

European Puts, Credit Protection, and Endogenous Default*

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June 26, 2019

Abstract

In a default corridor $[0, B]$ that the stock price can never enter, a deep out-of-the-money American put replicates a pure credit contract (Carr and Wu, 2011). Assuming discrete (one-period-ahead predictable) cash flows, we show that an endogenous credit-risk model generates, along with the default event, a default corridor at the *cash-outflow* dates, where $B > 0$ is given by these outflows (i.e., debt service and negative earnings minus dividends). In this endogenous setting, however, the credit-contract replicating put is not American, but rather European. Specifically, the crucial assumption that determines an endogenous default corridor at the cash-outflow dates is equityholders' deep pockets absorb these outflows; that is, no equityholders' fresh money, no endogenous corridor.

*I am grateful to Carlos Bellón, Carlos Velasco, and especially Peter Carr and Eduardo Schwartz for helpful comments. Any remaining errors are my own.

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1 Introduction

In Merton (1974), a key result is the link between put options and credit protection. For a leveraged firm, a corporate bond is the sum of a riskless bond and a short put on the firm’s assets, which is an insight extended along many avenues (see Carr and Wu, 2011). A derivative contract that directly provides exposition to a firm’s credit risk also exists, namely, credit default swaps. In a striking work, Carr and Wu (2011) introduce a default corridor in which deep out-of-the-money (DOOM) equity puts mirror pure credit contracts—and provide the same direct credit-risk exposition. Specifically, in a default corridor, “the stock price stays above a barrier B before default but drops below a lower barrier A after default, thus generating a default corridor $[A, B]$ that the stock price can never enter”. In this scenario, “a spread between two co-terminal American put options struck within the corridor replicates a pure credit contract, paying off when and only when default occurs prior to the option expiry” (p. 474).

As Carr and Wu show, a default corridor $[0, B]$ occurs if A vanishes ($A = 0$). In this simple case, a single DOOM American put (i.e., struck within this corridor $[0, B]$) becomes a digital put, which replicates a pure credit contract. That is, the price of a DOOM American put is linear in the strike price falling within the corridor $[0, B]$; the forward price of the DOOM European counterpart, scaled by the strike price, gives the one-period risk-neutral default probability. Both American and European DOOM puts essentially provide the same credit protection in a default corridor, only that exchange-listed options in the US are American style. If equity prices can jump to zero (Merton’s jump-to-default model, 1976), a default corridor $[0, B]$ can be readily accommodated, if the jump to zero is from above the barrier B (Carr and Wu).

In this paper, we study whether endogenous credit-risk models also generate a default corridor. Assuming discrete cash flows, we show that an endogenous credit-risk model generates, along with the (endogenous) default event, a default corridor $[0, B]$ in a natural way. In this endogenous setting, however, the default corridor only necessarily

happens at the cash-*outflow* dates, where $B > 0$ is given by these outflows, which we require to be one-period-ahead predictable. It follows the equity put that replicates a pure credit contract is not DOOM American, but rather European, expires at outflow dates, holds for any moneyness, and has a strike price lower than the outflows (B).

In an *endogenous* model, besides limited liability, which implies nonnegative equity prices, equityholders optimize the default decision and absorb (with their deep pockets) the firm's net cash flows—in Leland's tradition. This discrete-time cash flow is either an inflow (e.g., a cash dividend) or an outflow. Then an endogenous setting generates, along with the default event, a default corridor at the outflow dates. That is, in contrast to the ex-dividend equity price, which falls on dividend days, if $B_1 > 0$ is the first outflow, either the ex-cash-flow equity price increases and is larger than B_1 at the first-outflow date if the firm survives, or it is equal to zero if the firm defaults (i.e., the equity continuation value C_1 is less than or equal to B_1). Specifically, the ex-cash-flow equity price equals $C_1 \times 1_{\{C_1 > B_1\}}$, which is either zero or larger than $B_1 > 0$.

Therefore, this model simultaneously generates a default event $\{C_1 \leq B_1\}$ and a corridor $[0, B_1]$ at the first-outflow date (i.e., $C_1 \times 1_{\{C_1 > B_1\}} \notin (0, B_1]$), in which the stock price cannot enter and in which B_1 is equal to the outflow. By contrast, if equityholders do not absorb the outflow B_1 , but instead B_1 is subtracted from the firm's assets or is refinanced, default is delayed until the assets are entirely depleted and a corridor does not exist (i.e., the equity price can be arbitrarily small). Because cash flows are random, that B_1 is one-period predictable (or paid in arrears or a lower bound) is a necessary requirement in our setting; otherwise, the corridor $[0, B_1]$ is random. Other than being one-period predictable, cash flows are unconstrained. Hence, the corridor $[0, B_1]$ is robust to the process followed by them and the endogenous equity price.

An example in which B_1 is deterministic is when B_1 is given by a bond's coupon or principal (Merton, 1974; Geske, 1977). In this scenario, if equityholders absorb all coupons, a default corridor is present at all coupon dates. In general, B_1 includes both financial and operational leverage, namely, debt service and negative earnings minus

dividends. Naturally, if the predictable outflow is rather zero or an *inflow* $B_1 \leq 0$, such as a cash dividend, defaulting is entirely irrational (i.e., assuming $C_1 > 0$), implying no credit protection is necessary and the corridor $[0, B_1]$ is empty.

Indeed, as in the standard case of call options, we assume endogenous equity prices are strictly positive between outflow dates, which has two implications. First, default is never optimal, and hence a corridor does not exist (is empty) between two outflow dates. Namely, equity prices can be arbitrarily small without triggering default. Second, a deep in-the-money American put is optimally exercised before expiration (Duffie, 2001). Therefore, only the European put counterpart replicates a pure credit contract, with a maturity equal to the first-outflow date and a strike price (K) less than or equal to the outflow— B_1 . That is, the binary payoff of this low strike-price put is given by

$$\max \{0, K - C_1 \times 1_{\{C_1 > B_1\}}\} = K \times 1_{\{C_1 \leq B_1\}} \text{ if } K \leq B_1.$$

Moreover, although in Carr-Wu’s model, in-the-money American/European puts struck within the corridor $[0, B_1]$ cannot exist because the equity price can never enter in the corridor, in our setting, in-the-money puts exist; that is, the credit-contract replicating put is not necessarily DOOM, but rather a low strike-price (LSP) European put.

The distortion created by an endogenous default corridor implies that only the price of European (not the American) puts that expire at the first-outflow date is linear in the strike price falling within $[0, B_1]$. This linear price yields an implied-volatility skew for low strike prices $K \leq B_1$, whereby this implied volatility soars because, within the corridor, the put payoff is not the capped difference between the strike and equity prices as in a benchmark setting, but rather the strike price (i.e., $\max \{0, K - C_1\} < K$). It follows that for riskier firms, such as speculative-grade firms, we can have linear-in-the-strike-price (falling within $[0, B_1]$) put prices and a steep skew at maturities matching cash-outflows dates. These firms are more leveraged, which potentially leads to larger outflows, $B_1 > 0$. By contrast, for investment-grade firms, these linear-in-

the-strike put prices should correspond to a thinner corridor.

For puts with a shorter maturity than the first-outflow date, a default corridor is not there (i.e., equity prices can be small without triggering default). By contrast, for puts with slightly larger maturity than this first-outflow date, in which a corridor $[0, B_1]$ means the ex-cash-flow equity price is either 0 or larger than $B_1 > 0$, the equity price is also likely larger than B_1 conditional on nondefault, and the low strike-price put can still provide a good approximation to a digital put—a pure credit contract. Naturally, after the first period ends and (conditional on nondefault) the second period becomes a new first period, we again have a default corridor (if a new outflow $B_2 > 0$ exists).

For a same interval $[0, B_1]$, DOOM European puts expiring at the first outflow date provide the same credit protection in Carr and Wu's as in our endogenous corridor. Then, if we see a credit default swap as a multiperiod insurance contract, the protection leg of this long-term contract is replicated by rolling over LSP European puts.¹ However, if not corridor exists, and default happens (i.e., the equity price equals zero) before the option maturity, although the American put could be exercised in the money prior to default, the European put always pays the largest amount, that is, the strike price. Conversely, if the European put ends out-of-the-money, the firm survives. It follows the price of LSP European puts and credit-default-swap rates must be linked, and empirically correlated (Carr and Wu).²

Finally, although endogenous default is standard in credit-risk modelling (Duffie, 2001), firms also get short-term financing or rollover expiring debt in practice. For this reason, we provide an extension of the endogenous corridor. Two other extensions are provided latter; one of them is motivated by the (tentative) takeover of a distress firm.

Perhaps in normal times, that is, when defaulting is not an option for equityholders, the firm can easily absorb/finance any outflows. However, perhaps in bad times, that is, when defaulting is an option, debt markets are tapered, and therefore only fresh

¹And, if necessary, entering in a swap on the recovery rates of corporates bonds on default.

²For low interest rates, the prices of DOOM European and American puts are similar too.

injections of new capital can support the outflows and daily operations of the firm, keeping the same firm alive.³ In this scenario, that is, in the bad times, a default corridor exists. For example, consider equityholders have some liquidity issues, and cannot absorb the outflow, $B_1 > 0$. It is logical to assume they will transfer the firm to the debtholders, perhaps for an amount distinct from $C_1 - B_1$, rather than default. Then, if debtholders cannot refinance B_1 with a second (and new) lender, they have to absorb B_1 as well, and therefore the same default corridor $[0, B_1]$ exists.

Section 2 motivates our endogenous-default corridor in a coupon-bond model. Section 3 prices European puts in a default corridor. Section 4 provides two extensions of the corridor, and relates our work to the literature. Section 5 concludes. An appendix contains omitted proofs and shows the corridor in a general discrete-time setting.

2 A coupon-bond model

We consider a two-period model $N = 2$, $n \in \{1, 2\}$ and respective times $0 < T_1 < T_2$. We denote by V_t the value of the firm's assets, $0 \leq t \leq T_2$. The firm issues a two-period coupon bond (Merton, 1974; Geske, 1977), where T_2 is the maturity, $D > 0$ is the face value, and $c \times D > 0$ is the coupon. In this structural setting, we denote by r the riskless rate and assume a Q risk-neutral measure exists.

As in any endogenous credit-risk model, the debt service (coupon or face value) is absorbed by equityholders' deep pockets (e.g., Leland, 1994; Leland and Toft, 1996's rollover model; Manso et al., 2010's performance-sensitive debt model; Carr and Wu's structural model; Duffie, 2001). Otherwise, if the debt service is subtracted from the firm's assets or is simply refinanced, default is delayed until these assets are entirely depleted. Then, $B_1 = cD > 0$ and $B_2 = (1 + c)D > 0$ are the two respective cash outflows at T_1 and T_2 .

³In bad times, retained earnings are exhausted and short-term financing soars. Rolling over the full face value of debt leads to a maturity rat race (Brunnermeier and Oehmke, 2013). Binding covenants may limit further indebtedness. Directly selling the firm's assets may be expensive, carrying a discount.

We denote by C_t the equity continuation value. We assume equityholders' limited liability, which implies $C_t \geq 0$, $0 \leq t \leq T_2$. It follows that because $B_n > 0$, defaulting at T_n is optimal if and only if

$$C_n \leq B_n, n = 1, 2,$$

with indifference between defaulting and paying the cash outflow if $C_n = B_n$. This default choice maximizes equity value; that is, it is endogenous.

2.1 Endogenous equity pricing

Given the terminal assets value ($V_2 \geq 0$), the continuation value of equity is defined recursively as follows (where, in an abuse of notation, $C_n = C_{T_n}$, $n = 1, 2$):

$$\begin{aligned} C_2 &= V_2 \geq 0, \\ C_t &= E_t^Q [e^{-r(T_2-t)} \max \{0, C_2 - (1+c) \times D\}] \geq 0, T_1 \leq t < T_2, \\ C_t &= E_t^Q [e^{-r(T_1-t)} \max \{0, C_1 - c \times D\}] \geq 0, 0 \leq t < T_1. \end{aligned} \tag{1}$$

In particular, C_1 is the price of a European call, whereas C_0 is the price of a compound option. The definition of C_1 and C_2 recognizes the debt service (i.e., the coupon cD and $(1+c)D$, respectively) is absorbed by equityholders' deep pockets, and is never subtracted from the firm's assets (i.e., from V_1 and V_2).

Importantly, the process C_t is always discontinuous at T_1 (and T_2); that is,

$$\lim_{t \uparrow T_1} C_t \rightarrow \max \{0, C_1 - c \times D\} < C_1 \text{ if } c \times D > 0. \tag{2}$$

Although the left-hand-side limit is only well defined if C_t does not jump at $t = T_1$ (where $t \uparrow T_1$ means $t \rightarrow T_1$, $t < T_1$), the inequality is always correct (if $C_1 > 0$).

In addition, we assume the equity continuation value is strictly positive; that is,

$$C_t > 0, t \in [0, T_2]. \quad (3)$$

As shown next, this assumption implies equity prices are also strictly positive between cash-outflow dates (i.e., conditional on no previous default), and default is never optimal outside the outflow dates.

To define the ex-cash-flow equity price (denoted by E), which is subject to default risk, we introduce an auxiliary binary process, $a \in \{0, 1\}$. Namely, $a_0 = 1$ and

$$a_n = a_{n-1} \times 1_{\{C_n > B_n\}}, n = 1, 2, \dots, N, \quad (4)$$

and hence $a_n = 0$ indicates the company has defaulted (i.e., $a_j = 0, j = n, n+1, \dots, N$).

Consequently,

$$E_2 = a_2 \times C_2 = 1_{\{C_1 > B_1\}} \times 1_{\{C_2 > B_2\}} \times C_2, \quad (5)$$

$$E_t = a_1 \times C_t = 1_{\{C_1 > B_1\}} \times C_t, T_1 \leq t < T_2,$$

$$E_t = a_0 \times C_t = C_t, 0 \leq t < T_1.$$

The (ex-cash-flow) equity-price function $E_n(C_n)$ is discontinuous at T_1 and T_2 .

That is,

$$E_1 = 0 \text{ if } C_1 \leq c \times D, \quad (6)$$

$$E_1 = C_1 > c \times D > 0 \text{ otherwise,}$$

both with positive probability.⁴ Likewise, either

$$E_2 = 0 \quad \text{or} \quad E_2 = C_2 > (1 + c) \times D > 0.$$

In particular, because equityowners absorb the entire debt service, conditional on non-default ($a_2 = 1$), the equity value equals the asset value at T_2 ; that is, $E_2 = V_2$.

Moreover, the discontinuity at T_1 implies the equity-price function is also close to discontinuous right after T_1 . That is, if $a_1 = 1$ and $T_1 < t < T_2$, $E_t = C_t$, where $C_t > 0$ can be arbitrarily close to zero. However, in the limit $t \downarrow T_1$, $E_1 = C_1 > c \times D$, which implies that for $t > T_1$ and t not far away from T_1 , the probability that $C_t \in [0, c \times D]$ should be relatively small (i.e., a function of a small $t - T_1$).

Importantly, $C_t > 0$ implies $E_t = C_t > 0$ between outflow dates, conditional on no previous default. Strictly positive equity prices imply default is not optimal between outflow dates. Hence, we next focus on outflow dates.

2.2 The endogenous default event and the default corridor

Implicit in the definition of the equity value (i.e., equation (5)) are two endogenous-default events at periods T_1 and T_2 , that is,

$$\{C_1 \leq c \times D\} \quad \text{and} \quad \{C_2 \leq (1 + c) \times D\}, \quad (7)$$

respectively. These two events are endogenous because they maximize equity value, and are default events because they imply equity value becomes zero. These two default

⁴The process E_t is also discontinuous at T_1 if the firm survives (otherwise, is zero); that is,

$$\lim_{t \uparrow T_1} E_t \rightarrow \max\{0, C_1 - c \times D\} < 1_{\{C_1 > c \times D\}} \times C_1 \quad \text{if } C_1 > c \times D \text{ and } c \times D > 0,$$

where the limit is only well defined if C_t does not jump at $t = T_1$.

events define two respective optimal default thresholds, Y_1 and Y_2 . That is,

$$\begin{aligned} V_1 &= Y_1 : C_1(Y_1) = c \times D \quad \text{and} \quad \{C_1(V_1) \leq c \times D\} \equiv \{V_1 \leq Y_1\}, \\ V_2 &= Y_2 : C_2(Y_2) = (1 + c) \times D \quad \text{and} \quad \{C_2(V_2) \leq (1 + c) \times D\} \equiv \{V_2 \leq Y_2\}. \end{aligned}$$

In general, $Y_1 > 0$ and $Y_2 > 0$ are unique because (i.e., we assume if necessary that) call-type payoffs are increasing functions in V . If V depends on stochastic parameters, Y_1 and Y_2 are threshold functions.

Moreover, because the outflows are absorbed by equityholders' deep pockets, from equation (6), these two endogenous default events lead to two default corridors,

$$[0, c \times D] \quad \text{and} \quad [0, (1 + c) \times D], \quad (8)$$

respectively, in which the ex-cash-flow equity price cannot enter. That is, $E_1 \notin (0, c \times D]$ and $E_2 \notin (0, (1 + c) \times D]$, at periods T_1 and T_2 , respectively. Thus, at T_1 , the following four default events are equivalent:

$$\{V_1 \leq Y_1\}, \quad \{C_1 \leq c \times D\}, \quad \{E_1 \leq c \times D\}, \quad \text{and} \quad \{E_1 = 0\},$$

which depend, respectively, on the firm's low asset value, low continuation value, low equity value, and exhausted equity value.

Remark 1. We provide an example in which a default corridor is empty. If $c \times D$ is exclusively paid from the firm's assets, the default event at T_1 is trivially given in terms of the asset value, namely, by $\{V_1 \leq c \times D\}$. This specific default event implies a default corridor does not exist at T_1 , because the equity-price function is continuous (i.e., zero or larger than zero). That is, if $\lim_{t \uparrow T_1} V_t = c \times D + \xi$, with $\xi > 0$ and $t < T_1$, $V_1 = \xi$ (after the coupon-payment date), and we assume the value of equity E is arbitrarily close to zero if $\xi \rightarrow 0$.

Moreover, if $C_1 \leq V_1$, that is, if equity value is bound by the value of the assets,

endogenous default leads to an earlier default than if the firm's managers entirely deplete the firm's assets or holdings previous to default.

Remark 2. As in Merton, $C_1 > 0$ is the premium of a European call that expires at T_2 (with a strike price equal to $(1 + c) \times D$). However, if we consider that $C_2 = \{V_2 - D\}^+$, a default corridor exists at T_2 , but it is very thin; that is, $[0, c \times D]$. Note that if we define

$$C_t = E_t^Q [e^{-r(T_2-t)} \times \max \{0, C_2 - c \times D\}], \quad T_1 \leq t < T_2,$$

the continuation value (C) is the same process as in equation (1), because

$$\{V_2 - (1 + c) \times D\}^+ = \{\{V_2 - D\}^+ - c \times D\}^+,$$

implying the definition of the equity price, default events, and default corridor are robust and carry over for $t < T_2$.

If $C_2 = \{V_2 - (1 + c) D\}^+$, that is, if the entire debt service (coupon and face value) is depleted from the firm's assets, $E_2 = a_1 \times C_2$ and a default corridor is empty at T_2 .

2.3 European puts, digital puts, and pure credit contracts

We denote by K the strike price of puts and calls. At T_1 , we show a low strike-price European equity put becomes a digital put, which replicates a pure credit contract. That is, for a put with maturity T_1 , the payoff reduces to

$$\begin{aligned} \max \{0, K - E_1\} &= (K - E_1) \times 1_{\{E_1 \leq K\}} & (9) \\ &= K \times 1_{\{E_1=0\}} + (K - E_1) \times 1_{\{c \times D < E_1 \leq K\}} \\ &= K \times 1_{\{E_1=0\}} \quad \text{if } K \leq c \times D, \end{aligned}$$

which is a digital option in the case of low strike-price (LSP) puts, namely, $K \leq c \times D$.

The second equality follows from equation (6).

Then, from equation (6), $E_n = E_n \times 1_{\{E_n > B_n\}}$, from which follows $1_{\{E_1=0\}} = 1_{\{E_1 \leq c \times D\}}$, and hence

$$\max \{0, K - E_1\} = K \times 1_{\{E_1 \leq c \times D\}} \quad \text{if } K \leq c \times D, \quad (10)$$

which replicates a pure credit contract, in which $\{E_1 \leq c \times D\}$ is the endogenous-default event. In particular, in this endogenous setting, a DOOM put (that replicates a pure credit contract) is rather an LSP put, for all moneyness.

A similar result follows for T_2 , in which $B_2 = (1 + c) \times D$; namely,

$$\max \{0, K - E_2\} = K \times 1_{\{E_2 \leq (1+c) \times D\}} \quad \text{if } K \leq (1 + c) \times D. \quad (11)$$

3 The price of European puts in a default corridor

We study the pricing of European puts/calls in a default corridor.

1) Under the Q -measure, from equation (10), the price of an LSP put with maturity T_1 is given by

$$E_0^Q [e^{-rT_1} K \times 1_{\{E_1 \leq c \times D\}}] = e^{-rT_1} K \times Q(V_1 \leq Y_1), \quad K \leq c \times D. \quad (12)$$

Like DOOM puts in Carr and Wu, in our setting, LSP European-put prices are linear in the strike price falling within the corridor (i.e., $K \leq c \times D$), a straightforward empirical prediction. The forward price of this European put, scaled by the strike price, gives the one-period risk-neutral default probability.

2) In terms of the implied volatility σ , where $P^{BS}(E_0, \sigma)$ denotes the Black-Scholes-

Merton put-price formula, we have that

$$e^{-rT_1} K \times Q(V_1 \leq Y_1) = P^{BS}(E_0, \sigma). \quad (13)$$

Then the implied-volatility curve $\sigma(K)$ holds that (see the Appendix)

$$\sigma'(K) \times \sqrt{T_1} = \frac{-N(-d_1)}{K \times N'(d_1)} < 0, \quad K \leq c \times D, \quad (14)$$

where $N(\cdot)$ is the cumulative Gaussian-distribution function.

A negative skew, $\sigma'(K) < 0$, implies LSP puts are expensive. First, they are more expensive in a default corridor than in the standard case of no corridor. That is, compared to a benchmark setting, LSP European puts that expire at T_1 are overpriced by the following amount:

$$e^{-rT_1} \times E_0^Q [e^{-rT_1} C_1 \times 1_{\{C_1 \leq \min\{K, B_1\}\}}] + e^{-rT_1} K \times E_0^Q [1_{\{K < C_1 \leq B_1\}}] > 0.$$

Conditional on $C_1 \in [0, B_1]$, the first (second) term covers the in-the-money (out-of-the-money) part of the put. Second, the deeper out of the money the put is, the more expensive this put is in implied-volatility units. For instance, in our numerical exercise, $\sigma(K)$ is unbounded when $K \rightarrow 0$.

3) The same linear (in the strike price) result happens in the case of the second-outflow date T_2 , in which $B_2 = (1+c) \times D$. The price of an LSP put with maturity T_2 is given by

$$\begin{aligned} & E_0^Q [e^{-rT_2} K \times 1_{\{E_2 \leq (1+c) \times D\}}] \\ &= E_0^Q [e^{-rT_2} K \times 1_{\{1_{\{C_1 > c \times D\}} \times 1_{\{C_2 > (1+c) \times D\}} \times V_2 \leq (1+c) \times D\}}] \\ &= e^{-rT_2} K \times (Q(V_1 \leq Y_1) + Q(V_1 > Y_1) \times Q(V_2 \leq Y_2)), \quad K \leq (1+c) \times D. \end{aligned} \quad (15)$$

Scaled by the discounted strike price $e^{-rT_n} K$, the price difference of two LSP European

puts with the same strike but different maturity equals the probability of default at T_2 , namely,

$$Q(V_1 > Y_1) \times Q(V_2 \leq Y_2) > 0, K \leq c \times D,$$

given that $c \times D < (1 + c) \times D$.

4) For no LSP European puts at T_1 (i.e., $K > c \times D$),

$$\begin{aligned} & E_0^Q [e^{-rT_1} \max\{0, K - E_1\}] \\ &= e^{-rT_1} K \times Q(V_1 \leq Y_1) + E_0^Q [e^{-rT_1} (K - E_1) \times 1_{\{c \times D < E_1 < K\}}], K > c \times D, \end{aligned} \quad (16)$$

and no LSP puts are also more expensive than in a benchmark setting.

5) Lastly, for European equity puts and calls (with respective prices p_t and c_t), with the same strike price (K) and expiring at the first-outflow date (T_1),

$$K + \max\{0, E_1 - K\} = E_1 + \max\{0, K - E_1\},$$

which follows from put-call parity at T_1 . Then, from the law of one price (see the Appendix), put-call parity at T_0 becomes

$$e^{-rT_1} K + c_0 = E_0 + e^{-rT_1} B_1 \times Q(V_1 > Y_1) + p_0, \quad (17)$$

where $B_1 = c \times D$ and $Q(V_1 > Y_1) = Q(E_1 > B_1)$. Similar to the case of a paying-dividend stock, put-call parity is also adjusted, in this case, by $e^{-rT_1} B_1 \times Q(V_1 > Y_1)$.

From the last equation, the call price is given by

$$c_0 = E_0 + e^{-rT_1} B_1 \times Q(V_1 > Y_1) + p_0 - e^{-rT_1} K. \quad (18)$$

Specifically, for DOOM puts (i.e., $K \leq B_1$), from equation (12),

$$c_0 = E_0 - e^{-rT_1} (K - B_1) \times Q(V_1 > Y_1). \quad (19)$$

For example, consider a spread between two co-terminal European calls struck within the corridor, with respective strike prices K_1 and K_2 , $K_1 < K_2 \leq B_1$. Then,

$$e^{rT_1} \times \frac{c_0(K_1) - c_0(K_2)}{K_2 - K_1} = Q(V_1 > Y_1), \quad (20)$$

which is the one-period risk-neutral surviving probability.

3.1 Numerical example: Merton's structural model

Following Merton (1974), we consider a one-period setting in which equity is a European call, with a strike price of $B > 0$ and a maturity of T_1 (which are the face value and maturity of a zero-coupon bond). In particular, Merton's model is widely used to infer a firm's distance to default.

First, we emphasize that although puts with the same maturity as debt are expensive if a default corridor exists, in which case $B_1 = B$, they are more expensive in the original Merton's noncorridor model. That is, if V_1 is the asset value, although the put payoff in a benchmark setting is $\max\{0, K - V_1\}$, this payoff, which equals the strike price K within the corridor $[0, B]$ if $K \leq B$, is increased by $(K + B - V_1) \times 1_{\{B < V_1 \leq K+B\}}$ in the latter noncorridor model. Namely, if $K \leq B$, these three payoffs, where $\{V_1 \leq B\}$ is the unique default event, hold:

$$\underbrace{(K - V_1) \times 1_{\{V_1 \leq K\}}}_{\substack{\text{noncorridor exists} \\ \text{benchmark setting} \\ (V_1 \text{ is the asset value;} \\ K \text{ is the put strike price)}}} < \underbrace{K \times 1_{\{V_1 \leq B\}}}_{\substack{\text{here, a default corridor} \\ \text{exists } (B_1=B) \\ \text{equityholders absorb debt}}} \leq \underbrace{K \times 1_{\{V_1 \leq B\}} + (K + B - V_1) \times 1_{\{B < V_1 \leq K+B\}}}_{\substack{\text{this part implies noncorridor exists } (B_1=0) \\ \text{debt is repaid by selling the firm's assets (original model)}}}. \\ \text{Merton's structural model } (B \text{ is the face value of a zero-coupon bond})$$

For maturities shorter than the debt maturity, a corridor does not exist—the two put payoffs in Merton's model are the same.

Second, we assume a lognormal asset value, $\ln \frac{V_1}{V_0} \sim \mathcal{N}\left(\left(r - \frac{\sigma^2}{2}\right)T_1, \sigma\sqrt{T_1}\right)$, where r is the riskless rate and σ is volatility. From the Black-Scholes-Merton formula, the

equity price is equal to

$$\begin{aligned} E_0(V_0) &= V_0 \times N(d_{1B}) - e^{-rT_1} B \times N(d_{2B}), \\ d_{1B} &= \frac{\ln \frac{V_0}{B} + (r + \sigma^2/2) \times T_1}{\sigma \sqrt{T_1}} \text{ and } d_{2B} = d_{1B} - \sigma \sqrt{T_1}, \end{aligned}$$

However, we assume equityholders absorb the outflow $B > 0$, which implies a default corridor $[0, B]$ exists at T_1 . That is,

$$E_1 = \max\{0, V_1 - B\} + B \times 1_{\{V_1 > B\}} = V_1 \times 1_{\{V_1 > B\}}.$$

Consider a European equity put, with strike price $K > 0$ and maturity T_1 as well. Given the same maturity of the equity claim (or Merton's call) and this equity-put derivative, and given that $E_1 = V_1 \times 1_{\{V_1 > B\}}$, the price of this equity put simplifies to

$$\begin{aligned} p_0 &= E_0^Q [e^{-rT_1} \max\{0, K - E_1\}] & (21) \\ &= e^{-rT_1} K \times E_0^Q [1_{\{V_1 \leq B\}}] + E_0^Q [e^{-rT_1} (K - V_1) \times 1_{\{B < V_1 \leq K\}}] \\ &= e^{-rT_1} K \times N(-d_{2B}) \\ &\quad + (e^{-rT_1} K \times (N(-d_{2K}) - N(-d_{2B})) - V_0 \times (N(d_{1K}) - V_0 \times N(d_{1B}))) \times 1_{\{K > B\}}, \end{aligned}$$

where $d_{1,2K}$ are defined akin to $d_{1,2B}$ with K instead of B . In particular,

$$p_0 = e^{-rT_1} K \times N(-d_{2B}), \quad K \leq B.$$

Conversely, for a reciprocal call with the same maturity and strike price, the payoff in terms of the asset value is given by

$$\begin{aligned} \max\{0, E_1 - K\} &= \max\{0, V_1 \times 1_{\{V_1 > B\}} - K\} \\ &= \max\{0, V_1 - K\} \times 1_{\{K \geq B\}} + (V_1 - K) \times 1_{\{K < B < V_1\}}, \end{aligned}$$

where the first (second) term corresponds to strike prices higher (lower) than B . It follows that, in contrast to a benchmark setting, low strike-price (i.e., $K < B$) calls are underpriced, because they pay nothing if $V_1 \in (K, B]$.

Following Carr and Wu, we define B as a low strike price, $B = 3$. We assume $\sigma = 30\%$, $r = 2\%$, a maturity of $T_1 = 6$ months, and four equity prices $E_0 = \{2.03, 3.03, 4.03, 6.03\}$, which are associated with the asset values of $V_0 = \{5, 6, 7, 9\}$, respectively. Each price implies a risky, healthier, sound, and super sound firm. For asset values lower than 5, the implied volatility of low strike-price equity puts quickly soars above 100%. In Figure 1, we show the four implied-volatility curves, for a range of strike prices $K \in [1, 20]$. Hence, the volatility-smile function, $\sigma(K)$, solves

$$p_0(\sigma = 0.3) = e^{-rT_1}K \times N(-d_{2K}, \sigma(K)) - V_0 \times N(-d_{1K}, \sigma(K)), \quad (22)$$

and in particular, for $K \leq B$,

$$e^{-rT_1}K \times N(-d_{2B}, \sigma = 0.3) = e^{-rT_1}K \times N(-d_{2K}, \sigma(K)) - V_0 \times N(-d_{1K}, \sigma(K)).$$

From Figure 1, for risky firms (i.e., $V_0 \leq 6$), the default corridor generates a clear volatility smile, with large implied-volatility levels away from the money. However, for sound firms (i.e., $V_0 \geq 7$), the default corridor generates more of a volatility smirk, where the implied volatility approximates the asset volatility for strike prices higher than $B = 3$. We also see the riskier the firm, namely, the lower V_0 , the larger the implied volatility, which is an example of a leverage effect. These results are robust to the maturity, $T_1 \in \{3, 6, 12\}$ months.

***** to include Figure 1 *****

4 Extensions of the Endogenous Default Corridor

We now provide two extensions. First, that equityholders absorb cash outflows in bad times is sufficient for the existence of an endogenous default corridor. Specifically, because we assume a firm cannot refinance any outflow $B_1 > 0$, the default corridor and the defaulting region are given by the same interval $[0, B_1]$. However, if we split the surviving region (B_1, ∞) into two complementary regions, $(B_1, b]$ and $[b, \infty)$, where $B_1 \leq b$, but we allow refinancing in the good-times region $[b, \infty)$, the default corridor is given by $[0, \min(B_1, b - B_1)]$. It follows that, if good times start soon (i.e., if $b \leq 2B_1$), we have a thinner corridor $[0, b - B_1]$ at the first outflow date; otherwise, the corridor is $[0, B_1]$. Again, how equityholders' deep pockets absorb the cash outflows explains the *size* of the default corridor.

Second, as any endogenous model assumes, substantial evidence shows default is not entirely random, but rather firms default in poor economic conditions or with expired debt (Asquith et al., 1994; Duffie et al., 2007; Campbell et al., 2008; Davydenko, 2012). However, because a firm's bankruptcy has severe economic consequences from layoffs to large distress costs, creditors may have a say in default (Carey and Gordy, 2007).

Motivated by this strategic say, assume the following scenario in which endogenous default is rather delayed. In hard times, in the interval $[0, B_1]$, the equity price is either 0 or larger than B_{\min} ($0 < B_{\min} \leq B_1$) until all uncertainty is solved—in which case either $E_1 = 0$ or, if the firm survives, $E_1 > B_1$. This setting supports Carr-Wu's corridor $[0, B_{\min}]$ after the outflow date and until uncertainty is solved, as well as an endogenous corridor $[0, B_1]$ if the uncertainty has an expiring time. In this setting, the key assumption is the equity price cannot slip in the corridor, namely, if B_{\min} is meaningful. We illustrate this example with the (tentative) takeover of a distress firm.⁵

The distress firm needs fresh capital, and a large shareholder (L1) announces a

⁵Specifically, Dia, which is a Spanish retailer, with 46,000 employees, and is present in several countries. Like many other retailers, because of serious competition in this sector, it is in difficult economic (poor sales) and financial (large expiring debt) conditions. See, for example, the following article, https://elpais.com/economia/2019/04/26/actualidad/1556263740_217086.html

capital-injection plan, but only if (in addition to banks extending a new credit line and bondholders increasing the maturity of expiring loans so all stakeholders contribute) it gets the control of the firm ($>50\%$ of shares). L1 offers a price per share of 0.67; an offer expiring in six months. In six months, two mutually exclusive scenarios are possible, either L1 does not get control and hence (no capital injection but firm default and) the equity price sinks to zero, or L1 gets control and the price per share is 0.67 or above.

However, because other equityholders want a better deal than 0.67, they not only put at risk the L1-control plans and solvency of the firm, but also push down the price per share to a low 0.34 during this six-month period. Associated to this potential takeover, two corridors exist for the troubled firm. First, a Carr and Wu's corridor $[0, 0.34]$ during the six months. Second, a corridor in six months, at the offer expiring, in which the price is either zero or above 0.67. The following (DIA Spanish retailer) stock price in Figure 2 seems consistent with both corridors.

***** to include Figure 2 *****

For this distressed firm, in a perfect world, the quotes of low strike-price American/European puts or credit default swaps expiring in six months can be used to get the risk-neutral probability of L1 not gaining control, and the firm stock price jumping to zero. Actually, a price per share of 0.34 represents a 50% risk-neutral probability of each of the two scenarios.

Related literature This paper is related to two strands of the literature. First, it is related to the literature on the link between tail risk, credit risk, and equity derivatives and on the spanning property of option markets (Cremers et al., 2008; Coval et al., 2009; Carr and Wu, 2009 and 2010; Collin-Dufresne et al., 2012; He et al., 2017; Cheng et al., 2018). Kelly et al. (2016) study the US financial-sector tail risk during the 2007-2009 crisis from the price of out-of-the-money puts. Culp et al. (2018) empirically extend Merton's put insights. Siriwardane (2018) uses Carr and Wu's default corridor to infer credit-risk spreads. Ibáñez (2018) develops a measure of default risk based on

Leland-type models. This measure is linked to the default corridor/event of the same endogenous model.

Second, it is related to the literature on the valuation of derivative securities in structural models, a problem that until recently (Geske et al., 2016; Bai et al., 2018) has received limited attention (Toft and Prucik, 1997). Bai et al. show Merton's model explains puts on a risky financial sector better than a benchmark setting. Specifically, all these works emphasize that even if the asset volatility is constant as in Merton's model, structural models generate a leverage effect in a natural way, because equity is a call option on leveraged assets. In all extant models, however, a default corridor does not exist, which is in contrast to Carr and Wu as well as this paper.

5 Concluding remarks

This paper shows an endogenous credit-risk model generates a default corridor $[0, B]$. This corridor is linked to the endogenous-default event, $\{E \leq B\}$, in which E is the ex-cash-flow equity price and $B > 0$ is given by the *outflows* (i.e., debt service and negative earnings minus cash dividends), which we only require to be one-period-ahead predictable. In this setting, the default corridor only necessarily happens at the outflow dates, which implies the low strike-price put (that replicates a pure credit contract) is not DOOM American but rather European style, and expires in the first outflow with a strike less than or equal to the outflow. The corridor $[0, B_1]$ especially applies to speculative-grade firms; these firms are more leveraged, which implies a larger outflow $B_1 > 0$. If the early-exercise premium is a small fraction of the American price, such as in a DOOM case, the exchanged-listed American price can provide a good approximation to the European counterpart, namely, a pure credit contract.

In sum, although endogenous credit-risk models are preferred by academicians, reduced-form models and credit-risk sensitive securities, such as credit default swaps or DOOM puts, are the workhorse for practitioners. In this paper, we bridge the gap

between both worlds by deriving a default corridor in the former setting.

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6 Online Appendix

The implied volatility of low strike-price European equity puts

For a put option, from the Black-Scholes-Merton formula,

$$\begin{aligned} P^{BS}(E_0) &= e^{-rT_1}K \times N(-d_2) - E_0 \times N(-d_1), \\ d_1 &= \frac{\ln \frac{E_0}{K} + (r + \sigma^2/2) \times T_1}{\sigma \sqrt{T_1}} \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{T_1}, \end{aligned}$$

which implies the following two Greeks:

$$\frac{\partial P^{BS}}{\partial K} = e^{-rT_1} \times N(-d_2) \quad \text{and} \quad \frac{\partial P^{BS}}{\partial \sigma} = E_0 N'(d_1) \sqrt{T_1},$$

where $N(\cdot)$ is the cumulative Gaussian-distribution function.

In terms of the implied volatility σ , the low strike-price (LSP) put price verifies that

$$e^{-rT_1}K \times Q(V_1 \leq Y_1) = P^{BS}(E_0), \quad K \leq c \times D.$$

Then, the implied-volatility function, $\sigma(K)$, holds that

$$e^{-rT_1} \times Q(V_1 \leq Y_1) = e^{-rT_1} \times N(-d_2) + \frac{\partial P^{BS}(E_0)}{\partial \sigma} \times \sigma'(K), \quad K \leq c \times D, \quad (23)$$

implying

$$\begin{aligned}\sigma'(K) \times \sqrt{T_1} &= e^{-rT_1} \times \frac{Q(V_1 \leq Y_1) - N(-d_2)}{E_0 \times N'(d_1)} \\ &= \frac{-N(-d_1)}{K \times N'(d_1)} < 0, K \leq c \times D,\end{aligned}\tag{24}$$

which is equation (14). ■

European put-call parity and call pricing

We derive the initial value of the ex-cash-flow equity price at T_1 , where $B_1 = cD$ and $a_1 = 1_{\{C_1 > B_1\}}$. From equation (5), where $E_1 = a_1 C_1$,

$$\begin{aligned}E_0^Q [e^{-rT_1} E_1] &= E_0^Q [e^{-rT_1} a_1 C_1] \\ &= E_0^Q [e^{-rT_1} a_1 (C_1 - B_1)] + e^{-rT_1} B_1 \times E_0^Q [a_1] \\ &= E_0 + e^{-rT_1} B_1 \times Q(V_1 > Y_1),\end{aligned}\tag{25}$$

because B_1 is predictable at T_0 .

For European equity puts and calls (with respective prices p_t and c_t), with the same strike price (K) and expiring at the first-outflow date (T_1), from put-call parity at T_1 ,

$$K + \underbrace{\max\{0, E_1 - K\}}_{=c_1} = E_1 + \underbrace{\max\{0, K - E_1\}}_{=p_1}.$$

Then, from the law of one price, put-call parity at $t = 0$ becomes

$$e^{-rT_1} K + c_0 = E_0 + e^{-rT_1} B_1 \times Q(V_1 > Y_1) + p_0,\tag{26}$$

The call price is given by

$$c_0 = E_0 + e^{-rT_1} B_1 \times Q(V_1 > Y_1) + p_0 - e^{-rT_1} K.$$

As in the case of a paying-dividend stock, put-call parity is also adjusted, in this case,

by $e^{-rT_1} B_1 \times Q(V_1 > Y_1)$.

Specifically, for LSP puts (i.e., $K \leq B_1$), from equation (12),

$$\begin{aligned}
c_0 &= E_0 + e^{-rT_1} B_1 \times Q(V_1 > Y_1) + e^{-rT_1} K \times Q(V_1 \leq Y_1) - e^{-rT_1} K \quad (27) \\
&= E_0 + e^{-rT_1} B_1 \times Q(V_1 > Y_1) - e^{-rT_1} K \times Q(V_1 > Y_1) \\
&= E_0 - e^{-rT_1} (K - B_1) \times Q(V_1 > Y_1).
\end{aligned}$$

For example, consider a spread between two co-terminal European calls struck within the corridor, with respective strike prices K_1 and K_2 , $K_1 < K_2 \leq B_1$. Then,

$$c_0(K_1) - c_0(K_2) = e^{-rT_1} \times (K_2 - K_1) \times Q(V_1 > Y_1), \quad (28)$$

which implies the following surviving probability, $\frac{c_0(K_1) - c_0(K_2)}{K_2 - K_1} = e^{-rT_1} \times Q(V_1 > Y_1)$.

For no LSP puts (i.e., $K > B_1$), from equation (16),

$$\begin{aligned}
c_0 &= E_0 + e^{-rT_1} B_1 \times Q(V_1 > Y_1) \quad (29) \\
&\quad + e^{-rT_1} K \times Q(V_1 \leq Y_1) + E_0^Q [e^{-rT_1} (K - E_1) \times 1_{\{B_1 < E_1 < K\}}] - e^{-rT_1} K \\
&= E_0 - e^{-rT_1} (K - B_1) \times Q(V_1 > Y_1) + E_0^Q [e^{-rT_1} (K - E_1) \times 1_{\{B_1 < E_1 < K\}}]. \blacksquare
\end{aligned}$$

6.1 A default corridor in an endogenous setting

We show the link between European puts and credit protection is given by endogenous default.

1. The endogenous credit-risk model We study a general discrete-time setting with stochastic cash flows. Consider a model with N periods, $n = 1, 2, \dots, N$, and respective times $0 < T_1 < T_2 < \dots < T_N$. We denote by B_n the negative payout rate of the firm. Thus, B_n is an outflow that is paid if $B_n > 0$ (an inflow that is collected if $B_n < 0$) by equityholders' deep pockets, in Leland's tradition.

The equity continuation value is denoted by C_t . We assume equityholders' limited liability, implying $C_t \geq 0$, $0 \leq t$. It follows that if $B_n > 0$, defaulting at T_n is optimal if and only if

$$C_n \leq B_n, n = 1, 2, \dots, N,$$

with indifference between defaulting and paying the cash outflow if $C_n = B_n$. This default choice maximizes equity value; that is, it is endogenous.

In addition, we also assume strictly positive values for the process C_t ; that is,

$$C_t > 0, t \in [0, T_N],$$

which implies default is never optimal between outflow dates.

2. Endogenous equity pricing Given a terminal value $C_N \geq 0$, the equity continuation value is defined recursively as follows:

$$C_t = E_t^Q [e^{-r(T_n - T_{n-1})} \{C_n - B_n\}^+] \geq 0, T_{n-1} \leq t < T_n, \quad (30)$$

for $n = 1, 2, \dots, N$. The process C_t is discontinuous at T_n ; that is,

$$\lim_{t \uparrow T_n} C_t \rightarrow \{C_n - B_n\}^+ < C_n \text{ if } B_n > 0, n = 1, 2, \dots, N. \quad (31)$$

Similarly, $\{C_n - B_n\}^+ > C_n > 0$ if $B_n < 0$.

The ex-cash-flow equity price, as in the previous coupon-bond model, is given by $E_N = a_N \times C_N$ and

$$E_t = a_{n-1} \times C_t, T_{n-1} \leq t < T_n, \quad (32)$$

for $n = 1, 2, \dots, N$, where $a_0 = 1$ and a_n is in equation (4). In particular, $E_t = C_t$, $0 \leq t < T_1$. The (ex-cash-flow) equity-price function $E_n(C_n)$ is also discontinuous at

T_n . That is, either

$$E_n = 0 \text{ or } E_n = C_n > B_n > 0 \text{ if } B_n > 0, n = 1, 2, \dots, N, \quad (33)$$

both with positive probability. By contrast, in the case of a cash inflow (i.e., $B_n < 0$), and conditional on no previous default (i.e., $a_{n-1} = 1$), $E_n = C_n > 0$ if $B_n < 0$.

Then $C_t > 0$ implies $E_t = C_t > 0$ between outflow dates, conditional on no previous default. It follows that default is never optimal between outflow dates. Hence, we focus on outflow dates. First, we provide an example.

Example In Figure 3, we provide a typical equity path that ends in default at T_3 . The firm's assets mature in four periods. Three deterministic cash flows exist, namely, $B_1 = -5$, $B_2 = 4$, and $B_3 = 4$ (i.e., $B_1 < 0$ is a cash dividend and $B_2, B_3 > 0$ are debt service or outflows). The assets are risky and have an expected value of 7.5 at T_4 . The value of the firm is $C_0 = E_0 = 6$, which equals the intrinsic value (i.e., $5 - 4 - 4 + 7.5 = 4.5$) plus some option/upside value (i.e., $6 - 4.5 = 1.5$).

***** to include Figure 3 *****

Equity value increases in the first period from $E_0 = 6$ to 7. Namely, right before T_1 , the value of equity is 7, that is, $C_1 = E_1 = 7 - 5 = 2$, which is the (downward-jumping) ex-dividend equity price. After this large dividend, most of the firm value is option value ($C_1 = 2$). From T_1 to T_2 , the firm remains stable. Right before T_2 , the value of equity is also 2, that is, $C_2 = E_2 = 2 + 4 = 6$, and the equity price jumps upwards. However, after T_2 and a high-volatility period, the firm quickly loses value and defaults at T_3 because $C_3 = 3$ is less than $B_3 = 4$. Hence, we advance that a default corridor $[0, 4]$ exists at T_2 and T_3 .

At T_3 , it is optimal to equityholders to not absorb the debt service but default, which implies $E_t = 0$, $t \geq T_3$. From $C_3 = 3$, and given no additional cash-flows, V_3 is also close to 3 (which is less than the initial expected value of 7.5). In brief, after this

poor path/performance, all stakeholders lose. Equityholders get $5 - 4 = 1$, which is less than the initial equity value of $E_0 = 6$, and debtholders get 4 and 3, instead of the promised cash flows of 4 and 4, which implies a loss of $1/8$ for them.

3. The endogenous default event and the default corridor Implicit in the definition of the equity value (i.e., equation (32)) are N endogenous-default events; that is,

$$\{C_n \leq B_n\}, n = 1, 2, \dots, N. \quad (34)$$

These N endogenous-default events lead to the N default corridors, in which the ex-cash-flow equity price cannot enter. That is, from equation (33),

$$E_n \notin (0, B_n], n = 1, 2, \dots, N.$$

In particular, the event $\{C_n \leq B_n\}$ is equivalent to $\{E_n \leq B_n\}$, $n = 1, 2, \dots, N$. Naturally, the event/corridor is empty if the cash flow is an inflow, that is, if $B_n < 0$.

However, because the payout rate B_n follows a random process, the N default corridors are conditional on B_n . Therefore, assuming B_n is one-period predictable, the only possible default corridor is at time T_1 and is given by $[0, B_1]$ (i.e., $E_1 \notin (0, B_1]$). In a general setting with operational leverage, earnings are stochastic; hence, we assume predictability. In the case of financial leverage, (no floating) coupons and principal are known since issuance time, and only refinancing costs are random.

Naturally, time advances, and after the first period ends (and conditional on non-default), the second period becomes a new first period, and we again have a default corridor. That is, if $C_1 > B_1$ at T_1 , we have a new default corridor at T_2 if $B_2 > 0$, because $E_2 \notin (0, B_2]$. So, without loss of generality, we assume $B_1 > 0$.

4. European puts, digital puts, and pure credit contracts At T_1 , an LSP European put becomes a digital put, which replicates a pure credit contract. That is,

for a put with maturity T_1 , the payoff reduces to

$$\begin{aligned}
\max\{0, K - E_1\} &= (K - E_1) \times 1_{\{E_1 \leq K\}} \\
&= K \times 1_{\{E_1=0\}} + (K - E_n) \times 1_{\{B_1 < E_1 \leq K\}} \\
&= K \times 1_{\{E_1=0\}} \text{ if } K \leq B_1,
\end{aligned} \tag{35}$$

which is a binary option in the case of LSP puts, namely, $K \leq B_1$. The second equality follows from equation (33). As emphasized above, the latter result only happens for $n = 1$, because B_n is predictable yet stochastic for $n > 1$.

Then, from equation (33), $E_n = E_n \times 1_{\{E_n > B_n\}}$, from which follows $1_{\{E_1=0\}} = 1_{\{E_1 \leq B_1\}}$ and hence

$$\max\{0, K - E_1\} = K \times 1_{\{E_1 \leq B_1\}} \text{ if } K \leq B_1, \tag{36}$$

which replicates a pure credit contract, where $\{E_1 \leq B_1\}$ is the endogenous-default event at T_1 . In this setting, the DOOM put (that replicates a pure credit contract) is rather an LSP put.

5. The price of European puts in a default corridor Similar to the coupon-bond model, in which leverage is only financial (and $B_1 = cD$), from equation (36), the price of an LSP European put with maturity T_1 is given by

$$E_0^Q [e^{-rT_1} K \times 1_{\{E_1 \leq B_1\}}] = e^{-rT_1} K \times E_0^Q [1_{\{E_1 \leq B_1\}}], K \leq B_1, \tag{37}$$

and the same implications follow as in section 3.

That is, LSP European-put prices are linear in the strike price, and the implied-volatility skew is negative, $\sigma'(K) < 0$, $K \leq B_1$. Put options are more expensive in a default corridor than in a benchmark setting of no corridor. Put-call parity is adjusted by the cash outflow (for $t < T_1$), and from this parity link, we price call options. All

these results happen for a maturity T_1 that is equal to the first-outflow date, in which the outflow is assumed to be predictable.

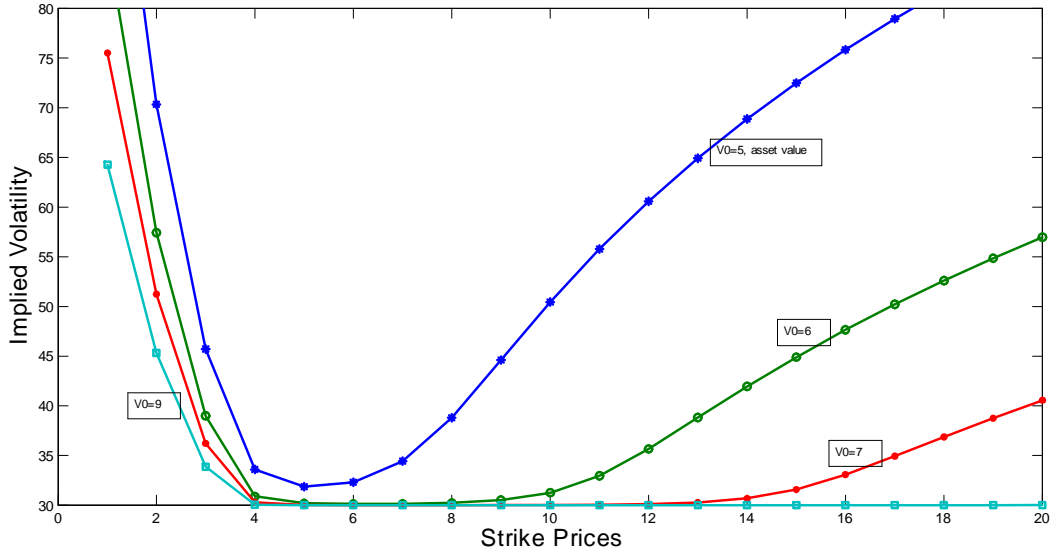


Figure 1: From a default corridor $[0, B]$ at T_1 , we show the implied-volatility curves generated by European put options. At T_1 , the value of equity equals $V_1 \times 1_{\{V_1 > B\}}$. We define the corridor by $B = 3$. The lognormal assets volatility is $\sigma = 30\%$, the interest rate is $r = 2\%$, and the maturity is $T_1 = 6$ months. Equity and the European equity put have the same maturity. We consider four equity prices, $E_0 = \{2.03, 3.03, 4.03, 6.03\}$, corresponding to the four asset values, $V_0 = \{5, 6, 7, 9\}$, respectively.

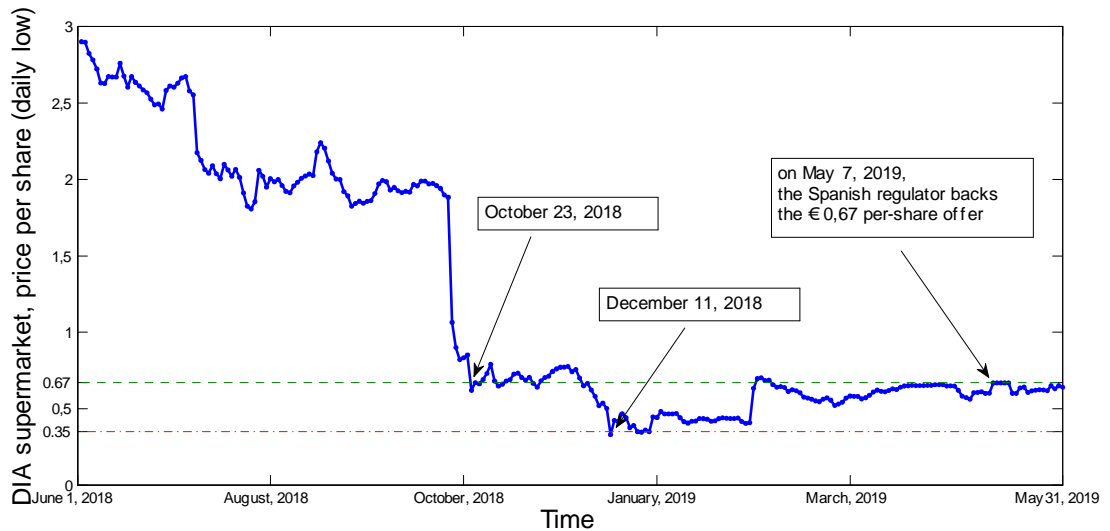


Figure 2: Price per share of DIA retailer (in Euros) from June 1, 2018 to May 31, 2019. Since October 23, 2018, DIA stock price looks consistent with a default corridor.

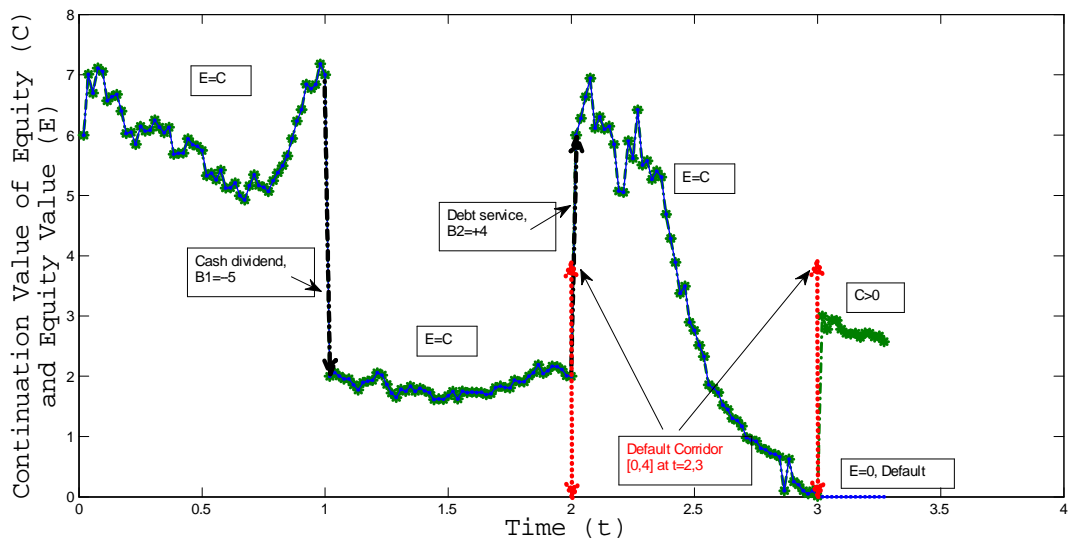


Figure 3: In a default corridor, a typical equity path that ends in default. The firm's assets mature in four periods. Three deterministic cash outflows exist, $B_1 = -5$, $B_2 = 4$, and $B_3 = 4$, where $B_1 < 0$ is a dividend and $B_2, B_3 > 0$ are debt payments. The initial value of the firm is $E_0 = 6$. At T_1 , equityholders receive a dividend of 5, and equity falls from 7 to 2. T_1 to T_2 is a calm period, equityholders pay a debt service of 4, and equity jumps from 2 to 6. After T_2 and a high-volatility period, the firm quickly loses value, and equityholders choose defaulting at T_3 because the value of the assets is 3, which is less than the debt service of 4. That is, $E_t = 0$, $t \geq T_3$.