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GRADO EN INGENIERÍA EN TECNOLOGÍAS INDUSTRIALES

TRABAJO FIN DE GRADO

Optimal Headways for Flexible-Route Bus Services with Many-to-Many Demand Patterns

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GRADO EN INGENIERÍA EN TECNOLOGÍAS INDUSTRIALES

TRABAJO FIN DE GRADO

Optimal Headways for Flexible-Route Bus Services with Many-to-Many Demand Patterns

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OPTIMIZACIÓN DE FRECUENCIAS PARA RUTAS FLEXIBLES DE AUTOBUSES CON PATRONES DE DEMANDA VARIABLES

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RESUMEN DEL PROYECTO

Introducción

El transporte público juega un papel de gran importancia en la sociedad actual, ya que proporciona a sus usuarios una forma de viajar cómoda a la par que sostenible. No obstante, los servicios de autobús convencionales con rutas programadas pueden resultar costosas en ineficientes cuando se sirven áreas con baja demanda o con carreteras en malas condiciones. De esta manera, áreas suburbanas o de baja densidad poblacional podrían beneficiarse enormemente de servicios flexibles de autobuses en cuanto a tiempos de espera o en costes de viaje. Asimismo, el transporte a la demanda serviría a los medios rurales como medio de conexión con poblaciones mayores. Otra aplicación de los servicios flexibles de autobús sería su uso como transporte paratránsito, orientado a ciudadanos con discapacidades o de la tercera edad para facilitar sus desplazamientos mediante puntos de recogida y horarios adaptados a sus necesidades. Otra aportación del transporte a la demanda es la eliminación -o en su defecto la minimización- del fenómeno conocido como 'bus bunching' o amontonamiento de autobuses. Este acontecimiento tiene lugar cuando dos vehículos de transporte público coinciden en una misma parada porque uno de ellos no ha podido ajustarse a su horario, lo cual deriva en frustración por parte de los pasajeros e ineficiencias en el servicio.

Por todas estas razones, el objetivo de este estudio es determinar si los servicios flexibles de transporte público son realmente rentables y se adaptan con éxito a demandas variables en el tiempo y el espacio. Para ello, se proponen dos escenarios diferentes a los que se aplicará el método de resolución ideado para este proyecto. En primer lugar, se ha diseñado una red de transporte conformada por tres regiones con características similares conectadas a una estación central. En este caso, se estudia si programar frecuencias de autobuses comunes a las tres rutas resultaría más o menos costoso que implementar rutas no coordinadas, esto es, que cada región presente una frecuencia de servicio diferente en función de sus características. Esta frecuencia o 'headway' dependería de variables tales como la demanda, la distancia a la estación central o el tamaño del área de servicio. En segundo lugar, se plantea un escenario más complejo a través de una red formada por nueve regiones con características dispares a la que se aplica el mismo método de resolución que en el caso previo. De esta manera, se pretende comprobar si los resultados anteriores son concluyentes además de determinar si la formulación desarrollada es aplicable a situaciones de mayor complicación. Con todo, estos dos escenarios se muestran en las Figuras 1 y 2 en el siguiente apartado.

Metodología

El objetivo de este proyecto reside en minimizar el coste medio por pasajero mediante la optimización de la frecuencia con que los autobuses pasan por la estación central, es decir, obtener el valor de dicho parámetro que garantice el mínimo coste al viajero. Para conseguir esto, el problema se formula como una función de costes que engloba los costes de operador y de usuario -tanto los correspondientes al tiempo de viaje como al tiempo de espera. A continuación, la función objetivo se calcula dividiendo los costes totales entre el flujo total de pasajeros para obtener el coste medio por persona en cada región. En la formulación se tendrá en cuenta si la red es coordinada o no coordinada -frecuencias idénticas o ajustadas a cada ruta- para hacer las variaciones pertinentes.

De este modo, la función objetivo de costes se resuelve como un problema de optimización restringido no linear. Por medio del Solver de Excel, se aplicará el algoritmo de Gradiente Reducido Generalizado o algoritmo GRG como herramienta de resolución. Se hace uso de este método puesto que es reconocido por su eficiencia a la hora se resolver problemas de optimización no lineares con restricciones no lineares. Este algoritmo analizará la pendiente de la función a partir de unos valores iniciales introducidos y los irá variando hasta que el gradiente de la función alcance el valor cero, lo cual indicará que ha encontrado un mínimo local. Por esta razón, se debe incluir la condición necesaria que garantice la optimalidad global de la variable (i.e. frecuencia) para obtener el mínimo absoluto de la función en su lugar. Además, se incluirá un estudio gráfico de la convexidad de las funciones.

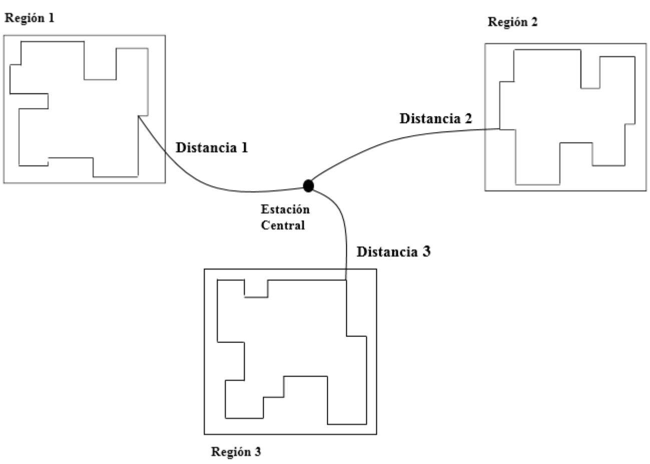

Figura 1. Servicio flexible de transporte para el Caso Práctico. [Fuente: Elaboración propia]

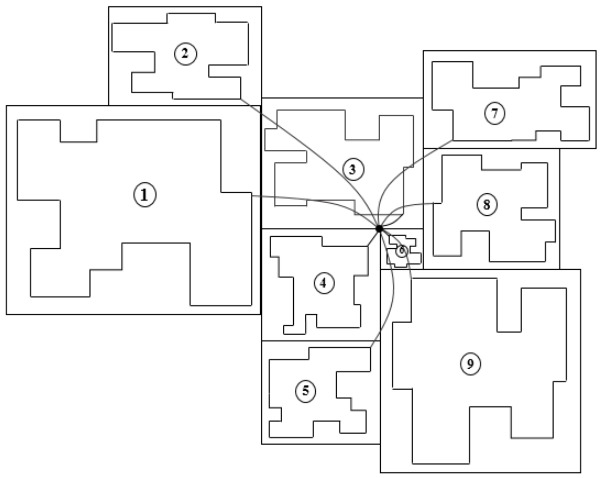

Figura 2. Servicio flexible de transporte para la Extensión del Caso Práctico. [Fuente: Elaboración propia]

Resultados

Caso Práctico: Patrones de demanda en una red de muchos a muchos

En este caso, los usuarios pueden viajar desde diferentes regiones a cualquier destino de la red. Para permitir los trasbordos de pasajeros, dicha red incluye una estación intermedia que conecta las tres rutas. Tras aplicar en algoritmo GRG pertinente a cada tipo de red (coordinada o no coordinada), se obtienen los resultados mostrados en las Tablas 1 y 2.

Debe tenerse en cuenta que la frecuencia del autobús se mantiene con la denominación en inglés (i.e. headway). Como se puede observar, los costes de la red coordinada son inferiores a los de la no coordinada, siendo por lo tanto la opción más rentable. Por un lado, los costes del operador y los de tiempo de espera superan a las del caso anterior. Los costes de operador son mayores puesto que el número de vehículos necesarios para satisfacer la demanda es significativamente superior -36 autobuses frente a 42. En el caso de los costes de espera, si no hay coordinación entre rutas los tiempos de espera en la estación central se disparan y con ello los correspondientes costes. Por otro lado, los costes medios de viaje se reducen puesto que la frecuencia media es menor, y por consiguiente el tiempo de ida y vuelta es menor también.

Tabla 1. Resultados del Case Práctico 1.

h* (horas)	**C_A ($/pasajero)**	**$C_{O,A}$ ($/pasajero)**	**$C_{V,A}$ ($/pasajero)**	**$C_{E,A}$ ($/pasajero)**	**Flota**
0.185	21.00	4.60	15.23	0.185	36

Tabla 2. Resultados del Case Práctico 2.

h_{AV}* (horas)	**C_A ($/pasajero)**	**$C_{O,A}$ ($/pasajero)**	**$C_{V,A}$ ($/pasajero)**	**$C_{E,A}$ ($/pasajero)**	**Flota**
0.156	22.07	5.27	14.82	1.97	42

Tras realizar un exhaustivo análisis numérico, se lleva a cabo un análisis de sensibilidad con el objetivo de comparar los dos tipos de redes propuestas al variar cuatro parámetros clave: demanda, distancia a estación central, y el valor que el usuario da al tiempo que emplea viajando y al tiempo de espera al realizar el transbordo. En este contexto, se comparan los costes por pasajero en cada tipo de red a través de los valores medios de las frecuencias obtenidas con la formulación del Caso Práctico. Con todo ello se aspira a determinar qué tipo de sistema es económicamente más rentable desde diferentes puntos de vista, además de averiguar qué parámetros son más decisivos a la hora de diseñar una red flexible de transporte.

Si estudiamos detenidamente los resultados y las gráficas obtenidas de la Figura 3, se concluye que las variables más importantes a considerar son la distancia que conecta cada región con la terminal central (D) y el valor del tiempo de viaje (v_V). En cuanto a los otros dos parámetros, se observa que las diferencias de costes son más pequeñas a medida que la demanda total es mayor, y que el valor del tiempo de espera tiene un impacto poco significativo en el coste medio por pasajero puesto que representa el porcentaje más bajo de los costes totales.

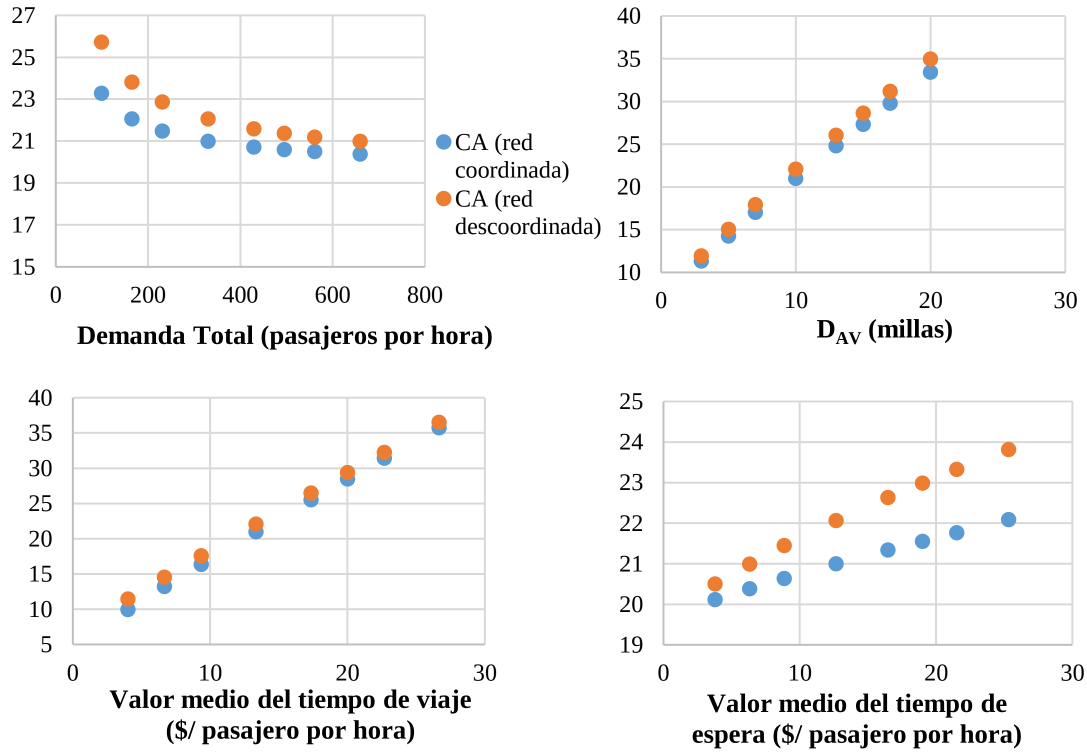

Figura 3. Análisis de sensibilidad para el Caso Práctico.
[Fuente: Elaboración propia]

Extensión del Caso Práctico

En el supuesto de una red conformada por nueve regiones diferentes conectadas a la estación central, los resultados confirman que un sistema con frecuencias comunes (red coordinada) resulta más rentable a la hora de diseñar una red flexible de transporte. La Tabla 3 presenta los resultados obtenidos al aplicar la formulación, siendo los superiores correspondientes al caso de red no coordinada y los inferiores a la red coordinada. En este escenario, a pesar de que las zonas presentan características muy dispares y los valores de frecuencias obtenidos están bastante alejados entre sí, se confirma que los resultados del Caso Práctico son realmente concluyentes y que por lo tanto los sistemas de transporte no coordinados son menos eficientes y más costosos.

Tabla 3. Resultados de la Extensión del Caso Práctico.

	Regiones								
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
h* (hrs)	0.171	0.443	0.268	0.284	0.326	0.221	0.332	0.546	0.547
	0.352	0.352	0.352	0.352	0.352	0.352	0.352	0.352	0.352
C_A^Z	24.57	30.79	12.98	14.68	20.34	7.95	22.36	26.36	29.04
	24.44	**29.23**	**11.12**	**12.40**	**17.73**	**6.41**	**20.09**	**25.71**	**28.21**
Diferencia de costes	0.54	5.35	16.67	18.37	14.69	23.97	11.28	2.54	2.92

Conclusiones

Los sistemas de transporte público flexibles proporcionan innumerables beneficios a los viajeros procedentes de regiones de baja demanda. Por una parte, el transporte a la demanda mejora la conexión entre zonas rurales, suburbanas y de baja densidad poblacional y urbes mayores, facilitando así el desplazamiento de sus habitantes en lugar de obstaculizarlo. Por otra parte, aporta comodidad y confort al servicio público de transporte. Al realizar este estudio, se concluye que el método de resolución desarrollado es aplicable a escenarios de mayor complejidad. Esto implica que el número de regiones puede aumentarse para conseguir una aproximación de los valores de las frecuencias y de los costes medios por pasajero en dicha red. Asimismo, se observa que tanto en el Caso Práctico como en su Extensión las redes de transporte coordinadas con frecuencias comunes supera a los sistemas no coordinados desde un punto de vista financiero. Por último, se cree que este proyecto podría ser mejorado si se considerara examinar con mayor detenimiento las siguientes sugerencias: adición de restricciones, uso de 'integer-ratios' (i.e. valores de frecuencias proporcionales) para aumentar la coordinación de los autobuses en la estación central y/o el establecimiento de estaciones de transbordo intermedias entre zonas.

OPTIMAL HEADWAYS FOR FLEXIBLE-ROUTE BUS SERVICES WITH MANY-TO-MANY DEMAND PATTERNS

Introduction

Public transportation plays a key role in social environment. When adequately developed and implemented, it provides users with a sustainable and handy means of transportation. However, conventional bus services traveling fixed routes become costly and inefficient when areas serviced present low demand densities or unfavorable geometric characteristics of roads. In this light, suburban or low-populated areas would largely benefit from accommodating flexible bus services by reducing waiting times and money to users and making in-time demand adjustments. Rural population, who are not able to commute to larger cities on account of mainly economic reasons, also fit into this category. Flexible-route services may also be useful as a paratransit service so as to facilitate transportation for disabled, handicapped and elderly passengers and meet their special mobility needs. Due to their accessibility difficulties to pre-determined bus stops, adaptable pick-up times and locations would improve their quality of life and well-being. Additionally, flexible-route bus services would prevent bus bunching caused by conventional bus systems, where two or more vehicles run in the same place at the same time when they are unable to stick to their fixed schedule. This situation leads to passenger frustration and a loss of confidence in public transportation, as well as an incurrence of additional waiting costs.

In this light, there is a great burden placed on public transportation along with a significant room for improvement in the area. On the one hand, flexible-route bus services can be developed as an alternative to fixed routes in order to reduce passengers' traveling costs and in-vehicle and waiting times, all the while maximizing flexibility and thus providing higher user satisfaction. On the other hand, flexible-route buses can also be an interesting and cheaper option to those passengers that use fully demand-responsive services such as taxis. From previous researches in this matter and the observation of current public transportation operations, we have come to the conclusion that there has not yet been an extensive implementation of flexible-route bus services nor a profound investigation on how headways impact a multi-zonal region bus network.

The ultimate aim of this paper is to reveal if there is enough evidence to support the feasibility of flexible-route bus services for many-to-many demand systems. In order to do so, we present two many-to-many demand scenarios in which we will apply our mathematical solution method. First of all, we minimize system-wide costs for two types of networks -coordinated and uncoordinated- in order to find which one would be more cost-effective as a flexible service option. The resulting system costs are contrasted by conducting a sensitivity analysis to several key parameters. Secondly, we propose a nine-region network to determine if the methodology developed would be applicable to situations of higher complexity, as well as to prove whether the outcomes found in the previous case are conclusive. Those two scenarios are shown in Figures 1 and 2 in the following section.

Methodology

The focus of the project is on minimizing the average passenger traveling cost by optimizing the headway of the bus routes, i.e., finding the best value of that variable to ensure maximum cost saving. In order to do this, the problem will be formulated as a cost function that will comprise supplier and user costs (in-vehicle and waiting costs). Hereafter, the objective function of the study will be calculated from total costs divided by the total passenger flow in order to find the average passenger cost of the network at issue. The type of network will depend on whether the headways of the bus routes are coordinated or not.

In this light, the objective cost function will be solved as a nonlinear constrained optimization problem. By means of Excel Solver, we will apply the Generalized Reduce Gradient (GRG) solving method. GRG method is proven to be a precise and accurate method for solving non-linear programming problem with non-linear constraints. It analyses the slope of the objective function starting from the initial input values and continues by changing them until the function's gradient reaches a zero value. Since the results obtained from this algorithm are locally optimal solutions, we must add the necessary condition restraints for the global optimality of the variable (i.e. headway) in order to obtain the absolute extremum solutions instead. Additionally, a graphical representation of the convexity of the functions would be provided.

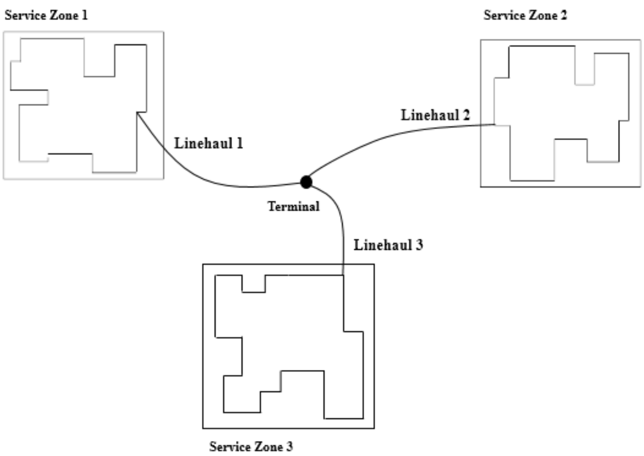

Figure 1. Flexible-routes Bus Service for Case Study.
[Source: prepared by the author]

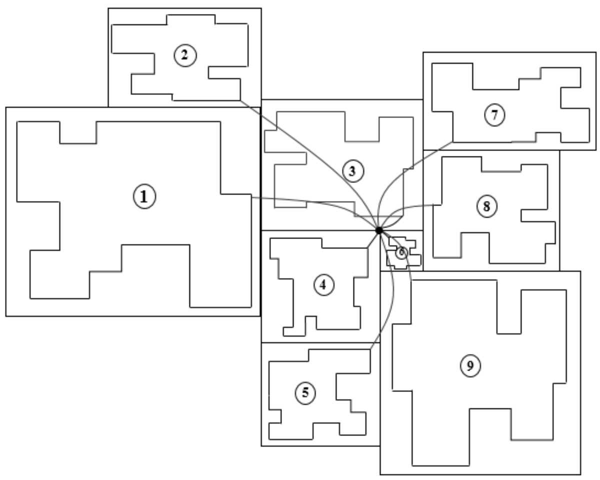

Figure 2. Flexible-routes Bus Service for the Extension of Case Study.
[Source: prepared by the author]

Results

Case Study: Many-to-Many demand scenario

After applying the GRG algorithm to the two types of networks, results found are shown in Tables 1 and 2. As it is noted, a system with common headways (coordinated network) is less costly than adjusting bus frequencies to each region's characteristics (uncoordinated network). On the one hand, both supplier and waiting costs are higher than in the previous scenario. Supplier costs rise because the number of vehicles required to service the regions increases significantly. For waiting costs, as transfers are not coordinated at the central terminal, commuters must wait to pick up the next bus to arrive at their final destination and thus boosting costs. On the other hand, average in-vehicle costs decrease for uncoordinated systems because the average value of headway is lower, causing the average round trip time to decrease as well.

Table 1. Results for Case Study 1.

h* (hours)	C_A ($/passenger)	$C_{S,A}$ ($/passenger)	$C_{V,A}$ ($/passenger)	$C_{W,A}$ ($/passenger)	Fleet size
0.185	21.00	4.60	15.23	0.185	36

Table 2. Results for Case Study 2.

h_{AV}* (hours)	C_A ($/passenger)	$C_{S,A}$ ($/passenger)	$C_{V,A}$ ($/passenger)	$C_{W,A}$ ($/passenger)	Fleet size
0.156	22.07	5.27	14.82	1.97	42

After performing a numerical discussion, we conduct a sensitivity analysis to compare the two types of scenarios when varying four key parameters of the regions: demand, line-haul distance, value of in-vehicle time and value of waiting time. We will compare the average costs per traveler in the network by using the optimal value of headway found in Case Study 1 and the average optimal value of headway found in Case Study 2. By doing this, we seek to determine from different fronts which type of network is less costly and hence more feasible, as well as which variables are important to consider when designing this type of networks. Results found are shown in Figure 3.

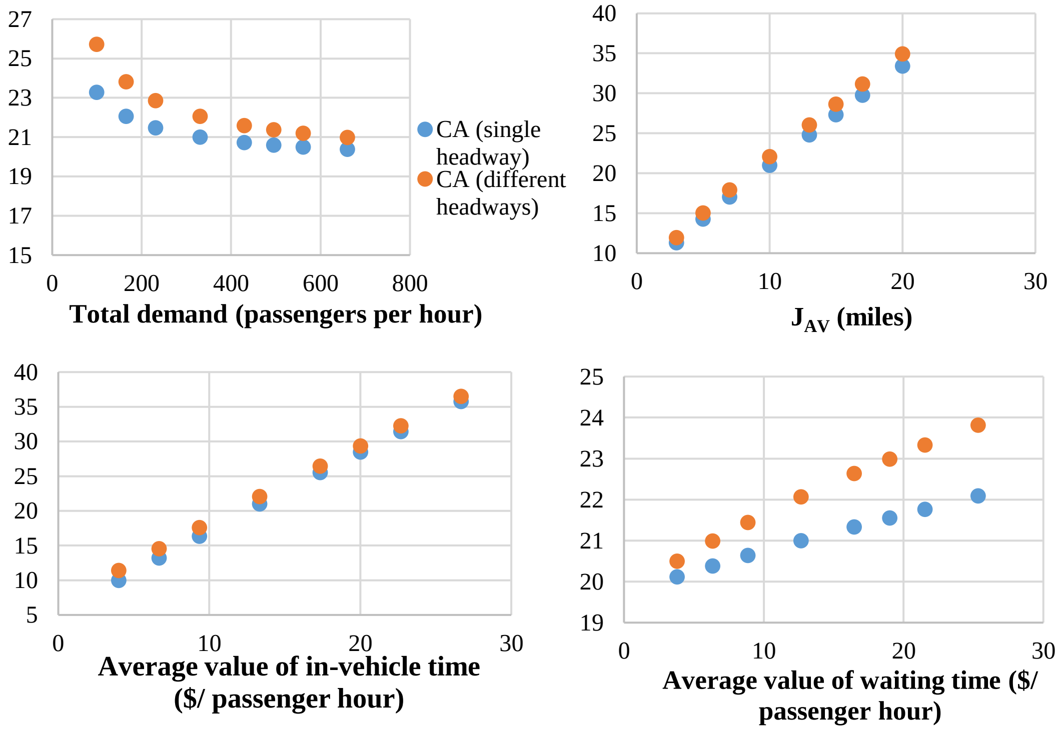

Figure 3. Sensitivity analysis for Case Study. [Source: prepared by the author]

Taking a close look to the graphs and the numerical results, we have noted that the key parameters to take into consideration are the line haul distance and the value of in-vehicle time. Regarding the other two variables, we have found that differences in costs are higher for lower values of demand thus favoring the common headways users, and that the value of waiting time has a rather low impact on the average cost per passenger.

Extension of Case Study

For the nine-region scenario, results found in Table 3 confirm that a common headway network is the most cost-efficient option when designing a flexible-route bus system. It should be noted that the top results correspond to the uncoordinated system and the bottom results to the uncoordinated system. In this case, even though the network proposed was comprised of regions with extremely disparate characteristics and the headways obtained differ greatly, the final conclusion is that uncoordinated headways are more expensive due to the amount of buses needed and the rise in waiting costs.

Table 3. Results for Extension of Case Study.

	Regions								
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
h* (hrs)	0.171	0.443	0.268	0.284	0.326	0.221	0.332	0.546	0.547
	0.352	0.352	0.352	0.352	0.352	0.352	0.352	0.352	0.352
C_A^Z	24.57	30.79	12.98	14.68	20.34	7.95	22.36	26.36	29.04
	24.44	**29.23**	**11.12**	**12.40**	**17.73**	**6.41**	**20.09**	**25.71**	**28.21**
Cost difference	0.54	5.35	16.67	18.37	14.69	23.97	11.28	2.54	2.92

Conclusions

Flexible-route bus systems provide travelers from low demand density regions with countless benefits. On the one hand, helps connecting rural regions, suburban areas and small town to larger cities, facilitating the commuting of their inhabitants rather than hampering it. On the other hand, it brings convenience and comfort to the service. By conducting this study, we have found that the solution method developed in this paper can be applied to scenarios of higher levels of complexity, meaning that the number of regions can be increased in order to achieve an approximation for the values of the headways and the average costs per passenger. Additionally, we have come to the conclusion -in both the Case Study and its Extension- that coordinated networks with common headways outperform the uncoordinated systems from a financial standpoint. Lastly, we believe that this work could see some improvement if the following suggestions are examined more in depth: addition of constraints, use of integer-ratios to increase coordination at the central terminal or the implementation of intermediate transfer stations.

MEMORIA

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Introduction

Public transportation plays a key role in social environment. When adequately developed and implemented, it provides users with a sustainable and handy means of transportation. However, conventional bus services traveling fixed routes become costly and inefficient when areas serviced present low demand densities or unfavorable geometric characteristics of roads. In this light, suburban or low-populated areas would largely benefit from accommodating flexible bus services by reducing waiting times and money to users and making in-time demand adjustments. Rural population, who are not able to commute to larger cities on account of mainly economic reasons, also fit into this category. Flexible-route services may also be useful as a paratransit service to facilitate transportation for disabled, handicapped and elderly passengers and meet their special mobility needs. Due to their accessibility difficulties to pre-determined bus stops, adaptable pick-up times and locations would improve their quality of life and well-being. Additionally, flexible-route bus services would prevent bus bunching caused by conventional bus systems, where two or more vehicles run in the same place at the same time when they are unable to stick to their fixed schedule. This situation leads to passenger frustration and a loss of confidence in public transportation, as well as an incurrence of additional waiting costs. In this light, there is a great burden placed on public transportation along with a significant room for improvement in the area. On the one hand, flexible-route bus services can be developed as an alternative to fixed routes in order to reduce passengers' traveling costs and in-vehicle and waiting times, all the while maximizing flexibility and thus providing higher user satisfaction. On the other hand, flexible-route buses can also be an interesting and cheaper option to those passengers that use fully demand-responsive services such as taxis.

The ultimate aim of this paper is to minimize system-wide costs for many-to-many demand patterns by optimizing the headway of both a coordinated and an uncoordinated network designed for this study, in order to determine which system is less costly. The starting point of the research is a many-to-one scenario -a local service zone connected to major central terminal- to showcase the basic formulation required to develop a flexible-route bus system. On the basis of this insight, the study culminates with the development of a three-area system connected to a CBD (or central business district) with disparate demand densities followed by a comparison of the optimal headways and costs found. Ultimately, the resulting system costs are contrasted by conducting a sensitivity analysis to several key parameters.

All in all, this paper aims to reveal if there is enough evidence to support the feasibility of flexible-route bus services for many-to-many demand systems. Hence, the formulation developed for the study is also suitable for networks with higher number of regions (higher than three). For this reason, the study closes with a nine-area scenario -of higher complexity than the previous ones- to put into practice the methodology used and determine if the results found are conclusive. In view of the results, some suggestions will be made to improve the network's performance.

The structure of this study is as follows. Section 2 is reserved for bibliographic review to help contextualize our study and to analyze the current state of the art in this matter. Section 3 is dedicated to formulating the mathematical approach that will be followed to solve the scenarios posed. In Section 4, we perform a numerical analysis of the results along with a sensitivity study to several key parameters. This section also includes the outcomes obtained from the complex practical example proposed. Lastly, Section 5 provides conclusions, and Sections 6 and 7 are dedicated to References and Appendixes.

Literature Review

At present, fixed bus services are still the norm in every transit network. This is so because if route planning does not depend on demand fluctuations and zone characteristics, bus itineraries and passenger transfers are easier to design and coordinate. On the one hand, sticking to conventional public transportation minimizes risk, as well as money expenditure on field study to analyze demand patterns and traveler habits within the region concerned. On the other hand, society can largely benefit from flexible-route bus systems by having a demand-reactive service that adjusts to passengers' traveling needs in a timely manner. For that reason, flexible transportation services have been explored in-depth by other authors and have even been implemented in practice, as the following literature reveals.

There are many studies that have investigated the benefits of flexible services. Koffman (2004) conducted a thorough research on bus services that were not fully demand responsive nor fixed, and broke those services down into several categories. The kind of system that would be analyzed in this paper somewhat falls into the category of Demand-responsive connector, in which buses travel within zone in a demand-responsive form, with one or more transfer stations that connect to other routes -presumably fixed routes. According to this study, a total of 25% transit systems of the sample used this mode of operation. A similar study performed later by the National Academies of Sciences, Engineering, and Medicine (2010) showed that Demand-responsive connector services were operated by 24.2% of the respondents of the survey. It should be noted that in the latter case the sample was significantly higher and thus more representative. This study also stated that the principal users of flexible transportation were senior citizens (29%) and persons with disabilities (27%), and both studies agreed that agencies operating this type of service targeted rural areas, small towns and suburban regions (low-density areas).

An important parameter strongly relevant for transportation matters is distance decay. This term is translated in passengers' willingness to travel a certain distance to reach their final destination. Iacono, Krizek, & El-Geneidy (2008) provide a high set of distance decay functions that show how users are less likely to use a service when that distance is greater. Thus, this fact supports the idea that door-to-door services would gain popularity as this distance reduced.

Many researches consider the variability of passenger flow and even develop models to predict changes in demand. Fan & Machemehl (2008) applied a tabu search approach as the optimization method to design a public transportation network with variable demand. Along these lines, the algorithm proposed by Pacheco, Álvarez, Casado, & González-Velarde (2009) for route design and bus assignment in the city of Burgos showed robust results when applied to a flexible demand scenario. Zhang et al. (2018) integrate smart cards and GPS tracing systems to formulate a model to forecast demand and implement flexible transit lines during special periods. A practical case study was conducted, and results showed that the lines designed were effective and presented high attendance.

There are also many authors that compare fixed and flexible networks in terms of costs, demand patterns, zone features, vehicle types and customer satisfaction. Many of these studies concur with the fact that flexible-route bus services are cost-effective and perform better in low densely populated areas with low levels of demand (Chang & Schonfeld, 1991), (Kim & Schonfeld, 2012), (Kim & Schonfeld, 2013), an idea that serves as a baseline to several publications later. Kim, Schonfeld, & Kim (2018) integrate conventional and flexible bus services by jointly optimizing vehicle-size, route spacing, service area and headways to a multi-region scenario over multiple time periods. This paper presents key improvements to (Kim & Schonfeld, 2012), as it takes into consideration a larger number of zones and different vehicle types, as well as to (Kim & Schonfeld, 2013) by adding complexity to the problem. The study performed by Nourbakhsh & Ouyang (2012) demonstrates how flexible services incurred lower costs compared to other transportation options (taxis and fixed-scheduled services) under low-to-moderate demand needs. The research also proved that these services provide users with increased safety (e.g. at night or during off-peak hours) and reduced discomfort (e.g. inclement weather), along with cutting the need for some infrastructure (e.g. bus stops or bus lines). Quadrifoglio, Hall, & Dessouky (2006) determine that system-wide costs can be minimized by merging the strengths of both types of networks (fixed and flexible) in low demand areas. By doing this, commuters would benefit from the flexibility and convenience of demand responsive transit systems (DRT) combined with the low cost provided by conventional services.

In addition to the examples in the United States, there has also been a minor on-demand-transport development in some countries of the EU that connects depopulated and ageing rural areas to higher densely populated areas. Delgado Urrecho & Martínez Fernández (2016) show that some examples in various Spanish Communities are the living proof of the likelihood of success of this project given the right region conditions and inhabitants' necessities.

Transfer station locations and coordination of different transportation services have also been profoundly examined as a way of improving passenger commutes and cutting down waiting times. Ting & Schonfeld (2005) determine that integer-ratio headways are the most efficient way to coordinate a multiple hub network when demand or distances differ greatly among routes -as opposed to using common headways. They use a heuristic algorithm to jointly optimize headways and slack times of the routes, and they state that the lower the demand, the higher the effectiveness of an integer ratio approach. Following this reasoning, Tuzun Aksu & Akyol (2014) devise a genetic algorithm aimed to reduce passengers' transfer times by means of a clustering strategy that incorporates the integer-ratio concept. Kim & Schonfeld (2014) support headway coordination in a multiple region system (either common headways or integer-ratios) when demand varies considerably over time and space. In this study, the authors propose merging fixed and flexible services in order to adjust each of them to the characteristics or the regions served -and thus meeting the users' particular needs. Lastly, Wu et al. (2019) incorporate the concept of demand assignment and rerouting as a way of improving coordination, which ultimately lead to the reduction of user costs. To do this, they develop a bi-level programming model which first minimizes system-wide costs and then analyzes the passenger route choice behavior in case of a missed connection.

Regarding the bus bunching phenomenon, we have found that most publications consulted are optimized for fixed-route bus networks and aimed to ensure that buses stay on schedule. Ibarra-Rojas & Rios-Solis (2012) focus on accurate timetable generation as a way of minimizing it. Daganzo (2009), Daganzo & Pilachowski (2011) and Argote-Cabanero, Daganzo, & Lynn (2015) develop adaptative headway-based or scheduled-based control schemes to help driver respond to unexpected disruptions when traveling their routes. According to some of these studies, mitigating bus bunching is more challenging in fixed bus services because they must adjust to specific timetables.

In the United States, one area that has seen some progress in developing adaptable networks is the paratransit realm. As a result of the Americans with Disabilities Act of

1990 (ADA), public transportation agencies must not discriminate against individuals with physical or mental impairment by providing an efficient door-to-door service (U.S. Department of Justice, 2005). Because of this, ensuring flexible public transit systems for disabled and elderly passengers has become a major requirement for public transportation organizations. Attanasio et al. (2004) state that due to the ageing of the population and the focus on cost reduction on western societies, there is an increasing need for improvement in the Dial-a-ride services. These systems are currently at the service of elderly and handicapped citizens, and can be divided into two modes: static, in which requests are known in advance, and dynamic, which incorporates real-time requests in the best way possible satisfying operational constraints. Results found show that using parallel computing is an effective technique to solve immediate bus routing problems. Similarly, Fraga Neto & Alcântara Cardoso (2017) address the weelchair transportation issue as a dynamic Dial-a-ride problem and test their algorithm in a real transportation network in the region of Grande Vitória (ES, Brasil). The outcome of this study showed how 99.05% of the applicants were accommodated by processing the new requests in the system with a mean response time of five seconds per request.

From previous researches in this matter and the observation of current public transportation operations, we have come to the conclusion that there has not yet been an extensive implementation of flexible-route bus services nor a profound investigation on how headways impact a multi-zonal region bus network. With that in mind, we proceed to analyze the feasibility of such model here in this paper.

Flexible-route Service Formulation and Methodology

To enable a better comprehension of more complex scenarios, the Base Case of this study analyses a single service zone connected to a central terminal (many-to-one demand pattern). Users in this type of networks travel from their local area to a main central terminal and vice versa, as shown in Figure 1. Basing on this exemplification, this paper extends the scope to a multi-zone system that connects several zones to a central station and allows passengers to travel within the network from one area to another indistinctly. Depending on whether or not there is coordination of headways, this research will explore the possibility of arranging the values of the results in order to minimize transfer times – and thus waiting costs- at the central terminal. The problem designed to reflect this many-to-many demand pattern is represented in Figure 2 as a three-area system with converging routes at a central station. Nevertheless, the formulation developed for this example is applicable to larger systems with larger number of zones or even for subdivisions of the original areas proposed -depending on the result of the sensitivity analysis that will be performed later for zone sizes.

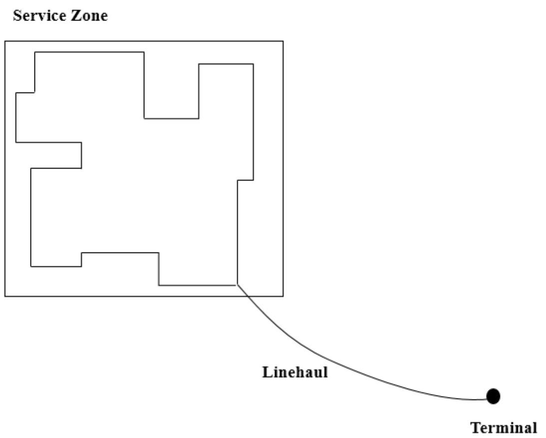

Figure 1. Flexible-route Bus Service for many-to-one demand patterns. [Source: prepared by the author]

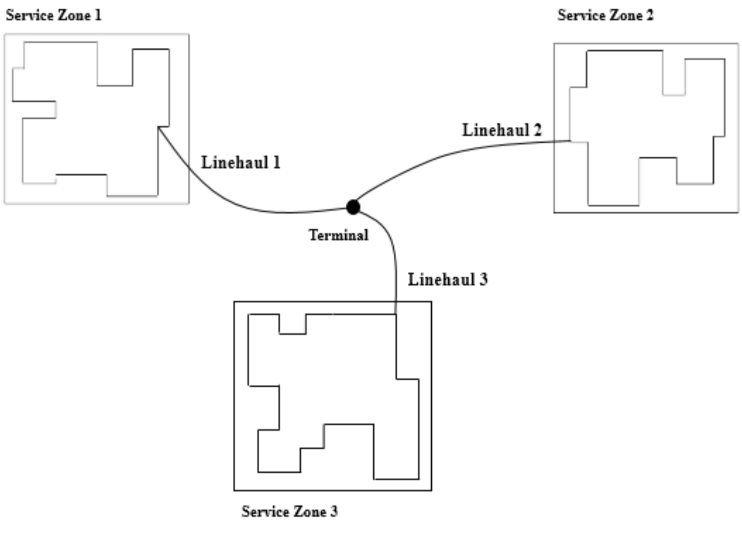

Figure 2. Flexible-route Bus Service for Case Study. [Source: prepared by the author]

Prior to cost formulation, some assumptions must be made in order to provide a relatively simple and meaningful model. These assumptions are adapted for consistency and based on previous studies that addressed relevant issues to this paper. Kim & Schonfeld (2012) performed a thorough analysis on one route services (as such in our base case), which was later modified to take multiple zones into account Kim & Schonfeld (2013):

1. The demand is fixed with service quality and price.
2. Destinations and origins are randomly and uniformly distributed over time and space within each region.
3. Service zones are fairly complex and fairly convex.
4. Stein's (1978) formula is assumed to provide an acceptable approximation of the length of the tour within each zone, in which $D \approx \emptyset\sqrt{nA}$ and $\emptyset = 1.15$ for rectilinear movements (Daganzo, 1984).
5. The number of stopping points per zone for each vehicle is at least five.
6. Passenger pickups and drop-offs are intermingled within each tour.
7. Within each local region, the average speed includes stopping times and dwell times.
8. The estimated average waiting time is approximately half of the headway.
9. The estimated average passenger's travel time is assumed to be half of the round travel time.
10. Bus layover time is negligible.
11. External costs are assumed to be negligible.

The cost function for flexible-route bus network comprises operator costs (supplier´s costs) and user costs. On the one hand, bus supplier costs include costs of links, terminals, vehicles and control systems. On the other hand, user's costs entail: in-vehicle costs, calculated on the basis of the total traveling time of the passenger from origin to destination; waiting costs, resulting from total time spent at the bus stop and possibly at the transfer station; and access costs, which are negligible as the model is meant to provide a personalized door-to-door service where passengers are picked up and dropped off at a desired location. Lastly, external costs such as the estimated costs of noise, pollution and other external costs are not taken into consideration in this study.

In this light, the objective cost function will be solved as a nonlinear constrained optimization problem. By means of Excel Solver, we will apply the Generalized Reduced Gradient (GRG) solving method. GRG method is proven to be a precise and accurate method for solving Non-linear Programming problem with non-linear constraints (Chen, Kang, & Lee, 2004). It analyses the slope of the objective function starting from the initial input values and continues by changing them until the function's gradient reaches a zero value. Since the result obtained from this algorithm is a locally optimal solution, we must add the necessary condition restraint for the global optimality of the variable (i.e. headway) in order to obtain the absolute extremum solution instead.

Hereafter, we proceed to develop the formulation for the two cases analyzed in the paper.

Base Case: Many-to-one scenario

Baseline values and notation for a sole area connected to a terminal are provided in Table 1 and can also be found in (Kim, Levy, & Shonfeld, 2018). A detailed explanation of the mathematical expressions for each cost are described there as well.

Total costs for this network comprise supplier, in-vehicle and waiting costs:

$$C_T = C_S + C_V + C_W \tag{1}$$

Supplier, in-vehicle and waiting costs can be formulated as follows:

$$C_S = c \cdot \frac{R}{h} = \frac{2Jc}{hV_X} + \frac{\emptyset c}{yV_x}\sqrt{\frac{qA}{uh}} \tag{2}$$

$$C_V = qv_V\frac{R}{2} = \frac{qJv_v}{V_X} + \frac{\emptyset v_v}{2yV_X}\sqrt{\frac{q^3hA}{u}} \tag{3}$$

$$C_W = qv_W\frac{h}{2} = \frac{qv_Wh}{2} \tag{4}$$

By substituting these expressions in (1), total costs result in:

$$C_T = \frac{2Jc}{hV_X} + \frac{\emptyset c}{yV_x}\sqrt{\frac{qA}{uh}} + \frac{qJv_v}{V_X} + \frac{\emptyset v_v}{2yV_X}\sqrt{\frac{q^3hA}{u}} + \frac{qv_Wh}{2} \tag{5}$$

In order to have a clearer understanding, we would analyze the costs incurred by each passenger traveling within the network. The average cost per passenger is found from dividing total costs (C_T) by the demand (q):

$$C_A = \frac{2Jc}{hV_XQA} + \frac{\emptyset c}{yV_x}\sqrt{\frac{1}{uhQ}} + \frac{Jv_v}{V_X} + \frac{\emptyset v_vA}{2yV_X}\sqrt{\frac{Qh}{u}} + \frac{v_Wh}{2} \tag{6}$$

Equation (6) is the objective function of the base case. The ultimate purpose is to minimize the average cost per passenger in the network by finding the optimal value of the headway. By means of Excel Solver and applying the GRG algorithm, we find just the local extremum of the function. Therefore, in order to obtain the global optimal solution, the second derivative of C_A respect to h must have a positive value for the entire domain of the function (necessary condition for convexity). As the function is strictly convex, any local minimum is also the global minimum. We add the following constraint to the algorithm to ensure the condition of convexity is satisfied:

$$\frac{\partial^2 C_A}{\partial h^2} = \frac{4Jc}{h^3 V_X QA} - \frac{\emptyset c}{2yV_x\sqrt{Quh^5}} - \frac{\emptyset v_v A\sqrt{Q}}{8yV_X\sqrt{uh^3}} > 0 \tag{14}$$

We will analyze the convexity of the function later on this paper.

Another condition that must be considered is that the optimal value of the headway (h*) must be smaller than the maximum allowable headway for the system:

$$h_{MAX} = \frac{Sl}{QA} \tag{15}$$

or else its value will be that of h_{MAX}:

$$h^* = min\{h_{MAX}, h_{GRG}\} \tag{16}$$

Lastly, depending on the value of the headway, we will be able to determine the amount of buses needed (fleet size) to serve the network (Kim, Levy, & Shonfeld, 2018):

$$N = \frac{R}{h} = \frac{2J}{hV_X} + \frac{\emptyset}{yV_x}\sqrt{\frac{qA}{uh}} \tag{17}$$

Table 1. Notation for Base Case.

Symbol	**Variable**	**Units**	**Base Value**	**Range for Sensitivity Analysis**
A	Zone Size (Area)	Square miles	4	-
a	Parameter for bus operating cost	$ per bus hour	30	-
b	Parameter for bus operating cost	$ per seat hour	0.3	-
c	Unit Bus Operating Cost (=a+b*S)	$ per bus hour	-	-
C_A	Average Cost	$ per passenger	-	-
C_T	Total Cost	$ per hour	-	-
C_V	Supplier Cost	$ per hour	-	-
C_W	Waiting Cost	$ per hour	-	-
D_C	Tour Length within Zone	Kilometers	-	-
h	Headway	Hours	-	-
$h*$	Optimal Headway	Hours	-	-
J	Line Haul Distance	Miles	10	-
l	Load factor	Dimensionless	1.0	-
$\emptyset$	Stein's Constant	Dimensionless	1.15	-
Q	Demand Density	$\frac{Trips}{SquareMiles * Hour}$	10	-
q[1]	Demand	Trips per Hour	-	-
S	Bus Capacity	Seats per Bus	45	-
u	Number of passengers per Stop	Number of Passengers	1.6	-
V_L	Average Local Speed	Miles per Hour	-	-

[1] passengers traveling inbound

Symbol	Variable	Units	Base Value	Range for Sensitivity Analysis
v_V	Value of in-vehicle time	\$ per passenger Hour	12	-
v_W	Value of waiting time	\$ per passenger hour	15	-
V_X	Line Haul Speed	Miles per Hour	30	-
y	Ratio of local speed to express speed	Dimensionless	0.9	-
Z	Number of Zones	Dimensionless	1	-

Case Study: Many-to-many demand scenario

The main goal of this study is to analyze many-to-many demand patterns in which several zones are connected to each other and converge in a central station. Passengers can hop on the bus at any point of the system and travel with no restriction to any other destination of the network. It should be noted that, in the long run, the number of commuters traveling from an area must equal the number of commuters going to that area, as passengers will probably end up return to their place of departure. For this reason, we designate the total number of users leaving area Z as q_Z^{IN} and the total number traveling back to area Z as q_Z^{OUT}.

Baseline values for demand and the characteristics of each region Z are shown in Table 2 and Table 3 respectively.

Table 2. Demand values for Case Study.

	Area 1	**Area 2**	**Area 3**	**Central station**
Area 1	0	70	30	10
Area 2	80	0	30	20
Area 3	20	40	0	30
Central station	10	20	30	0

Table 3. Notation for Case Study.

Symbol	Variable	Units	Base Value (A_1)	Base Value (A_2)	Base Value (A_3)	Range for Sensitivity Analysis
a^Z	Parameter for bus operating cost per region	$ per bus hour	30	30	30	-
A^Z	Zone Size (Area)	Square miles	4	5	6	-
b^Z	Parameter for bus operating cost per region	$ per seat hour	0.3	0.3	0.3	-
C_A	Average Cost in the network	$ per passenger	-	-	-	-
C_A^Z	Average Cost per region	$ per passenger	-	-	-	-
C_S^Z	Supplier Cost per region	$ per hour	-	-	-	-
C_T	Total Cost	$ per hour	-	-	-	-
C_T^Z	Total Cost per region	$ per hour	-	-	-	-
C_V^Z	In-vehicle Cost per region	$ per hour	-	-	-	-
C_W^Z	Waiting Cost per region	$ per hour	-	-	-	-
c^Z	Unit Bus Operating Cost per region	$ per bus hour	-	-	-	-
h	Headway[2]	Hours	-	-	-	-
$h*$	Optimal Headway	Hours	-	-	-	-
h^Z	Headway for each region[3]	Hours	-	-	-	-
J^Z	Line Haul Distance of region Z	Miles	10	12	8	3-20

[2] Value of headway for Case Study 1
[3] Values of headways for Case Study 2

Symbol	Variable	Units	Base Value (A_1)	Base Value (A_2)	Base Value (A_3)	Range for Sensitivity Analysis
l^Z	Load factor per region	Dimensionless	1.0	0.7	1.3	-
N^Z	Fleet Size per region	Number of vehicles	-	-	-	-
ϕ	Stein's Constant	Dimensionless	1.15	1.15	1.15	-
q^Z	Demand	Trips per Hour	110	130	90	99-660
S^Z	Bus Capacity per region	Seats per Bus	45	50	40	-
u^Z	Number of passengers per Stop per region	Number of Passengers	1.6	1	0.8	-
${v_V}^Z$	Value of in-vehicle time per region	$ per passenger Hour	12	15	13	4-27
${v_W}^Z$	Value of waiting time per region	$ per passenger hour	15	11	12	4-25
${v_X}^Z$	Line Haul Speed for region Z	Miles per Hour	30	25	35	-
y^Z	Ratio of local speed to express speed for region Z	Dimensionless	0.9	1	1.2	-

The formulation for the case study depends on whether the bus routes share the same headway or not. We will explore both scenarios and determine which would deliver higher benefits (lower average costs per passenger).

1. *Single headway optimization*

The cost function for this case results from the addition of individual cost functions for each area. Each of these cost functions comprises two different demands (outbound and inbound passengers), as both groups will have to stop at the central station either to take another bus or leave the network. Passengers leaving each region to any other destination in the network would be designated as q_Z^{IN} (users traveling inbound), and those arriving at each area from the central station are q_Z^{OUT} (users traveling outbound).

The total cost of supplier within one area includes users traveling inbound from area Z (q_Z^{IN}) and coming to area Z (q_Z^{OUT}). As both amounts math in the long run, as shown in equation (18), costs for each area are double than in the Base Case:

$$q_{IN}^Z = q_{OUT}^Z = q^Z \tag{18}$$

$${C_S}^Z = 2 * \left[\frac{2J^Z c^Z}{hV_X^Z} + \frac{\emptyset c^Z}{y^Z V_X^Z} \sqrt{\frac{q^Z A^Z}{u^Z h}} \right] \tag{19}$$

Total in-vehicle costs for each area are also double that of the Base Case:

$${C_V}^Z = 2 * \left[\frac{J^Z v_V^Z q^Z}{V_X^Z} + \frac{\emptyset v_{V,Z}}{2y^Z V_X^Z} * \sqrt{\frac{{q^Z}^3 h A^Z}{u^Z}} \right] \tag{20}$$

If the routes present coordinated headways, buses arrive at the same time at the central station. Thus, waiting time at the terminal is assumed to be negligible as passengers traveling outbound do not have to wait to take the next bus. For this, total waiting costs can be formulated as follows:

$${C_W}^Z = v_W^Z q^Z \frac{h}{2} \tag{21}$$

Total costs for each area result from adding up equations (19), (20) and (21):

$${C_T}^Z = {C_S}^Z + {C_V}^Z + {C_W}^Z \tag{22}$$

and the total cost function for the system can be found by adding the individual cost functions:

$$C_T = \sum_{Z=1}^{n} {C_T}^Z \tag{23}$$

The average cost per passengers is the result of dividing total costs by total demand traveling within the network:

$$q_T = \sum_{Z=1}^{n} q^Z \tag{24}$$

$$C_A = \frac{C_T}{q_T} \tag{25}$$

As in the Base Case, equation (25) is the objective function of this scenario. To minimize C_A, we apply the GRG algorithm and add the necessary condition for convexity by making sure that the second derivative C_A respect to headway has a positive value:

$$\frac{\partial^2 C_A}{\partial h^2} = \sum_{Z=1}^{n} \frac{8J^Z c^Z}{h^3 {V_X}^Z Q^Z A^Z} - \frac{\emptyset c^Z}{y^Z {V_X}^Z \sqrt{Q^Z u^Z h^5}} - \frac{\emptyset {v_v}^Z A^Z \sqrt{Q^Z}}{4y^Z V_X \sqrt{u^Z h^3}} > 0 \tag{26}$$

We must ensure as well that the optimal value of the headway (h^*) is below the maximum allowable headways for each individual route ($h_{MAX}{}^Z$)

$$h_{MAX}^Z = \frac{S^Z l^Z}{q^Z} \tag{27}$$

or else its value will be that of $h_{MAX}{}^Z$:

$$h^* = min\{h_{MAX}^Z, h_{GRG}\} \tag{28}$$

Lastly, the required fleet size for each region is formulated as follows:

$$N^Z = \frac{R^Z}{h} = 2 * \left[\frac{2J^Z}{hV_X^Z} + \frac{\emptyset}{y^Z V_X^Z}\sqrt{\frac{q^Z A^Z}{u^Z h}}\right] \tag{29}$$

2. *Many headway optimizations*

In this scenario, we optimize each region individually. Each route has with its own independent headway, so there is no coordination of buses in the transfer station, and thus there is an additional waiting time at the central terminal. Both supplier and in-vehicle costs are same as in the previous section, and waiting costs differ in that now they include the transferring time:

$$C_S{}^Z = 2 * \left[\frac{2J^Z c^Z}{h^Z V_X^Z} + \frac{\emptyset c^Z}{y^Z V_X^Z}\sqrt{\frac{q^Z A^Z}{u^Z h^Z}}\right] \tag{30}$$

$$C_V{}^Z = 2 * \left[\frac{J^Z v_V^Z q^Z}{V_X^Z} + \frac{\emptyset v_{V,Z}}{2y^Z V_X^Z}\sqrt{\frac{{q^Z}^3 h^Z A^Z}{u^Z}}\right] \tag{31}$$

$$C_W{}^Z = v_W^Z q_{IN}^Z \frac{h^Z}{2} + v_W^Z q_{OUT}^Z \frac{h^Z}{2} = v_W^Z (2q^Z)\frac{h^Z}{2} = v_W^Z q^Z h^Z \tag{32}$$

Total costs for each route are as follows:

$$C_T{}^Z = C_S{}^Z + C_V{}^Z + C_W{}^Z \tag{33}$$

The average cost per passenger in each route Z can be found by diving the total cost function by the passenger flow of the route. This flow is comprised by the sum of the passengers leaving or arriving at each area Z (q^Z). Thus, the objective function is the following:

$$C_A{}^Z = \frac{C_T{}^Z}{q^Z} \tag{34}$$

This function will be minimized and analyzed for each region separately.

In this case, we follow the same procedure to optimize $C_A{}^Z$ by means of the GRG algorithm. Nevertheless, since we are minimizing average costs separately, the constraint for global optimality is that as in Case Study 1 -equation (26)- but performed for each region. We also check the values for h^Z obtained do not surpass that of the maximum allowable headway for each individual route ($h_{MAX}{}^Z$). The required fleet size for each route is formulated as in the single headway scenario -equation (29).

Once we find the optimal values for the headways, we compute the average costs per passenger in the network so that we can make comparisons between scenarios:

$$C_A = \frac{\sum_{Z=1}^{n} C_T{}^Z}{\sum_{Z=1}^{n} q^Z} = \frac{C_T}{q_T} \tag{35}$$

$$C_{S,A} = \frac{\sum_{Z=1}^{n} C_S{}^Z}{\sum_{Z=1}^{n} q^Z} = \frac{C_S}{q_T} \tag{36}$$

$$C_{V,A} = \frac{\sum_{Z=1}^{n} C_V{}^Z}{\sum_{Z=1}^{n} q^Z} = \frac{C_V}{q_T} \tag{37}$$

$$C_{W,A} = \frac{\sum_{Z=1}^{n} C_W{}^Z}{\sum_{Z=1}^{n} q^Z} = \frac{C_W}{q_T} \tag{38}$$

Extension of Case Study

In order to demonstrate how this same exact formulation can be used with networks of higher complexity, we propose a nine-region system all connected to a central terminal as shown in Figure 4. Zones are listed from 1 to 9. The input values of demand in are reflected in Table 4 together with the characteristics of each zone Table 5. It should be noted that we have generated highly random data and that we have given special consideration to making them as dissimilar as possible. In this manner, we can truly test the formulation proposed and analyze what the results look like when increasing the difficulty of the problem, thus giving a better approximation to a real-life situation.

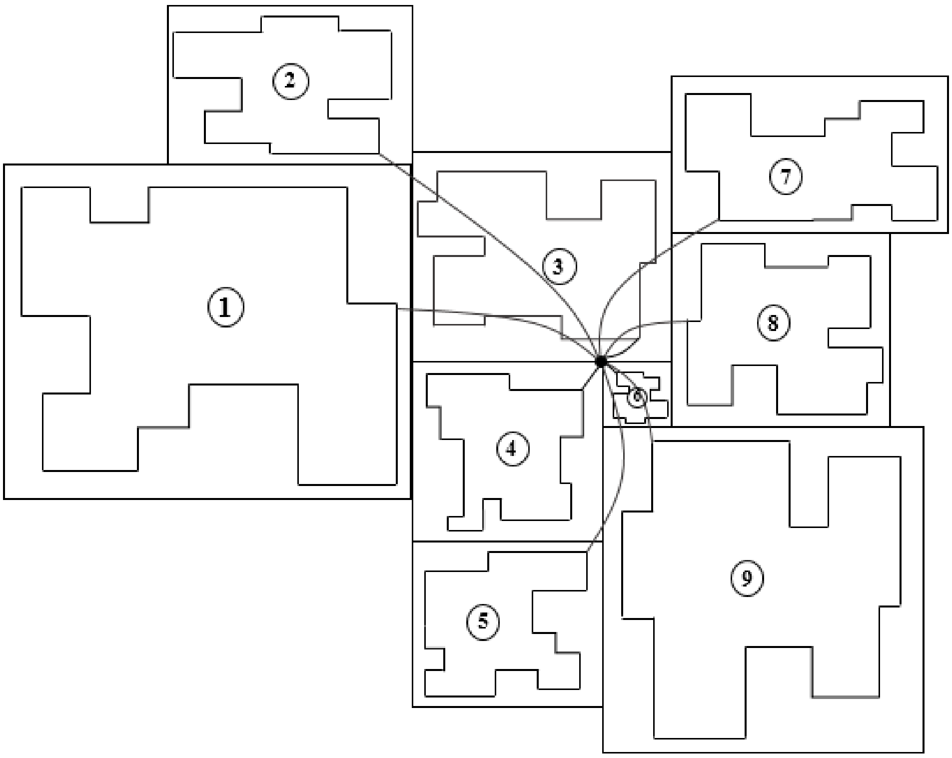

Figure 3. Flexible-routes Bus Service for Extension of Case Study. [Source: prepared by the author]

As Figure 3 and Base Values reveal, the features of each region differ greatly from one another, especially in zone size, line-haul distance or number of seats per bus. Regarding demand, we have focused on maintaining a significantly low demand density because, as we know, flexible-route bus systems prove to be cost-effective under low-demand scenarios. Same as in previous cases, demand inputs represent one-way passenger trip - either leaving region Z o arriving at region Z- and it is also stable in the long run, meaning that commuters traveling inbound are assumed to return to their point of departure at some point of time.

Table 4. Demand values for the Nine-region scenario.

	R. 1	R. 2	R. 3	R. 4	R. 5	R. 6	R. 7	R. 8	R. 9	Central Terminal
Region 1	0	2	6	3	6	0	4	0	0	29
Region 2	5	0	0	0	3	1	0	0	1	5
Region 3	5	0	0	1	0	0	4	0	0	0
Region 4	3	0	0	0	1	1	0	0	1	1
Region 5	6	5	2	0	0	0	0	1	1	0
Region 6	0	1	0	0	2	0	0	1	0	1
Region 7	0	1	1	2	3	1	0	1	1	5
Region 8	0	0	0	0	0	1	2	0	1	2
Region 9	2	1	1	0	0	0	0	1	0	0
Central Terminal	29	5	0	1	0	1	5	2	0	0

Table 5. Base Values for the Nine-region scenario.

	Region 1	Region 2	Region 3	Region 4	Region 5	Region 6	Region 7	Region 8	Region 9
a	30	30	30	30	30	30	30	30	30
A	59.5	13.08	23.69	15.04	13.78	1.53	18.93	14.57	45.56
b	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
c	45	37.5	42	37.5	37.5	33	39	39	44
Height	8.5	4.54	4.37	3.76	3.47	1.3	3.28	3.57	6.7
J	4.4	6.37	0.94	0.68	4.4	0.4	3.9	2	2
l	1	0.7	1.3	1.2	1	0.8	0.9	0.9	1.1
Q	0.840	1.147	0.422	0.465	1.089	3.259	0.793	0.412	0.110
S	50	25	40	25	25	10	30	30	45
u	1.6	1	1.1	1.5	1.4	1.3	1.2	1	1
$\mathbf{v_V}$	12	15	13	12	11	13	15	12	14
$\mathbf{v_W}$	15	11	12	14	15	12	13	11	13
$\mathbf{V_x}$	30	25	35	28	32	31	29	27	33
Width	7	2.88	5.42	4	3.97	1.18	5.77	4.08	6.8
y	1	0.9	1.5	1.1	0.8	0.9	1.1	0.7	1.1

In this light, we now proceed to follow the mathematical formulation we applied in the Case Study. We intend to minimize user costs by optimizing both the common headway and the multiple headway scenarios. The relevance of this extension lies in the need for demonstrating that the solution method proposed in this paper is strong enough to solve matters of high levels of complexity by means of a rather simple approach.

In addition to this, we also aim to analyze what happens when the areas serviced present very disparate features. This is so because the preceding case is solved for quite similar regions, hence further from an actual public transportation problem. By solving this nine-region scenario, we can get a better approximation of what a flexible-route bus system looks like, as well as determining whether the type of network obtained in the Case Study is conclusive or not.

Numerical Analysis

The results found after applying the GRG algorithm are presented below. After discussing these outcomes, we perform an elasticity analysis for sensitivity and as a way of crosschecking the mathematical formulation used. The base values used along the formulations can be found in Table 1, Table 2 and Table 3.

Base Case: Many-to-one demand scenario

The optimal solution for the headway is found at 0.289 hours (17.36 minutes), with an average cost of $11.6 per passenger. This value for the headway satisfies the condition of global optimality, as the result of the second derivative of the objective function (equation 14) results in 47.77 (i.e. a positive value). Additionally, it doesn't exceed the value of the maximum allowable headway of the network, which is found at 1.125 hours. Lastly, the number of buses required to serve the system results in 4 vehicles.

To show the convexity of the curve, we have graphed average costs (C_A) against headway (h) The figure shows how optimal headway (0.289 hours) is found at the minimum of the curve at the baseline values of the parameters noted at Table 1.

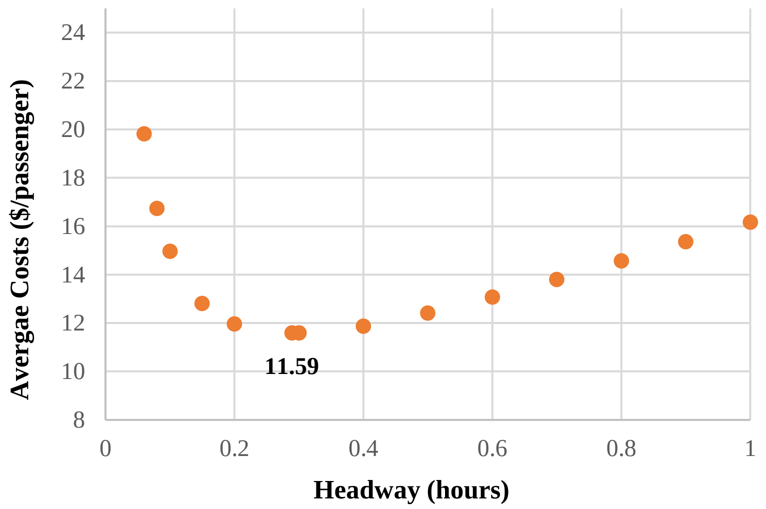

Figure 4. Average cost function against headway. [Source: prepared by the author]

All in all, these results are aimed to serve as reference points for the Study Case.

Case Study: Many-to-many demand scenario

1. *Common headways*

The results of headway and costs obtained for this scenario are shown in Table 6, and Table 7 shows an average of those results in order to contrast them with the next scenario. Average costs are obtained from equations (35) to (38). The optimal value for the headway is 0.185 hours (11.07 minutes) -slightly smaller than that of the Base Case- as the network serves a higher number of users and regions. For this reason, buses must travel the routes more frequently to meet demand requirements. It should be noted that average costs are significantly higher than in the previous case. Supplier costs, in-vehicle costs and waiting costs account for the 22%, 73% and 6% of the average system cost, being in-vehicle costs the considerably higher than the rest and hence the most transcendent. The number of vehicles needed to satisfy the entire demand of the network is found at 36 buses: 11 for Route 1, 16 for Route 2 and 9 for Route 3. This number is significantly higher than that in then Base Case because of the greater number of areas served, the increase in commuters from each region (inbound and outbound demand) and the decrease in the value of the headway.

Table 6. Results for Case Study 1.

	h* (hours)	**C_A^Z ($/passenger)**	**$C_{S,A}^Z$ ($/passenger)**	**$C_{V,A}^Z$ ($/passenger)**	**$C_{W,A}^Z$ ($/passenger)**	**Fleet size**
Region 1	0.185	17.18	4.16	11.64	1.38	11
Region 2	0.185	28.46	5.49	21.96	1.02	16
Region 3	0.185	14.88	3.86	9.92	1.11	9

Table 7. Narrowed results for comparison purposes for Case Study 1.

h* (hours)	**C_A ($/passenger)**	**$C_{S,A}$ ($/passenger)**	**$C_{V,A}$ ($/passenger)**	**$C_{W,A}$ ($/passenger)**	**Fleet size**
0.185	21.00	4.60	15.23	0.185	36

Table 8 shows how h* does not exceed the maximum allowable headways for each region (0.409 hours, 0.269 hours and 0.578 hours respectively). The notable differences in these values depend on the input values chosen randomly to conduct this analysis. Lastly, we conclude that the value of headway is globally optimal as the value of equation (26) is positive at 333.93, and thus the function is convex.

Table 8. Conditions of optimality for Case Study 1.

h* (hours)	**h_{MAX}^1 (hours)**	**h_{MAX}^2 (hours)**	**h_{MAX}^3 (hours)**	**Condition of global optimality**
0.185	0.409	0.269	0.578	333.93

The graph of the average cost function for Case Study 1 is shown in Appendix A. Figure 4 shows how the global minimum is found when the value of headway is 0.185 hours, matching both graphical and analytical solutions, and how the function is strictly convex in the entire domain.

2. Many headways

In this scenario, we optimize each headway individually for comparisons with the previous case and to determine which network would be less costly and more efficient. The values obtained for each region are shown in Table 9. It is relevant to point out that since each headway is optimized individually, the results are tailored to each particular route and derive strongly from the characteristics of each zone. In order to compare the results of both cases with ease -and because they are highly dependable on the input data chosen randomly- we have narrowed them to a single output as shown in Table 10. The optimal value of the headway is found averaging the results of each region, and each average cost is obtained from equations (35), (36), (37) and (38). In this table, we can also obtain an approximation on the weight of each cost in the average cost per passenger for this type of network. Service, in-vehicle and waiting costs constitute roughly 24%, 67% and 9% of the user´s costs. Compared to a system with a sole headway, operator costs increase mainly because of the rise in the vehicles needed to satisfy the demand of the system, which goes from 36 buses up to 42. In-vehicle costs are slightly lower due to a fall in the value of the headway, causing the roundtrip time (R^Z) to decrease as well. Regarding waiting costs, since passengers must wait both at their pick-up location and sometimes at the central station as well -since there may not be coordination of bus arrivals-, they rise respect to Case Study 1.

Table 9. Results for Case Study 2.

	h* (hrs)	**C_A^Z ($/passenger)**	**$C_{S,A}^Z$ ($/passenger)**	**$C_{V,A}^Z$ ($/passenger)**	**$C_{W,A}^Z$ ($/passenger)**	**Fleet size**
Region 1	0.158	18.48	4.75	11.37	2.36	13
Region 2	0.158	29.38	6.25	21.39	1.74	19
Region 3	0.153	15.88	4.49	9.56	1.83	10

Table 10. Narrowed results for comparison purposes for Case Study 2.

h_{AV}* (hours)	**C_A ($/passenger)**	**$C_{S,A}$ ($/passenger)**	**$C_{V,A}$ ($/passenger)**	**$C_{W,A}$ ($/passenger)**	**Fleet size**
0.156	22.07	5.27	14.82	1.97	42

Table 11 shows how the optimal solutions found satisfy both the condition of global optimality and the maximum allowable headway constraint. It should be noted that since the areas of the network share the same characteristics in both scenarios, h_{MAX}^Z has the same exact values. Lastly, the convexity of the cost function of each region is shown in Appendix A. Figures 5 to 7 illustrate how the C_A function presents a single minimum in its domain, which ultimately makes it the global minimum we are looking for.

Table 11. Conditions of optimality for Case Study 2.

	h* (hours)	h_{MAX}^{Z} (hours)	Global optimality condition
Region 1	0.158	0.409	206.92
Region 2	0.158	0.269	226.70
Region 3	0.153	0.578	164.06

Sensitivity Analysis

In this section we aim to compare the two types of scenarios when varying four key parameters of the regions: demand, line-haul distance, value of in-vehicle time and value of waiting time. We will compare the average costs per traveler in the network by using the optimal value of headway found in Case Study 1 and the average optimal value of headway found in Case Study 2. By doing this, we seek to determine from different fronts which type of network is less costly and hence more feasible. A visual comparison of these costs is provided at the end of this section in Figure 5.

<u>*Demand*</u>

For this comparison, we have calculated the optimal headway of each scenario for the total amount of passengers traveling in the network. This number of users ranges from half to double the original value of the total demand (Table 2). Results found prove that a network with a single value of headway is more cost-effective than stablishing a headway for each individual route -given the characteristics chosen or similar. They also reveal that, as demand increases, both the value of headway and average cost per passenger decrease. This fall is more pronounced for low demand densities and stabilizes when demand is high. Additionally, cost difference between scenarios also falls from 10.5% to 3% as there are more passengers traveling the network.

Table 13 presents a breakdown of the variation of each cost from the lowest to highest demand input of the range used. The weight of each cost on the average system cost remains similar as stated above in this paper, being in-vehicle costs the greatest and waiting costs the lowest. In this light, it is noted that waiting costs show a significant decrease when demand rises due to their high dependency on the value of the headway, whereas in-vehicle costs barely change. Service cost decrease as there are more users sharing the costs, which ultimately reduces C_A.

Table 12. Sensitivity analysis results for demand inputs.

q_{TOT}	C_A (common headways)	h*	C_A (different headways)	h^*_{AV}	Type of network	Cost difference
99	23.28	0.447	25.72	0.344	Common headways	10.5%
165	22.07	0.313	23.82	0.250	Common headways	7.9%
231	21.48	0.244	22.86	0.200	Common headways	6.5%
330	21.00	0.185	22.07	0.156	Common headways	5.1%
429	20.72	0.149	21.59	0.129	Common headways	4.2%
495	20.60	0.132	21.37	0.116	Common headways	3.8%
561	20.50	0.119	21.20	0.105	Common headways	3.4%
660	20.38	0.103	20.99	0.093	Common headways	3.0%

Table 13. Cost variation with demand increase.

Cost increase	C_A	$C_{S,A}$	$C_{V,A}$	$C_{W,A}$
Case Study 1	-12.44%	-30.13%	7.48%	-76.92%
Case Study 2	-18.38%	-37.87%	8.94%	-72.95%

Line Haul Distance

Table 14 shows the sensitivity if average cost per user and its corresponding headway based on average line haul distance variation. For both scenarios, the distance to the central station used to conduct the analysis is the result of averaging the values of each region, ranging from roughly half to double their original values. Results found show that a network coordinated by means of a single headway is again the most economical alternative. They also indicate that, as distance becomes greater, average costs grow by almost 200% within the interval considered, making them extremely cost-sensitive to changes in the value of the line haul. If we take a closer look to Table 15, it is also noticeable that there is a boost of in-vehicle costs by approximately 280% in both cases. Since in-vehicle costs account for nearly 70% of user's costs, it is safe to say that this is the reason why average costs suffer such rise. The value of headway progressively increases when line haul becomes larger as buses would have to cover greater distances and hence spend more time traveling the routes. Lastly, it is important to point out that while there is a significant variation in average costs, the cost difference among scenarios remains steady at around only a 5%.

Table 14. Sensitivity analysis results for line haul distance inputs.

J_{AV}	C_A (common headways)	h*	C_A (different headways)	h^*_{AV}	Type of network	Cost difference
3	11.33	0.105	11.94	0.091	Common headways	5.39%
5	14.26	0.131	15.03	0.113	Common headways	5.36%
7	17.03	0.154	17.93	0.132	Common headways	5.26%
10	21.00	0.185	22.07	0.156	Common headways	5.08%
13	24.83	0.211	26.05	0.178	Common headways	4.91%
15	27.32	0.228	28.63	0.191	Common headways	4.80%
17	29.79	0.243	31.18	0.204	Common headways	4.70%
20	33.43	0.266	34.95	0.221	Common headways	4.56%

Table 15. Cost variation with average line haul distance increase.

Cost increase	C_A	$C_{S,A}$	$C_{V,A}$	$C_{W,A}$
Case Study 1	194.99%	48.41%	277.01%	153.50%
Case Study 2	192.65%	55.73%	285.53%	144.93%

Value of In-vehicle time

The value that time represents to travelers is a tough parameter to consider. The challenge lies in the difficulties to accurately measure it and in its fluctuation depending on the region considered. For this paper, we have chosen these values randomly and assumed that the zones don’t differ much from one to another. Same as in the previous case, we use an average of the value of time of the regions. The findings of this analysis are shown in Table 16, and they demonstrate how a single headway network is still the most beneficial option. In this way, as the perceived value of time increases, headways must decrease and traveling becomes more expensive. Since in-vehicle costs represent 70% of total costs, a rise in the value of time has the strongest repercussion on them. In this study, the in-vehicle cost increase amounts to 485 % for Case Study 1 and 504% for Case Study 2 in the range considered, making the value of in-vehicle time a critical parameter for network design.

Regarding the rest of the costs, we should highlight that waiting costs decrease as a response of the fall in the value of headway, and that supplier costs rise because a larger number of buses would be needed to satisfy passengers' necessities. To conclude, cost differences among systems are higher for low values of time, ranging from 12.64% down to 2.14%.

Table 16. Sensitivity analysis results for value of in-vehicle time inputs.

v_{AV}	**C_A (common headways)**	**h***	**C_A (different headways)**	**h^*_{AV}**	**Type of network**	**Cost difference**
4.00	9.98	0.273	11.44	0.204	Common headways	14.64%
6.67	13.21	0.240	14.53	0.187	Common headways	10.00%
9.33	16.37	0.214	17.58	0.173	Common headways	7.37%
13.33	21.00	0.185	22.07	0.156	Common headways	5.08%
17.33	25.52	0.163	26.48	0.142	Common headways	3.74%
20.00	28.49	0.152	29.38	0.135	Common headways	3.13%
22.67	31.42	0.142	32.25	0.128	Common headways	2.66%
26.67	35.76	0.129	36.52	0.119	Common headways	2.14%

Table 17. Cost variation with average in-vehicle time value increase.

Cost increase	**C_A**	**$C_{S,A}$**	**$C_{V,A}$**	**$C_{W,A}$**
Case Study 1	258.41%	85.17%	485.26%	-52.62%
Case Study 2	219.34%	60.51%	504.42%	-41.46%

Value of waiting time

The last parameter contemplated for sensitivity purposes is the value of waiting time. As with value of in-vehicle time, this last variable is also challenging to measure with precision and is also found from averaging the data of each region. It should be noted that waiting times were found to be worth between two and three times in-vehicle value (Quarmby, 1967). Nevertheless, as waiting costs are the least representative component of users' average costs, we observe in Table 19 that even though they show an outstanding rise within the range considered (383% for Case Study 1 and 306% for Case Study 2) the average cost per user hardly varies. The implications of this behavior are to acknowledge that the value of waiting time is not as an important parameter as others for this particular study. Notwithstanding this discovery, a single headway network still emerges victorious as the most profitable choice.

Table 18. Sensitivity analysis results for value of waiting time inputs.

v_{WAV}	C_A (common headways)	h*	C_A (different headways)	h^*_{AV}	Type of network	Cost difference
3.80	20.12	0.218	20.50	0.205	Common headways	1.91%
6.33	20.38	0.206	20.99	0.186	Common headways	2.99%
8.87	20.64	0.197	21.45	0.172	Common headways	3.91%
12.67	21.00	0.185	22.07	0.156	Common headways	5.08%
16.47	21.34	0.175	22.63	0.144	Common headways	6.07%
19.00	21.55	0.169	22.99	0.138	Common headways	6.66%
21.53	21.76	0.164	23.33	0.132	Common headways	7.20%
25.33	22.09	0.156	23.82	0.125	Common headways	7.83%

Table 19. Cost variation with average waiting time value increase.

Cost increase	C_A	$C_{S,A}$	$C_{V,A}$	$C_{W,A}$
Case Study 1	9.79%	31.68%	-5.60%	382.94%
Case Study 2	16.18%	47.43%	-7.09%	306.06%

Hereunder there is a graphical representation of the results of this analysis. At first glance, it can be inferred from the figures that the highest change in costs occur when the line haul or the in-vehicle time value are varied. In those two cases, average costs per passenger range from nearly 10$ up to almost 35$, with costs increases in the order of 190% and 200% respectively. In response to demand changes, there is a slight decrease in the average costs per passenger (around -12% for coordinated networks and -18% for uncoordinated). In addition to this, the graph shows higher costs differences for lower values of demand, favoring the common headways users. Lastly, average costs per passenger do not experience a significant change when the value of waiting time is modified, even though waiting costs do vary in a considerable way (costs increases of around 380% and 300%). However, as explained above, waiting costs account for a rather low percentage of total costs, causing a small impact on the overall cost context.

All in all, with the visual aid of the following figures we aim to facilitate the comprehension and comparison of costs when these four parameters are varied.

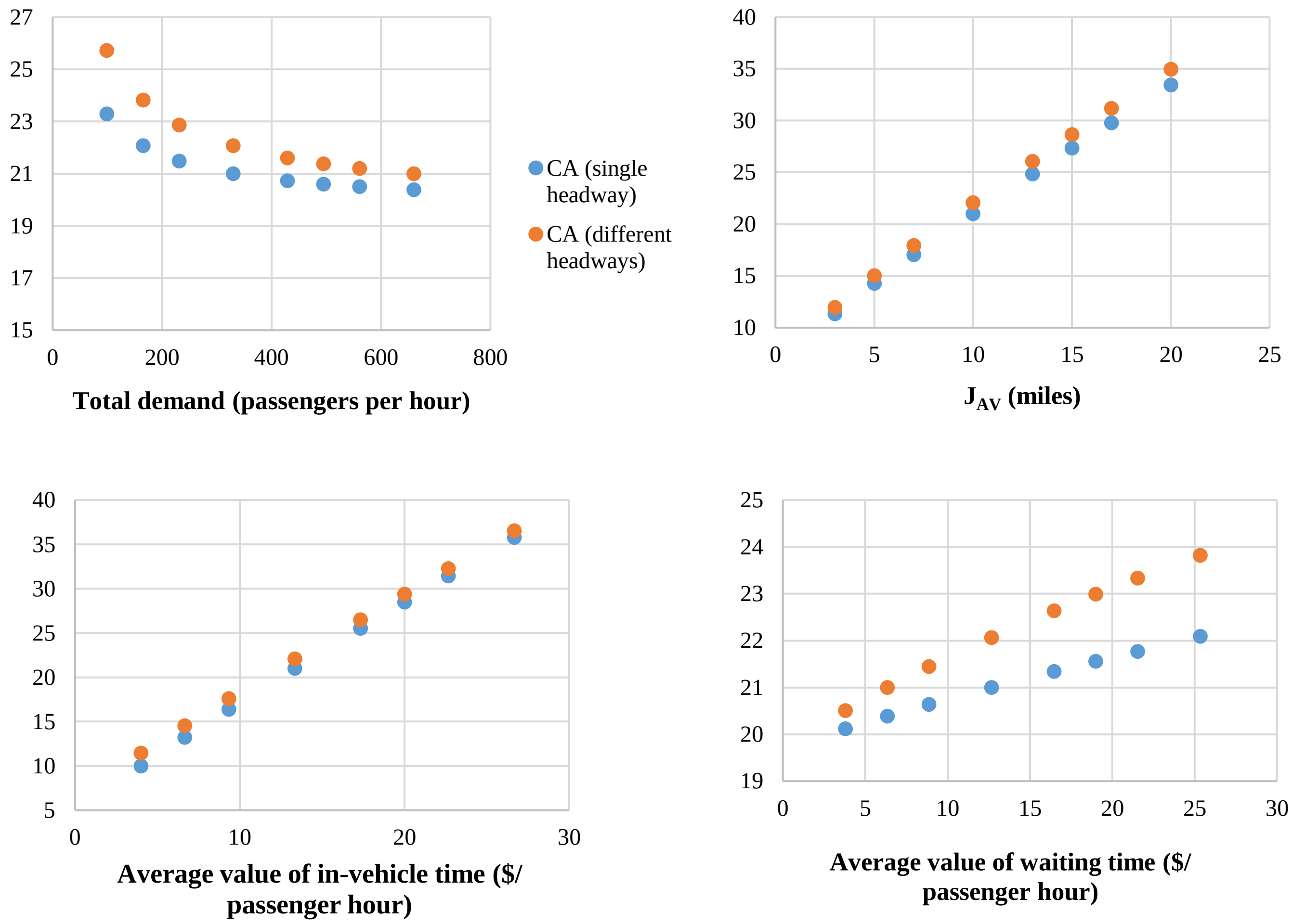

Figure 5. Sensitivity analysis for Case Study 1 and 2. [Source: prepared by the author]

Extension of Case Study

Following the formulation of the Case Study and applying the GRG algorithm, the outcomes of the proposed scenario are shown in Table 20 and Table 21. In order to facilitate the comparison of the results, we decided to display them together at the same time. The results shown include the values of the constraints that must be met to guarantee optimal solutions as well as the values of the headways, a breakdown of the average costs, and the fleet size required to service each region. As footnotes indicate, the top data corresponds to the uncoordinated situation, whereas the bottom refers to the common headway case.

First, we should underline how the results found satisfy the restrictions posed in this paper. On the one hand, we confirm that the bus frequencies obtained do not surpass the maximum allowable headway limitation of the regions. On the other hand, we prove that these solutions found are globally optimal -and thus the global minimum of the objective function- since the values of equation (26) yield positive values in both cases. A graphical representation of the convexity of the objective functions can be found in Appendix A.

Contrary to what it could be thought at first, the system with common headways turns out to be the most cost-effective solution for the design of this flexible-route bus network again. It could be expected that, since the values of the headways found for the uncoordinated network are adjusted specifically to each region's parameters and are widely distinct both from one another and the common headway solution, arranging different bus frequencies for every route would be more profitable. Nevertheless, calculations are determinant and show the opposite.

Table 20. Results for Extension of Case Study.

	Regions									**Sum**
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	
h* (hrs)	0.171[4]	0.443	0.268	0.284	0.326	0.221	0.332	0.546	0.547	
	0.352[5]	0.352	0.352	0.352	0.352	0.352	0.352	0.352	0.352	
h* (min)	10.28	26.57	16.07	17.04	19.57	13.23	19.92	32.77	32.83	
	21.09	21.09	21.09	21.09	21.09	21.09	21.09	21.09	21.09	
hmax	1.00	1.17	5.20	4.29	1.67	1.60	1.80	4.50	9.90	
	1.00	1.17	5.20	4.29	1.67	1.60	1.80	4.50	9.90	
C_S^Z	513.47	166.99	69.03	56.85	134.92	21.80	138.23	81.20	76.38	1258.86
	326.00	199.26	58.37	49.93	127.70	15.99	132.62	107.70	101.31	1118.90
C_V^Z	586.53	221.83	28.60	18.08	96.80	4.73	132.41	40.93	33.24	1163.17
	764.03	210.15	31.76	19.66	98.77	5.54	134.49	34.95	28.33	1327.67
C_W^Z	128.51	73.06	32.13	27.83	73.37	13.23	64.76	36.04	35.56	484.50
	131.83	29.00	21.09	17.23	39.55	10.55	34.28	11.60	11.43	306.55
C_T^Z	1228.51	461.88	129.77	102.76	305.10	39.76	335.39	158.17	145.19	2911.27
	1221.87	438.42	111.22	86.82	266.02	32.07	301.39	154.25	141.07	2753.12
N	12	5	2	2	4	1	4	3	2	35
	8	6	2	2	4	1	4	3	3	33
Global optimal condition	78.80	42.72	21.28	19.74	60.01	45.92	54.18	11.03	12.42	194.215

[4] Uncoordinated network
[5] Common headways

Table 21. Results for Average Costs per passenger.

	Regions								
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
C_A^Z	24.57[4]	30.79	12.98	14.68	20.34	7.95	22.36	26.36	29.04
	24.44[5]	**29.23**	**11.12**	**12.40**	**17.73**	**6.41**	**20.09**	**25.71**	**28.21**
Cost difference	0.54	5.35	16.67	18.37	14.69	23.97	11.28	2.54	2.92

On this basis, we move on to breaking down the costs to reach a deeper understanding of the results obtained. We ought to bear in mind that the solutions found must be compared within the same region and between scenarios. This is so because the major differences among zones impede their comparability and can lead to wrong conclusions.

- **Supplier costs**. For each region, the lower the value of headway, the greater the operator's costs. The main reason behind this is the rise in the number of vehicles required when the frequency of the service demanded is greater. From an economic standpoint, passengers would benefit more from higher headways but, on the other side, that would lead to them giving up convenience. Overall, as the fleet needed to cover each type of network is higher for the uncoordinated scheme -35 buses vs 33-, the coordinated system proves to be less expensive with a cost difference of 12.51%.
- **In-vehicle costs.** In this case, as headway decreases so does the in-vehicle costs. This is so because the passenger's travel time is assumed to be to half of the roundtrip time, R (Kim & Schonfeld, 2012). The round travel time is comprised by the time spent by the bus since it leaves the central station until it returns, this is, the sum of the time spent traveling the distance between each zone and the central station (bi-directional tour) and the travel time in the region. This trip time is directly proportional to the tour length within the zone, which in turn depends on the service frequency or headway. In this situation, in-vehicle costs turn out to be 14.14% higher when common headways are implemented in the network. The main reason for that is that Region 1 presents a significantly low value of headway in comparison with the rest of the zones and the common headway result, reducing the total in-vehicle costs of the system. Nevertheless, this fall in regional in-vehicle costs are counterbalanced with higher supplier costs.
- **Waiting costs.** Since the areas served by flexible public transportation systems are characterized by low demand [(Kim & Schonfeld, 2012), (Kim & Schonfeld, 2013), (Nourbakhsh & Ouyang, 2012)], headways tend to present rather high values. Additionally, our study formulates waiting costs differently for uncoordinated and coordinated scenarios. In the first scenario, passengers will probably have to wait in the transfer station to take their next bus as there is no coordination between lines. For that reason, waiting costs for a common headway network are obtained as in equation (21) and for an uncoordinated system as in equation (32), which considers the headway twice. Taking a closer look to the

results, it is noted that the greatest costs differences occur when the optimal headway of an uncoordinated route is substantially higher than the optimized common headway, in accordance with the reasoning explained above. This situation arises in Regions 2, 8 and 9. When the values of the headways are fairly close -as in Regions 5 and 7- this gap is also remarkable. For Regions 1 and 6, waiting costs almost match because the uncoordinated headway is way below the common headway, and for Regions 3 and 4 the cost difference is higher but not as marked as in for the higher headways. All in all, waiting costs are 58.05% higher in uncoordinated systems thereby becoming its major drawback.

To conclude, Table 21 integrates all these costs and displays the average cost per passenger in each region. As is apparent, these results show how implementing common headways is the most cost-effective choice in every case, and thus being the most economical solution for its passengers. The greatest cost difference corresponds to Region 6 (23.97%), as the lower the cost per passenger the higher its sensitivity. Conversely, Region 1 holds the lowest cost gap at 0.54% of difference. In light of these findings together with the results from Case Study, it is safe to say that users will benefit more from a common headways network than they would from an uncoordinated system. Additionally, it should not be forgotten that both scenarios are designed for flexible-route transportation services.

Conclusions

Conclusions on Formulation and Methodology

Flexible-route bus systems provide travelers from low demand density regions with countless benefits. On the one hand, helps connecting rural regions, suburban areas and small town to larger cities, facilitating the commuting of their inhabitants rather than hampering it. On the other hand, it brings convenience and comfort to the service. In this paper, we formulate a nonlinear constrained problem that we solve by means of the GRG algorithm. For each scenario, we developed an objective function that comprised supplier and user costs either for the entire network (common headways scenario) or for each region in particular (uncoordinated scenario). Those functions are formulated in equation (25) and in equation (34), respectively. The objective for each case was to minimize average costs per user in each region by finding the optimal value of the headway using a local search. Nevertheless, the results obtained are proven to be globally optimal because of the results of equation (26) and supported by graphical representation of the functions in Appendix A, where their convexity is demonstrated.

In addition to the Base Case and the Case Study, we also applied our formulation to a practical case of higher complexity -Extension of Case Study- in order to determine whether or not our approach could be used for larger number of regions. In that sense, the outcomes found here achieved a satisfactory result, so that Z can take any positive integer value.

Conclusions on the Results

Numerical analysis leads to the ineluctable conclusion that common headways are at the same time more economically feasible and easier to implement than uncoordinated systems -both for the Case Study and the Extension of it. This way, commuters are saved from dwell times at transfer stations and thus the convenience of the service is increased.

Concerning the sensitivity analysis performed in the Case Study, results found showed that the most important parameters to consider when designing these networks are the line haul distance and the value of in-vehicle time. This is so because they have the highest impact on in-vehicle costs, which as we know carry the largest weight on total costs and hence they make substantial difference as they wary. Moving on to the next parameter under discussion, we also found out that the value of waiting time is not as vital as the rest of the variables. The main reason for that lies in that waiting costs hold the lowest share in total costs. Finally, we must point out that costs decrease as demand increase. This behavior is in line with our expectations, as greater number of people means that costs are shared between more passengers. However, from the literature review we have learned that if demand is high and areas are densely populated, then flexible transportation services become unprofitable and more expensive than fixed-route bus systems [(Kim & Schonfeld, 2012), (Nourbakhsh & Ouyang, 2012), (Kim & Schonfeld, 2013)]. For the purpose of this paper, we decide to assume low demand densities in the regions served.

In order to cross-check the results of the Case Study, the nine-region scenario proposed in this paper also verifies that common headways outperform the uncoordinated network from and financial standpoint.

Recommendations for Future Research

In view of the results, we believe that this study could see some improvement when the following suggestions are examined in greater depth or even combined:

- **Addition of constraints.** One possible way of making our solving method more precise and capable of solving real life situations is to incorporate actual limitations of the transportation system. Among them are human factors, such as the operating hours of the drivers; external factors, such as adverse weather conditions, traffic jams or road accidents; and internal factors, such as the vehicle's own traffic
- **Integer-ratios.** As some authors state (Ting & Schonfeld, 2005), integer ratios can be a useful solution for increasing coordination at transfer stations and thus lead to a decrease in costs. Under certain conditions, this approach manages to reduce waiting times at intermediate terminals by converting headways into an integer multiple of a base cycle. This way, vehicles serving each route would arrive in phase at the central terminal, facilitating the commuting of passengers. As you have gathered, this improvement would only be relevant to the uncoordinated network scenario.
- **Intermediate transfer stations.** Depending on the characteristics of the regions served and their geographical layout, it may be efficient to locate intermediate stops in the boundaries of inner and outer zones of the network. An exemplification of this solution is shown in Figure 6 using the nine-region scenario from our study. As can be appreciated, by implementing this measure passengers would not have to travel longer distances than necessary and would be prevented from incurring additional costs.

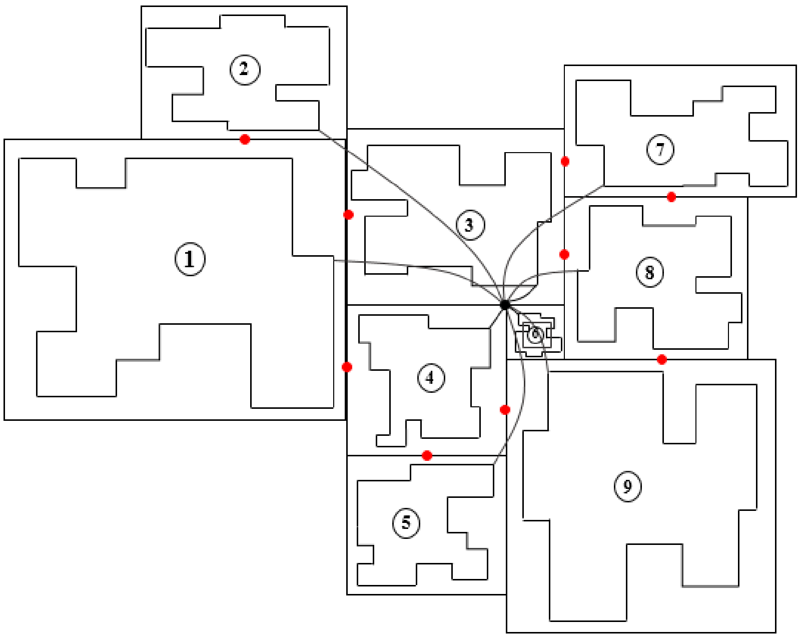

Figure 6. Extension of Case Study with intermediate transfer stations [Source: prepared by the author].

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Appendix A – Convexity graphs

Case Study

1. *Single headway optimization*

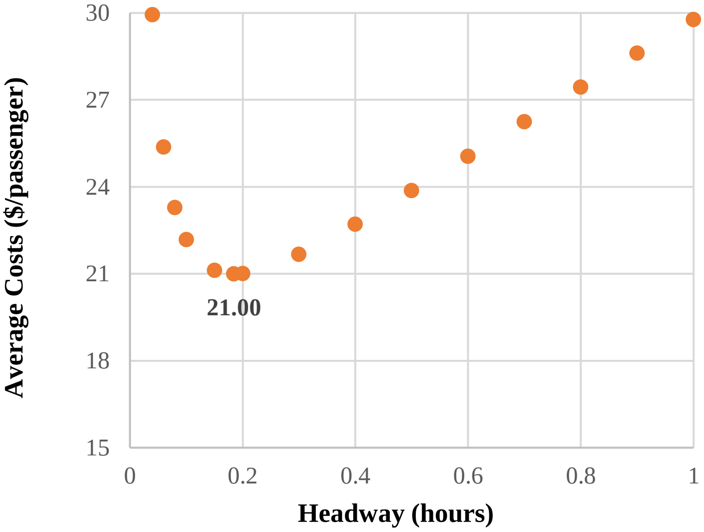


Figure 7. Average costs against headway in coordinated network. [Source: prepared by the author]

2. *Many headway optimizations*

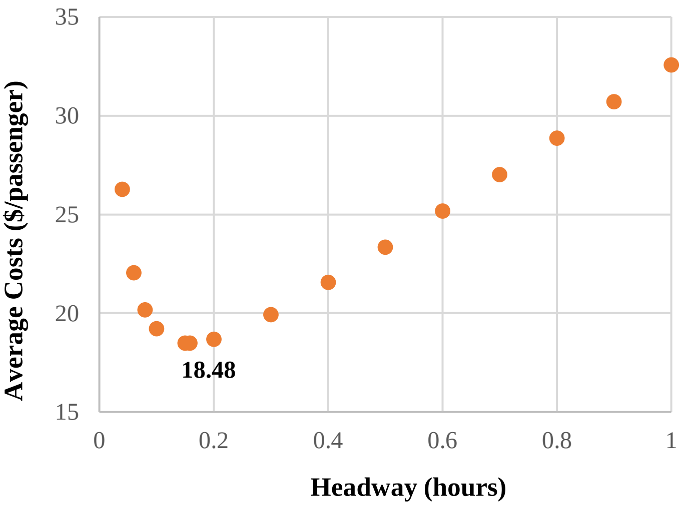


Figure 9. Average costs against headway for Region 1. [Source: prepared by the author]

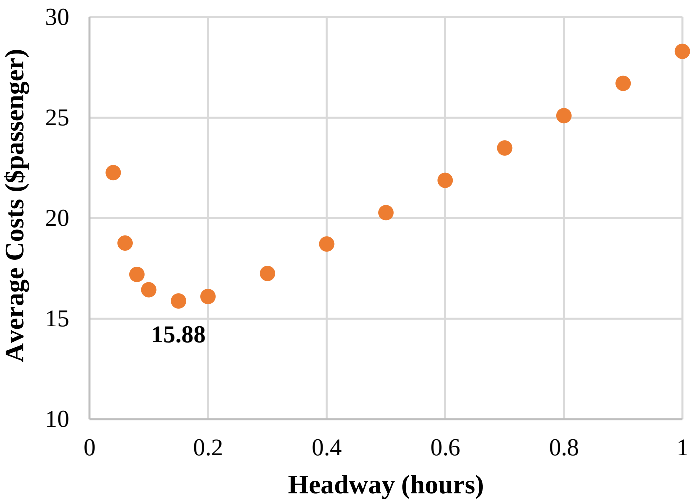


Figure 8. Average costs against headway for Region 2. [Source: prepared by the author]

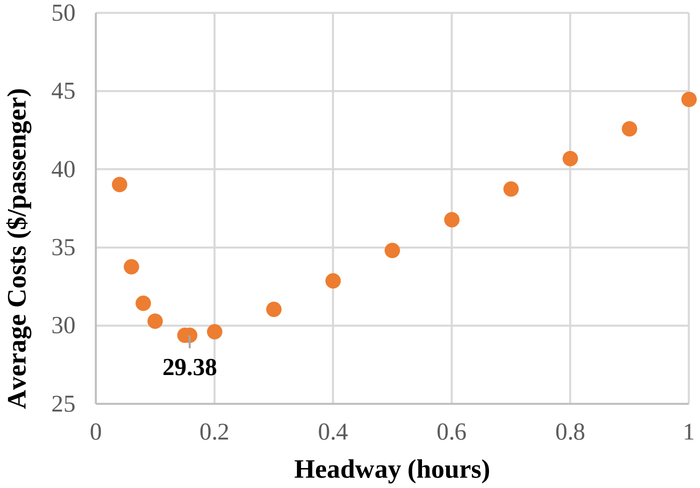


Figure 10. Average costs against headway for Region 3.
[Source: prepared by the author]

Extension of Case Study

1. Single headway optimization

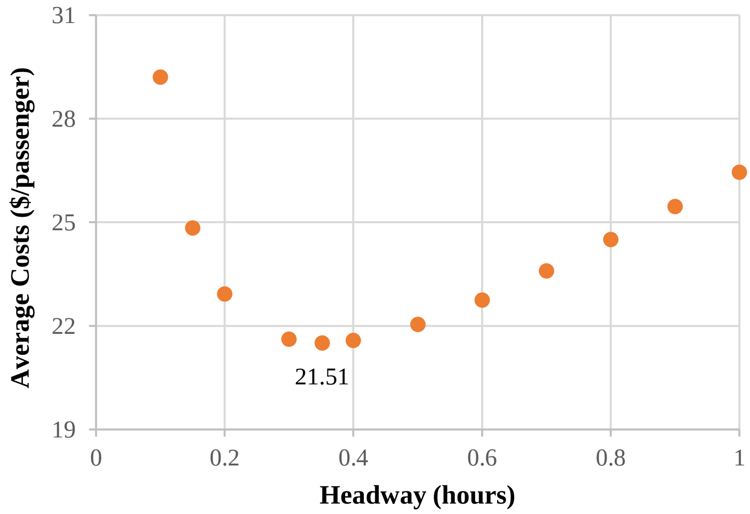
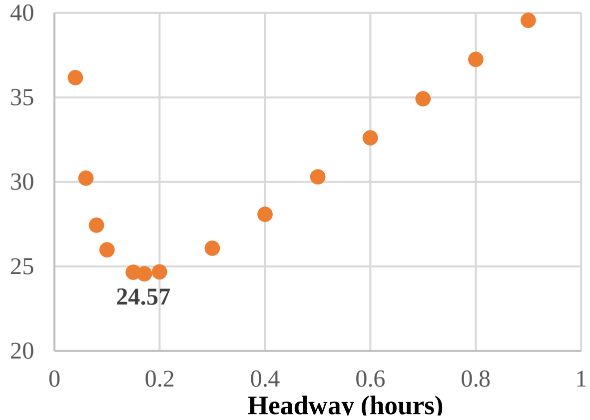
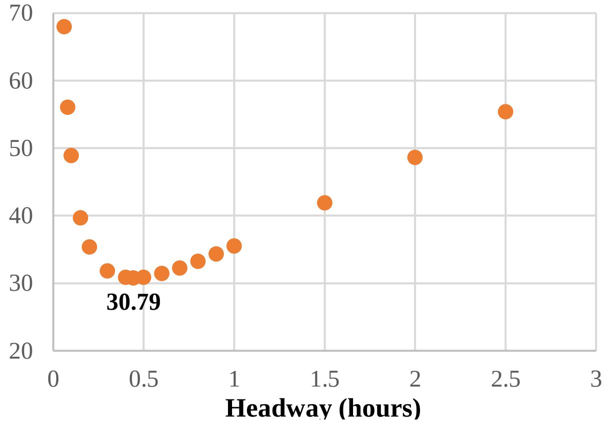
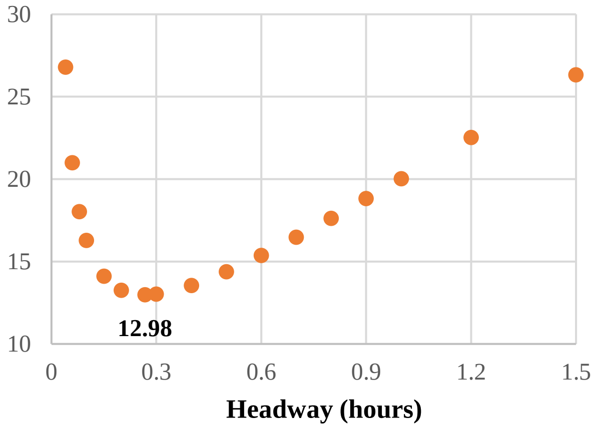


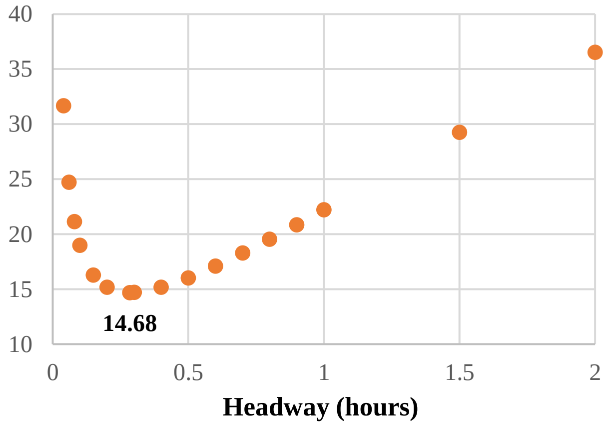
Figure 11. Average costs against headway in coordinated nine-region network.
[Source: prepared by the author]

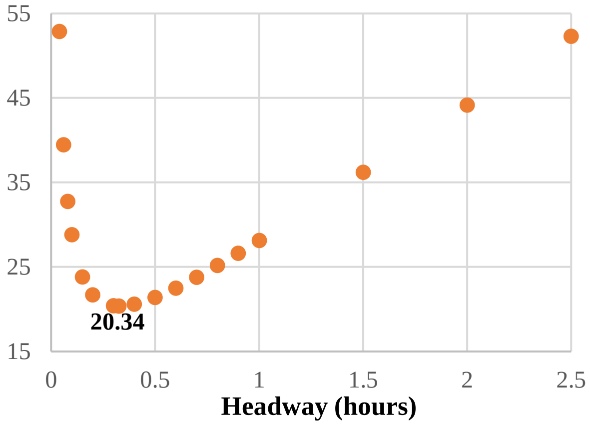
1. Many headway optimization

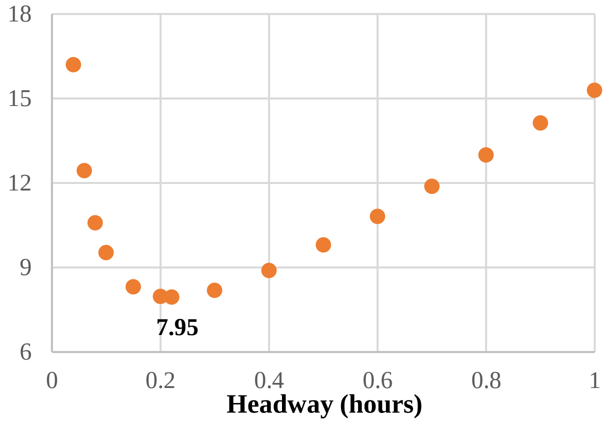


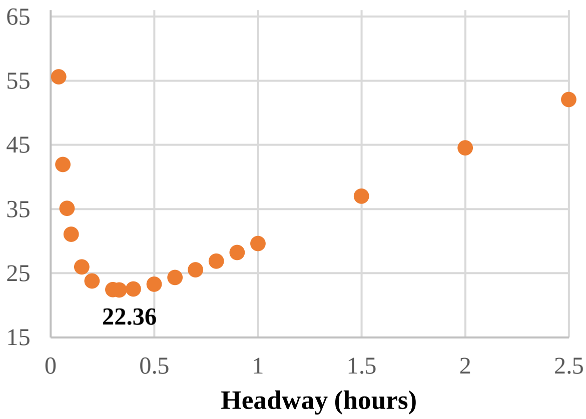


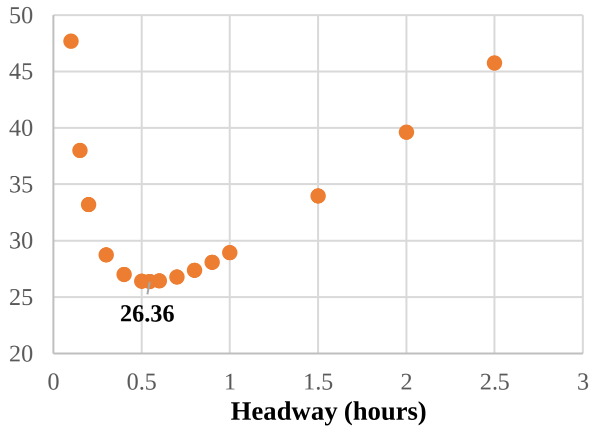












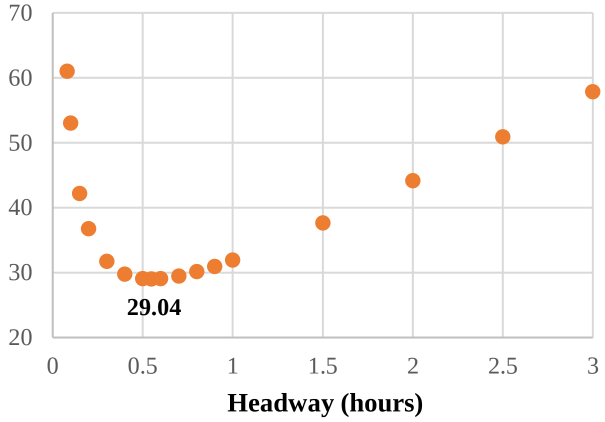


Figure 12. Average costs against headway in uncoordinated nine-region network.
[Source: prepared by the author]