)}{l(T)} - (R + d_L)}{d_S - d_L}.
$$

Then, for $u = t$ and $t = T$, $\frac{d\phi_u}{d\phi_t} = 1$ and $\frac{d\phi_T}{d\phi_t} > 0$ (see the two remarks next),

$$
\left. \frac{1}{l(T)} \times \frac{d}{d\phi_t} I(u, \phi_u; f_u, T) \right|_{u=t, t=T} = -(d_S - d_L) + (d_S - d_L) \times m(\phi_T) \times (f_T - f_T^{nc}) \times T_\phi^* \times \left. \frac{d\phi_T}{d\phi_t} \right|_{t=T},
$$

$$(52)$$

from which follows equation (23).

Two remarks. (i) $\left. \frac{d\phi_u}{d\phi_t} \right|_{u=t} = 1$ follows from equation (11). From a first-order Taylor approximation, $t \leq u \leq T$,

$$
\phi_u \approx \phi_t + (-\phi_t \times d_S + m(\phi_t) \times f_t(\phi_t)) \times (u - t),
$$

and if $u \to t$,

$$
\frac{d\phi_u}{d\phi_t} \approx 1 + \left(-d_S + \frac{d[m(\phi_t) \times f_t(\phi_t)]}{d\phi_t}\right) \times (u - t) \to 1.
$$

(ii) From a similar approximation from the default boundary, $\phi_T = \phi^*(T)$ and

$$\phi_t \approx \phi^*(T) - \underbrace{(-\phi^*(T) \times d_S + m(\phi^*(T)) \times f_T)}_{=\phi'_T} \times (T-t),$$

and if $\tau = T - t \to 0$,

$$
\begin{aligned}
\frac{d\phi_t}{dT} &\approx \frac{d\phi^*(T)}{dT} - \phi'_T - \frac{d[\phi'_T]}{dT} \times \tau \\
&\to \frac{d\phi^*(T)}{dT} - \underbrace{(-\phi^*(T) \times d_S + m(\phi^*(T)) \times f_T)}_{=\phi'_T} < 0,
\end{aligned}
\tag{53}
$$

because no issuance policy pulls the firm away from the default boundary.

Further, note that

$$
\begin{aligned}
\left.\frac{dT}{d\phi_t}\right|_{t=T} &= T_\phi^* \times \left.\frac{d\phi_T}{d\phi_t}\right|_{t=T} = T_\phi^* \times \frac{1}{1 - \phi'_T \times T_\phi^*} \\
&= T_\phi^* \times \frac{\frac{d\phi^*(T)}{dT}}{\frac{d\phi^*(T)}{dT} - \phi'_T} = \frac{1}{\frac{d\phi^*(T)}{dT} - \phi'_T} < 0,
\end{aligned}
$$

which is consistent with the previous equation. It follows that $\left.\frac{d\phi_T}{d\phi_t}\right|_{t=T} > 0$.

Boundary conditions for equation (28). We check that value-matching and smooth-pasting conditions also hold. From $m(\phi_T) = \phi_T \times d_S + (1 - \phi_T) \times d_L$ and $A(T) = m(\phi_T) \times l(T)$,

$$
\begin{aligned}
\left. F(t, \phi_t; T) \right|_{t=T} &= (1 - l(T)) - (\phi_T \times (1 - l(T)) + (1 - \phi_T) \times (1 - l(T))) = 0, \\
\left. F_\phi(t, \phi_t; T) \right|_{t=T} &= (A(T) - (\phi_T \times d_S + (1 - \phi_T) \times d_L) \times l(T)) \times \left.\frac{dT}{d\phi_t}\right|_{t=T} = 0, \\
\left. F_t(t, \phi_t; T) \right|_{t=T} &= (A(T) - (\phi_T \times d_S + (1 - \phi_T) \times d_L) \times l(T)) \times \left(\left.\frac{dT}{dt}\right|_{t=T} - 1\right) = 0,
\end{aligned}
$$

where the other terms (as in equation (54)) cancel (and are omitted for brevity).

Proof of Proposition 3. Derivation of the equilibrium nonconstrained rollover policy. First, given that $c = r$, $R = r + \zeta$, and $A(t) = y(t) - c + \zeta(X - 1)$,

$$F_\phi(t, \phi_t; T) \tag{54}$$
$$= e^{-R \times \tau} \times A(T) \times \frac{dT}{d\phi_t} + \left(e^{-(R+d_S) \times \tau} - e^{-(R+d_L) \times \tau}\right) \times l(T)$$
$$- \left(-e^{-R \times \tau} \times R + \phi_t \times e^{-(R+d_S) \times \tau} \times (R + d_S)\right.$$
$$\left. + (1 - \phi_t) e^{-(R+d_L) \times \tau} \times (R + d_L)\right) \times l(T^*) \times \frac{dT}{d\phi_t}$$
$$+ \left(-e^{-R \times \tau} + \phi_t \times e^{-(R+d_S) \times \tau} + (1 - \phi_t) \times e^{-(R+d_L) \times \tau}\right) \times l'(T) \times \frac{dT}{d\phi_t}.$$

It follows

$$IC_t \sim -m(\phi_T) + \left(-R + \phi_t \times e^{-d_S \times \tau} \times (R + d_S) + (1 - \phi_t) \times e^{-d_L \times \tau} \times (R + d_L)\right)$$
$$- \left(-1 + \phi_t \times e^{-d_S \times \tau} + (1 - \phi_t) \times e^{-d_L \times \tau}\right) \times \frac{l'(T)}{l(T)}, \tag{55}$$

where the price difference between the two bonds cancels and $-e^{-R \times \tau} \times l(T^*) \times \frac{dT}{d\phi_t}$ factorizes out (see remark (i) below). In particular, $IC_T = m(\phi_T) - m(\phi_T) = 0$.

Equivalently to equation (55),

$$IC_t \sim -m(\phi_T) - R + \frac{l'(T)}{l(T)}$$
$$- (d_S - d_L) \times \left(\phi_t \times e^{-d_S \times \tau} \times (f_T^{nc} - 1) + (1 - \phi_t) \times e^{-d_L \times \tau} \times f_T^{nc}\right)$$
$$= - (d_S - d_L) \times \left(\phi_T - f_T^{nc} + \phi_t \times e^{-d_S \times \tau} \times (f_T^{nc} - 1) + (1 - \phi_t) \times e^{-d_L \times \tau} \times f_T^{nc}\right).$$

Therefore, $IC_t = 0$ implies ϕ_t^{nc} in equation (30). Also, from IC_t we can obtain IC_T', and from $IC_T' = 0$, along with a Taylor expansion, follows equation (23) as well.

Second,

$$
\begin{aligned}
\phi'_t &= \frac{\partial}{\partial t}\left[\frac{f_T^{nc} + (\phi_T - f_T^{nc}) \times e^{d_L \times \tau}}{f_T^{nc} + (1 - f_T^{nc}) \times e^{-(d_S - d_L) \times \tau}}\right] \\
&= -d_L \times \frac{(\phi_T - f_T^{nc}) \times e^{d_L \times \tau}}{f_T^{nc} + (1 - f_T^{nc}) \times e^{-(d_S - d_L) \times \tau}} \\
&\quad -\phi_t \times (d_S - d_L) \times \frac{(1 - f_T^{nc}) \times e^{-(d_S - d_L) \times \tau}}{f_T^{nc} + (1 - f_T^{nc}) \times e^{-(d_S - d_L) \times \tau}} \\
&= -d_L \times \frac{-f_T^{nc} + f_T^{nc} + (\phi_T - f_T^{nc}) \times e^{d_L \times \tau}}{f_T^{nc} + (1 - f_T^{nc}) \times e^{-(d_S - d_L) \times \tau}} \\
&\quad -\phi_t \times (d_S - d_L) \times \frac{-f_T^{nc} + f_T^{nc} + (1 - f_T^{nc}) \times e^{-(d_S - d_L) \times \tau}}{f_T^{nc} + (1 - f_T^{nc}) \times e^{-(d_S - d_L) \times \tau}} \\
&= -d_L \times \phi_t - \phi_t \times (d_S - d_L) + f_T^{nc} \times \frac{d_L + \phi_t \times (d_S - d_L)}{f_T^{nc} + (1 - f_T^{nc}) \times e^{-(d_S - d_L) \times \tau}} \\
&= -\phi_t \times d_S + f_T^{nc} \times \frac{m(\phi_t)}{f_T^{nc} + (1 - f_T^{nc}) \times e^{-(d_S - d_L) \times \tau}}.
\end{aligned}
$$
(56)

Therefore, from $\phi'_t = -d_S \times \phi_t + m(\phi_t) \times f_t$,

$$
f_t^{nc} = \frac{d_S \times \phi_t + \phi'_t}{m(\phi_t)} = \frac{f_T^{nc}}{f_T^{nc} + (1 - f_T^{nc}) \times e^{-(d_S - d_L) \times \tau}}. \quad \blacksquare
$$

Two remarks about equation (55). (i) If $IC_t = 0$, the sign of $\frac{dT}{d\phi_t}$ is not relevant, $\frac{dT}{d\phi_t}$ factorizes out. This derivative $\frac{dT}{d\phi_t}$ can be computed from equation (30).[8] (ii) In the case of corner paths, which (if same type) are parallel and do not cross between each other, $\frac{dT}{d\phi_t} < 0$ and $IC_t \neq 0$ hold for both corner-shortening and -lengthening paths. For example, for a lengthening path, $f_t = 0$, $t \leq T$, implies $\phi_T = \phi_t \times e^{-d_S \times \tau}$ and

$$
\begin{aligned}
\frac{d\phi_t}{dT} &= \frac{d}{dT}\left[\phi^*(T) \times e^{d_S \times (T-t)}\right] = \frac{d\phi^*(T)}{dT} \times e^{d_S \times (T-t)} + d_S \times \phi^*(T) \times e^{d_S \times (T-t)} \\
&= \left(\frac{d\phi^*(T)}{dT} + d_S \times \phi^*(T)\right) \times e^{d_S \times (T-t)} < 0
\end{aligned}
$$

because paths are parallel and are below the default boundary—in the waiting region.

---

[8]For example, consider the default times T and $T + \Delta T$ are linked to a lengthening and hump-shaped path, respectively; hence, $\Delta T < 0$. For a sufficiently early time $t < T$, the two ϕ_t's paths will cross, which implies $\frac{\Delta T}{\Delta \phi_t}$ is positive. Yet it is less straightforward the result in the limit, $\lim_{\Delta T \to 0} \frac{\Delta T}{\Delta \phi_t}$.

That is, because of the following constraint, which says no issuance policy pulls the firm away from the default boundary,

$$\frac{d\phi^*(T)}{dT} < \phi'_T = -\phi_T d_S + m(\phi_T) f_T, \ f_T = 0.$$

Next, $f_T^{nc} \notin [0,1]$ implies $IC_t \neq 0$ if $f_t \in \{0,1\}$, $t \leq T$. For example, in the case of a lengthening path (i.e., $\phi_T = \phi_t \times e^{-d_S \times \tau}$), equation (29) reduces to

$$
\begin{aligned}
IC_t \quad &\sim \quad -(d_S - d_L) \times \left(-f_T^{nc} + \phi_t \times e^{-d_S \times \tau} \times f_T^{nc} + (1 - \phi_t) \times e^{-d_L \times \tau} \times f_T^{nc}\right) < 0 \\
&\Longleftrightarrow \quad f_T^{nc} \times \left(1 - \left(\phi_t \times e^{-d_S \times \tau} + (1 - \phi_t) \times e^{-d_L \times \tau}\right)\right) < 0 \Longleftrightarrow f_T^{nc} < 0,
\end{aligned}
$$

given that $\phi_t \in [0,1]$. It follows a corner-lengthening path is an equilibrium if $f_T^{nc} < 0$.

Proof of Proposition 5. The case $\phi'_t < 0$, $t \leq T$, and the case ϕ_t is hump-shaped, $t \leq T$, are clear. We prove the shortening equilibrium, $\phi'_t > 0$. We have

$$
\begin{aligned}
\frac{d\phi_t}{d\tau} &= -\frac{\partial}{\partial t}\left[\frac{f_T + (\phi_T - f_T) \times e^{d_L \times \tau}}{f_T + (1 - f_T) \times e^{-(d_S - d_L) \times \tau}}\right] \\
&= (\phi_T - f_T) \times e^{d_L \times \tau} \times \frac{d_L f_T + d_S (1 - f_T) \times e^{-(d_S - d_L) \times \tau}}{\left(f_T + (1 - f_T) \times e^{-(d_S - d_L) \times \tau}\right)^2} \\
&\quad + \frac{f_T \times (d_S - d_L) \times (1 - f_T) \times e^{-(d_S - d_L) \times \tau}}{\left(f_T + (1 - f_T) \times e^{-(d_S - d_L) \times \tau}\right)^2},
\end{aligned}
\tag{57}
$$

by using the second equality in equation (56) and the definition of ϕ_t in equation (30). Equation (57) implies $\phi'_T = 0 \Leftrightarrow \phi_T = \frac{d_L f_T}{d_L f_T + d_S (1 - f_T)}$ (from $\tau = 0$ and $\frac{d\phi_t}{d\tau} = 0$).

Next,

$$\frac{d\phi_t}{d\tau} - \left(\phi_T - \frac{d_L f_T}{d_L f_T + d_S (1 - f_T)}\right) \times e^{d_L \times \tau} \times \frac{d_L f_T + d_S (1 - f_T) \times e^{-(d_S - d_L) \times \tau}}{(f_T + (1 - f_T) \times e^{-(d_S - d_L) \times \tau})^2}$$

$$= -f_T \times \frac{(d_S - d_L) \times (1 - f_T)}{d_L f_T + d_S (1 - f_T)} \times e^{d_L \times \tau} \times \frac{d_L f_T + d_S (1 - f_T) \times e^{-(d_S - d_L) \times \tau}}{(f_T + (1 - f_T) \times e^{-(d_S - d_L) \times \tau})^2}$$

$$+ \frac{f_T \times (d_S - d_L) \times (1 - f_T) \times e^{-(d_S - d_L) \times \tau}}{(f_T + (1 - f_T) \times e^{-(d_S - d_L) \times \tau})^2}$$

$$\sim -f_T \times (d_S - d_L) \times (1 - f_T) \left(\frac{e^{d_L \times \tau} \times \left(d_L f_T + d_S (1 - f_T) \times e^{-(d_S - d_L) \times \tau}\right)}{d_L f_T + d_S (1 - f_T)} - e^{-(d_S - d_L) \times \tau}\right)$$

$$= -f_T \times (d_S - d_L) \times (1 - f_T)$$

$$\times \frac{d_L f_T \times \left(e^{d_L \times \tau} - e^{-(d_S - d_L) \times \tau}\right) + d_S (1 - f_T) \times e^{-(d_S - d_L) \times \tau} \times \left(e^{d_L \times \tau} - 1\right)}{d_L f_T + d_S (1 - f_T)} \leq 0.$$

It follows that

$$\frac{d\phi_t}{d\tau} < 0 \ \text{ if } \phi_T < \frac{d_L f_T}{d_L f_T + d_S (1 - f_T)}, \text{ for all } \tau \geq 0,$$

and $\phi'_t = -\frac{d\phi_t}{d\tau} > 0$. $\blacksquare$

10.1 Linear cash flows

We assume a linear declining cash-flow process, $y'(t) = -\mu < 0$ and $y''(t) = 0$, $t \geq 0$.
In addition, $R = r + \zeta$, $a_y \geq 1$ if $y(T) \leq 0$ (and $0 < a_y \leq 1$ if $y(T) \geq 0$), and
$0 \leq a_X \leq 1$. It follows that the bond-recovery value equals

$$
\begin{aligned}
1 - l(T) &= \int_0^{T_a - T} e^{-R \times u} \times (a_y y(T) - \mu \times u + a_X \zeta X) \times du \qquad (58) \\
&= (a_y y(T) + a_X \zeta X) \times \frac{1 - e^{-R \times (T_a - T)}}{R} \\
&\quad - \mu \times \left((T_a - T) \times \frac{-e^{-R \times (T_a - T)}}{R} + \frac{1 - e^{-R \times (T_a - T)}}{R^2}\right) \\
&= \frac{\mu}{R} \times \left((T_a - T) - \frac{1 - e^{-R \times (T_a - T)}}{R}\right).
\end{aligned}
$$

The abandonment time, T_a, is given by

$$a_y y\left(T\right) - \mu \times \left(T_a - T\right) + a_X \zeta X = 0 \iff T_a - T = \frac{a_y y\left(T\right) + a_X \zeta X}{\mu}.$$

In particular, $T_a = T$ if $a_y y\left(T\right) + a_X \zeta X \le 0$, and $T_a > T$ if $a_y y\left(T\right) + a_X \zeta X > 0$.

If $l\left(T\right) < 1$ (i.e., bond-recovery value is positive),

$$a_y y\left(T\right) + a_X \zeta X \quad > \quad 0,$$
$$a_y y\left(T\right) - \mu \times \left(T_a - T\right) + a_X \zeta X \quad = \quad 0 \quad \text{and} \quad \frac{d\left[T_a - T\right]}{dT} = -a_y.$$

Then
$$l'\left(T\right) = a_y \mu \times \frac{1 - e^{-R \times \left(T_a - T\right)}}{R} > 0$$

and
$$l''\left(T\right) = -a_y^2 \mu \times e^{-R \times \left(T_a - T\right)} < 0.$$

The constraints on the parameters a_y and a_x also imply that $l\left(T\right) > 0$. Next we show that a_y and a_x determine the types and fractions of equilibria as well.

10.1.1 The fraction of lengthening equilibria

Because the intersection point is given by $\phi^*\left(T\right) = f_T^{nc}$, we study the tradeoff between these two functions, $\phi^*\left(T\right)$ and f_T^{nc}. First,

$$\begin{aligned}
\frac{\phi^*\left(T\right) - f_T^{nc}}{l\left(T\right)^{-1}} &= \zeta X + y\left(T\right) - \left(l'\left(T\right) + \left(r + \zeta\right) \times \left(1 - l\left(T\right)\right)\right) \\
&= \zeta X + y\left(T\right) - \left(a_y \mu \times \frac{1 - e^{-R \times \left(T_a - T\right)}}{R} + \mu \times \left(\left(T_a - T\right) - \frac{1 - e^{-R \times \left(T_a - T\right)}}{R}\right)\right) \\
&= \left(1 - a_X\right) \zeta X - \left(y\left(T\right) + \mu \times \frac{1 - e^{-R \times \left(T_a - T\right)}}{R}\right) \times \left(a_y - 1\right).
\end{aligned}$$

For instance, $\phi^*\left(T\right) \ge f_T^{nc}$ if $a_y = a_X \le 1$ (if we assume $y\left(T\right) \ge 0$) and all equilibria

are lengthening. Another one-parameter scenario is

$$\frac{\phi^*(T) - f_T^{nc}}{l(T)^{-1}} = \left(\zeta X + y(T) + \mu \times \frac{1 - e^{-R\times(T_a - T)}}{R}\right) \times (1 - a_y) \quad \text{if } a_y = a_X,$$

and if $a_y > 1$ and $y(T) \leq 0$, the tradeoff can be both positive or negative (i.e., lengthening and nonlengthening paths, respectively, coexist).

Second, the slope of this function at the intersection point is given by

$$\left. \frac{d\left[\phi^*(T) - f_T^{nc}\right]}{dT} \right|_{\phi^*(T) = f_T^{nc}} = \mu\left(1 + a_y \times e^{-R\times(T_a - T)}\right) \times (a_y - 1) > 0 \quad \text{if } a_y > 1,$$

but this slope is negative if $0 \leq a_y < 1$.

Third, if $a_y \neq 1$, the intersection point (denoted by $T = T(\zeta X)$) is given by

$$\zeta X = \left(y(T) + \mu \times \frac{1 - e^{-R\times(T_a - T)}}{R}\right) \times \frac{a_y - 1}{1 - a_X}.$$

Computing $\frac{dT(\zeta X)}{d[\zeta X]}$,

$$1 = \left(-\mu \frac{dT(\zeta X)}{d[\zeta X]} + \left(-a_y\mu \frac{dT(\zeta X)}{d[\zeta X]} + a_X\right) e^{-R\times(T_a - T)}\right) \times \frac{a_y - 1}{1 - a_X}$$

and

$$\begin{aligned}
\frac{dT(\zeta X)}{d[\zeta X]} &= \frac{1}{\mu} \times \frac{-\frac{1 - a_X}{a_y - 1} + a_X e^{-R\times(T_a - T)}}{1 + a_y e^{-R\times(T_a - T)}} \\
&= \frac{1}{\mu} \times \frac{a_X \times \left(1 + (a_y - 1) e^{-R\times(T_a - T)}\right) - 1}{(a_y - 1) \times (1 + a_y e^{-R\times(T_a - T)})} \\
&= \frac{1}{\mu} \times \frac{e^{-R\times(T_a - T)}}{(1 + a_y e^{-R\times(T_a - T)})} > 0 \text{ if } a_X = 1.
\end{aligned}$$

The larger the upside-option expected payoff ζX, the later the default time $T(\zeta X)$. For example, $\frac{dT(\zeta X)}{d[\zeta X]} > 0$ if $a_y > 1$ and if $(a_y - 1) \times e^{-R\times(T_a - T)} > \frac{1 - a_X}{a_X}$, or $\frac{dT(\zeta X)}{d[\zeta X]} > 0$ if $a_y < 1$ and $a_X < 1$.

The case $a_X = 1$ and $a_y \neq 1$ The intersection point is given by

$$0 = y(T) + \mu \times \frac{1 - e^{-R \times (T_a - T)}}{R},$$

implying, as above,

$$
\begin{aligned}
0 &= -\mu \frac{dT(\zeta X)}{d[\zeta X]} + \left(-a_y \mu \frac{dT(\zeta X)}{d[\zeta X]} + 1 \right) e^{-R \times (T_a - T)}, \\
\frac{dT(\zeta X)}{d[\zeta X]} &= \frac{1}{\mu} \times \frac{e^{-R \times (T_a - T)}}{1 + a_y e^{-R \times (T_a - T)}} > 0.
\end{aligned}
$$

The fraction of lengthening equilibria solves $\phi^*(T) = f_T^{nc}$. That is,

$$
\begin{aligned}
f_T^{nc} &= \frac{1}{d_S - d_L} \times \left(\frac{l'(T)}{l(T)} - (R + d_L) \right) \\
&= \frac{1}{d_S - d_L} \times \left(\frac{a_y \mu \times \frac{1 - e^{-R \times (T_a - T)}}{R}}{1 - \frac{\mu}{R} \times \left((T_a - T) - \frac{1 - e^{-R \times (T_a - T)}}{R} \right)} - (R + d_L) \right) \\
&= \frac{-(R + d_L)}{d_S - d_L} + \frac{a_y \mu / R}{d_S - d_L} \times \frac{1 - e^{-R \times (T_a - T)}}{1 - \frac{\mu}{R} \times \left((T_a - T) - \frac{1 - e^{-R \times (T_a - T)}}{R} \right)},
\end{aligned}
$$

evaluated at the crossing point, where

$$
\begin{aligned}
T_a - T &= \frac{a_y y(T) + \zeta X}{\mu} \quad \text{and} \\
\frac{d[T_a - T]}{d[\zeta X]} &= -a_y \times \frac{dT(\zeta X)}{d[\zeta X]} + \frac{1}{\mu} = \frac{-a_y}{\mu} \times \frac{e^{-R \times (T_a - T)}}{1 + a_y e^{-R \times (T_a - T)}} + \frac{1}{\mu} \\
&= \frac{1}{\mu} \times \frac{1}{1 + a_y e^{-R \times (T_a - T)}} > 0.
\end{aligned}
$$

It follows

$$
\begin{aligned}
\frac{d\phi^*(T(\zeta X))}{d[\zeta X]} &= \frac{df_{T(\zeta X)}^{nc}}{d[\zeta X]} \\
&\sim \left(R \times e^{-R \times (T_a - T)} \times l(T) + \left(1 - e^{-R \times (T_a - T)} \right) \times \frac{\mu}{R} \times \left(1 - e^{-R \times (T_a - T)} \right) \right) \times \frac{d[T_a - T]}{d[\zeta X]} > 0,
\end{aligned}
$$

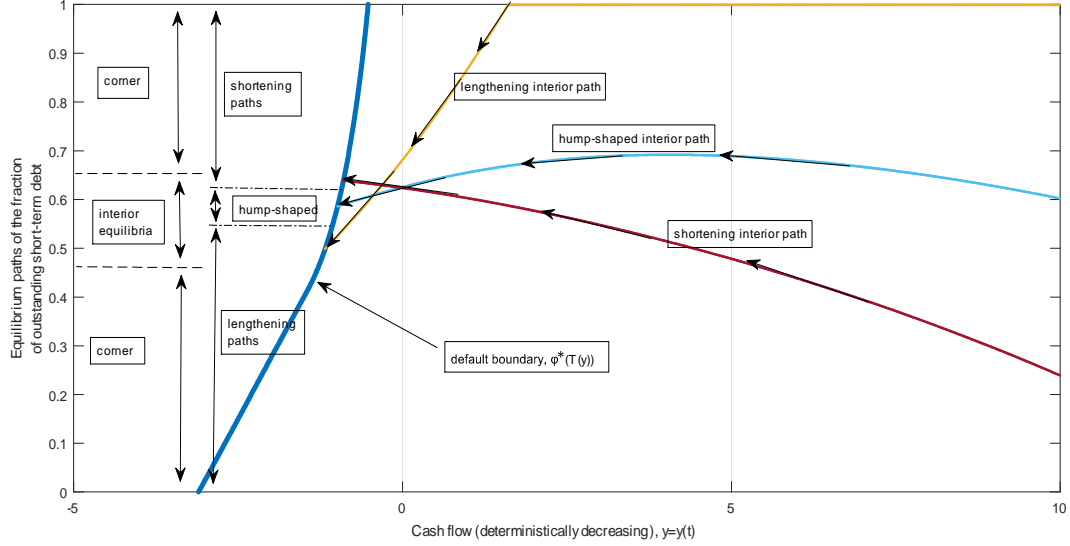which is equation (40), as we want to prove, and holds for all $a_y > 0$. ∎

56

Figure 1: Equilibrium paths of the fraction of outstanding short-term debt, ϕ_t, $t \leq T^*(\phi_{\mathcal{T}})$. We plot the optimal default boundary, $\phi^*(y)$. We then plot three interior paths: a lengthening, a hump-shaped, and a shortening one. Parameters are as follows (the same as in HM's figures): $c = 0.1$; $r = 0.1$; $X = 13$; $\mu = 13$; $\xi = 0.35$; $d_S = 5$; $d_L = 1$; $a_y = 3$; $a_X = 0.95$.
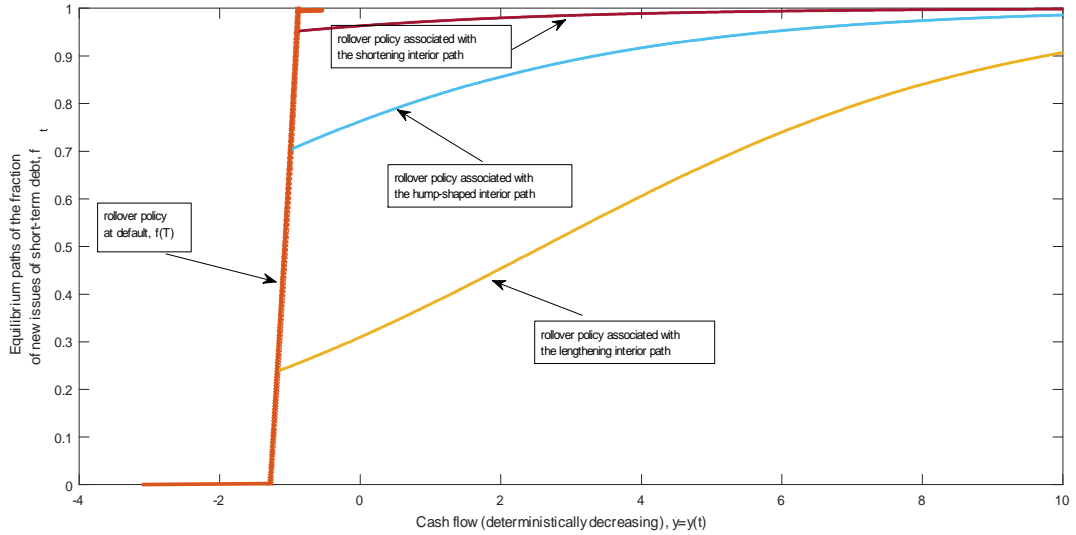

Figure 2: Equilibrium paths of the fraction of newly issued short-term debt, f_t, $t \leq T^*(\phi_{\mathcal{T}})$. We plot the equilibrium rollover policy at default, $f_T(y)$, and the three paths associated with the previous Figure 1. Parameters are as follows: $c = 0.1$; $r = 0.1$; $X = 13$; $\mu = 13$; $\xi = 0.35$; $d_S = 5$; $d_L = 1$; $a_y = 3$; $a_X = 0.95$.
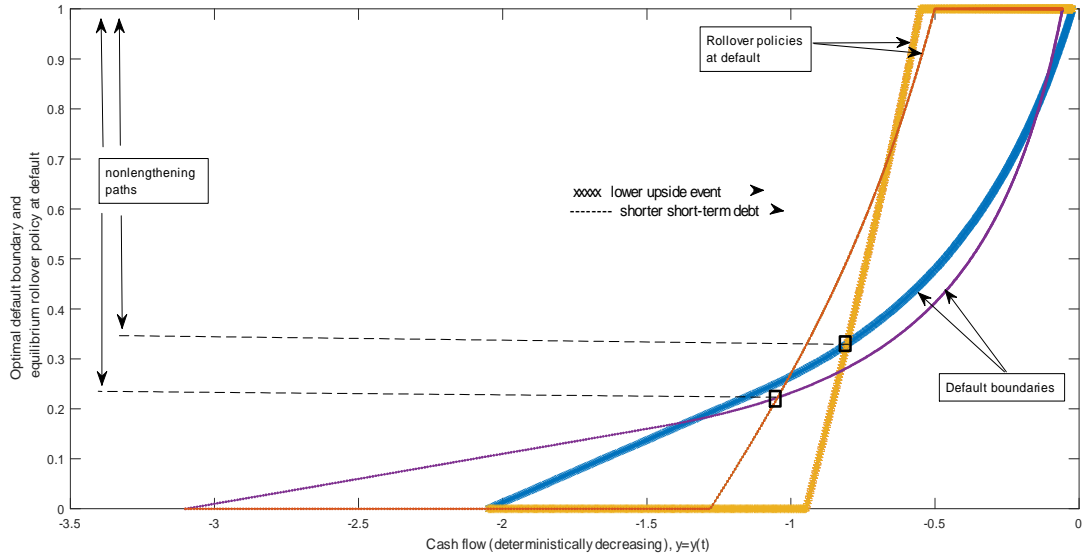
Figure 3: Optimal default boundary, $\phi^*(y)$, and equilibrium rollover policy at default, $f_T(y)$. Parameters are as follows: $c = 0.1$; $r = 0.1$; $X = 13$; $\mu = 13$; $\xi = 0.35$; $d_S = 5$; $d_L = 1$; $a_y = 3$; and $a_X = 0.95$. For the model of a lower upside event (the thick line), $X = 10$; for the model of a shorter short-term debt (the thin line), $d_S = 11$. A box signals the crossing points between $\phi^*(y)$ and $f_T(y)$.
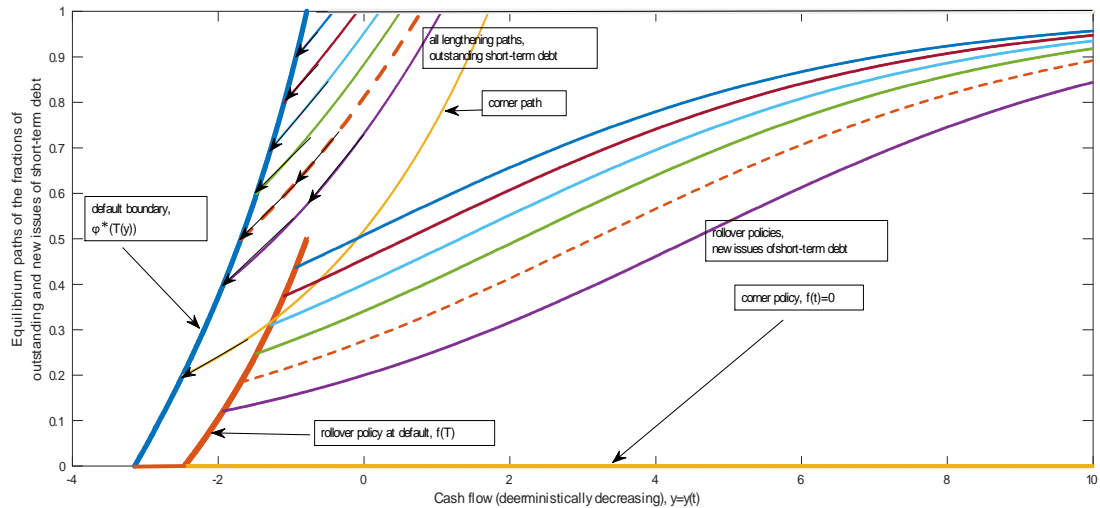

Figure 4: Equilibrium paths of the fractions of outstanding and new issues of short-term debt, ϕ_t and f_t, $t \leq T^*(\phi_{\mathcal{T}})$. We also plot the optimal default boundary, $\phi^*(y)$, and the optimal rollover policy at default, $f_T(y)$. Parameters are as follows: $c = 0.1$; $r = 0.1$; $X = 13$; $\mu = 13$; $\xi = 0.35$; $d_S = 5$; $d_L = 1$; and $a_y = 0.80$; $a_X = 0.80$. For the equilibrium path defaulting at $\phi_{\mathcal{T}} = 0.50$, we show the two paths (ϕ_t and f_t) in dash lines.