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PDE PROBLEMS WITH CONCENTRATING TERMS NEAR THE BOUNDARY

Dedicated to Professor Tomás Caraballo on occasion of his Sixtieth Birthday

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Abstract. In this paper we study several PDE problems where certain linear or nonlinear termsin the equation concentrate in the domain, typically (but not exclusively) near the boundary. We analyze some linear and nonlinear elliptic models, linear and nonlinear parabolic ones as well as some damped wave equations. We show that in all these singularly perturbed problems, the concentrating terms give rise in the limit to a modification in the original boundary condition of the problem. Hence we describe in each case which is the singular limit problem and analyze the convergence of solutions.

1. **Introduction.** In this paper we consider several types of PDE problems in which certain terms in the equations *concentrate*, as a parameter $\varepsilon \to 0$. Typically, near the boundary of the domain. This implies that the problem under consideration are subjected to singular perturbations that drastically change the nature of the problem, when passing to the limit.

In all the problems considered the goal is then to identify the form of the limit problem and to describe the process of convergence of solutions, as $\varepsilon \to 0$.

To make the notations apparent, we will consider Ω, an open bounded smooth set in $I\!R^N$ with a C^2 boundary $\partial\Omega$. Let $\Gamma = \partial\Omega$ and consider the subset of Ω

$$\omega_\varepsilon = \{x \in \Omega \ : \ \mathrm{dist}(x, \Gamma) < \varepsilon\}$$

for sufficiently small ε, say $0 < \varepsilon < \varepsilon_0$. Notice that this set has measure $|\omega_\varepsilon| \sim \varepsilon|\Gamma|$ for small values of ε. For sufficiently small $\sigma \geq 0$ we can define the "parallel" interior boundary $\Gamma_\sigma = \{y - \sigma\vec{n}(y),\ y \in \Gamma\}$ where $\vec{n}(x)$ denotes the outward normal vector. Note that $\Gamma_0 = \Gamma$ and that for small ε, the set ω_ε is a neighborhood of Γ in $\bar{\Omega}$, that collapses to the boundary when the parameter ε goes to zero.

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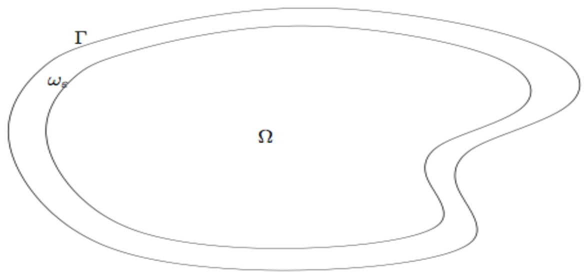

FIGURE 1. The set ω_ε

Then, assuming regularity of Γ, we can describe ω_ε as

$$\omega_\varepsilon = \{x = y - \sigma\vec{n}(y),\ y \in \Gamma,\ \sigma \in (0,\varepsilon)\} = \bigcup_{0<\sigma<\varepsilon} \Gamma_\sigma \tag{1.1}$$

for sufficiently small ε, say $0 < \varepsilon < \varepsilon_0$.

To illustrate the type of concentrated terms we will consider, assume we have a family of functions $\{j_\varepsilon\}_\varepsilon$ such that

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |j_\varepsilon|^r \leq C,$$

for some $1 \leq r < \infty$ (for $r = \infty$ we assume $\|j_\varepsilon\|_{L^\infty(\omega_\varepsilon)}$ is bounded uniformly in ε). Then we prove that, taking subsequences if necessary, there exists a function $j_0 \in L^r(\Gamma)$ (or a bounded Radon measure on Γ, $j_0 \in \mathcal{M}(\Gamma)$ if $r = 1$) such that for any smooth function φ, we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{\omega_\varepsilon} j_\varepsilon\varphi = \int_\Gamma j_0\varphi. \tag{1.2}$$

Thus say that we have a "L^r–concentrated convergent subsequence" and write

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} j_\varepsilon \to j_0 \quad cc - L^r \tag{1.3}$$

and say $\{j_\varepsilon\}_\varepsilon$ is an "L^r–concentrated (sequentially) compact family". See Section 2.1.

We will consider several types of PDEs with concentrating terms as in (1.2), (1.3). These terms could be non-homogeneous data, linear potentials or even linear or nonlinear terms depending on the unknowns themselves. For example, in Section 2.2, we will consider general elliptic problems in divergence form of the type

$$\begin{cases} -\mathrm{div}(a(x)\nabla u^\varepsilon) + c(x)u^\varepsilon + \lambda u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x)u^\varepsilon = \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} f_\varepsilon + g_\varepsilon & \text{in } \Omega, \\ a(x)\frac{\partial u^\varepsilon}{\partial n} + b(x)u^\varepsilon = h_\varepsilon & \text{on } \Gamma, \end{cases} \tag{1.4}$$

where $\lambda \in I\!R$. Observe that here a non-homogeneous data, f_ε and a linear potential, V_ε, concentrate as in (1.3).

We will find conditions on the concentrating terms to prove that assuming that the terms g_ε and h_ε converge in certain weak sense to g_0 and h_0, respectively, then the solutions of (1.4) converge to the unique solution of

$$\begin{cases} -\mathrm{div}(a(x)\nabla u) + c(x)u + \lambda u = g_0 & \text{in } \Omega, \\ a(x)\frac{\partial u}{\partial n} + (b(x) + V_0(x))u = f_0 + h_0 & \text{on } \Gamma. \end{cases} \tag{1.5}$$

Notice that the singular concentrating terms in (1.4) transfer to boundary terms in the limit problem (1.5).

Similar type of singular, concentrating, terms will be considered in further sections. For example, in Section 2.3 we will considers some elliptic nonlinear eigenvalue problems related to optimal constants in Sobolev embeddings. In Section 3 we will analyze the natural linear parabolic problems associated to (1.4) and (1.5). For these problems we will prove strong results on the convergence of the associated linear semigroups, in optimal families of Bessel-type spaces. In Section 4 we will consider nonlinear parabolic problems in which the concentrating term is the nonlinear one. Hence in this case the limit problem is a parabolic problem with nonlinear boundary conditions. Under natural dissipativity assumptions we show that the asymptotic behavior of solutions are close as $\varepsilon \to 0$, by showing the upper semicontinuity of the family of global attractors. Then, in Section 5 we turn back to linear parabolic problems with now concentrating linear potentials and where the time derivative of the unknown also concentrates. This implies that the approximating problem is a coupled elliptic–parabolic transmission problem while the limit one has dynamic boundary conditions; that is, an evolution problem on the boundary Γ which has a nonlocal coupling with an elliptic equation in Ω. Our goal is then to give sufficient conditions on the data to prove convergence and to describe the limit process. The last two sections are devoted to damped type wave equations. As we lose the parabolic structure that implies regularization of solutions, we are bound then to rely on energy estimates and compactness arguments. First, in Section 6, the main feature is that the damping is concentrated and, therefore, the limit problem is a wave equation with boundary feedback boundary condition; in particular, the damping acts only on the boundary in the limit. In such a situation the type of approximating problems we consider, appear naturally in control theory/stabilization of waves, see [15, 44, 59, 47, 48], or in homogenization of vibration problems with inclusions near the boundary, see [51, 28, 29, 17] and references therein. On the other hand, the limit problem appears in the boundary control theory, see [40, 41, 39, 67, 50, 19] and references therein. Second, in Section 7 we explore the situation in which the damping and linear potential concentrate around a smooth, compact orientable hypersurface without boundary, $\mathcal{M}$, contained in Ω, that is not touching the boundary Γ. In that case we show that the limit problem is then a wave–wave transmission problem on *both sides* of $\mathcal{M}$ coupled with an evolution problem on $\mathcal{M}$. The limit problem can also be considered as a wave equation in Ω with damping concentrated only on $\mathcal{M}$, see [32] for a two dimensional case. The results in this section appear for the first time in print.

Problems with concentrating terms near the boundary have been considered in the literature, some of which we are reporting about in this paper. For example linear elliptic problems have been studied in [10]. Nonlinear elliptic problems, some including oscillations in the boundary, have been considered in [12], [7], [4], [11]; see also [42]. Linear parabolic problems can be found in [55] while nonlinear ones where considered in [34], [53], [5]. Delay nonlinear parabolic problems can be found in [6], while parabolic dynamic boundary conditions can be found in [35]. Also, asymptotic behavior of non-autonomous damped wave equations have been studied in [3].

In all these examples a common feature is that concentrating terms near the boundary give rise, in the limit, to a boundary term. The form of the boundary term depends on the problem under consideration. Also, this influences the way the solutions of the approximate problem converge to those of the limit one. Notice that one source of difficulties is that in (1.2) one term is defined in $\omega_\varepsilon \subset \Omega$ while the limit one is defined on $\Gamma = \partial\Omega$ so that convergence has to be seen in a dual space of regular test functions.

2. **Stationary problems.** In this section we present the basic results and tools for stationary problems before approaching evolution ones.

2.1. **Concentrating integrals.** In this section we prove several results that describe how different concentrated integrals converge to surface integrals. Most of the results are taken from [10] where full details can be found.

We will extensively use the scale of Bessel Potential spaces $H^{s,q}(\Omega)$, which are obtained via complex interpolation procedure of the usual Sobolev spaces $W^{k,q}(\Omega)$ with $k = 0, 1, ...$, see for instance [1, 64, 2]. Recall that Bessel spaces have the sharp continuous embeddings, see [1],

$$H^{s,q}(\Omega) \subset \begin{cases} L^r(\Omega),\ s - \frac{N}{q} \geq -\frac{N}{r},\ 1 \leq r < \infty, & \text{if } s - \frac{N}{q} < 0 \\ L^r(\Omega),\ \ 1 \leq r < \infty, & \text{if } s - \frac{N}{q} = 0 \\ C^\eta(\bar{\Omega}) & \text{if } s - \frac{N}{q} > \eta > 0. \end{cases}$$

Moreover, we have the dual spaces $H^{-2\alpha,q}(\Omega) = (H^{2\alpha,q'}(\Omega))'$, see [2] for details, which implies

$$H^{-s,q}(\Omega) \supset \begin{cases} L^r(\Omega),\ -s - \frac{N}{q} \leq -\frac{N}{r},\ 1 < r \leq \infty, & \text{if } s - \frac{N}{q'} < 0 \\ L^r(\Omega),\ \ 1 < r \leq \infty, & \text{if } s - \frac{N}{q'} = 0 \\ \mathcal{M}(\Omega) & \text{if } s - \frac{N}{q'} > 0. \end{cases}$$

Also, the regularity of Ω and standard trace theory, see [1], imply that for $s > \frac{1}{q}$, the trace operator, γ, is well defined on $H^{s,q}(\Omega)$ and

$$H^{s,q}(\Omega) \xrightarrow{\gamma} \begin{cases} L^r(\Gamma),\ s - \frac{N}{q} \geq -\frac{N-1}{r},\ 1 \leq r < \infty, & \text{if } s - \frac{N}{q} < 0 \\ L^r(\Gamma),\ \ 1 \leq r < \infty, & \text{if } s - \frac{N}{q} = 0 \\ C^\eta(\Gamma) & \text{if } s - \frac{N}{q} > \eta > 0. \end{cases}$$

The value ε_0 in (1.1) will be chosen small enough so that, for all $0 < \varepsilon < \varepsilon_0$, the strip ω_ε can be parametrized in a C^2 way by $\Gamma \times [0, \varepsilon)$, that is, the map

$$\begin{aligned} T_\varepsilon : \Gamma \times [0, \varepsilon) &\longrightarrow \omega_\varepsilon \\ (x, \sigma) &\longrightarrow x - \sigma \vec{n}(x) \end{aligned}$$

is a C^2 diffeomorphism. Notice that if we define $\Omega_\delta = \Omega \setminus \bar{\omega}_\delta$, for $0 < \delta < \varepsilon_0$, then we can construct the following C^2 diffeomorphism $\tau_\delta : \overline{\Omega} \longrightarrow \overline{\Omega_\delta}$ defined by

$$\tau_\delta(x) = \begin{cases} x & \text{if } \operatorname{dist}(x, \Gamma) \geq \varepsilon_0 \\ z - \psi_\delta(\sigma)\vec{n}(z) & \text{if } x = z - \sigma\vec{n}(z),\ \sigma \in [0, \varepsilon_0) \end{cases}$$

where the function $\psi_\delta : [0, \varepsilon_0] \to [\delta, \varepsilon_0]$ is a C^2 function such that $\psi_\delta(\varepsilon_0) = \varepsilon_0$, $\psi'_\delta(\varepsilon_0) = 1$, $\psi''_\delta(\varepsilon_0) = 0$, $\psi_\delta(0) = \delta$, it is strictly increasing, $|\psi_\delta(\sigma) - \sigma| + |\psi'_\delta(\sigma) - 1| + |\psi''_\delta(\sigma)| \to 0$ uniformly in $\sigma \in [0, \varepsilon_0]$ as $\delta \to 0$ and the map $\delta \to \psi_\delta \in C^2([0, \varepsilon_0])$ is continuous.

Observe that τ_δ is a C^2 diffeomorphism between Ω and Ω_δ which satisfies $\|\tau_\delta\|_{C^2}$, $\|\tau_\delta^{-1}\|_{C^2} \leq C$ with C independent of $\delta \in (0, \varepsilon_0)$, the map $\delta \to \tau_\delta \in C^2(\bar{\Omega})$ is continuous for $\delta \in [0, \varepsilon_0]$

$$\|\tau_\delta - I\|_{C^2(\overline{\Omega})} \to 0, \quad \text{as} \quad \delta \to 0,$$

and also τ_δ is a C^2 diffeomorphism between Γ and Γ_δ and $\tau_\delta(x) = x - \delta \vec{n}(x)$ for $x \in \Gamma$; see [12, 10].

This diffeomorphism induce isomorphisms $\tau_\delta^* : H^{s,q}(\Omega_\delta) \longrightarrow H^{s,q}(\Omega)$ for all $0 \leq s \leq 2$ and $1 \leq q \leq \infty$, which are defined by $\tau_\delta^*(u) = u \circ \tau_\delta$. The C^2-bounds obtained above for τ_δ and τ_δ^{-1} and the fact that $\|\tau_\delta - I\|_{C^2(\Omega)} \to 0$ as $\delta \to 0$ imply that the isomorphisms τ_δ^* and $(\tau_\delta^*)^{-1}$ are uniformly bounded in $\delta \in (0, \varepsilon_0)$. Moreover, we also have that for $u \in H^{s,q}(\Omega_\delta)$, we get $\|\tau_\delta^*(u) - u\|_{H^{s,q}(\Omega_\delta)} \to 0$. They also induce the isomorphisms $\hat{\tau}_\delta : L^r(\Gamma_\delta) \to L^r(\Gamma)$, for $1 \leq r \leq \infty$, defined by $\hat{\tau}_\delta(v) = v \circ \tau_\delta$. Similarly, as we have argued for τ_δ^* we will have that $\hat{\tau}_\delta$ and $\hat{\tau}_\delta^{-1}$ are also uniformly bounded. It is not difficult to prove now that if we denote by γ_δ the trace operator from $H^{s,q}(\Omega)$ to $L^r(\Gamma_\delta)$ and γ the trace operator from $H^{s,q}(\Omega)$ to $L^r(\Gamma)$ then

$$\hat{\tau}_\delta \circ \gamma_\delta \to \gamma, \qquad \text{as } \delta \to 0$$

and this convergence is pointwise from $H^{s,q}(\Omega)$ to $L^r(\Gamma)$ if $s > 1/q$, $s - \frac{N}{q} \geq -\frac{N-1}{r}$ and in the operator norm if $s > 1/q$, $s - \frac{N}{q} > -\frac{N-1}{r}$. Notice also that we have $\hat{\tau}_\delta \circ \gamma_\delta = \gamma \circ \tau_\delta^*$

In particular, for any H defined on ω_ε and for $\varepsilon < \varepsilon_0$, we have

$$\int_{\omega_\varepsilon} H \, dx = \int_0^\varepsilon \int_{\Gamma_\delta} H \, dS_\delta \, d\delta,$$

and

$$\int_{\Gamma_\delta} H dS_\delta = \int_\Gamma H\big(\tau_\delta(x)\big) J(\tau_\delta(x)) \, dS_0(x),$$

where dS_δ is the surface measure associated to Γ_δ and $J(\tau_\delta(x)) := J(x, \delta)$ is the surface Jacobian of the transformation τ_δ. Note that in particular there exists constants $0 < J_1 \leq J_2$ such that for all $x \in \Gamma$ and for all $\delta \in [0, \varepsilon_0]$

$$J_1 \leq J(x, \delta) \leq J_2 \quad \text{and} \quad \|J_\delta - 1\|_{L^\infty(\Gamma)} \to 0 \text{ as } \delta \to 0.$$

With these tools we can show the following lemma.

Lemma 2.1. *Assume that $v \in H^{s,q}(\Omega)$ with $\frac{1}{q} < s \leq 2$ and $s - \frac{N}{q} \geq -\frac{(N-1)}{r}$, or $v \in H^{1,1}(\Omega)$, i.e, $s = 1 = q$ and $r = 1$ below. Then for sufficiently small ε_0, we have*

i) The map $[0, \varepsilon_0] \ni \sigma \mapsto \displaystyle\int_{\Gamma_\sigma} |v|^r$ is continuous.

ii) There exist a positive constant C independent of ε and v such that for any $\varepsilon \leq \varepsilon_0$, we have

$$sup_{\sigma \in [0,\varepsilon)} \|v\|_{L^r(\Gamma_\sigma)} \leq C \|v\|_{H^{s,q}(\Omega)}, \qquad \int_{\omega_\varepsilon} |v|^r = \int_0^\varepsilon \Big(\int_{\Gamma_\sigma} |v|^r \Big) d\sigma.$$

In particular

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |v|^r \leq C \|v\|^r_{H^{s,q}(\Omega)} \tag{2.1}$$

and $\displaystyle\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |v|^r = \int_\Gamma |v|^r$.

We can now analyze how concentrating integrals converge for certain families of functions which vary with ε and have weak regularity properties.

Lemma 2.2. *Assume that a given family f_ε defined on ω_ε is an "L^r–concentrated bounded family" near Γ, that is, for some $1 \le r < \infty$ and a constant C independent of ε,*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |f_\varepsilon|^r \le C,$$

or $\sup_{x\in\omega_\varepsilon} |f_\varepsilon(x)| \le C$ for the case $r = \infty$. Then,
i) For any $s > \frac{1}{q}$ and $s - \frac{N}{q} \ge -\frac{N-1}{r'}$, $\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} f_\varepsilon$ is bounded in $H^{-s,q'}(\Omega)$.
ii) For every sequence converging to zero (that we still denote $\varepsilon \to 0$) there exist a subsequence (that we still denote the same) and a function $f_0 \in L^r(\Gamma)$ (or a bounded Radon measure on Γ, $f_0 \in \mathcal{M}(\Gamma)$ if $r = 1$) such that

a) For any smooth function φ, defined in $\overline{\Omega}$, we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{\omega_\varepsilon} f_\varepsilon \varphi = \int_\Gamma f_0 \varphi$$

b) If $u^\varepsilon \to u^0$ weakly in $H^{s,q}(\Omega)$ with $s > 1/q$ and

$$s - \frac{N}{q} > -\frac{N-1}{r'}, \tag{2.2}$$

or strongly in case of equal sign in (2.2), then

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{\omega_\varepsilon} f_\varepsilon u^\varepsilon = \int_\Gamma f_0 u^0.$$

In other words $\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} f_\varepsilon \to f_0$ in $H^{-s,q'}(\Omega)$.

Also the following consequence will be used further below.

Corollary 1. *i) Assume $\varphi \in H^\sigma(\Omega)$ with $\sigma > \frac{1}{2}$, and denote φ_0 the trace of φ on Γ. Then*

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}\varphi \to \varphi_0 \quad \text{in } H^{-s}(\Omega) \text{ as } \varepsilon \to 0 \tag{2.3}$$

for any s such that $s > \frac{1}{2}$ and

$$(s - \frac{N}{2})_- + (\sigma - \frac{N}{2})_- > -N + 1, \tag{2.4}$$

where x_- denotes the negative part of x. Finally if $\varphi \in C(\overline{\Omega})$, (2.3) holds for any $s > \frac{1}{2}$.
ii) Assume

$$\|u_0^\varepsilon\|^2_{H^1(\Omega)} \le C.$$

Then, by taking subsequences if necessary, there exists $u_0 \in H^1(\Omega)$ such that, as $\varepsilon \to 0$,

$$u_0^\varepsilon \to u_0 \quad \text{weakly in } H^1(\Omega), \qquad \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_0^\varepsilon \to u_{0|\Gamma} \quad \text{weakly in } H^{-1}(\Omega)$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u_0^\varepsilon|^2 = \int_\Gamma |u_0|^2.$$

We can also prove,

Proposition 1. *Assume we have a family of functions V_ε, $0 \le \varepsilon \le \varepsilon_0$, satisfying the hypotheses of Lemma 2.2. Moreover, assume that (taking subsequences if necessary) there exits a function $V_0 \in L^r(\Gamma)$ (or a bounded Radon measure on Γ, $V_0 \in \mathcal{M}(\Gamma)$ if $r = 1$) such that for any smooth function φ, we have*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\omega_\varepsilon} V_\varepsilon \varphi = \int_\Gamma V_0 \varphi.$$

Then, for $s > \frac{1}{p}$, $\sigma > \frac{1}{q}$ and $(s - \frac{N}{p})_- + (\sigma - \frac{N}{q})_- > -\frac{N-1}{r'}$, if we define the operators, $P_\varepsilon : H^{s,p}(\Omega) \to H^{-\sigma,q}(\Omega)$ for $0 \le \varepsilon \le \varepsilon_0$ by

$$< P_\varepsilon(u), \varphi >= \frac{1}{\varepsilon} \int_{\omega_\varepsilon} V_\varepsilon u^\varepsilon \varphi^\varepsilon, \qquad < P_0(u), \varphi >= \int_\Gamma V_0 u \varphi,$$

then $P_\varepsilon \to P_0$ in $\mathcal{L}(H^{s,p}(\Omega), H^{-\sigma,q}(\Omega))$.

2.2. **Elliptic problems and resolvent estimates.** In this section we analyze the behavior, as $\varepsilon \to 0$, of the solutions of the elliptic problem

$$\begin{cases} -\mathrm{div}(a(x)\nabla u^\varepsilon) + \lambda u^\varepsilon = m_\varepsilon(x) u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x) u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} h_\varepsilon + g_\varepsilon & \text{in } \Omega \\ a(x)\frac{\partial u^\varepsilon}{\partial n} + b(x) u^\varepsilon = m_{0,\varepsilon} u^\varepsilon + j_\varepsilon & \text{on } \Gamma \end{cases} \tag{2.5}$$

where $\Gamma = \partial\Omega$ with smooth coefficients $a \in C^1(\overline{\Omega})$, $b \in C^1(\Gamma)$, with suitable non-homogeneous given terms g_ε, j_ε and concentrating potentials V_ε and concentrating non-homogeneous terms h_ε. We present here results from [10] and [55].

We show below that the corresponding limit problem is the elliptic problem

$$\begin{cases} -\mathrm{div}(a(x)\nabla u) + \lambda u = m_0 u + g_0 & \text{in } \Omega, \\ a(x)\frac{\partial u}{\partial n} + b(x) u = m_{0,0} u + V_0(x) u + h_0 + j_0 & \text{on } \Gamma \end{cases} \tag{2.6}$$

where the concentrating terms in (2.5) turn into boundary terms in (2.6).

For the setting of problems (2.5) and (2.6) we define the elliptic operator A_0 by

$$A_0 u = -\mathrm{div}(a(x)\nabla u)$$

regarded as an unbounded operator in $L^q(\Omega)$, for $1 < q < \infty$, with domain given by

$$D(A_0) = H^{2,q}_{bc}(\Omega) := \{u \in W^{2,q}(\Omega) : \quad a(x)\tfrac{\partial u}{\partial n} + b(x) u = 0 \text{ on } \Gamma\}.$$

Using the complex interpolation–extrapolation procedure in [2], for which the reader is referred for further details, one can construct the scale of Banach spaces associated to this operator, which will be denoted $H^{2\alpha,q}_{bc}(\Omega)$ for $\alpha \in [-1, 1]$, which are closed subspaces of $H^{2\alpha,q}(\Omega)$ incorporating some boundary conditions. In particular, we have $H^{0,q}_{bc}(\Omega) = L^q(\Omega)$, and $H^{1,q}_{bc}(\Omega) = H^{1,q}(\Omega)$. Note that the scale with negative exponents satisfies $H^{-2\alpha,q}_{bc}(\Omega) = (H^{2\alpha,q'}_{bc}(\Omega))'$, for $0 < \alpha < 1$ $H^{-2\alpha,q}(\Omega) \hookrightarrow H^{-2\alpha,q}_{bc}(\Omega)$. See [2] for details.

Therefore we consider now nonsmooth perturbations of the operator A_0. More precisely we consider a nonsmooth potential $m(x)$ in Ω, a nonsmooth perturbation, $m_0(x)$ of the boundary coefficient $b(x)$ in Γ as well as a family of concentrated perturbations near Γ.

In order to treat all perturbations in a unified form, we define for $0 < \varepsilon \le \varepsilon_0$,

$$< P_\varepsilon u, \varphi >= \frac{1}{\varepsilon} \int_{\omega_\varepsilon} V_\varepsilon u \varphi, \quad < Q_0 u, \varphi >= \int_\Omega m u \varphi, \qquad < R_0 u, \varphi >= \int_\Gamma m_0 u \varphi$$

for suitable u and φ.

Theorem 2.3. *Assume that m lies in a bounded set in $L^p(\Omega)$, with $p > N/2$, m_0 lies in a bounded set in $L^r(\Gamma)$ and also that the family of potentials V_ε is a L^r–concentrated bounded family, for $r > N-1$, that is*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon|^r \leq C, \quad r > N-1.$$

Then, for any $1 < q < \infty$, there exists some $\omega_0 > 0$ independent of m, m_0 and ε, and some interval $J = (\sigma_0, \sigma_1) \subset (\frac{1}{q'}, 2 - \frac{1}{q})$, with

$$\sigma_0 = \max\{(\frac{N}{p} - \frac{N}{q})_+, \frac{1}{q'} + (\frac{N-1}{r} - \frac{N-1}{q})_+\} \tag{2.7}$$

$$\sigma_1 = \min\{2 - (\frac{N}{p} - \frac{N}{q'})_+, 1 + \frac{1}{q'} - (\frac{N-1}{r} - \frac{N-1}{q'})_+\} \tag{2.8}$$

such that for any $Re(\lambda) \geq \omega_0$ and any $\sigma \in J$ the elliptic operator $A_0 + \lambda I - (P_\varepsilon + Q_0 + R_0)$, between $H^{2-\sigma,q}_{bc}(\Omega)$ and $H^{-\sigma,q}_{bc}(\Omega)$, is invertible and

$$\|\big(A_0 + \lambda I - (P_\varepsilon + Q_0 + R_0)\big)^{-1}\|_{\mathcal{L}(H^{-\sigma,q}_{bc}(\Omega), H^{-\sigma,q}_{bc}(\Omega))} \leq \frac{C}{|\lambda|}, \qquad Re(\lambda) \geq \omega_0$$

and

$$\|\big(A_0 + \lambda I - (P_\varepsilon + Q_0 + R_0)\big)^{-1}\|_{\mathcal{L}(H^{-\sigma,q}_{bc}(\Omega), H^{2-\sigma,q}_{bc}(\Omega))} \leq C, \qquad Re(\lambda) \geq \omega_0$$

where C is independent of m, m_0, ε and λ.

We have then the following consequences.

Corollary 2.
i) Assume

$$\begin{aligned} m_\varepsilon \to m \quad & in\ L^p(\Omega), \quad p > \frac{N}{2}, \\ m_{0,\varepsilon} \to m_0 \quad & in\ L^r(\Gamma), \quad r > N-1, \\ \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \to V_0, \quad & cc - L^r \quad for\ some\ r > N-1. \end{aligned} \tag{2.9}$$

Assume moreover that

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} h_\varepsilon \to h_0 \quad cc - L^q \quad for\ some\ q > 1.$$

and $g_\varepsilon \to g_0$ weakly in $L^z(\Omega)$, $j_\varepsilon \to j_0$ weakly in $L^t(\Gamma)$ for some $z \geq Nq/(N-1+q)$ and $t \geq q$.

Then, there exists some $\omega_0 > 0$ independent of ε, such that for $Re(\lambda) \geq \omega_0$ there exists a unique solution, u^ε, of

$$\left\{\begin{array}{ll} -div(a(x)\nabla u^\varepsilon) + \lambda u^\varepsilon = m_\varepsilon(x)u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x)u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} h_\varepsilon + g_\varepsilon & in\ \Omega, \\ a(x)\frac{\partial u^\varepsilon}{\partial \vec{n}} + b(x)u^\varepsilon = m_{0,\varepsilon}(x)u^\varepsilon + j_\varepsilon & on\ \Gamma, \end{array}\right.$$

which converges

$$u^\varepsilon \to u \quad in \quad H^{s,q}(\Omega)$$

for any $s < 2 - \sigma_0$, with σ_0 as in (2.7), where u is the unique solution of the limiting problem

$$\left\{\begin{array}{ll} -div(a(x)\nabla u) + \lambda u = m(x)u + g_0 & in\ \Omega, \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u = \big(m_0(x) + V_0(x)\big)u + h_0 + j_0 & on\ \Gamma. \end{array}\right.$$

In particular, if $q > N-1$, $z > N/2$ and $t > N-1$, then

$$u^\varepsilon \to u \quad in \quad C^\beta(\overline{\Omega}),$$

for some $\beta > 0$.
ii) If $m \in L^p(\Omega)$, *with* $p > \frac{N}{2}$ *and* $m_0 \in L^r(\Gamma)$ *with* $r > N-1$ *then for any* $1 < q < \infty$, *the operator* $A_0-(Q_0+R_0)$ *in Theorem 2.3 is resolvent positive. That is, there exists some* $\omega_0 > 0$, *such that for any* $\lambda \geq \omega_0$ *and* $\sigma \in J = (\sigma_0, \sigma_1) \subset (\frac{1}{q'}, 2-\frac{1}{q})$

$$\text{if} \quad 0 \leq g \in H^{-\sigma,q}_{bc}(\Omega) \qquad \text{then} \qquad 0 \leq \big(A_0 + \lambda I - (Q_0+R_0)\big)^{-1} g \in H^{2-\sigma,q}_{bc}(\Omega).$$

The constant ω_0 *can be taken uniform for* m *lying in a bounded set in* $L^p(\Omega)$, *with* $p > N/2$ *and* m_0 *lying in a bounded set in* $L^r(\Gamma)$, *with* $r > N-1$.

The convergence above implies also that the spectrum of the operators are close. See Corollary 4.2 and Remark 4.3 in [10] or [37] for a precise statement. In particular, we have the following

Corollary 3. *Assume (2.9) and denote by* λ_1^ε *the first eigenvalue of the eigenvalue problem*

$$\begin{cases} -div(a(x)\nabla\varphi^\varepsilon) = m_\varepsilon(x)\varphi^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x)\varphi^\varepsilon + \lambda\varphi^\varepsilon & in\ \Omega \\ a(x)\frac{\partial\varphi^\varepsilon}{\partial\vec{n}} + b(x)\varphi^\varepsilon = m_{0,\varepsilon}(x)\varphi^\varepsilon & on\ \Gamma. \end{cases}$$

Then, as $\varepsilon \to 0$,

$$\lambda_1^\varepsilon \to \lambda_1^0$$

which is the first eigenvalue of the limit eigenvalue problem

$$\begin{cases} -div(a(x)\nabla\varphi) = m(x)\varphi + \lambda\varphi & in\ \Omega, \\ a(x)\frac{\partial\varphi}{\partial\vec{n}} + b(x)\varphi = \big(m_0(x) + V_0(x)\big)\varphi & on\ \Gamma. \end{cases}$$

2.3. **Sharp embeddings and nonlinear eigenvalue problems.** The existence of a trace $H^1(\Omega) \hookrightarrow L^q(\Gamma)$ for $1 \leq q \leq 2_* = 2(N-1)/(N-2)$, implies that we have the Sobolev trace inequality: there exists a constant C such that

$$C\left(\int_\Gamma |v|^q\, dS\right)^{2/q} \leq \int_\Omega (|\nabla v|^2 + v^2)\, dx,$$

for all $v \in H^1(\Omega)$. The best Sobolev trace constant is the largest C such that the above inequality holds, that is,

$$T_q = \inf_{v\in H^1(\Omega)\setminus H^1_0(\Omega)} \frac{\int_\Omega |\nabla v|^2 + v^2\, dx}{\left(\int_\Gamma |v|^q\, dS\right)^{2/q}}. \tag{2.10}$$

For subcritical exponents, $1 \leq q < 2_*$, the embedding is compact, so we have existence of extremals, i.e. functions where the infimum is attained. These extremals can be taken strictly positive in $\overline{\Omega}$ and smooth up to the boundary. If we normalize the extremals with

$$\int_\Gamma |u|^q dS = 1, \tag{2.11}$$

it follows that they are weak solutions of the following problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial\vec{n}} = T_q|u|^{q-2}u & \text{on } \Gamma. \end{cases} \tag{2.12}$$

In the special case $q = 2$ (2.12) is a linear eigenvalue problem of Steklov type, see [62]. In the rest of this section we will assume that the extremals are normalized according to (2.11).

Let us consider the usual Sobolev embedding associated to the set ω_ε, that is,

$$H^1(\Omega) \hookrightarrow L^q\left(\omega_\varepsilon, \frac{dx}{\varepsilon}\right).$$

which is continuous for exponents q such that $1 \le q \le 2^* = 2N/(N-2)$, see Lemma 2.1. Note that $2^* = 2N/(N-2)$ is larger than $2_* = 2(N-1)/(N-2)$. The best constant associated to this embedding is given by

$$S_q(\varepsilon) = \inf_{v \in H^1(\Omega)} \frac{\int_\Omega |\nabla v|^2 + v^2\,dx}{\left(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |v|^q\,dx\right)^{2/q}} \tag{2.13}$$

and for $q < 2^*$, by compactness, the infimum is attained. The extremals, normalized by

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u|^q\,dx = 1, \tag{2.14}$$

are weak solutions of

$$\begin{cases} -\Delta u + u = \frac{S_q(\varepsilon)}{\varepsilon}\chi_{\omega_\varepsilon}(x)|u|^{q-2}u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \vec{n}} = 0 & \text{on } \Gamma, \end{cases} \tag{2.15}$$

where $\chi_{\omega_\varepsilon}$ denotes the characteristic function.

Therefore we are bound to study the convergence of the solutions of (2.15), (2.14) to those of (2.12), (2.11) and so the convergence of the optimal constant (2.10) to (2.13). This was analyzed in [12].

Theorem 2.4. *Let Ω be a bounded, C^2 domain and let T_q and $S_q(\varepsilon)$ be the best Sobolev constants given by (2.10) and (2.13).*
(1) *For critical or subcritical q, $1 \le q \le 2_* = 2(N-1)/(N-2)$, we have*

$$\lim_{\varepsilon\to 0} S_q(\varepsilon) = T_q.$$

Moreover, for subcritical q, $1 \le q < 2_* = 2(N-1)/(N-2)$, the extremals of $S_q(\varepsilon)$ normalized according to (2.14) converge strongly (along subsequences) in $H^1(\Omega)$ and in $C^\beta(\Omega)$, for some $\beta > 0$, to an extremal of (2.10),

$$\lim_{\varepsilon\to 0} u_\varepsilon = u_0, \qquad \textit{strongly in } H^1(\Omega) \textit{ and in } C^\beta(\Omega).$$

In the critical case, $q = 2_* = 2(N-1)/(N-2)$, the extremals of $S_q(\varepsilon)$ converge weakly (along subsequences) in $H^1(\Omega)$ to a limit, u_0, that is a weak solution of (2.12). This convergence is strong in $H^1(\Omega)$ if and only if the limit verifies $\int_\Gamma u_0^q = 1$ and in this case u_0 is an extremal for T_{2_*}.
(2) *For supercritical q, $2_* = 2(N-1)/(N-2) < q < 2^* = 2N/(N-2)$, we have*

$$\lim_{\varepsilon\to 0} S_q(\varepsilon) = 0.$$

Remark 1. Observe that in the critical case, using a sequence of minimizers and subsequences if necessary we have $u_\varepsilon \to u_0$ weakly in $H^1(\Omega)$ and $S_\varepsilon(q) \to T_q$. Also, we have

$$\|u_0\|^2_{H^1(\Omega)} \le \liminf_{\varepsilon\to 0} \|u_\varepsilon\|^2_{H^1(\Omega)} \le \limsup_{\varepsilon\to 0} \|u_\varepsilon\|^2_{H^1(\Omega)} = \limsup_{\varepsilon\to 0} S_q(\varepsilon) = T_q$$

and

$$T_q \le \frac{\int_\Omega |\nabla u_0|^2 + u_0^2 dx}{\left(\int_\Gamma |u_0|^q dS\right)^{2/q}}.$$

Hence if u_0 is a minimizer, then $\int_\Gamma |u_0|^q dS \le 1$. Conversely, if $\int_\Gamma |u_0|^q dS \ge 1$ then the argument above shows that this integral is actually equal to 1 and u_0 is a minimizer. Moreover in such a case, we get the convergence of the $H^1(\Omega)$ norms and hence the strong convergence in this space.

Thus, u_0 is a minimizer if and only if $\int_\Gamma |u_0|^q dS = 1$ which in turn is equivalent to the strong convergence.

Also, in the critical case it may happen then that one has (2.14) and $\int_\Gamma |u_0|^q dS < 1$.

In [12] the question of radial symmetry of minimizers was also discussed and the following result was proved.

Theorem 2.5. *Let $S_q(\varepsilon)$ be the best Sobolev constant given by (2.13) with $\Omega = B(0,R)$.*
(1) *For $1 \le q \le 2$ and for every $R, \varepsilon > 0$, the extremals of (2.13) in a ball are radial functions that do not change sign. In particular, there exists a unique non negative extremal of (2.13) satisfying (2.14).*
(2) *For $2 < q < 2_* = 2(N-1)/(N-2)$, there exist $0 < R_0 \le R_1 < \infty$ such that:*
 (2.1) *for $0 < R \le R_0$ and ε small (possibly depending on R) the extremals of (2.13) are radial.*
 (2.2) *for $R \ge R_1$ and ε small (possibly depending on R) the extremals of (2.13) are not radial.*

Remark 2. As a consequence of our results we get that extremals for the Sobolev trace embedding in small balls are radial. For symmetry results of extremals of Sobolev inequalities see for example, [22], [38] and references therein.

3. **Linear parabolic problems.** In this section we are interested in the behavior, for small ε, of the solutions of the linear parabolic problem

$$\left\{\begin{array}{ll} u^\varepsilon_t - \operatorname{div}(a(x)\nabla u^\varepsilon) = m(x)u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x) u^\varepsilon + f(x) + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} h_\varepsilon(x) & \text{in } \Omega \\ a(x)\frac{\partial u^\varepsilon}{\partial \vec{n}} + b(x)u^\varepsilon = m_0(x)u^\varepsilon & \text{on } \Gamma \\ u^\varepsilon(0) = u_0 & \text{in } \Omega \end{array}\right.$$

where $a \in C^1(\overline{\Omega})$ with $a(x) \ge a_0 > 0$ in Ω, and $b(x)$ a $C^1(\partial\Omega)$ function, with $m \in L^p(\Omega)$, $p > N/2$ and $m_0 \in L^r(\Gamma)$, $r > N-1$ and $\mathcal{X}_{\omega_\varepsilon}$ denotes the characteristic function of the set ω_ε.

Following [55], we will show in this section that the "limit problem" for the singularly perturbed problem above is given by

$$\left\{\begin{array}{ll} u_t - \operatorname{div}(a(x)\nabla u) = m(x)u + f(x) & \text{in } \Omega \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u = (m_0(x) + V_0(x))u + h_0(x) & \text{on } \Gamma \\ u(0) = u_0 & \text{in } \Omega \end{array}\right.$$

where h_0, V_0 are obtained as the limits of the concentrating terms

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} h_\varepsilon \to h_0, \qquad \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \to V_0, \quad cc - L^r \quad \text{for some } r > N-1. \tag{3.1}$$

Since from the results in Section 2, the solutions of the elliptic problems (2.5) converge to the unique solution of the elliptic limit problem (2.6), see Corollary 2, then it is enough to consider here the linear homogeneous problems

$$\left\{\begin{array}{ll} u^\varepsilon_t - \operatorname{div}(a(x)\nabla u^\varepsilon) = m(x)u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x) u^\varepsilon & \text{in } \Omega \\ a(x)\frac{\partial u^\varepsilon}{\partial \vec{n}} + b(x)u^\varepsilon = m_0(x)u^\varepsilon & \text{on } \Gamma \\ u^\varepsilon(0) = u_0 & \text{in } \Omega \end{array}\right. \tag{3.2}$$

and

$$\left\{\begin{array}{ll} u_t - \operatorname{div}(a(x)\nabla u) = m(x)u & \text{in } \Omega \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u = \big(m_0(x) + V_0(x)\big)u & \text{on } \Gamma \\ u(0) = u_0 & \text{in } \Omega \end{array}\right. \tag{3.3}$$

with $m \in L^p(\Omega)$, $p > N/2$ and $m_0 \in L^r(\Gamma)$, $r > N-1$.

Theorem 3.1. *Assume that m lies in a bounded set in $L^p(\Omega)$, with $p > N/2$, m_0 lies in a bounded set in $L^r(\Gamma)$ and also that the family of potentials V_ε is a L^r–concentrated bounded family, for $r > N-1$, that is*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon|^r \leq C, \quad r > N-1.$$

Then, for any $1 < q < \infty$, the problem (3.2) defines a strongly continuous, order preserving, analytic semigroup, $S_{m,m_0,\varepsilon}(t)$ in the space $H^{2\gamma,q}_{bc}(\Omega)$ for any

$$\gamma \in I(q) := (-1 + \frac{1}{2q}, 1 - \frac{1}{2q'}).$$

Moreover the semigroup satisfies the smoothing estimates

$$\|S_{m,m_0,\varepsilon}(t)u_0\|_{H^{2\gamma',q}_{bc}(\Omega)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{2\gamma,q}_{bc}(\Omega)}, \qquad t > 0, \quad u_0 \in H^{2\gamma,q}_{bc}(\Omega)$$

for every $\gamma, \gamma' \in I(q)$, with $\gamma' \geq \gamma$, for some $M_{\gamma',\gamma}$ and $\mu \in I\!R$ independent of m, m_0 and $0 < \varepsilon \leq \varepsilon_0$ and $\gamma, \gamma' \in I(q)$. In particular, one has

$$\|S_{m,m_0,\varepsilon}(t)u_0\|_{L^\tau(\Omega)} \leq \frac{M_{\rho,\tau}e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{\rho}-\frac{1}{\tau})}}\|u_0\|_{L^\rho(\Omega)}, \qquad t > 0, \quad u_0 \in L^\rho(\Omega)$$

for $1 \leq \rho \leq \tau \leq \infty$ with $M_{\rho,\tau}$ and μ independent of m, m_0 and $0 < \varepsilon \leq \varepsilon_0$.

Finally, for every $u_0 \in H^{2\gamma,q}_{bc}(\Omega)$, with $\gamma \in I(q)$, the function $u^\varepsilon(t;u_0) := S_{m,m_0,\varepsilon}(t)u_0$ is in $C^\nu(\overline{\Omega})$ for any $0 < \nu < 1$ and is a weak solution of (3.2) in the sense that

$$\int_\Omega u^\varepsilon_t \varphi + \int_\Omega a(x)\nabla u^\varepsilon \nabla\varphi + \int_\Gamma (b(x) - m_0(x))u^\varepsilon\varphi = \frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon(x)u^\varepsilon\varphi + \int_\Omega m(x)u^\varepsilon\varphi$$

for all sufficiently smooth φ.

Note that if $V_0 \in L^r(\Gamma)$, for $r > N-1$, with the choice $V_\varepsilon = 0$ and $m_0 + V_0$ replacing m_0, the result above allows to define the semigroup $S_{m,m_0+V_0}(t)$ such that for every $u_0 \in H^{2\gamma,q}_{bc}(\Omega)$, with γ as above, the function $u(t;u_0) := S_{m,m_0+V_0}(t)u_0$ is a weak solution of (3.3) in the sense that

$$\int_\Omega u_t\varphi + \int_\Omega a(x)\nabla u\nabla\varphi + \int_\Gamma (b(x) - m_0(x))u\varphi = \int_\Gamma V_0(x)u\varphi + \int_\Omega m(x)u\varphi$$

for all sufficiently smooth φ. With these notations we have

Theorem 3.2. *Assume that as $\varepsilon \to 0$, (2.9) holds true and for any $1 < q < \infty$, consider the semigroups $S_{m_\varepsilon,m_{0,\varepsilon},\varepsilon}(t)$ and $S_{m,m_0+V_0}(t)$ as above.*

Then for every $\gamma, \gamma' \in I(q) := (-1 + \frac{1}{2q}, 1 - \frac{1}{2q'})$, $\gamma' \geq \gamma$, and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{m_\varepsilon,m_{0,\varepsilon},\varepsilon}(t) - S_{m,m_0+V_0}(t)\|_{\mathcal{L}(H^{2\gamma,q}_{bc}(\Omega), H^{2\gamma',q}_{bc}(\Omega))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \text{for all} \quad 0 < t \leq T.$$

In particular, for any $0 < \nu < 1$ the solutions $u^\varepsilon(t;u_0) := S_{m,m_{0,\varepsilon},\varepsilon}(t)u_0$ of (3.2) converge to solutions $u(t;u_0) := S_{m,m_0+V_0}(t)u_0$ of (3.3) in $C^\nu(\overline{\Omega})$ uniformly on bounded time intervals away from $t = 0$.

Remark 3. i) The constant $C(\varepsilon)$ in Theorem 3.2 can be estimated in terms of

$$\|m_\varepsilon - m\|_{L^p(\Omega)} + \|m_{0,\varepsilon} - m_0\|_{L^r(\Gamma)} + \|\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon - V_0\|$$

where the last norm is a suitable norm in a space $\mathcal{L}(H^{s,q}(\Omega), H^{-\sigma,q}(\Omega))$ for suitable s, σ see e.g. Proposition 1.
ii) From Theorem 3.2 and Corollary 3 we have that for sufficiently small ε, in Theorem 3.1 we can take any $\mu > -\lambda_1^0$.

4. **Nonlinear parabolic problems.** We analyze now the behavior, for small ε, of the solutions of the nonlinear parabolic problem

$$\begin{cases} u_t^\varepsilon - \operatorname{div}(a(x)\nabla u^\varepsilon) = f(x, u^\varepsilon) + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon(x, u^\varepsilon) & \text{in } \Omega \\ a(x)\frac{\partial u^\varepsilon}{\partial \vec{n}} + b(x)u = 0 & \text{on } \Gamma \\ u^\varepsilon(0) = u_0 & \text{in } \Omega \end{cases} \tag{4.1}$$

where $a \in C^1(\overline{\Omega})$ with $a(x) \geq a_0 > 0$ in Ω and $b(x)$ a $C^1(\partial\Omega)$ function and $\mathcal{X}_{\omega_\varepsilon}$ denotes the characteristic function of the set ω_ε. Note that without loss of generality we can assume that g_ε is defined on $\overline{\Omega} \times I\!R$.

We will show below that the "limit problem" for the singularly perturbed problem (4.1) is given by

$$\begin{cases} u_t - \operatorname{div}(a(x)\nabla u) = f(x, u) & \text{in } \Omega \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u = g_0(x, u) & \text{on } \Gamma \\ u(0) = u_0 \end{cases} \tag{4.2}$$

where g_0 is obtained as the limit of the concentrating terms

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon(\cdot, u) \to g_0(\cdot, u)$$

as we now explain. To be more precise, observe that the nonlinear terms in (4.1) may contain zero and first order terms in u, so they can be written as

$$f(x, u) = h(x) + m(x)u + f_0(x, u) \quad \text{with} \quad f_0(x, 0) = 0, \quad \frac{\partial}{\partial u} f_0(x, 0) = 0 \tag{4.3}$$

and

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon(x, u) = \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}\Big(h_\varepsilon(x) + V_\varepsilon(x)u + g_\varepsilon^0(x, u)\Big) \tag{4.4}$$

with $g_\varepsilon^0(x, 0) = 0$, $\frac{\partial}{\partial u} g_\varepsilon^0(x, 0) = 0$, with certain regularity properties that will be made precise below.

Analogously for (4.2) we will assume

$$g_0(x, u) = h_0(x) + V_0(x)u + g_0^0(x, u), \quad x \in \Gamma \tag{4.5}$$

where h_0, V_0 and $g_0^0(x, u)$ are obtained as the limits of the concentrating terms

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} h_\varepsilon \to h_0, \qquad \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \to V_0, \quad cc - L^r \quad \text{for some } r > N - 1. \tag{4.6}$$

while

$$g_\varepsilon^0(x, u) \to g_0^0(x, u) \quad \text{uniformly in } x \in \Gamma, \text{ for } u \text{ in bounded sets of } I\!R. \tag{4.7}$$

Our goal is to prove that under assumptions (4.7) and (4.6), plus some growth and dissipativity conditions on the nonlinear terms, problems (4.1) and (4.2) have globally defined solutions for certain classes of initial data. Moreover, we are going to show that the solutions of both problems have enough compactness so that they are attracted to the global attractors, $\mathcal{A}_\varepsilon$, $0 \leq \varepsilon \leq \varepsilon_0$ respectively. The global attractor for each problem contains all information about the asymptotic behavior of all solutions.

Furthermore, we are going to show that the asymptotic dynamics of (4.1) and (4.2) are close in the sense that the family of attractors $\mathcal{A}_\varepsilon$ is upper semicontinuous at $\varepsilon = 0$. That is,

$$dist(\mathcal{A}_\varepsilon, \mathcal{A}_0) := \sup_{u^\varepsilon \in \mathcal{A}_\varepsilon} \inf_{u^0 \in \mathcal{A}_0} \{\|u^\varepsilon - u^0\|\} \to 0, \text{ as } \varepsilon \to 0,$$

in a suitable and strong norm which here implies, among others, uniform convergence in $\overline{\Omega}$ for the functions and convergence of the derivatives in Lebesgue spaces. The results in this section are taken from [33, 34].

Observe that the approach for upper semicontinuity has grounds in, e.g. Section 2.5. in [30], see also [61], and requires the following ingredients. First, we must prove that all problems have attractors and that they are uniformly bounded with respect to the parameter $0 \le \varepsilon \le \varepsilon_0$. Then we must prove that the nonlinear semigroups defined by (4.1) converge as $\varepsilon \to 0$ to the one defined by (4.2). This in turn, will be obtained from the convergence of solutions for the corresponding linear equations, see [55].

4.1. **Well posedness for nonlinear problems.** In this section we give some results on the well posedness for both problems (4.1) and (4.2). For these we use the results in [8] adapted to the particularities of problems (4.1) and (4.2) mentioned above. Also note that we will make use of the semigroups described in Section 3 with boundary potential $m_0 = 0$.

Hence we consider (4.1) and (4.2) in the space $X = L^q(\Omega)$ or $X = H^{1,q}_{bc}(\Omega) = H^{1,q}(\Omega)$, for $1 < q < \infty$. For either choice of X there exist suitable growth restrictions on the nonlinearities, such that problems (4.1) and (4.2) are locally well posed in X. For this we consider the following class of nonlinear terms $\mathcal{N}_X$

Definition 4.1. The class $\mathcal{N}_X$ is formed up with functions $j(x,u)$ such that
i) $j(x,\cdot) : I\!R \to I\!R$ is locally Lipschitz, uniformly on $x \in \overline{\Omega}$ or $x \in \Gamma$
ii) If $X = L^q(\Omega)$, assume that

$$|j(x,u) - j(x,v)| \le c|u-v|(|u|^{\rho-1} + |v|^{\rho-1} + 1), \tag{4.8}$$

iii) If $X = H^{1,q}_{bc}(\Omega)$ and
a) if $1 < q < N$, assume (4.8)
b) if $q = N$ assume that for every $\eta > 0$, there exists $c_\eta > 0$ such that

$$|j(x,u) - j(x,v)| \le c_\eta(e^{\eta|u|^{\frac{N}{N-1}}} + e^{\eta|v|^{\frac{N}{N-1}}})|u-v|, \tag{4.9}$$

c) if $q > N$, no further conditions are assumed.

Then the techniques from [8] applied here give the following result.

Theorem 4.2. *Assume the nonlinear terms $f(x,u)$, $g_\varepsilon(x,u)$ and $g_0(x,u)$ satisfy (4.3), (4.4) and (4.5) respectively such that for every fixed $0 < \varepsilon \le \varepsilon_0$ we have $h, \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} h_\varepsilon \in L^\infty(\Omega)$, $m, \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \in L^p(\Omega)$ for some $p > N/2$, and for $\varepsilon = 0$, $h_0 \in L^\infty(\Gamma)$ and $V_0 \in L^r(\Gamma)$, for some $r > N-1$.*

Also, assume $X = L^q(\Omega)$ or $X = H^{1,q}_{bc}(\Omega)$, with $f_0, g^0_\varepsilon, g^0_0 \in \mathcal{N}_X$ and either:
i) For (4.1), with fixed $0 < \varepsilon \le \varepsilon_0$,
a) if $X = L^q(\Omega)$ the exponents ρ_{f_0} and $\rho_{g^0_\varepsilon}$ in (4.8), are such that

$$\rho_{f_0}, \rho_{g^0_\varepsilon} \le \rho_\Omega := 1 + \frac{2q}{N},$$

b) if $X = H^{1,q}_{bc}(\Omega)$ exponents ρ_{f_0} and $\rho_{g^0_\varepsilon}$ in (4.8), are such that

$$\rho_{f_0}, \rho_{g^0_\varepsilon} \leq \rho_\Omega := 1 + \frac{2q}{N-q}.$$

ii) For (4.2)

a) if $X = L^q(\Omega)$ the exponents ρ_{f_0} and $\rho_{g^0_0}$ in (4.8), are such that with $N \geq 2$ (respectively $N = 1$)

$$\rho_{f_0} \leq \rho_\Omega := 1 + \frac{2q}{N}, \quad \textit{and} \quad \rho_{g^0_0} \leq \rho_\Gamma := 1 + \frac{q}{N}, \ (\textit{respectively}, \ \rho_{g^0_0} < \rho_\Gamma := 1 + q),$$

b) if $X = H^{1,q}_{bc}(\Omega)$ exponents ρ_{f_0} and $\rho_{g^0_0}$ in (4.8), are such that

$$\rho_{f_0} \leq \rho_\Omega := 1 + \frac{2q}{N-q} \quad \textit{and} \quad \rho_{g^0_0} \leq \rho_\Gamma := 1 + \frac{q}{N-q}.$$

Then for any $u_0 \in X$ there exists a unique (in certain sense) mild solution $u(\cdot, u_0) \in C([0,\tau), X)$, of problems (4.1) or (4.2), respectively, satisfying $u(0, u_0) = u_0$ in X. This solution depends continuously on the initial data $u_0 \in X$.

In order to ensure that the local solutions constructed above are globally defined, following [9], we will assume the following sign conditions on the nonlinear terms

Sign conditions (S) Assume in addition that the there exist $C \in L^p(\Omega)$, $0 \leq D \in L^p(\Omega)$ with $p > \frac{N}{2}$

$$uf(x,u) \leq C(x)u^2 + D(x)|u|, \quad x \in \Omega, \quad u \in I\!R, \tag{4.10}$$

and either

i) for (4.1), with fixed $0 < \varepsilon \leq \varepsilon_0$, there exist $E_\varepsilon \in L^p(\Omega)$, $0 \leq F_\varepsilon \in L^p(\Omega)$, $p > \frac{N}{2}$ such that

$$ug_\varepsilon(x,u) \leq E_\varepsilon(x)u^2 + F_\varepsilon(x)|u|, \quad x \in \omega_\varepsilon, \quad u \in I\!R, \tag{4.11}$$

ii) for (4.2), there exist $E_0 \in L^r(\Gamma)$, $0 \leq F_0 \in L^r(\Gamma)$, $r > N-1$ such that

$$ug_0(x,u) \leq E_0(x)u^2 + F_0(x)|u|, \quad x \in \Gamma, \quad u \in I\!R. \tag{4.12}$$

Remark 4. Observe that comparing (4.3) with (4.10), (4.4) with (4.11) and (4.5) with (4.12), we get

$$|h(x)| \leq D(x), \quad |h_\varepsilon(x)| \leq F_\varepsilon(x), \quad |h_0(x)| \leq F_0(x).$$

Then we have, see [9, Theorem 2.2] and also [56, Theorems 2.5 and 2.6].

Theorem 4.3. *Under the sign assumptions (S) above, the local solutions in Theorem 4.2 are defined for all $t \geq 0$ and each solution is bounded in $L^\infty(\Omega)$ and in X on bounded time intervals away from $t = 0$. In particular (4.1) and (4.2) define nonlinear semigroups*

$$T_\varepsilon(t)u_0 = u^\varepsilon(t; u_0), \qquad 0 \leq \varepsilon \leq \varepsilon_0, \quad u_0 \in X,$$

for either $X = L^q(\Omega)$ or $X = H^{1,q}_{bc}(\Omega)$.

4.2. **Existence of attractors and uniform bounds.** In this section we give conditions that allow to prove that the nonlinear semigroups defined by problems (4.1) and (4.2) in Theorem 4.3 have global attractors $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ respectively and to obtain suitable uniform bounds on $\mathcal{A}_\varepsilon$ independent of ε. For this, by Lemma 2.2, we will assume that in (4.11)

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |E_\varepsilon|^r \le C, \quad \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} E_\varepsilon \to E \quad cc - L^r, \quad r > N-1, \tag{4.13}$$

and

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |F_\varepsilon|^r \le C, \quad \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} F_\varepsilon \to F \quad cc - L^r, \quad r > N-1. \tag{4.14}$$

Therefore we will also assume the following dissipativity condition.

Dissipative condition (D) There exists $\delta > 0$ such that the first eigenvalue, λ_1, of the following problem

$$\begin{cases} -\mathrm{div}(a(x)\nabla\varphi) = C(x)\varphi + \lambda\varphi & \text{in } \Omega \\ a(x)\frac{\partial\varphi}{\partial\vec{n}} + b(x)\varphi = \tilde{E}(x)\varphi & \text{on } \Gamma \end{cases}$$

satisfies

$$\lambda_1 > \delta > 0 \tag{4.15}$$

for $\tilde{E} = E$ as in (4.13) and $\tilde{E} = E_0$ in (4.12).

Lemma 4.4. *Assume the sign conditions (4.10), (4.11) and (4.12), the concentrated bounds (4.13), (4.14) and the dissipativity condition (4.15).*

Then there exist a constant K_∞ and a function $R_\infty(M,t)$, for $M,t > 0$, independent of ε such that for each fixed $M > 0$, $R_\infty(M,t)$, is monotonically decreasing and converges to zero, as $t \to \infty$ and such that for sufficiently small $0 \le \varepsilon \le \varepsilon_0$, the global solutions of problems (4.1) and (4.2) in Theorem 4.3, satisfy that for initial data such that $\|u_0\|_{L^q(\Omega)} \le M$

$$\sup_{0\le\varepsilon\le\varepsilon_0}\ \sup_{\|u_0\|_{L^q(\Omega)}\le M} \|u^\varepsilon(t,\cdot;u_0)\|_{L^\infty(\Omega)} \le K_\infty + R_\infty(M,t).$$

In particular, for any $M > 0$,

$$\limsup_{t\to\infty}\ \sup_{0\le\varepsilon\le\varepsilon_0}\ \sup_{\|u_0\|_{L^q(\Omega)}\le M} \|u^\varepsilon(t,\cdot;u_0)\|_{L^\infty(\Omega)} \le K_\infty.$$

With this and the smoothing effect of the equations we get

Lemma 4.5. *Under the assumptions in Lemma 4.4 assume moreover that*

$$\sup_{x\in\omega_\varepsilon} |h_\varepsilon(x)| \le C, \quad \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon|^r \le C, \qquad r > N-1$$

and $\{g_\varepsilon^0(x,u)\}_\varepsilon$ is uniformly bounded in $\overline{\Omega}$ on bounded sets of $I\!R$, i.e. for any $R > 0$ there exists a positive constant $C(R)$ independent of ε such that

$$|g_\varepsilon^0(x,u)| \le C(R), \quad \text{for all} \quad x \in \overline{\Omega}, \quad \text{and} \quad |u| \le R.$$

Then, for any $1 < \rho < \infty$ and $\gamma' < \frac{1}{2} + \frac{1}{2\rho}$ there exists a constant $K_{\rho,\gamma'}$ and a function $R_{\rho,\gamma'}(M,t)$, for $M,t > 0$, independent of ε such that for each fixed $M > 0$, $R_{\rho,\gamma'}(M,t)$, is monotonically decreasing and converges to zero, as $t \to \infty$ and such that for sufficiently small $0 \le \varepsilon \le \varepsilon_0$, the global solutions of problems (4.1) and (4.2) in Theorem 4.3, satisfy that for initial data such that $\|u_0\|_{L^q(\Omega)} \le M$

$$\sup_{0\le\varepsilon\le\varepsilon_0}\ \sup_{\|u_0\|_{L^q(\Omega)}\le M} \|u^\varepsilon(t,\cdot;u_0)\|_{H_{bc}^{2\gamma',\rho}(\Omega)} \le K_{\rho,\gamma'} + R_{\rho,\gamma'}(M,t).$$

In particular,

$$\limsup_{t\to\infty} \sup_{0\le\varepsilon\le\varepsilon_0} \sup_{\|u_0\|_{L^q(\Omega)}\le M} \|u^\varepsilon(t,\cdot;u_0)\|_{H^{2\gamma',\rho}_{bc}(\Omega)} \le K_{\rho,\gamma'}.$$

Therefore, the global semigroups defined by problems (4.1) and (4.2) in Theorem 4.3, have global attractors $\mathcal{A}_\varepsilon$ in X which satisfy

$$\sup_{0\le\varepsilon\le\varepsilon_0} \sup_{v\in\mathcal{A}_\varepsilon} \|v\|_{H^{2\gamma',\rho}_{bc}(\Omega)} \le K_{\rho,\gamma'}.$$

In particular the attractors are uniformly bounded in $H^{1,\rho}_{bc}(\Omega)$ and $C^\nu(\overline{\Omega})$ for any $1<\rho<\infty$ and for any $0<\nu<1$.

Remark 5. From here on we can assume that the nonlinear terms are globally Lipschitz and the semigroups $T_\varepsilon(t)$ and $T_0(t)$ are defined on $L^\rho(\Omega)$ for any $1<\rho<\infty$. In particular, the attractors $\mathcal{A}_\varepsilon$ attract solutions in the norm of $H^{2\gamma',\rho}_{bc}(\Omega)$ for any $1<\rho<\infty$ and $\gamma'<\frac{1}{2}+\frac{1}{2\rho}$.

Now since the nonlinear semigroups $T_\varepsilon(t)$ and $T_0(t)$ are order preserving and the estimates above, from Theorem 3.2 in [56], see also [18], we get the existence of extremal equilibria for problems (4.1) and (4.2) which are the caps of the attractors

Proposition 2. *Under the above notations and hypotheses, for each $0\le\varepsilon\le\varepsilon_0$, there exists two ordered extremal equilibria $\varphi^\varepsilon_m \le \varphi^\varepsilon_M$ such that $\mathcal{A}_\varepsilon \subset [\varphi^\varepsilon_m, \varphi^\varepsilon_M]$, $\varphi^\varepsilon_m, \varphi^\varepsilon_M \in \mathcal{A}_\varepsilon$ and*

$$\varphi^\varepsilon_m \le \liminf_{t\to\infty} u^\varepsilon(t,x;u_0) \le \limsup_{t\to\infty} u^\varepsilon(t,x;u_0) \le \varphi^\varepsilon_M$$

uniformly in $x\in\Omega$ and for initial data u_0 such that $\|u_0\|_{L^q(\Omega)}\le M$.

4.3. **Concentrated nonlinear terms.** In this section, we prove two technical results that will allow to pass to the limit in nonlinear terms which are concentrating near the boundary as $\varepsilon\to 0$.

For this, we consider a family of functions

$$g^0_\varepsilon : \overline{\Omega}\times I\!R \longrightarrow I\!R,$$

for $0\le\varepsilon\le\varepsilon_0$, satisfying the following conditions

i) $\{g^0_\varepsilon(x,u)\}_\varepsilon$ is uniformly bounded in $\overline{\Omega}$ on bounded sets of $I\!R$, i.e. for any $R>0$ there exists a positive constant $C(R)$ independent of ε such that

$$|g^0_\varepsilon(x,u)| \le C(R), \quad \text{for all} \quad x\in\overline{\Omega}, \quad \text{and} \quad |u|\le R. \tag{4.16}$$

ii) $\{g^0_\varepsilon(x,u)\}_\varepsilon$ is uniformly continuous in $\overline{\Omega}$, uniformly on bounded sets of $I\!R$ and also uniformly Lipschitz on bounded sets of $I\!R$, i.e. for any $R>0$ there exists a positive constant $L(R)$ independent of ε such that

$$\left|g^0_\varepsilon(x,u) - g^0_\varepsilon(x,v)\right| \le L(R)|u-v|, \quad \text{for all} \quad x\in\overline{\Omega}, \quad |u|\le R,\ |v|\le R. \tag{4.17}$$

iii) $g^0_\varepsilon(x,u)$ converges to $g^0_0(x,u)$ uniformly on Γ and on bounded sets of $I\!R$, i.e. for any $R>0$

$$g^0_\varepsilon(x,u)\to g^0_0(x,u) \quad \text{as} \quad \varepsilon\to 0, \quad \text{uniformly on } x\in\Gamma \text{ and } |u|\le R \tag{4.18}$$

Then we have the following result. Note that here p and q are not meant to be the same as in previous sections. Also, the result below applies in the case $g^0_\varepsilon = g^0_0$, that is, when the family does not depend on ε.

Lemma 4.6. *Consider a family of functions*

$$g_\varepsilon^0 : \overline{\Omega} \times I\!R \longrightarrow I\!R$$

for $0 \le \varepsilon \le \varepsilon_0$. *Also, consider a family of functions,* $\mathcal{C}$, *in* Ω *such that, for some* $1 < p < \infty$ *and* $R > 0$

$$\|v\|_{H^{1,p}(\Omega)\cap L^\infty(\Omega)} \le R \quad \textit{for all } v \in \mathcal{C}. \tag{4.19}$$

i) If $\{g_\varepsilon^0\}_\varepsilon$ *satisfies (4.16), then there exists a positive constant,* $M(R)$, *independent of* ε *such that for every* $1 < q < \infty$ *and any* $\varphi \in H^{s,q'}(\Omega)$ *with* $s > \frac{1}{q'}$ *and every* $v \in \mathcal{C}$ *we have*

$$|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} g_\varepsilon^0(\cdot, v)\varphi| \le M(R)\|\varphi\|_{H^{s,q'}(\Omega)}.$$

In particular $\sup_{v\in\mathcal{C}} \|\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon^0(\cdot, v)\|_{H^{-s,q}(\Omega)} \le M(R)$.
ii) If $\{g_\varepsilon^0\}_\varepsilon$ *satisfies (4.16), (4.17) and (4.18), then there exists* $M(\varepsilon, R) \to 0$ *as* $\varepsilon \to 0$ *such that for every* $\varphi \in H^{1,q'}(\Omega)$ *and* $v \in \mathcal{C}$

$$\Big|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} g_\varepsilon^0(\cdot, v)\varphi - \int_\Gamma g_0^0(\cdot, v)\varphi\Big| \le M(\varepsilon, R)\|\varphi\|_{H^{1,q'}(\Omega)},$$

provided

$$p \ge \frac{q(N-1)}{N}. \tag{4.20}$$

In particular $\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon^0(\cdot, v) \to g_0^0(\cdot, v)$ *in* $H^{-1,q}(\Omega)$, *uniformly in* $v \in \mathcal{C}$.

In particular, we get

Corollary 4. *Assume (4.16), (4.17), (4.18) and*

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} h_\varepsilon \to h_0, \quad cc - L^\infty$$

and consider the nonlinear terms

$$H_\varepsilon(u) = h + f_0(\cdot, u) + \lambda u + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} h_\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon^0(\cdot, u)$$

and

$$H_0(u) = \big(h + f_0(\cdot, u) + \lambda u\big)_\Omega + \big(h_0 + g_0^0(\cdot, u)\big)_\Gamma$$

Finally, consider a family $\mathcal{C}$ *as in Lemma 4.6, that is satisfying (4.19). Then we have that for any* $1 < q < \infty$ *and* $\frac{1}{q'} < s \le 1$
i) There exists $C > 0$ *independent of* $\varepsilon > 0$ *such that*

$$\sup_{v\in\mathcal{C}}\{\|H_\varepsilon(v)\|_{H_{bc}^{-s,q}(\Omega)}, \|H_0(v)\|_{H_{bc}^{-s,q}(\Omega)}\} \le C.$$

ii) If (4.20) holds, that is $p \ge \frac{q(N-1)}{N}$, *there exists* $M(\varepsilon) \to 0$ *as* $\varepsilon \to 0$ *such that*

$$\sup_{v\in\mathcal{C}} \|H_\varepsilon(v) - H_0(v)\|_{H_{bc}^{-s,q}(\Omega)} \le M(\varepsilon).$$

4.4. **Upper semicontinuity of attractors.** With all the above we can then obtain the convergence of the nonlinear semigroups. Note that although the nonlinear problems (4.1) and (4.2) are set in the space $X = L^q(\Omega)$ or $X = H^{1,q}_{bc}(\Omega)$ as in Section 4.1, depending on the growth of the nonlinear term, the convergence results below always take place in $H^{1,\rho}_{bc}(\Omega)$ for any $1 < \rho < \infty$.

Lemma 4.7. *Fix any $M > 0$ and $t_0 > 0$ and consider any initial data such that $\|u_0\|_{L^q(\Omega)} \leq M$ and denote $u_\varepsilon = T_\varepsilon(t_0)u_0$.*

Then, for any $1 < \rho < \infty$ and any $T > 0$, there exists a constant $C(M, T, \varepsilon) \to 0$ if $\varepsilon \to 0$, such that for $\varepsilon \in (0, \varepsilon_0)$,

$$\|T_\varepsilon(t)u_\varepsilon - T_0(t)u_\varepsilon\|_{H^{1,\rho}_{bc}(\Omega)} \leq C(M, T, \varepsilon) \to 0, \quad \text{as } \varepsilon \to 0, \text{ for } t \in [0, T].$$

In particular

$$\sup_{v_\varepsilon \in \mathcal{A}_\varepsilon} \|T_\varepsilon(t)v_\varepsilon - T_0(t)v_\varepsilon\|_{H^{1,\rho}_{bc}(\Omega)} \leq C(M, T, \varepsilon) \to 0, \quad \text{as } \varepsilon \to 0, \text{ for } t \in [0, T].$$

We are now in a position to prove the upper semicontinuity of the family of attractors.

Theorem 4.8. *Under the above assumptions, for any $1 < \rho < \infty$, the family of global attractors of (4.1) and (4.2), $\mathcal{A}_\varepsilon$, is upper semicontinuous at $\varepsilon = 0$ in $H^{1,\rho}_{bc}(\Omega)$, that is*

$$dist_{H^{1,\rho}_{bc}(\Omega)}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \to 0, \ \text{if } \varepsilon \to 0$$

where

$$dist_{H^{1,\rho}_{bc}(\Omega)}(\mathcal{A}_\varepsilon, \mathcal{A}_0) := \sup_{u^\varepsilon \in \mathcal{A}_\varepsilon} \inf_{u^0 \in \mathcal{A}_0} \{\|u^\varepsilon - u^0\|_{H^{1,\rho}_{bc}(\Omega)}\}$$

In particular, we get the upper semicontinuity of equilibria

Corollary 5.
i) For every sequence ε_k with $\varepsilon_k \to 0$ as $k \to \infty$ and for every sequence of equilibria $\varphi^{\varepsilon_k} \in \mathcal{A}_{\varepsilon_k}$ there exists a subsequence (that we denote the same) and a equilibrium point $\varphi^0 \in \mathcal{A}_0$ such that

$$\varphi^{\varepsilon_k} \to \varphi^0, \quad k \to \infty \quad \text{in } H^{1,\rho}_{bc}(\Omega)$$

for any $1 < \rho < \infty$.
ii) In particular, considering the extremal equilibria in Proposition 2, we obtain that

$$\varphi^0_m \leq \liminf_{\varepsilon \to 0} \varphi^\varepsilon \leq \limsup_{\varepsilon \to 0} \varphi^\varepsilon \leq \varphi^0_M$$

5. **Dynamic boundary conditions.** Dynamic boundary conditions have the main characteristic of involving the time derivative of the unknown. They have been used, among others, as a model of "boundary feedback" in stabilization and control problems of membranes and plates, [15, 40, 41, 39, 44, 66], in phase transition problems, [65, 24, 25, 26, 49, 16], in some hydrodynamic problems, [27, 63] or in population dynamics, [21]. They have also been considered in the context of elliptic–parabolic problems, [20, 57]. Also several of so called "transmission problems" have been described and analyzed in [60], some of which lead, under some singular perturbation limits, to problems with dynamical boundary conditions.

In this section our goal is to prove that dynamic boundary conditions can be obtained as the singular limit of elliptic/parabolic problems in which the time derivative concentrates in a narrow region close to the boundary.

Hence we consider the following family of parabolic problems

$$\left\{\begin{array}{ll} \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon_t - \Delta u^\varepsilon + \lambda u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x) u^\varepsilon = f + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon & \text{in } \Omega \\ \frac{\partial u^\varepsilon}{\partial \vec{n}} = 0 & \text{on } \Gamma \\ u^\varepsilon(0,x) = u^\varepsilon_0(x) & \text{in } \Omega \end{array}\right. \tag{5.1}$$

where $\mathcal{X}_{\omega_\varepsilon}$ is the characteristic function of the set ω_ε and $\lambda \in I\!R$. Then, following [35], we show that the limit problem is the following parabolic problem with dynamic boundary conditions

$$\left\{\begin{array}{ll} -\Delta u^0 + \lambda u^0 = f & \text{in } \Omega \\ u^0_t + \frac{\partial u^0}{\partial \vec{n}} + V(x) u^0 = g & \text{on } \Gamma \\ u^0(0,x) = v_0(x) & \text{on } \Gamma \end{array}\right. \tag{5.2}$$

where v_0, V and g are obtained as the limits of the concentrating terms

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon_0 \to v_0, \qquad \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \to V, \qquad \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon \to g.$$

Notice that all concentrating terms in (5.1) are transferred, in the limit, to the boundary condition in (5.2).

Note that (5.1) is formally equivalent to solving

$$\left\{\begin{array}{ll} -\Delta u^\varepsilon + \lambda u^\varepsilon = f & \text{in } \Omega \setminus \bar{\omega}_\varepsilon \\ \frac{1}{\varepsilon} u^\varepsilon_t - \Delta u^\varepsilon + \lambda u^\varepsilon + \frac{1}{\varepsilon} V_\varepsilon u^\varepsilon = f + \frac{1}{\varepsilon} g_\varepsilon & \text{in } \omega_\varepsilon \\ \frac{\partial u^\varepsilon}{\partial \vec{n}} = 0 & \text{on } \Gamma \\ u^\varepsilon(0,x) = u^\varepsilon_0(x) & \text{in } \Omega \end{array}\right. \tag{5.3}$$

and that in (5.3) boundary conditions are missing on $\Gamma_\varepsilon = \partial\omega_\varepsilon \setminus \Gamma = \partial(\Omega \setminus \bar{\omega}_\varepsilon)$. Since there would be several ways of connecting the solutions of the elliptic and the parabolic equations in (5.3) along that boundary, we consider the boundary conditions on Γ_ε that ensure maximal smoothness of solutions. This is achieved by imposing the classical transmissions conditions on Γ_ε, that is, no jump of the u^ε and its normal derivate across Γ_ε, see [58],

$$[u_\varepsilon]_{\Gamma_\varepsilon} = [\frac{\partial u_\varepsilon}{\partial \vec{n}}]_{\Gamma_\varepsilon} = 0. \tag{5.4}$$

Hence, (5.3) and (5.4) is a formulation of an elliptic–parabolic transmission problem, see [46], Chapter 1, Section 9, for related problems.

On the other hand, (5.2) must be understood as an evolution problem on the boundary Γ, such that, for each time $t > 0$, the solution must be lifted to the interior of Ω by means of the elliptic equation in (5.2). In this way the term $\frac{\partial u^0}{\partial \vec{n}}$, which is the so called Dirichlet Neumann operator, becomes a linear nonlocal operator for functions defined on Γ.

Here and below $H^s(\Omega)$ denote, for $s \geq 0$, the standard Sobolev spaces and for $s > 0$ we denote

$$H^{-s}(\Omega) = \big(H^s(\Omega)\big)'.$$

Also $H^{-1}_0(\Omega)$ will denote the dual space of $H^1_0(\Omega)$.

Finally, we will consider below traces on Γ of functions defined in Ω. Hence, we will denote by $\gamma(u)$ the trace of a function u and denote by γ the trace operator on $H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma)$ for $s > \frac{1}{2}$, and $H^{-1/2}(\Gamma)$ will denote the dual space of $H^{1/2}(\Gamma)$. We will also use the embeddings

$$H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma) \subset H^{-1}(\Omega)$$

in the sense that for every $f \in H^1(\Omega)$, $\gamma(f) \in H^{-\frac{1}{2}}(\Gamma) \subset H^{-1}(\Omega)$ is defined as

$$\Big\langle \gamma(f), \phi \Big\rangle_{-1,1} := \int_\Gamma \gamma(f)\gamma(\phi), \quad \phi \in H^1(\Omega).$$

We will often find below some elements in $H^{-1}(\Omega)$ for which we will employ the notation

$$h = f_\Omega + g_\Gamma$$

where f and g are functions defined in Ω and on Γ respectively. This will denote the functional defined by

$$\Big\langle h, \phi \Big\rangle_{-1,1} = \int_\Omega f\phi + \int_\Gamma g\phi$$

for all sufficiently smooth function ϕ in $\overline{\Omega}$.

5.1. **The approximating parabolic problems.** Note that in [58] a very similar problem to (5.1) was considered. In fact in [58] Dirichlet boundary conditions were assumed on Γ instead as Neumann ones and also $V_\varepsilon = 0$. Therefore, we modify the arguments in [58] to apply them to (5.1). See Theorem 1.1, Theorem 4.9 and Proposition 4.10 in [58]. Also, temporarily, we remove the dependence on ε.

Hence, we identify $L^2(\Omega)$ with its dual and denote by $H^{-1}(\Omega)$ the dual space of $H^1(\Omega)$ and then $H^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$. Also, we assume

$$V \in L^\rho(\omega), \quad \rho > N/2, \tag{5.5}$$

and define the bilinear symmetric form in $H^1(\Omega)$

$$a(\varphi, \phi) = \int_\Omega \nabla\varphi\nabla\phi + \int_\omega V\varphi\phi + \lambda\int_\Omega \varphi\phi$$

for every $\varphi, \phi \in H^1(\Omega)$, which defines an linear mapping, L, between $H^1(\Omega)$ and its dual $H^{-1}(\Omega)$, given by

$$\Big\langle L(\varphi), \phi \Big\rangle_{-1,1} = a(\varphi, \phi).$$

Now we write (5.1) as

$$\left\{ \begin{array}{ll} \mathcal{X}_\omega u_t - \Delta u + \mathcal{X}_\omega V u + \lambda u = h(t) & \text{in } \Omega,\ t > 0, \\ \frac{\partial u}{\partial \vec{n}} = 0 & \text{on } \Gamma \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{array} \right. \tag{5.6}$$

with $h = f + \mathcal{X}_\omega g$.

Then in a similar fashion as in Theorems 1.1 and 4.9 in [58], we have the following result that states the well–posedness of (5.6).

Theorem 5.1. *Denote $\lambda_{\Omega\setminus\bar{\omega}}$ the first eigenvalue of the Laplacian operator with Dirichlet boundary conditions in $\Omega \setminus \bar{\omega}$. Assume $\lambda > -\lambda_{\Omega\setminus\bar{\omega}}$, $h \in L^2((0,T), L^2(\Omega))$ and $u_0 \in L^2(\omega)$.*

i) Then there exists a unique solution of (5.6), which satisfies

$$u \in C([0,T), L^2(\omega)) \cap L^2((0,T), H^1(\Omega)), \quad u(0) = u_0 \ \textit{in } \omega$$

and satisfies (5.6) in the sense that

$$\mathcal{X}_\omega u_t + L(u) = h \quad \textit{in} \quad H^{-1}(\Omega), \quad \textit{a.e. } t \in (0,T).$$

ii) Assume moreover that either

a) $h \in W^{1,1}((0,T), L^2(\Omega))$ or

b) $h \in L^2((0,T), L^2(\omega)) = L^2((0,T) \times \omega)$ and $h \in W^{1,1}((0,T), L^2(\Omega \setminus \bar{\omega}))$

and $u_0 \in H^1(\Omega)$ *satisfies*

$$-\Delta u_0 + \lambda u_0 = h(0) \quad \text{in } \Omega \setminus \bar{\omega}. \tag{5.7}$$

Then

$$u \in C([0,T), H^1(\Omega)) \cap L^2((0,T), H^2(\Omega)) \quad u(0) = u_0 \text{ in } \Omega$$

and $u(t)$ *satisfies (5.6) a.e.* $t \in (0,T)$.

Also, as in Proposition 4.10 in [58], we get

Proposition 3. *Assume, as above, that* $\lambda > -\lambda_{\Omega\setminus\bar{\omega}}$ *and* $u_0 \in H^1(\Omega)$ *satisfying (5.7) and* $h(t) \in L^2(\Omega)$ *a.e.* $t \in (0,T)$, *are given.*
i) If $h \in W^{1,1}((0,T), L^2(\Omega))$, *then*

$$\|\nabla u(t)\|^2_{L^2(\Omega)} + \int_\omega V u(t)^2 + \lambda \|u(t)\|^2_{L^2(\Omega)} + 2\int_0^t \int_\omega u_t^2$$
$$= \|\nabla u_0\|^2_{L^2(\Omega)} + \int_\omega V u_0^2 + \lambda\|u_0\|^2_{L^2(\Omega)} + 2\left(\int_\Omega h(t)u(t) - \int_\Omega h(0)u_0 - \int_0^t \int_\Omega h_t u\right).$$

Therefore, the mapping $(u_0, h) \longmapsto (u, u_t)$ *is Lipschitz from* $H^1(\Omega) \times W^{1,1}((0,T), L^2(\Omega))$ *into* $C([0,T], H^1(\Omega)) \times L^2((0,T) \times \omega)$.
ii) If $h \in L^2((0,T)\times\omega)$ *and* $h \in W^{1,1}((0,T), L^2(\Omega\setminus\overline{\omega}))$, *then*

$$\|\nabla u(t)\|^2_{L^2(\Omega)} + \int_\omega V u(t)^2 + \lambda\|u(t)\|^2_{L^2(\Omega)} + 2\int_0^t \int_\omega u_t^2 = \|\nabla u_0\|^2_{L^2(\Omega)} + \int_\omega V u_0^2$$
$$+ \lambda\|u_0\|^2_{L^2(\Omega)} + 2\left(\int_0^t \int_\omega h u_t + \int_{\Omega\setminus\overline{\omega}} h(t)u(t) - \int_{\Omega\setminus\overline{\omega}} h(0)u_0 - \int_0^t \int_{\Omega\setminus\overline{\omega}} h_t u\right).$$

Therefore, the mapping $(u_0, h_\omega, h_{\Omega\setminus\overline{\omega}}) \longmapsto (u, u_t)$ *is Lipschitz from* $H^1(\Omega) \times L^2((0,T)\times\omega) \times W^{1,1}((0,T), L^2(\Omega\setminus\overline{\omega}))$ *into* $C([0,T], H^1(\Omega)) \times L^2((0,T)\times\omega)$.

5.2. **The limit problem with dynamic BC.** We consider the parabolic problem (5.2), that is

$$\left\{\begin{array}{ll} -\Delta u^0 + \lambda u^0 = f & \text{in } \Omega \\ u^0_t + \frac{\partial u^0}{\partial \vec{n}} + V_0(x)u^0 = g & \text{on } \Gamma \\ u^0(0,x) = v_0(x) & \text{on } \Gamma \end{array}\right. \tag{5.8}$$

for which we adapt the results in [57] for which the reader is referred for full details. Hence, in this case we assume

$$V_0 \in L^\rho(\Gamma), \quad \rho > N-1$$

and define the bilinear symmetric form in $H^1(\Omega)$

$$a_0(\varphi,\phi) = \int_\Omega \nabla\varphi\nabla\phi + \int_\Gamma V_0\varphi\phi + \lambda\int_\Omega \varphi\phi$$

for every $\varphi,\phi \in H^1(\Omega)$, which defines a linear mapping, L_0, between $H^1(\Omega)$ and its dual $H^{-1}(\Omega)$.

Now as in Corollary 3.3 in [57] we have the following result that states the well–posedness of (5.8).

Proposition 4. *Denote by* λ_Ω *the first eigenvalue of the Laplace operator in* Ω *with Dirichlet boundary conditions and assume* $\lambda > -\lambda_\Omega$.

Assume $\lambda > -\lambda_\Omega$, $f \in L^2((0,T), L^2(\Omega))$, $g \in L^2((0,T)\times\Gamma)$ *and* $v_0 \in L^2(\Gamma)$ *are given.*

i) Then there exists a unique solution of (5.8) which satisfies

$$u^0 \in L^2((0,T), H^1(\Omega)), \quad \gamma(u^0)_t \in L^2((0,T)\times\Gamma)$$

and satisfies (5.8) in the sense that

$$\gamma(u^0)_t + L_0(u^0) = f_\Omega + g_\Gamma$$

as an equality in $H^{-1}(\Omega)$, a.e. $t \in (0,T)$. In particular $\gamma(u^0) \in C([0,T], L^2(\Gamma))$ and $\gamma(u^0)(0) = v_0$.
ii) Moreover, if $f \in C([0,T), L^2(\Omega))$ and $u_0 \in H^1(\Omega)$ satisfies

$$-\Delta u_0 + \lambda u_0 = f(0), \quad \text{in } \Omega$$

then with $v_0 = \gamma(u_0)$ we have $u^0 \in C([0,T), H^1(\Omega))$, $u^0(0) = u_0$.

5.3. **Time dependent concentrating integrals.** In this section we show several results that describe how different concentrated integrals converge to surface integrals. Hereafter we denote by $C > 0$ any positive constant such that C is independent of ε and t. This constant may change from line to line.

The results in Section 2.1 can now be extended to handle concentrating integrals including a time dependence.

Lemma 5.2. *A) Consider $v \in L^r((0,T), H^s(\Omega))$ with $1 \le r < \infty$, $s > \frac{1}{2}$ and $H^s(\Omega) \subset L^q(\Gamma)$, that is, $s - \frac{N}{2} \ge -\frac{(N-1)}{q}$. Then,*

$$\int_0^T \Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |v|^q\Big)^{r/q} \le C\int_0^T \|v(t,\cdot)\|^r_{H^s(\Omega)} dt = \|v\|^r_{L^r((0,T),H^s(\Omega))}$$

and

$$\lim_{\varepsilon\to 0}\int_0^T \Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |v|^q\Big)^{r/q} = \int_0^T \Big(\int_\Gamma |v|^q\Big)^{r/q} = \|v\|^r_{L^r((0,T),L^q(\Gamma))}.$$

B) Consider a family g_ε defined on $(0,T)\times\omega_\varepsilon$, such that for some $1 < q < \infty$, $1 \le r < \infty$ and a positive constant C independent of ε,

$$\int_0^T \Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g_\varepsilon(t,x)|^r dx\Big)^{\frac{q}{r}} dt \le C$$

or $\int_0^T sup_{x\in\omega_\varepsilon} |g_\varepsilon(t,x)|^q\, dt \le C$ for the case $r = \infty$.

Then, for every s satisfying $s - \frac{N}{2} > -\frac{N-1}{r'}$, and for every sequence converging to zero (that we still denote $\varepsilon \to 0$) there exists a subsequence (that we still denote the same) and a function $g \in L^q((0,T), L^r(\Gamma))$ (or a bounded Radon measure on $\Gamma, g \in L^q((0,T), \mathcal{M}(\Gamma))$ if $r = 1$) such that

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon \to g \quad \text{ in } L^q((0,T), H^{-s}(\Omega)), \text{ weakly as } \varepsilon \to 0,$$

where $\mathcal{X}_{\omega_\varepsilon}$ is the characteristic function of the set ω_ε. In particular, for any smooth function φ, defined in $[0,T]\times\bar{\Omega}$, we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} g_\varepsilon\varphi = \int_0^T\int_\Gamma g\varphi.$$

Also, if $u^\varepsilon \to u^0$ strongly in $L^{q'}((0,T), H^s(\Omega))$ then

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} g_\varepsilon u^\varepsilon = \int_0^T\int_\Gamma g u^0.$$

C) Consider a family g_ε defined on $(0,T)\times\omega_\varepsilon$, and assume that for some $1 < r, q < \infty$, there exist $h \in L^q(0,T)$, and $g \in L^q((0,T), L^r(\Gamma))$ such that

$$\Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g_\varepsilon(t,\cdot)|^r\Big)^{\frac{1}{r}} \le h(t), \quad a.e. \quad t \in [0,T]$$

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon(t,\cdot) \to g(t,\cdot) \ \text{in } H^{-s}(\Omega) \quad a.e. \quad t \in (0,T)$$

with $s-\frac{N}{2} > -\frac{N-1}{r'}$. Then $\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon \to g$ in $L^q((0,T), H^{-s}(\Omega))$.

In particular, if $\varphi \in L^q((0,T), H^\sigma(\Omega))$, with $\sigma > \frac{1}{2}$, we consider $\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}\varphi(t)$ and $\varphi_0(t) = \varphi|_\Gamma(t)$. Then

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}\varphi \to \varphi_0 \quad \text{in } L^q((0,T), H^{-s}(\Omega)) \ \text{as } \varepsilon \to 0 \tag{5.9}$$

for σ, s as in (2.4). If $\varphi \in C([0,T]\times\bar{\Omega})$, (5.9) holds for any $q > 1$ and $s > \frac{1}{2}$.

Now we prove the following result that will be used below in the analysis of (5.1) and (5.2). Note that the assumption on the potentials below is, not only uniform in ε, but more restrictive in ρ than the one needed for fixed ε, as in (5.5), i.e. $\rho > N/2$.

Lemma 5.3. *Assume that the potentials V_ε satisfy, for $\rho > N-1$*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon|^\rho \le C, \quad \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon\varphi = \int_\Gamma V\varphi$$

for any smooth function φ defined in $\bar{\Omega}$ and for some function $V \in L^\rho(\Gamma)$, see Lemma 2.2. Then

i) There exists some $\lambda_0 \in I\!R$, independent of $\varepsilon > 0$, such that for $\lambda > \lambda_0$ the elliptic operator, associated to the parabolic problems (5.1) and (5.2), are positive.

ii) If s is such that $\frac{1}{2}+\frac{N-1}{2\rho} < s \le 1$ and

$$u^\varepsilon \to u^0 \ \text{weakly in } L^2((0,T), H^s(\Omega)),$$

then for any function $\varphi \in L^2((0,T), H^s(\Omega))$ we have

$$\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} V_\varepsilon u^\varepsilon\varphi \to \int_0^T\int_\Gamma V u^0\varphi$$

We also have the following result.

Lemma 5.4. *We consider a family of functions $u^\varepsilon : [0,T] \to H^1(\Omega)$ such that for some positive constant C independent of ε and t, we have*

$$\|u^\varepsilon(t,\cdot)\|_{H^1(\Omega)} \le C, \quad t \in [0,T] \tag{5.10}$$

and $u^\varepsilon_t \in L^2((0,T)\times\omega_\varepsilon)$ with

$$\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} |u^\varepsilon_t|^2 \le C. \tag{5.11}$$

Then, there exists a subsequence (that we still denote the same) and a function $u^0 \in L^\infty((0,T), H^1(\Omega))$ with $u^0_{|\Gamma} \in H^1((0,T), L^2(\Gamma))$ such that as $\varepsilon \to 0$,

$$u^\varepsilon \to u^0 \quad w-* \quad \text{in } L^\infty((0,T), H^1(\Omega))$$

and

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon \to u^0_{|\Gamma} \quad \text{in } H^1((0,T), H^{-1}(\Omega)).$$

In particular, for every $\varphi \in L^2((0,T), H^1(\Omega))$ we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} u^\varepsilon \varphi = \int_0^T \int_\Gamma u^0 \varphi, \qquad \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} u_t^\varepsilon \varphi = \int_0^T \int_\Gamma u_t^0 \varphi.$$

Finally $\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon \to u^0_{|\Gamma}$ in $C([0,T], H^{-1}(\Omega))$ as $\varepsilon \to 0$ and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} |u^\varepsilon|^2 = \int_0^T \int_\Gamma |u^0|^2.$$

We will finally make use of the following result.

Lemma 5.5. *Assume the family of potentials V_ε is as in Lemma 5.3. Also, assume u^ε is as in Lemma 5.4, that is, satisfies (5.10) and (5.11), and let u^0 be as in the conclusion of Lemma 5.4.*

Then if s is such that $\frac{1}{2} + \frac{N-1}{\rho} < s$, we have

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon u^\varepsilon \to V u^0_{|\Gamma} \quad \text{in } C([0,T], H^{-s}(\Omega)).$$

If, additionally, $\rho > 2(N-1)$ then $\frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} V_\varepsilon |u^\varepsilon|^2 \to \int_0^T \int_\Gamma V|u^0|^2$.

5.4. **Passing to the limit.** We analyze the limit of the solutions of the parabolic problems (5.1), with $0 \le \varepsilon \le \varepsilon_0$. For this we will assume that the data of the problem satisfy, for each $\varepsilon > 0$ the assumptions in the first part of Theorem 5.1 with $h_\varepsilon = f_\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon$ and the following uniform bounds in $\varepsilon > 0$:

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon|^\rho \le C, \quad \text{some } \rho > N-1, \tag{5.12}$$

$$u_0^\varepsilon \in H^1(\Omega) \quad \text{and } \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u_0^\varepsilon|^2 \le C \tag{5.13}$$

$$f_\varepsilon \in L^2((0,T), L^2(\Omega)), \text{ and } \int_0^T \|f_\varepsilon\|^2_{L^2(\Omega)} \le C \tag{5.14}$$

and

$$\frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} |g_\varepsilon|^2 \le C \tag{5.15}$$

for some constant C independent of ε.

Observe that in Theorem 5.1 we require $\lambda > -\lambda_{\Omega\setminus\bar{\omega_\varepsilon}}$ and since $\lambda_{\Omega\setminus\bar{\omega_\varepsilon}} > \lambda_\Omega$ and $\lambda_{\Omega\setminus\bar{\omega_\varepsilon}} \to \lambda_\Omega$ as $\varepsilon \to 0$. Thus, if $\lambda > -\lambda_\Omega$, then for sufficently small ε we have $\lambda > -\lambda_{\Omega\setminus\bar{\omega_\varepsilon}}$. Hence we will also assume hereafter that

$$\lambda > -\lambda_\Omega. \tag{5.16}$$

Then, by Lemma 2.2 and 5.2, by taking subsequences if necessary, we can assume that there exists functions $V \in L^\rho(\Gamma)$, $v_0 \in L^2(\Gamma)$, $f \in L^2((0,T), L^2(\Omega))$ and $g \in L^2((0,T), L^2(\Gamma))$ such that, as $\varepsilon \to 0$

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \to V \text{ weakly in } H^{-s}(\Omega) \text{ with } s - \frac{N}{2} > -\frac{N-1}{\rho'}, \tag{5.17}$$

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_0^\varepsilon \to v_0 \text{ weakly in } H^{-s}(\Omega) \text{ with } s > \frac{1}{2} \tag{5.18}$$

$$f_\varepsilon \to f \text{ weakly in } L^2((0,T), L^2(\Omega)) \tag{5.19}$$

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon \to g \text{ weakly in } L^2((0,T), H^{-s}(\Omega)) \text{ with } s > \frac{1}{2}. \tag{5.20}$$

Also, observe that by (5.16), using the first part of Proposition 4, the problem (5.2) with initial data $v_0 \in L^2(\Gamma)$, potential $V \in L^\rho(\Gamma)$ and nonhomogeneous terms $f \in L^2((0,T), L^2(\Omega))$ and $g \in L^2((0,T), L^2(\Gamma))$ is well posed.

Then we have

Theorem 5.6. *Under the above notation, assume (5.17), (5.18), (5.19) and (5.20) and consider u^ε the solutions of (5.1) as in the first part of Theorem 5.1. Moreover assume $\lambda > -\lambda_\Omega$. Also, let u^0 be the solution of (5.2) as in the first part of Proposition 4 with initial data $v_0 \in L^2(\Gamma)$, potential $V \in L^\rho(\Gamma)$ and nonhomogeneous terms $f \in L^2((0,T), L^2(\Omega))$ and $g \in L^2((0,T), L^2(\Gamma))$.*

Then, as $\varepsilon \to 0$,

$$u^\varepsilon \to u^0 \text{ weakly in } L^2((0,T), H^1(\Omega))$$

and

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon \to u^0_{|\Gamma} \text{ in } L^2((0,T), H^{-1}(\Omega)) \text{ weakly},$$

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon u^\varepsilon \to V u^0_{|\Gamma} \text{ in } L^2((0,T), H^{-1}(\Omega)) \text{ weakly}.$$

In particular, for any $\varphi \in L^2((0,T), H^1(\Omega))$

$$\frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} u^\varepsilon \varphi \to \int_0^T \int_\Gamma u^0 \varphi, \qquad \frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} V_\varepsilon u^\varepsilon \varphi \to \int_0^T \int_\Gamma V u^0 \varphi. \square$$

Now we impose stronger assumptions than (5.12)–(5.15) on the data and obtain stronger convergence of solutions than in Theorem 5.6. More precisely, we assume now the initial conditions satisfy

$$\|u_0^\varepsilon\|^2_{H^1(\Omega)} \le C, \tag{5.21}$$

and also the compatibility conditions on the initial data, (5.7) with $h = f_\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_\varepsilon g_\varepsilon$, i.e.

$$-\Delta u_0^\varepsilon + \lambda u_0^\varepsilon = f_\varepsilon(0) \text{ in } \Omega \setminus \bar{\omega}_\varepsilon. \tag{5.22}$$

We also assume

$$f_\varepsilon \in H^1((0,T), L^2(\Omega)), \text{ and } \|f_\varepsilon\|_{H^1((0,T),L^2(\Omega))} \le C \tag{5.23}$$

and (5.15), where C is a positive constant independent of ε.

Hence using (2.1) in Lemma 2.2 we have that $\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u_0^\varepsilon|^2 \le C\|u_0^\varepsilon\|^2_{H^1(\Omega)}$ and therefore (5.21) and (5.23) imply (5.13), (5.14) respectively.

Then by taking subsequences if necessary, we can assume (5.17), (5.18), (5.19) and (5.20). Moreover from Corollary 1 we have that in this case

$$u_0^\varepsilon \to u_0^0 \text{ weakly in } H^1(\Omega) \quad \text{and} \quad \frac{1}{\varepsilon}\mathcal{X}_\varepsilon u_0^\varepsilon \to u_0^0|_\Gamma \text{ weakly in } H^{-1}(\Omega). \tag{5.24}$$

In particular $v_0 = u_0^0|_\Gamma$ in (5.18). Also, in (5.19) we have $f \in H^1((0,T), L^2(\Omega))$ and

$$f_\varepsilon \to f \text{ weakly in } H^1((0,T), L^2(\Omega)). \tag{5.25}$$

Then we first make the following remark.

Lemma 5.7. *Under the above assumptions, we have*

$$-\Delta u_0^0 + \lambda u_0^0 = f(0) \quad \text{ in } \Omega. \tag{5.26}$$

Therefore, for each $\varepsilon > 0$ we are under the assumptions in the second part of Theorem 5.1 with $h_\varepsilon = f_\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon$. Also, for the limit problem (5.2) we are under the assumptions of the second part of Proposition 4, with initial data $v_0 = u_0^0|_\Gamma \in L^2(\Gamma)$, potential $V \in L^\rho(\Gamma)$ and nonhomogeneous terms $f \in C([0,T], L^2(\Omega))$ and $g \in L^2((0,T), L^2(\Gamma))$.

Hence, we have the following result that improves the convergence in Theorem 5.6.

Theorem 5.8. *Under the above notation, assume (5.15), (5.21),(5.22) and (5.23). Moreover assume $\lambda > -\lambda_\Omega$. By taking subsequences if necessary, we can assume that the data satisfies (5.17), (5.18), (5.19) and (5.20) and moreover (5.24), (5.25) and (5.26).*

Then if u^ε and u^0 are as in Theorem 5.6, we have that in addition to the convergence in Theorem 5.6 we have now that u^ε converges to u^0, weak in $L^\infty((0,T), H^1(\Omega))$ and*

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon \to u^0_{|\Gamma} \in H^1((0,T), L^2(\Gamma))$$

weakly in $H^1((0,T), H^{-1}(\Omega))$ and strongly in $C([0,T], H^{-1}(\Omega))$. Also

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon u^\varepsilon \to V u^0_{|\Gamma} \quad \text{in } C([0,T], H^{-s}(\Omega))$$

for $\frac{1}{2} + \frac{N-1}{\rho} < s$. If additionally $\rho > 2(N-1)$ then u^ε converges to u^0 also in $L^2((0,T), H^1(\Omega))$.

6. **Damped wave equations.**

Now we consider some singular perturbation of a forced wave equation where the damping region to be concentrated in a neighborhood of the boundary that shrinks to the boundary as $\varepsilon \to 0$. To be more precise, we consider the following family of damped wave equations

$$\begin{cases} u^\varepsilon_{tt} + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon_t - \Delta u^\varepsilon + \lambda u^\varepsilon = f_\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon & \text{in } \Omega \times (0,T) \\ \frac{\partial u^\varepsilon}{\partial \vec{n}} = 0 & \text{on } \Gamma \times (0,T) \\ u^\varepsilon(0,x) = u^\varepsilon_0(x), u^\varepsilon_t(0,x) = v^\varepsilon_0(x) & \text{in } \Omega \end{cases} \tag{6.1}$$

for $\lambda > 0$ and $T > 0$ fixed. Then, we are going to show that the limit problem is the following damped wave equation with boundary feedback damping

$$\begin{cases} u^0_{tt} - \Delta u^0 + \lambda u^0 = f & \text{in } \Omega \times (0,T) \\ u^0_t + \frac{\partial u^0}{\partial \vec{n}} = g & \text{on } \Gamma \times (0,T) \\ u^0(0,x) = u_0(x), u^0_t(0,x) = v_0(x) & \text{in } \Omega \end{cases} \tag{6.2}$$

where u_0, v_0, f are obtained as the weak limits of initial data $u^\varepsilon_0, v^\varepsilon_0$ and f_ε, while g is obtained as the limit of the concentrating terms

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon \to g|_\Gamma.$$

Notice that again all concentrating terms in (6.1) are transferred, in the limit, to the boundary condition in (6.2). The results here are taken from [36].

6.1. **The approximating damped wave equations.** Here we consider (6.1) for $0 < \varepsilon \leq \varepsilon_0$ which we write as

$$\left\{ \begin{array}{ll} u^\varepsilon_{tt} + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon_t - \Delta u^\varepsilon + \lambda u^\varepsilon = h_\varepsilon & \text{in } \Omega \times (0,T) \\ \frac{\partial u^\varepsilon}{\partial \vec{n}} = 0 & \text{on } \Gamma \times (0,T) \\ u^\varepsilon(0,x) = u^\varepsilon_0(x), u^\varepsilon_t(0,x) = v^\varepsilon_0(x) & \text{in } \Omega \end{array} \right. \tag{6.3}$$

with $h_\varepsilon(t,x) = f_\varepsilon(t,x) + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon(t,x)$. This, in turn, can be written as

$$U_t + A_\varepsilon U = H_\varepsilon(t), \tag{6.4}$$

with $U = (u, u_t)^\perp$, $H_\varepsilon(t) = (0, h_\varepsilon(t))^\perp$, $U(0) = U_0 = (u_0, v_0)^\perp$ and the operator

$$A_\varepsilon = \begin{pmatrix} 0 & -I \\ -\Delta + \lambda I & \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} \end{pmatrix}$$

acting on $E = H^1(\Omega) \times L^2(\Omega)$ with domain given by $D(A_\varepsilon) = H^2_N(\Omega) \times H^1(\Omega)$ where

$$H^2_N(\Omega) = \{u \in H^2(\Omega), \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \Gamma\}.$$

Note that in [58] a very similar problem to (6.1) was considered but with Dirichlet boundary conditions on Γ instead of Neumann ones as in this section. Then in a similar fashion as in Theorem 5.1, Theorem 5.2 in [58], we have the following result that states the well–posedness of (6.4).

Theorem 6.1.
i) (Existence of solutions). If $h \in L^1((0,T), L^2(\Omega))$ and $U_0 = (u_0, v_0)^\perp \in E = H^1(\Omega) \times L^2(\Omega)$ then there exists a unique **mild solution**, *$U(t) = (u, v)^\perp$ of (6.4) satisfying $U(0) = U_0$, which is given by the variation of constants formula*

$$U(t) = U(t, U_0, h) = S_\varepsilon(t)U_0 + \int_0^t S_\varepsilon(t-s)H(s)\,ds \quad 0 < t < T \tag{6.5}$$

where $S_\varepsilon(t)$ is a C_0 semigroup of contractions generated by the operator $-A_\varepsilon$ in E and $H(t) = (0, h(t))^\perp$. In this case, $U \in C([0,T], E)$ and $U(0) = U_0$ or equivalently

$$u \in C([0,T], H^1(\Omega)),\ v \in C([0,T], L^2(\Omega)), \quad u(0) = u_0,\ v(0) = v_0.$$

Moreover, the mapping $(U_0, h) \mapsto U$ is Lipschitz between $E \times L^1((0,T), L^2(\Omega))$ and $C([0,T], E)$.

ii) (Further regularity). If $h \in W^{1,1}((0,T), L^2(\Omega))$ or $h \in C([0,T], H^1(\Omega))$, and $U_0 \in D(A_\varepsilon)$. Then the mild solution of (6.4) given in (6.5) is a **strict solution**, *that is, $U \in C([0,T], D(A_\varepsilon)) \cap C^1([0,T], E)$ and satisfies (6.4) point-wise. Therefore $v(t) = u_t(t)$ and u is a solution of (6.3) such that*

$$u \in C([0,T], H^2_N(\Omega)), \quad u_t \in C([0,T], H^1(\Omega)), \quad u_{tt} \in C([0,T], L^2(\Omega)).$$

Moreover, in the first case for h, $U_t = (u_t, u_{tt})^\perp$ with $u_t(0) = v_0$ and $u_{tt}(0) = -\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} v_0 + \Delta u_0 - \lambda u_0 + h(0)$, is a mild solution of (6.4) in E, with right hand side H_t, that is

$$U_t(t) = S_\varepsilon(t)U_t(0) + \int_0^t S_\varepsilon(t-s)H_t(s)\,ds \quad 0 < t < T.$$

Observe that in case ii) of Theorem 6.1, u satisfies the PDE (6.3) in Ω and the boundary condition in Γ in a point-wise sense.

Also, the mild solutions of (6.4) in part i) of Theorem 6.1 possess the following properties.

Proposition 5. *Assume, as above, $h \in L^1((0,T), L^2(\Omega))$ and $U_0 = (u_0, v_0)^\perp \in E = H^1(\Omega) \times L^2(\Omega)$ and consider $U = (u,v)^\perp$ be the mild solution of (6.4) given by (6.5), with $H = (0, h(t))^\perp$.*
i) Then, U is characterized by $U \in C([0,T], E)$, $v = u_t$ as a weak derivative in $L^2(\Omega)$ (that is, for every $\varphi \in L^2(\Omega)$, $\frac{d}{dt}\int_\Omega u\varphi = \int_\Omega v\varphi$ in distribution sense in $(0,T)$) and for every $\phi \in H^1(\Omega)$, $\int_\Omega u_t\phi$ is absolutely continuous with

$$\frac{d}{dt}\int_\Omega u_t\phi + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} u_t\phi + \int_\Omega \nabla u\nabla\phi + \lambda\int_\Omega u\phi = \int_\Omega h\phi$$

a.e. $t \in (0,T)$. In particular, $v_t = u_{tt}$ as a weak derivative in $H^{-1}(\Omega)$ and $u_{tt} \in L^1((0,T), H^{-1}(\Omega))$, that is

$$u_{tt} + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_t + L(u) = h \qquad \text{in } H^{-1}(\Omega) \text{ a.e. } t \in (0,T)$$

where L is the isometric isomorphism between $H^1(\Omega)$ and its dual $H^{-1}(\Omega)$, given by

$$\Big\langle L(u), \phi\Big\rangle_{-1,1} = \int_\Omega \nabla u\nabla\phi + \lambda\int_\Omega u\phi \tag{6.6}$$

for every $u, \phi \in H^1(\Omega)$. In particular, the mild solution of (6.4) given by (6.5) is a weak solution of (6.3).
ii) $U = (u, u_t)^\perp$ satisfies the energy equality

$$\frac{2}{\varepsilon}\int_0^\tau\int_{\omega_\varepsilon} |u_t|^2 + E_0(u(\tau), u_t(\tau)) = E_0(u_0, v_0) + 2\int_0^\tau\int_\Omega hu_t \tag{6.7}$$

for $0 < \tau < T$, where E_0 is the energy functional given by

$$E_0(u,v) = \int_\Omega v^2 + \int_\Omega |\nabla u|^2 + \lambda\int_\Omega u^2. \tag{6.8}$$

6.2. **The limit problem with boundary feedback.** Now we consider the problem (6.2), that is

$$\begin{cases} u^0_{tt} - \Delta u^0 + \lambda u^0 = f & \text{in } \Omega \times (0,T) \\ u^0_t + \frac{\partial u^0}{\partial \vec{n}} = g & \text{on } \Gamma \times (0,T) \\ u^0(0,x) = u_0(x) u^0_t(0,x) = v_0(x) & \text{in } \Omega. \end{cases} \tag{6.9}$$

Note that a very similar problem was considered [57] where the boundary Γ was assumed to be split into two regular subsets $\Gamma = \Gamma_1 \cup \Gamma_0$. Then Dirichlet boundary conditions were assumed on Γ_0 and dynamic boundary conditions on Γ_1. Therefore we adapt here the results in [57] to our setting. Thus, we define the normal derivative operator as follows: for $u \in Y_0 := \{z \in H^1(\Omega),\ \Delta z \in L^2(\Omega)\}$ the normal derivative $\frac{\partial u}{\partial \vec{n}} \in H^{-\frac{1}{2}}(\Gamma)$ is defined as

$$\Big\langle \frac{\partial u}{\partial \vec{n}}, \gamma(v)\Big\rangle_{-\frac{1}{2},\frac{1}{2}} = \int_\Omega \Delta u v + \int_\Omega \nabla u\nabla v, \quad v \in H^1(\Omega)$$

which, using L in (6.6), can be recast as

$$\Big\langle \frac{\partial u}{\partial \vec{n}}, \gamma(v)\Big\rangle_{-\frac{1}{2},\frac{1}{2}} = \Big\langle L(u), v\Big\rangle_{-1,1} + \int_\Omega (\Delta u - \lambda u)v, \quad u \in Y_0,\ v \in H^1(\Omega).$$

We consider now $E = H^1(\Omega) \times L^2(\Omega)$ and $E' = L^2(\Omega) \times H^{-1}(\Omega)$ its dual space with duality pairing

$$\Big\langle (u,v), (\phi,\varphi)\Big\rangle_{E\times E'} = \Big\langle \varphi, u\Big\rangle_{-1,1} + \int_\Omega v\phi.$$

In what follows we will denote by $U = (u, v)$ a generic element of E, while $U^* = (u, w)$ will denote a generic element in E'.

Then if $g = 0$ problem (6.9) can be written as

$$U_t + AU = F(t),$$

where $U = (u, u_t)^\perp$, $F(t) = (0, f(t))^\perp$ and $U(0) = U_0 = (u_0, v_0)^\perp$ with

$$A = \begin{pmatrix} 0 & -I \\ -\Delta+\lambda I & 0 \end{pmatrix} \text{ and } D(A) = \{(u,v), u \in Y_0,\ v \in H^1(\Omega),\ v+\frac{\partial u}{\partial \vec{n}}=0 \text{ on } \Gamma\}$$

such that $-A$ generates a C_0 semigroup $S(t)$ in $E = H^1(\Omega) \times L^2(\Omega)$.

To handle the case $g \neq 0$, following [57], we proceed as follows. By transposition, $-A^*$ generates in E' the C_0 semigroup $S^*(t)$, i.e. the transposed semigroup of $S(t)$, and is given by

$$A^* = \begin{pmatrix} \gamma & -I \\ L & 0 \end{pmatrix}, \text{ with } D(A^*) = \{(u,w) \in H^1(\Omega) \times H^{-1}(\Omega),\ \gamma(u) - w \in L^2(\Omega)\},$$

see Lemma 2.1 in [57]. In this way the solution of the limit problem (6.9) are given by the following result which relates them with the mild solutions in E' of the dual equation

$$U_t^* + A^*U^* = H(t), \quad \text{in } E' = L^2(\Omega) \times H^{-1}(\Omega) \tag{6.10}$$

with $H(t) = (0, h(t))^\perp$ and $h(t) := f_\Omega(t) + g_\Gamma(t)$. Observe that a strict solution $U^* = (u, w)^\perp$ of this equation satisfies

$$u_t = w - \gamma(u) \quad \text{in } L^2(\Omega), \qquad w_t + L(u) = f_\Omega(t) + g_\Gamma(t) \quad \text{in } H^{-1}(\Omega)$$

which can be written as

$$(u_t + \gamma(u))_t + L(u) = h = f_\Omega + g_\Gamma \quad \text{in } H^{-1}(\Omega)$$

and is a weak formulation of (6.9) in the sense that for every $\phi \in H^1(\Omega)$ and a.e. $t \in (0, T)$,

$$\frac{d}{dt}\Big(\int_\Omega u_t\phi + \int_\Gamma \gamma(u)\gamma(\phi)\Big) + \int_\Omega \nabla u \nabla\phi + \lambda\int_\Omega u\phi = \int_\Omega f\phi + \int_\Gamma g\phi.$$

Indeed, as in Theorem 2.3 in [57] we get the following.

Theorem 6.2. *Let, $f \in L^1((0,T), L^2(\Omega))$, $g \in L^2((0,T), L^2(\Gamma))$ and $U_0 = (u_0, v_0) \in E = H^1(\Omega) \times L^2(\Omega)$. Let $U^*(t) = (u, w)^\perp$ be the mild solution of the dual equation (6.10) in $E' = L^2(\Omega) \times H^{-1}(\Omega)$*

$$U^*(t) = S^*(t)U_0^* + \int_0^t S^*(t-s)H(s)\,ds \quad 0 < t < T \tag{6.11}$$

where $U_0^* = (u_0, w_0)^\perp$, $w_0 = v_0 + \gamma(u_0) \in H^{-1}(\Omega)$, $H(t) = (0, h(t))^\perp$ and $h := f_\Omega + g_\Gamma \in L^1((0,T), H^{-1}(\Omega))$.

Then $U^* \in C([0,T], E')$, $w = u_t + \gamma(u)$, and $U(t) = (u, u_t)^\perp$ satisfies

i) (Regularity). $U \in C([0,T], E)$, $\gamma(u) \in C([0,T], H^{\frac{1}{2}}(\Gamma)) \cap H^1((0,T), L^2(\Gamma))$ and $u_{tt} \in L^1((0,T), H^{-1}(\Omega))$.

ii) (Energy estimate). U satisfies the energy equality

$$E_0(u(\tau), u_t(\tau)) + 2\int_0^\tau \int_\Gamma \gamma(u)_t^2 = E_0(u_0, v_0) + 2\int_0^\tau \int_\Gamma g\gamma(u)_t + 2\int_0^\tau \int_\Omega f u_t \tag{6.12}$$

for $0 < \tau < T$, where E_0 is the energy functional given by (6.8).

iii) (The equation). The function $u(t)$ satisfies the equation

$$(u_t+\gamma(u))_t+L(u)=h=f_\Omega+g_\Gamma$$

a.e. $[0,T]$, as an equality in $H^{-1}(\Omega)$.

Observe that Theorem 6.2 suggest that when going from E into E' we employ the following linear injective (not onto) "change of variables", see [57] for more details,

$$E\ni U=(u,v)\longmapsto U^*=(u,w)\in E', \qquad w=v+\gamma(u)\in H^{-1}(\Omega).$$

From the above theorem we can make the following definition.

Definition 6.3. Let $f\in L^1((0,T),L^2(\Omega))$, $g\in L^2((0,T),L^2(\Gamma))$ and $U_0=(u_0,v_0)\in E=H^1(\Omega)\times L^2(\Omega)$.

A function $u\in C([0,T],L^2(\Omega))\cap L^1((0,T),H^1(\Omega))$ such that $u_t\in C([0,T],H^{-1}(\Omega))$ and $\gamma(u)\in C([0,T],H^{-1}(\Omega))$ is a **mild solution** of (6.9) if $u_t+\gamma(u)$ has a weak derivative in $H^{-1}(\Omega)$ and satisfies a.e. $t\in[0,T]$

$$(u_t+\gamma(u))_t+L(u)=h=f_\Omega+g_\Gamma \quad \text{in } H^{-1}(\Omega)$$

and $u(0)=u_0$, $u_t(0)=v_0$.

Now, we will show that mild solutions as in Definition 6.3 are given by Theorem 6.2.

Proposition 6. *For given $f\in L^1((0,T),L^2(\Omega))$, $g\in L^2((0,T),L^2(\Gamma))$ and $U_0=(u_0,v_0)\in E=H^1(\Omega)\times L^2(\Omega)$, a function u is a* **mild solution** *of (6.9) as Definition 6.3 if and only if $U^*(t)=(u,u_t+\gamma(u))^\perp$ is given by (6.11) in Theorem 6.2.*

In particular, this mild solution is unique and satisfies the energy equality (6.12).

Concerning further regularity, as in Theorem 2.4 in [57] we have the following result that allows to construct strict solutions of (6.9).

Proposition 7. *Under the assumptions of Theorem 6.1 above, assume moreover $(u_0,v_0)\in H^1(\Omega)\times H^1(\Omega)$, $f\in W^{1,1}((0,T),L^2(\Omega))$ and $g\in H^1((0,T),L^2(\Gamma))$ are such that*

$$\gamma(v_0)+L(u_0)-g(0)\in L^2(\Omega),$$

that is,

$$\gamma(v_0)+\frac{\partial u_0}{\partial\vec{n}}=g(0) \ \text{ in } H^{-1/2}(\Gamma). \tag{6.13}$$

Let $U=(u,u_t)^\perp$ be constructed as in Theorem 6.2. Then $U=(u,u_t)^\perp$ satisfies $U\in C^1([0,T],E)$ and $\gamma(u)_t\in C([0,T],L^2(\Gamma))$

Moreover, u_t is a mild solution of (6.9) in E, with right hand sides f_t and g_t in Ω and Γ, respectively.

In particular,

i) (Further regularity). $U_t=(u_t,u_{tt})^\perp\in C([0,T],E)$, $\gamma(u_t)\in C([0,T],H^{\frac12}(\Gamma))\cap H^1((0,T),L^2(\Gamma))$.

ii) (Energy estimate for U_t). U_t satisfies the energy equality

$$E_0(u_t(\tau),u_{tt}(\tau))+2\int_0^\tau\int_\Gamma\gamma(u_t)_t^2=E_0(v_0,u_{tt}(0))+2\int_0^\tau\int_\Gamma g_t\gamma(u_t)_t+2\int_0^\tau\int_\Omega f_tu_{tt}$$

for $0<\tau<T$ where E_0 is the energy functional given by (6.8) and $u_{tt}(0)=-\gamma(v_0)-L(u_0)+f(0)+g(0)\in L^2(\Omega)$.

iii) (The equation for U_t). Moreover, $u\in C([0,T],Y_0)$, where $Y_0=\{z\in H^1(\Omega),\Delta z\in L^2(\Omega)\}$, and a.e. $t\in[0,T]$

$$u_{tt} + \gamma(u_t) + L(u) = h(t) = f_\Omega(t) + g_\Gamma(t) \quad \textit{in } H^{-1}(\Omega),$$
$$\frac{\partial u}{\partial \vec{n}} = g - \gamma(u_t) \in L^2(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma),$$
$$(u_{tt} + \gamma(u_t))_t + L(u_t) = h_t(t) \quad \textit{in } H^{-1}(\Omega).$$

Remark 6. i) Note that (6.13) is a weak formulation, in $H^{-1}(\Omega)$, of the condition

$$\begin{cases} -\Delta u_0 + \lambda u_0 & \in L^2(\Omega) \\ v_0 + \frac{\partial u_0}{\partial \vec{n}} = g(0) & \text{on } \Gamma. \end{cases}$$

ii) Under the hypotheses of Proposition 7 we have $u(t) \in Y_0$ and $u_{tt}(t) \in L^2(\Omega)$ then u satisfies (6.9) in the sense that

$$\begin{cases} u_{tt} - \Delta u + \lambda u = f & \text{in } \Omega \times (0,T) \\ \gamma(u_t) + \frac{\partial u}{\partial \vec{n}} = g & \text{on } \Gamma \times (0,T). \end{cases}$$

6.3. **Passing to the limit.** We analyze here the limit of the solutions of the hyperbolic problems (6.1), with $0 < \varepsilon \le \varepsilon_0$.

6.3.1. *Convergence of mild solutions.* For this we will assume that the data of the problem satisfy the following uniform bounds in $0 < \varepsilon < \varepsilon_0$:

$$\|u_0^\varepsilon\|_{H^1(\Omega)} \le C, \quad \|v_0^\varepsilon\|_{L^2(\Omega)} \le C \tag{6.14}$$

$$\int_0^T \|f_\varepsilon\|_{L^2(\Omega)} \le C, \qquad \int_0^T \|f_\varepsilon\|^p_{H^{-1}(\Omega)} \le C \quad \text{for some } 1 < p < 2, \tag{6.15}$$

and

$$\frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} |g_\varepsilon|^2 \le C \tag{6.16}$$

for some constant C independent of ε.

Then, we can assume, by taking subsequences if necessary, the following convergences as $\varepsilon \to 0$:

(i)

$$u_0^\varepsilon \to u_0^0 \quad \text{weakly in } H^1(\Omega), \quad v_0^\varepsilon \to v_0^0 \quad \text{weakly in } L^2(\Omega)$$

and strongly in $L^2(\Omega)$ and $H^{-1}(\Omega)$ respectively,

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_0^\varepsilon \to \gamma(u_0^0) \quad \text{weakly in } H^{-1}(\Omega) \tag{6.17}$$

and

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u_0^\varepsilon|^2 \to \int_\Gamma |\gamma(u_0^0)|^2,$$

see Corollary 1.

(ii) $f \in L^1((0,T), L^2(\Omega)) \cap L^p((0,T), H^{-1}(\Omega))$ and

$$f_\varepsilon \to f \quad \text{weakly in } L^1((0,T), L^2(\Omega)), \quad f_\varepsilon \to f \quad \text{weakly in } L^p((0,T), H^{-1}(\Omega)). \tag{6.18}$$

(iii) $g \in L^2((0,T), L^2(\Gamma))$ and

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon \to g \quad \text{weakly in } L^2((0,T), H^{-s}(\Omega)) \text{ with } s > \frac{1}{2}, \tag{6.19}$$

see Lemma 5.2 ii).

With these assumptions consider the mild solutions, $u^\varepsilon(t)$ and $u^0(t)$ of (6.1) and (6.2), respectively, constructed in Sections 6.1 and 6.2. Then we have the following result concerning convergence of mild solutions of (6.1) to mild solutions of (6.2).

Theorem 6.4. *With the notations above, as $\varepsilon \to 0$, we have*

$$u^\varepsilon \to u^0 \quad w\text{-}* \text{ in } L^\infty((0,T), H^1(\Omega)) \quad \text{and strongly in } C([0,T], L^2(\Omega)), \tag{6.20}$$

$$u_t^\varepsilon \to u_t^0 \quad w\text{-}* \text{ in } L^\infty((0,T), L^2(\Omega)) \quad \text{and strongly in } C([0,T], H^{-1}(\Omega)), \tag{6.21}$$

$$u_{tt}^\varepsilon \to u_{tt}^0 \quad \text{weakly in } L^p((0,T), H^{-1}(\Omega)), \tag{6.22}$$

with $1 < p < 2$ as in (6.15) and

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon \to \gamma(u^0) \quad \text{in } H^1((0,T), H^{-1}(\Omega)). \tag{6.23}$$

Additionally

$$\frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} |u^\varepsilon|^2 \to \int_0^T \int_\Gamma |\gamma(u^0)|^2. \tag{6.24}$$

In particular,

$$\int_0^T E_0(u^0, u_t^0) \le \liminf_{\varepsilon \to 0} \int_0^T E_0(u^\varepsilon, u_t^\varepsilon) \tag{6.25}$$

with E_0 the energy functional defined by (6.8).

6.3.2. *Convergence of strict solutions.* Now we impose stronger assumptions than (6.14)–(6.16) on the data and obtain stronger convergence of solutions than in Theorem 6.4. In particular, we obtain convergence of strict solutions.

With the notations in Theorem 6.4, we consider the initial data $u_0^\varepsilon \in H_N^2(\Omega)$, $v_0^\varepsilon \in H^1(\Omega)$ satisfying the following uniform bounds

$$\|u_0^\varepsilon\|_{H^1(\Omega)} \le C, \qquad \|v_0^\varepsilon\|_{H^1(\Omega)} \le C.$$

We also assume that the nonhomogenous terms satisfy

$$\|f_\varepsilon\|_{W^{1,1}((0,T),L^2(\Omega))} \le C \quad \text{and} \quad \|f_\varepsilon\|_{W^{1,p}((0,T),H^{-1}(\Omega))} \le C,$$

where $1 < p < 2$ and $g_\varepsilon \in H^1((0,T), L^2(\omega_\varepsilon))$ with

$$\frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} |g_\varepsilon|^2 \le C \quad \text{and} \quad \frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} |(g_\varepsilon)_t|^2 \le C,$$

where C is a positive constant independent of ε.

We will also assume the compatibility condition of the initial data

$$-\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} v_0^\varepsilon - \Delta u_0^\varepsilon - \lambda u_0^\varepsilon + f_\varepsilon(0) + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon(0) \in L^2(\Omega),$$

and

$$\| - \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} v_0^\varepsilon - \Delta u_0^\varepsilon - \lambda u_0^\varepsilon + f_\varepsilon(0) + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon(0)\|_{L^2(\Omega)} \le C.$$

Then by taking subsequences if necessary, we can assume (6.17) and moreover

$$u_0^\varepsilon \to u_0^0, \quad v_0^\varepsilon \to v_0^0 \quad \text{weakly in } H^1(\Omega)$$

strongly in $L^2(\Omega)$ and

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_0^\varepsilon \to \gamma(u_0^0), \quad \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} v_0^\varepsilon \to \gamma(v_0^0) \quad \text{weakly in } H^{-1}(\Omega),$$

see Corollary 1.

On the other hand, on f_ε and g_ε by taking subsequences if necessary, we can assume (6.18) and (6.19) and moreover the convergence given in the following result.

Lemma 6.5. *i) $f \in W^{1,1}((0,T), L^2(\Omega)) \cap W^{1,p}((0,T), H^{-1}(\Omega))$ and*

$$f_\varepsilon \to f \quad \text{weakly in } W^{1,1}((0,T), L^2(\Omega)) \cap W^{1,p}((0,T), H^{-1}(\Omega))$$

and strongly in $C([0,T], H^{-1}(\Omega))$.
ii) $g \in H^1((0,T), L^2(\Gamma))$ and for any $\frac{1}{2} < s < 1$

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon \to g \quad \text{weakly in } H^1((0,T), H^{-s}(\Omega))$$

and strongly in $C([0,T], H^{-s}(\Omega))$.

The following result shows that the mild solutions $u^\varepsilon(t)$ and $u^0(t)$ of (6.1) and (6.2), respectively, constructed in Sections 6.1 and 6.2, are actually strict solutions.

Lemma 6.6. *With the assumptions above, the function $u^\varepsilon(t)$ in Theorem 6.4 is a strict solution of (6.1) as in part ii) in Theorem 6.1.*

Also, the function $u^0(t)$ in Theorem 6.4 is a strict solution of (6.2) as in Proposition 7.

Hence, we have the following result that improves the convergence in Theorem 6.4.

Theorem 6.7. *With the notations above, as $\varepsilon \to 0$,*

$$u^\varepsilon \to u^0, \quad w\text{-}* \text{ in } L^\infty((0,T), H^1(\Omega)),$$

and strongly in $C([0,T], H^s(\Omega))$, $s < 1$,

$$u_t^\varepsilon \to u_t^0 \quad w\text{-}* \text{ in } L^\infty((0,T), H^1(\Omega)),$$

and strongly in $C([0,T], L^2(\Omega))$,

$$u_{tt}^\varepsilon \to u_{tt}^0 \quad w\text{-}* \text{ in } L^\infty((0,T), L^2(\Omega)), \quad \text{weakly in } W^{1,p}((0,T), H^{-1}(\Omega)),$$

with $1 < p < 2$ as in (6.15) and strongly in $C([0,T], H^{-1}(\Omega))$.

Also

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon \to \gamma(u^0), \quad \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_t^\varepsilon \to \gamma(u_t^0) \quad \text{in } H^1((0,T), H^{-1}(\Omega))$$

$$\frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} |u^\varepsilon|^2 \to \int_0^T \int_\Gamma |\gamma(u^0)|^2, \qquad \frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} |u_t^\varepsilon|^2 \to \int_0^T \int_\Gamma |\gamma(u_t^0)|^2.$$

In particular,

$$\int_0^T E_0(u_t^0, u_{tt}^0) \le \liminf_{\varepsilon\to 0} \int_0^T E_0(u_t^\varepsilon, u_{tt}^\varepsilon)$$

with E_0 the energy functional defined by (6.8).

Now we show that strong convergence of the initial data in the energy space, $E = H^1(\Omega) \times L^2(\Omega)$ implies convergence of the solution $(u^\varepsilon, u_t^\varepsilon) \to (u^0, u_t^0)$ in $L^2((0,T), E)$. From the convergence in Theorem 6.7, it is enough to show the convergence $u^\varepsilon \to u^0$ in $L^2((0,T), H^1(\Omega))$.

Proposition 8. *Under the notation and hypothesis of Theorem 6.7, we also assume the initial data satisfies $u_0^\varepsilon \to u_0$ strongly in $H^1(\Omega)$, $v_0^\varepsilon \to v_0$ strongly in $L^2(\Omega)$.*

Then $u^\varepsilon \to u^0$ in $L^2((0,T), H^1(\Omega))$.

7. **Concentrating terms away from the boundary.** In this section we explore the potential use of the techniques in previous sections to analyze problems with singular terms concentrating away from the boundary. These cases will reflect a different nature in the limit problem which does not influence the boundary conditions.

For this we will consider in an open bounded smooth set in $I\!R^N$, Ω, with a C^2 boundary, $\Gamma = \partial\Omega$, an embedded smooth, compact, orientable hypersurface $\mathcal{M} \subset \Omega$ such that $\Omega \setminus \mathcal{M}$ has two components, an *interior* one Ω_1 that does not touch Γ and $\partial\Omega_1 = \mathcal{M}$ and an *exterior* one Ω_2 such that $\partial\Omega_2 = \Gamma \cup \mathcal{M}$.

Next, we define, for sufficiently small $\varepsilon > 0$, $0 < \varepsilon \le \varepsilon_0$, a neighborhood of $\mathcal{M}$

$$\omega_\varepsilon = \{x - \sigma\vec{n}(x),\ x \in \mathcal{M},\ \sigma \in (-\varepsilon, \varepsilon)\} \subset \Omega$$

where $\vec{n}(x)$ denotes the normal vector at a point $x \in \mathcal{M}$ outwards from Ω_1. As before $\mathcal{X}_{\omega_\varepsilon}$ denotes the characteristic function of the set ω_ε.

We will also denote by $\gamma(u)$ the trace on $\mathcal{M}$ of a regular enough function defined in either Ω_1 or Ω_2.

Hence, we consider the following family of damped wave equations

$$\left\{\begin{array}{ll} u^\varepsilon_{tt} + \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon_t + \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon - \Delta u^\varepsilon + \lambda u^\varepsilon = f_\varepsilon + \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon & \text{in } \Omega \times (0,T) \\ u^\varepsilon = 0 & \text{on } \Gamma \times (0,T) \\ u^\varepsilon(0,x) = u^\varepsilon_0(x), u^\varepsilon_t(0,x) = v^\varepsilon_0(x) & \text{in } \Omega \end{array}\right. \tag{7.1}$$

with $\lambda > 0$ and $T > 0$ fixed.

We will show that the corresponding limit problem, as $\varepsilon \to 0$ is now given by the wave equation with damping on $\mathcal{M}$

$$\left\{\begin{array}{lll} u^0_{tt} - \Delta u^0 + \lambda u^0 = f & & \text{in } (\Omega \setminus \mathcal{M}) \times (0,T) \\ u^0 = 0 & & \text{on } \Gamma \times (0,T) \\ u^0_t - [\frac{\partial u^0}{\partial n}] + u^0 = g, & [u^0] = 0 & \text{on } \mathcal{M} \times (0,T) \\ u^0(0,x) = u_0(x), u^0_t(0,x) = v_0(x) & & \text{in } \Omega \end{array}\right. \tag{7.2}$$

where $[F]$ denotes the jump of the function F across $\mathcal{M}$, u_0, v_0, f are obtained as the weak limits of initial data $u^\varepsilon_0, v^\varepsilon_0$ and f_ε, while g is obtained as the limit of the concentrating terms

$$\frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon \to g|_{\mathcal{M}}.$$

This last convergence will be achieved in a similar manner as in Sections 2.1 and 5.3.

We start with the following technical lemma that will be used below. Recall that we assume $\lambda > 0$.

Lemma 7.1. *In the space $E = H^1_0(\Omega) \times L^2(\Omega)$ the following two expressions, $\left(E^\varepsilon_0\right)^{1/2}$ and $E_0^{1/2}$ define Hilbertian norms, equivalent to the usual one, where*

$$E^\varepsilon_0(u,v) = \int_\Omega |\nabla u|^2 + \lambda \int_\Omega u^2 + \frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |u|^2 + \int_\Omega |v|^2. \tag{7.3}$$

and

$$E_0(u,v) = \int_\Omega |\nabla u|^2 + \lambda \int_\Omega |u|^2 + \int_{\mathcal{M}} |\gamma(u)|^2 + \int_\Omega |v|^2. \tag{7.4}$$

Proof. For (7.3) just note that from Lemma 2.1 we have, for some positive constant C independent of ε, $\frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |u|^2 \le C\|u\|^2_{H^1_0(\Omega)}$.

For (7.4) just note that from trace theory and regularity of $\mathcal{M}$ we have $\int_{\mathcal{M}} |\gamma(u)|^2 \leq C\|u\|^2_{H^1_0(\Omega)}$. $\square$

7.1. **Analysis of the approximating damped hyperbolic problems.** Here we consider (7.1) for $0 < \varepsilon \leq \varepsilon_0$

$$\begin{cases} u^\varepsilon_{tt} + \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon_t + \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon - \Delta u^\varepsilon + \lambda u^\varepsilon = h_\varepsilon & \text{in } \Omega \times (0,T) \\ u^\varepsilon = 0 & \text{on } \Gamma \times (0,T) \\ u^\varepsilon(0,x) = u^\varepsilon_0(x), u^\varepsilon_t(0,x) = v^\varepsilon_0(x) & \text{in } \Omega \end{cases} \tag{7.5}$$

with $h_\varepsilon(t,x) = f_\varepsilon(t,x) + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon(t,x)$ which we write as

$$U_t + A_\varepsilon U = H_\varepsilon(t), \tag{7.6}$$

with $U = (u, u_t)^\perp$, $H_\varepsilon(t) = (0, h_\varepsilon(t))^\perp$, $U(0) = U_0 = (u_0, v_0)^\perp$ and the operator

$$A_\varepsilon = \begin{pmatrix} 0 & -I \\ -\Delta + \lambda I + \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} & \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} \end{pmatrix}$$

acting on $E = H^1_0(\Omega) \times L^2(\Omega)$ with domain given by

$$D(A_\varepsilon) = (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega).$$

Notice the similarities with the operator considered in Section 6.1.

Our first result now concerns the homogeneous problem (7.6) with $h_\varepsilon = 0$.

Proposition 9. *The operator $-A_\varepsilon$ generates a C_0 semigroup in E, denoted $S_\varepsilon(t)$ which is a semigroup of contractions with respect to the norm in E given in (7.3). Therefore, if $U_0 \in E$, then $U(t) = S_\varepsilon(t)U_0 = (u(t), v(t))^\perp$ is a* **mild solution** *of (7.6), with $h_\varepsilon = 0$, and satisfies $U \in C([0,\infty); E)$ or equivalently $u \in C([0,\infty); H^1_0(\Omega))$, $v \in C([0,\infty); L^2(\Omega))$.*

If moreover, $U_0 \in D(A_\varepsilon)$, then $U(t) = S_\varepsilon(t)U_0 = (u(t), v(t))^\perp$ is a **strict solution** *of (7.6), with $h_\varepsilon = 0$, and satisfies $U \in C([0,\infty); D(A_\varepsilon)) \cap C^1([0,\infty); E)$ and satisfies (7.6), with $h_\varepsilon = 0$, pointwise. Therefore $v(t) = u_t$ and u is a solution of (7.5) with $h_\varepsilon = 0$,such that*

$$u \in C([0,\infty); H^1_0(\Omega) \cap H^1_0(\Omega)),\ u_t \in C([0,\infty); H^1_0(\Omega)),\ u_{tt} \in C([0,\infty); L^2(\Omega)).$$

Proof. Observe that for $U = (u, v)^\perp \in D(A_\varepsilon)$ using the scalar product associated to the norm (7.3), we have

$$A_\varepsilon(U) \cdot U = \Big(-v, (-\Delta + \lambda I)u + \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u + \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} v \Big) \cdot (u, v) = \frac{1}{2\varepsilon}\int_{\omega_\varepsilon} v^2 \geq 0.$$

so A_ε is dissipative.

Since $D(A_\varepsilon)$ is clearly dense in E then to conclude the proof it is enough to show that $R(A_\varepsilon + I) = E$. For this $A_\varepsilon U + U = (j,k)^\perp \in E$ is equivalent to $u - v = j$ and

$$-\Delta u + \lambda u + \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u + \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} v + v = k.$$

Hence $-\Delta u + (\lambda + 1)u + \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u = k + j - \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} j$ which has a unique solution $u \in H^2(\Omega) \cap H^1_0(\Omega))$. Hence $v \in H^1_0(\Omega)$ and we get the result. $\square$

On the other hand if $h_\varepsilon \neq 0$ using general results on semigroups as in [52] we obtain the following result.

Theorem 7.2.
i) (Existence of solutions). If $h_\varepsilon \in L^1((0,T), L^2(\Omega))$ and $U_0 = (u_0, v_0)^\perp \in E = H_0^1(\Omega) \times L^2(\Omega)$ then there exists a unique **mild solution**, *$U(t) = (u,v)^\perp$ of (7.6) satisfying $U(0) = U_0$, which is given by the variation of constants formula*

$$U(t) = U(t, U_0, h) = S_\varepsilon(t)U_0 + \int_0^t S_\varepsilon(t-s)H_\varepsilon(s)\,ds \quad 0 < t < T. \tag{7.7}$$

In this case, $U \in C([0,T], E)$, or equivalently $u \in C([0,T], H_0^1(\Omega))$, $v \in C([0,T], L^2(\Omega))$, $u(0) = u_0$, $v(0) = v_0$.

Moreover, the mapping $(U_0, h) \mapsto U$ is Lipschitz between $E \times L^1((0,T), L^2(\Omega))$ and $C([0,T], E)$.

ii) (Further regularity). If $h_\varepsilon \in W^{1,1}((0,T), L^2(\Omega))$ or $h_\varepsilon \in C([0,T], H_0^1(\Omega))$, and $U_0 \in D(A_\varepsilon)$. Then the mild solution of (7.6) given in (7.7) is a **strict solution**, *that is, $U \in C([0,T], D(A_\varepsilon)) \cap C^1([0,T], E)$ and satisfies (7.6) point-wise. Therefore $v(t) = u_t(t)$ and u is a solution of (7.5) such that*

$$u \in C([0,T], H^2(\Omega) \cap H_0^1(\Omega)), \quad u_t \in C([0,T], H_0^1(\Omega)), \quad u_{tt} \in C([0,T], L^2(\Omega)).$$

Observe that in case ii) of Theorem 7.2, u satisfies the PDE (7.5) in Ω and the boundary condition in Γ in a point-wise sense.

We also get the following characterization of the mild solutions of (7.6) in part i) of Theorem 7.2. This results uses the characterization of the functions given by the variations of constant formula (7.7) in [14] and is similar to Proposition 5.3 in [58].

Proposition 10 (Characterization of mild solutions). *Assume, as above, $h \in L^1((0,T), L^2(\Omega))$ and $U_0 = (u_0, v_0)^\perp \in E = H_0^1(\Omega) \times L^2(\Omega)$ and consider $U = (u,v)^\perp$ be the mild solution of (7.6) given by (7.7), with $H = (0, h(t))^\perp$.*

Then, U is characterized by $U \in C([0,T], E)$, $v = u_t$ as a weak derivative in $L^2(\Omega)$ (that is, for every $\varphi \in L^2(\Omega)$, $\frac{d}{dt}\int_\Omega u\varphi = \int_\Omega v\varphi$ in distribution sense in $(0,T)$) and for every $\phi \in H_0^1(\Omega)$, $\int_\Omega u_t\phi$ is absolutely continuous with

$$\frac{d}{dt}\int_\Omega u_t\phi + \frac{1}{2\varepsilon}\int_{\omega_\varepsilon} u_t\phi + \frac{1}{2\varepsilon}\int_{\omega_\varepsilon} u\phi + \int_\Omega \nabla u \nabla\phi + \lambda\int_\Omega u\phi = \int_\Omega h\phi \tag{7.8}$$

a.e. $t \in (0,T)$. In particular, $v_t = u_{tt}$ as a weak derivative in $H^{-1}(\Omega)$ and $u_{tt} \in L^1((0,T), H^{-1}(\Omega))$, that is

$$u_{tt} + \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_t + \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u + L(u) = h \quad \text{in } H^{-1}(\Omega) \text{ a.e. } t \in (0,T) \tag{7.9}$$

where L is the isometric isomorphism between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$, given by (6.6).

Observe that (7.8) or equivalently, (7.9) and (7.5), implies that the mild solution of (7.6) given by (7.7) is a weak solution of (7.5).

We now show that mild solutions in Theorem 7.2 also satisfy a natural energy equality.

Proposition 11 (Energy equality). *Assume, as above, $h \in L^1((0,T), L^2(\Omega))$, $U_0 = (u_0, v_0)^\perp \in E = H_0^1(\Omega) \times L^2(\Omega)$ and let $U = (u,v)^\perp$ be the mild solution of (7.6) given by (7.7), with $H = (0, h(t))^\perp$.*

Then $U = (u, u_t)^{\perp}$ satisfies the energy equality

$$\frac{1}{\varepsilon}\int_0^{\tau}\int_{\omega_\varepsilon} |u_t|^2 + E_0^{\varepsilon}(u(\tau), u_t(\tau)) = E_0^{\varepsilon}(u_0, v_0) + 2\int_0^{\tau}\int_{\Omega} h u_t \tag{7.10}$$

for $0 < \tau < T$, where E_0^{ε} is given by (7.3).

Proof. As usual, we argue by density. First, assume the solution is smooth enough such that $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_t \in H_0^1(\Omega)$. Part ii) in Theorem 7.2 gives sufficient conditions on the data for this assumption to hold true.

Then multiplying in (7.5) by u_t in $L^2(\Omega)$ and integrating by parts we have

$$\frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |u_t|^2 + \frac{1}{2}\frac{d}{dt}\Big(\int_{\omega_\varepsilon} |u_t|^2 + \int_{\Omega} |\nabla u|^2 + \lambda\int_{\Omega} |u|^2 + \frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |u|^2\Big) = \int_{\Omega} h u_t \tag{7.11}$$

and integrating (7.11) in $(0, \tau)$, with $\tau \in [0, T]$ we get (7.10).

Now for $h \in L^1((0,T), L^2(\Omega))$, $U_0 = (u_0, v_0)^{\perp} \in E = H_0^1(\Omega) \times L^2(\Omega)$ consider a sequence of data such that $U_0^n \in D(A_\varepsilon) \subset E$, $h_n \in C_c^2((0,T), L^2(\Omega))$, such that $U_0^n \to U_0$ in E, $h_n \to h$ in $L^1((0,T), L^2(\Omega))$. From part ii) in Theorem 7.2, $U^n = (u^n, u_t^n) \in C([0,T], D(A_\varepsilon)) \cap C^1([0,T], E)$, in particular $U_t^n = (u_t^n, u_{tt}^n) \in C([0,T], E)$ and, as above U^n satisfies (7.10) for every n.

By the Lipschitz dependence of mild solutions in part i) of Theorem 7.2 we have

$$U^n \to U = U(\cdot, U_0, h) \ \text{ in } \ C([0,T], E)$$

which implies in particular $\|U^n(t)\|_E \to \|U(t)\|_E$, and taking into account that $E_0^{\varepsilon}(u, v)$ is an equivalent norm in E, see Lemma 7.1, we get

$$E_0^{\varepsilon}(u^n, u_t^n)(t) \to E_0^{\varepsilon}(u, u_t)(t) \text{ as } n \to \infty.$$

Also, $\frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_t^n \to \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_t$ in $C([0,T], L^2(\Omega))$ as $n \to \infty$ and

$$\frac{1}{2\varepsilon}\int_0^{\tau}\int_{\omega_\varepsilon} |u_t^n|^2 \longrightarrow \frac{1}{2\varepsilon}\int_0^{\tau}\int_{\omega_\varepsilon} |u_t|^2.$$

Finally, since $u_t^n \to u_t$ in $C([0,T], L^2(\Omega))$ and $h_n \to h$ in $L^1((0,T), L^2(\Omega))$ we get

$$\int_0^{\tau}\int_{\Omega} h_n u_t^n \to \int_0^{\tau}\int_{\Omega} h u_t,$$

and passing to the limit as $n \to \infty$ in (7.10) we obtain the energy equality for the mild solution $U(\cdot, U_0, h)$. $\square$

7.2. **Convergence of mild solutions.** We analyze here the limit of the solutions of the hyperbolic problems (7.1), with $0 < \varepsilon \le \varepsilon_0$. For this we will obtain uniform energy estimates and, by compactness, study the limt as $\varepsilon \to 0$.

For this we will assume that the data of the problem satisfy the following assumptions:

(i)

$$\|u_0^{\varepsilon}\|_{H^1(\Omega)} \le C, \quad \|v_0^{\varepsilon}\|_{L^2(\Omega)} \le C \tag{7.12}$$

and, by taking subsequences if necessary, as $\varepsilon \to 0$,

$$u_0^{\varepsilon} \to u_0^0 \quad \text{weakly in } H^1(\Omega), \quad v_0^{\varepsilon} \to v_0^0 \quad \text{weakly in } L^2(\Omega)$$

and strongly in $L^2(\Omega)$ and $H^{-1}(\Omega)$ respectively,

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_0^{\varepsilon} \to \gamma(u_0^0) \quad \text{weakly in } H^{-1}(\Omega), \qquad \frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |u_0^{\varepsilon}|^2 \to \int_{\mathcal{M}} |\gamma(u_0^0)|^2.$$

(ii)

$$\int_0^T \|f_\varepsilon\|_{L^2(\Omega)} \le C, \qquad \int_0^T \|f_\varepsilon\|^p_{H^{-1}(\Omega)} \le C \quad \text{for some } 1 < p < 2, \tag{7.13}$$

and, by taking subsequences if necessary, as $\varepsilon \to 0$,

$$f_\varepsilon \to f \quad \text{weakly in } L^1((0,T), L^2(\Omega)), \qquad f_\varepsilon \to f \quad \text{weakly in } L^p((0,T), H^{-1}(\Omega)). \tag{7.14}$$

(iii)

$$\frac{1}{2\varepsilon}\int_0^T \int_{\omega_\varepsilon} |g_\varepsilon|^2 \le C \tag{7.15}$$

and, by taking subsequences if necessary, as $\varepsilon \to 0$,

$$\frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon \to g \quad \text{weakly in } L^2((0,T), H^{-s}(\Omega)) \text{ with } s > \frac{1}{2}. \tag{7.16}$$

In particular, for every φ smooth defined in $[0,T] \times \bar{\Omega}$ we have

$$\int_0^T \int_{\omega_\varepsilon} g_\varepsilon \varphi \to \int_0^T \int_{\mathcal{M}} g\varphi.$$

With these assumptions consider the mild solutions, $u^\varepsilon(t)$ of (7.1) constructed in Section 7.1. That is, $u^\varepsilon(t)$ is as in Theorem 7.2 and Proposition 10 with with $(u_0^\varepsilon, v_0^\varepsilon) \in E$, $h_\varepsilon = f_\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon \in L^1((0,T), L^2(\Omega))$.

Then we have the following result concerning convergence of mild solutions of (7.1). As we will show, the limit is naturally a mild solutions of (7.2).

Theorem 7.3. *With the notations above, as $\varepsilon \to 0$, we have, by taking sequences if necessary, that there exists $u^0 \in L^\infty((0,T), H_0^1(\Omega))$ with $u_t^0 \in L^\infty((0,T), L^2(\Omega))$ such that*

$$u^\varepsilon \to u^0 \quad w\text{-}* \text{ in } L^\infty((0,T), H_0^1(\Omega)) \quad \text{and strongly in } C([0,T], L^2(\Omega)), \tag{7.17}$$

$$u_t^\varepsilon \to u_t^0 \quad w\text{-}* \text{ in } L^\infty((0,T), L^2(\Omega)) \quad \text{and strongly in } C([0,T], H^{-1}(\Omega)), \tag{7.18}$$

$$u_{tt}^\varepsilon \to u_{tt}^0 \quad \text{weakly in } L^p((0,T), H^{-1}(\Omega)), \tag{7.19}$$

with $1 < p < 2$ as in (7.13) and

$$\frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon \to \gamma(u^0) \quad \text{in} \quad H^1((0,T), H^{-1}(\Omega)). \tag{7.20}$$

Additionally

$$\frac{1}{2\varepsilon}\int_0^T \int_{\omega_\varepsilon} |u^\varepsilon|^2 \to \int_0^T \int_{\mathcal{M}} |\gamma(u^0)|^2. \tag{7.21}$$

In particular

$$\int_0^T E_0(u^0, u_t^0) \le \liminf_{\varepsilon \to 0} \int_0^T E_0^\varepsilon(u^\varepsilon, u_t^\varepsilon) \tag{7.22}$$

where E_0^ε and E_0 are given by (7.3) and (7.4) respectively.

Proof. We proceed in several steps. Below we will use K or C a generic constant that does not depend on ε.

Step 1. We will prove that for any $\tau \in [0,T]$

$$\frac{1}{2\varepsilon}\int_0^\tau \int_{\omega_\varepsilon} |u_t^\varepsilon|^2 + \frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |u^\varepsilon|^2 + \|u_t^\varepsilon(\tau)\|^2_{L^2(\Omega)} + \|u^\varepsilon(\tau)\|^2_{H_0^1(\Omega)} \le K \tag{7.23}$$

with $K > 0$ independent of ε.

For this note from Proposition 11, with $\tau \in [0,T]$, the solutions $(u^\varepsilon(t), u^\varepsilon_t(t))$ satisfy the energy equality (7.10), i.e.

$$\frac{1}{\varepsilon}\int_0^\tau \int_{\omega_\varepsilon} |u^\varepsilon_t|^2 + E^\varepsilon_0(u^\varepsilon(\tau), u^\varepsilon_t(\tau)) = E^\varepsilon_0(u^\varepsilon_0, v^\varepsilon_0) + 2\int_0^\tau \int_\Omega h_\varepsilon u^\varepsilon_t, \tag{7.24}$$

with

$$2\int_0^\tau \int_\Omega h_\varepsilon u^\varepsilon_t = 2\int_0^\tau \int_\Omega f_\varepsilon u^\varepsilon_t + \frac{1}{\varepsilon}\int_0^\tau \int_{\omega_\varepsilon} g_\varepsilon u^\varepsilon_t.$$

Now we obtain some upper bounds to the terms in the right hand side of (7.24). First, using the hypotheses on the initial data (7.12) we get

$$E^\varepsilon_0(u^\varepsilon_0, v^\varepsilon_0) \le \|(u^\varepsilon_0, v^\varepsilon_0)\|^2_E + C\|u^\varepsilon_0\|^2_{H^1_0(\Omega)} \le K,$$

since from Lemma 2.1, there exits $C > 0$ independent of ε, such that $\frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |u|^2 \le C\|u\|^2_{H^1_0(\Omega)}$.

Next, applying Young's inequality, we obtain

$$\Big|\frac{1}{2\varepsilon}\int_{\omega_\varepsilon} g_\varepsilon u^\varepsilon_t\Big| \le \Big(\frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |g_\varepsilon|^2\Big)^{\frac12}\Big(\frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |u^\varepsilon_t|^2\Big)^{\frac12} \le \frac{1}{4\varepsilon}\int_{\omega_\varepsilon} |u^\varepsilon_t|^2 + \frac{1}{4\varepsilon}\int_{\omega_\varepsilon} |g_\varepsilon|^2.$$

Thus, taking into account (7.15) we obtain

$$\Big|\frac{1}{\varepsilon}\int_0^\tau \int_{\omega_\varepsilon} g_\varepsilon u^\varepsilon_t\Big| \le \frac{1}{2\varepsilon}\int_0^\tau \int_{\omega_\varepsilon} |u^\varepsilon_t|^2 + K.$$

Now, for every $\tau \in [0,T]$ we have that, using (7.13),

$$\Big|2\int_0^\tau \int_\Omega f_\varepsilon u^\varepsilon_t\Big| \le 2\int_0^\tau \|f_\varepsilon\|_{L^2(\Omega)}\|u^\varepsilon_t\|_{L^2(\Omega)} \le 2\|f_\varepsilon\|_{L^1((0,T),L^2(\Omega))} y(\tau) \le K y(\tau)$$

where $y(\tau) = \sup_{0\le t\le\tau} \|u^\varepsilon_t\|_{L^2(\Omega)}$.

Then, from the energy equality (7.24) we get

$$\frac{1}{2\varepsilon}\int_0^\tau \int_{\omega_\varepsilon} |u^\varepsilon_t|^2 + \frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |u^\varepsilon|^2 + \|u^\varepsilon_t(\tau)\|^2_{L^2(\Omega)} + \|u^\varepsilon(\tau)\|^2_{H^1_0(\Omega)} \le K(y(\tau)+1).$$

In particular for every $0 \le t \le \tau$ we have that

$$\|u^\varepsilon_t(t)\|^2_{L^2(\Omega)} \le K(\sup_{0\le s\le t} \|u^\varepsilon_t\|_{L^2(\Omega)} + 1) \le K(y(\tau)+1)$$

and then $y^2(\tau) \le K(y(\tau)+1)$ from where $y^2(\tau) \le K$, which gives (7.23).

Step 2. As a consequence we obtain the following uniform estimates

$$\sup_{0\le\tau\le T} \|u^\varepsilon(t)\|_{H^1_0(\Omega)} = \|u^\varepsilon\|_{L^\infty((0,T),H^1_0(\Omega))} \le K, \tag{7.25}$$

$$\sup_{0\le\tau\le T} \|u^\varepsilon_t(t)\|_{L^2(\Omega)} = \|u^\varepsilon_t\|_{L^\infty((0,T),L^2(\Omega))} \le K, \tag{7.26}$$

$$\sup_{0\le t\le T} \frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |u^\varepsilon(t)|^2 = \|\frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon\|_{L^\infty((0,T),L^2(\Omega))} \le K \tag{7.27}$$

$$\frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} |u^\varepsilon_t|^2 \le K, \tag{7.28}$$

$$\|\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon_t\|_{L^2((0,T),H^{-1}(\Omega))} \le K, \tag{7.29}$$

$$\|u^\varepsilon_{tt}\|_{L^p((0,T),H^{-1}(\Omega))} \le K \tag{7.30}$$

with $1 < p < 2$ as in (7.13).

First, note that we get (7.25), (7.26), (7.27) and (7.28) straight from (7.23). To prove (7.29) observe that for every $\phi \in H^1_0(\Omega)$, using again Lemma 2.1, there exits $C > 0$ independent of ε, such that we have $\frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |\phi|^2 \le C\|\phi\|^2_{H^1_0(\Omega)}$. Then,

$$\Big|\Big\langle \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon_t, \phi\Big\rangle_{-1,1}\Big| = \Big|\frac{1}{2\varepsilon}\int_{\omega_\varepsilon} u^\varepsilon_t \phi\Big| \le \Big(\frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |u^\varepsilon_t|^2\Big)^{\frac{1}{2}}\Big(\frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |\phi|^2\Big)^{\frac{1}{2}}$$

$$\le C\Big(\frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |u^\varepsilon_t|^2\Big)^{\frac{1}{2}}\|\phi\|_{H^1_0(\Omega)}$$

and then

$$\|\frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon_t\|_{H^{-1}(\Omega)} \le C\Big(\frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |u^\varepsilon_t|^2\Big)^{\frac{1}{2}} \le K$$

and thus (7.29) follows.

Finally, we prove (7.30). In effect, since u^ε satisfies a.e. $t \in [0, T]$

$$u^\varepsilon_{tt} = h_\varepsilon - L(u^\varepsilon) - \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon - \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon_t, \quad \text{in } H^{-1}(\Omega) \tag{7.31}$$

with $h_\varepsilon = f_\varepsilon + \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon$, we get

$$\begin{aligned}\|u^\varepsilon_{tt}\|_{H^{-1}(\Omega)} \le &\|f_\varepsilon\|_{H^{-1}(\Omega)} + \|\frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon\|_{H^{-1}(\Omega)} + \|L(u^\varepsilon)\|_{H^{-1}(\Omega)}\\ &+ \|\frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon\|_{H^{-1}(\Omega)} + \|\frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon_t\|_{H^{-1}(\Omega)},\end{aligned}$$

and from (7.13) and (7.15), (7.25) and (7.29), we get (7.30).

In the remaining steps, we will pass to the limit as $\varepsilon \to 0$.

Step 3. Here we will study the convergence of u^ε to u^0.

i) From (7.25), (7.28), (7.27) and Lemma 5.4 (replacing Γ with $\mathcal{M}$) there exists a subsequence (that we still denote the same) and a function $u^0 \in L^\infty((0, T), H^1_0(\Omega))$ with $\gamma(u^0) \in H^1((0, T), L^2(\mathcal{M})) \cap L^\infty((0, T), L^2(\mathcal{M}))$ such that as $\varepsilon \to 0$

$$u^\varepsilon \to u^0 \quad \text{w-* in } L^\infty((0, T), H^1_0(\Omega)),$$

$$\frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon \to \gamma(u^0) \quad \text{in } H^1((0, T), H^{-1}(\Omega));$$

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^T\int_{\omega_\varepsilon} |u^\varepsilon|^2 = \int_0^T\int_{\mathcal{M}} |\gamma(u^0)|^2.$$

Thus, we get the weak* convergence in (7.17), (7.20) and (7.21).

ii) Now we prove

$$u^\varepsilon \to u^0 \text{ in } C([0, T], L^2(\Omega))$$

which ends the proof of (7.17). For this, note that from (7.26) then $u^\varepsilon : [0, T] \to L^2(\Omega)$ is equicontinuous. Also, for every $t \in [0, T]$ fixed, from (7.25) we have $u^\varepsilon(t, \cdot)$ is uniformly bounded in $H^1_0(\Omega) \subset L^2(\Omega)$ with compact embedding. Hence, using Ascoli-Arzela's Theorem we have a new subsequence such that $u^\varepsilon \to u^0$ in $C([0, T], L^2(\Omega))$. In particular, for $t = 0$, $u^\varepsilon_0 = u^\varepsilon(0, \cdot) \to u^0(0, \cdot)$ in $L^2(\Omega)$ and then $u^0(0) = u^0_0$.

Step 4. In this part, we study the convergence of u^ε_t to u^0_t and prove (7.18).

i) Now, we will prove

$$u^\varepsilon_t \to u^0_t \quad \text{w-* in } L^\infty((0, T), L^2(\Omega)) \text{ as } \varepsilon \to 0. \tag{7.32}$$

First, from (7.26) and taking another subsequence if necessary, there exists $v^* \in L^\infty((0,T), L^2(\Omega))$ such that

$$u_t^\varepsilon \to v^* \quad \text{w-* in } L^\infty((0,T), L^2(\Omega)) \text{ as } \varepsilon \to 0.$$

Second, we will prove that $v^* = u_t^0$, and we get (7.32). In effect, for every $\phi(t,\cdot)$ smooth such that $\phi(T,\cdot) = \phi_t(T,\cdot) = 0$ and integrating by parts we have

$$\int_0^T \int_\Omega u_t^\varepsilon \phi = -\int_\Omega u_0^\varepsilon \phi(0) - \int_0^T \int_\Omega u^\varepsilon \phi_t. \tag{7.33}$$

From the convergence of u_0^ε, u^ε and u_t^ε we get

$$\int_0^T \int_\Omega v^* \phi = -\int_\Omega u_0 \phi(0) dx - \int_0^T \int_\Omega u^0 \phi_t.$$

Thus, we have $v^* = u_t^0 \in L^\infty((0,T), L^2(\Omega))$ and we conclude (7.32).
ii) In what follows we will prove

$$u_t^\varepsilon \to u_t^0 \quad \text{in } C([0,T], H^{-1}(\Omega)), \text{ as } \varepsilon \to 0.$$

First, from (7.30), we obtain that for $t_i \in [0,T]$

$$\begin{aligned} \|u_t^\varepsilon(t_1) - u_t^\varepsilon(t_2)\|_{H^{-1}(\Omega)} &\leq \int_{t_1}^{t_2} \|u_{tt}^\varepsilon\|_{H^{-1}(\Omega)} \\ &\leq \Big(\int_{t_1}^{t_2} \|u_{tt}^\varepsilon\|^p_{H^{-1}(\Omega)} \Big)^{\frac{1}{p}} |t_2 - t_1|^{\frac{1}{p'}} \leq C |t_2 - t_1|^{\frac{1}{p'}}, \end{aligned}$$

and we get $\{u_t^\varepsilon\}_\varepsilon$ is equicontinuous with values in $H^{-1}(\Omega)$.

Next, we also note that for every $t \in [0,T]$ fixed, we have $u_t^\varepsilon(t,\cdot)$ is uniformly bounded in $L^2(\Omega) \subset H^{-1}(\Omega)$ with compact embedding. Therefore from Ascoli-Arzela's Theorem there exists a subsequence which converge to a limit function in $C([0,T], H^{-1}(\Omega))$. Finally, we note that this limit function must be u_t^0 and we conclude (7.18).

In particular, for $t = 0$, we have $v_0^\varepsilon = u_t^\varepsilon(0,\cdot) \to u_t^0(0,\cdot)$. Thus $u_t^0(0) = v_0^0$.

Step 5. Now we study the convergence of u_{tt}^ε to u_{tt}^0 and prove (7.19).

In fact from (7.30) we obtain a subsequence that

$$u_{tt}^\varepsilon \to v^* \quad \text{weakly in } L^p((0,T), H^{-1}(\Omega)),$$

with $1 < p < 2$ as in (7.14). Analogously to (7.33) we have $v^* = u_{tt}^0$,

Step 6. Now we prove (7.22). For this observe that from the weak convergence of $(u^\varepsilon, u_t^\varepsilon) \to (u^0, u_t^0)$ in $L^2((0,T), E)$ we get

$$\|(u^0, u_t^0)\|^2_{L^2((0,T),E)} \leq \liminf_{\varepsilon \to 0} \|(u^\varepsilon, u_t^\varepsilon)\|^2_{L^2((0,T),E)},$$

which we can write, from (7.3), (7.4) as

$$\int_0^T E_0(u^0, u_t^0) - \int_0^T \int_{\mathcal{M}} |\gamma(u^0)|^2 \leq \liminf_{\varepsilon \to 0} \Big[\int_0^T E_0^\varepsilon(u^\varepsilon, u_t^\varepsilon) - \frac{1}{2\varepsilon} \int_0^T \int_{\omega_\varepsilon} |u_\varepsilon|^2 \Big].$$

Now, using account that (7.21) we conclude (7.22). $\square$

Now we identify the limit function above as a weak solution of (7.2). First we obtain the following.

Proposition 12. *The limit function $u^0(t)$ in Theorem 7.3 satisfies*

$$\frac{d}{dt}\Big(\int_\Omega u_t^0\phi + \int_{\mathcal{M}} \gamma(u^0)\,\phi\Big) + \int_{\mathcal{M}} \gamma(u^0)\,\phi + \int_\Omega \nabla u^0 \nabla\phi + \lambda\int_\Omega u^0\phi = \int_\Omega f\phi + \int_{\mathcal{M}} g\phi \tag{7.34}$$

for every $\phi \in H_0^1(\Omega)$. In particular it is a weak solution of (7.2).

Proof. First, we note from (7.17), (7.18), (7.19) and (7.20) we have $\gamma(u^0) \in L^2(\mathcal{M})$ and

$$u^0 \in L^\infty((0,T), H^1(\Omega)) \cap C([0,T], L^2(\Omega)), \quad \gamma(u^0) \in H^1([0,T], H^{-1}(\Omega))$$

$$u_t^0 \in L^\infty((0,T), L^2(\Omega)) \cap C([0,T], H^{-1}(\Omega)) \text{ and } u_{tt}^0 \in L^p((0,T), H^{-1}(\Omega)).$$

Now since $u^\varepsilon(t)$ is a mild solution of (7.5) as in Proposition 10, then from (7.31) if $\varphi(t,x)$ is a smooth function such that $\varphi \in L^{p'}((0,T), H_0^1(\Omega))$, we have

$$\int_0^T < u_{tt}^\varepsilon, \varphi >_{-1,1} + \int_0^T \int_\Omega \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon\varphi + \int_0^T \int_\Omega \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_t^\varepsilon\varphi + \int_0^T \int_\Omega \nabla u^\varepsilon \nabla\varphi$$
$$+ \lambda\int_0^T \int_\Omega u^\varepsilon\varphi = \int_0^T \int_\Omega f_\varepsilon\varphi + \int_0^T \int_\Omega \frac{1}{2\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon\varphi. \tag{7.35}$$

Then passing to the limit as $\varepsilon \to 0$ in (7.35), from (7.19), (7.20) and (7.17), together with (7.14) and (7.16) respectively, we get

$$\int_0^T \Big\langle u_{tt}^0, \varphi\Big\rangle_{-1,1} + \int_0^T \int_{\mathcal{M}} \gamma(u^0)\,\varphi + \int_0^T \int_{\mathcal{M}} \gamma(u^0)_t\,\varphi$$
$$+ \int_0^T \int_\Omega \nabla u^0 \nabla\varphi + \lambda\int_0^T \int_\Omega u^0\varphi = \int_0^T \int_\Omega f\varphi + \int_0^T \int_{\mathcal{M}} g\varphi. \tag{7.36}$$

Now, if we consider $\varphi(t,x) = \xi(t)\phi(x) \in L^{p'}((0,T), H_0^1(\Omega))$ with $\phi \in H_0^1(\Omega)$ and $\xi \in L^{p'}(0,T)$ in (7.36), then we get

$$\int_0^T \xi(t)\Big\langle u_{tt}^0, \phi\Big\rangle_{-1,1} + \int_0^T \xi(t)\int_{\mathcal{M}} (\gamma(u^0) + \gamma(u^0)_t)\,\phi + \int_0^T \xi(t)\int_\Omega \nabla u^0\nabla\phi +$$
$$+\lambda\int_0^T \xi(t)\int_\Omega u^0\phi = \int_0^T \xi(t)\int_\Omega f\phi + \int_0^T \xi(t)\int_{\mathcal{M}} g\phi.$$

for every $\xi(t) \in L^{p'}(0,T)$ and we obtain that

$$\Big\langle u_{tt}^0, \phi\Big\rangle_{-1,1} + \int_{\mathcal{M}} (\gamma(u^0) + \gamma(u^0)_t)\,\phi + \int_\Omega \nabla u^0\nabla\phi + \lambda\int_\Omega u^0\phi = \int_\Omega f\phi + \int_{\mathcal{M}} g\phi.$$

a.e. t $\in [0,T]$ and for every $\phi \in H_0^1(\Omega)$. Thus we get (7.34).

Finally observe that if we take a test function $\phi \in \mathcal{D}(\Omega)$ in (7.2), integrate by parts in both sides of $\mathcal{M}$ and use Lemma 7.4 then we get (7.34) as well. □

Now we prove the result used above.

Lemma 7.4. *For $u \in H^2(\Omega \setminus \mathcal{M}) \cap H_0^1(\Omega)$ and $\phi \in H_0^1(\Omega)$, we have*

$$\int_\Omega (-\Delta u)\phi = \int_\Omega \nabla u \nabla\phi - \int_{\mathcal{M}} \Big[\frac{\partial u}{\partial \vec{n}}\Big]\phi.$$

Proof. Notice that for u, ϕ as above

$$\int_{\Omega_1} (-\Delta u)\phi = \int_{\Omega_1} \nabla u \nabla \phi - \int_{\mathcal{M}} \frac{\partial u}{\partial \vec{n}} \phi$$

since $\vec{n}$ is the outer normal, while

$$\int_{\Omega_2} (-\Delta u)\phi = \int_{\Omega_2} \nabla u \nabla \phi - \int_{\Gamma} \frac{\partial u}{\partial \vec{n}} \phi + \int_{\mathcal{M}} \frac{\partial u}{\partial \vec{n}} \phi = \int_{\Omega_2} \nabla u \nabla \phi + \int_{\mathcal{M}} \frac{\partial u}{\partial \vec{n}} \phi.$$

So, adding up, we get the result. □

7.3. **Analysis of the limit problem.** In this section we prove that the limit problem (7.2) is well posed. Indeed from Proposition 12 we are going to study the problem

$$\left\{ \begin{array}{ll} \left(u_t^0 + \gamma(u^0)\right)_t + \gamma(u^0) + Lu^0 = h = f + g_{\mathcal{M}} & \text{in } \Omega \times (0,T) \\ u^0 = 0 & \text{on } \Gamma \times (0,T) \\ u^0(0,x) = u_0(x) u_t^0(0,x) = v_0(x) & \text{in } \Omega \end{array} \right. \tag{7.37}$$

where $f \in L^2(\Omega)$, $g \in L^2(\mathcal{M})$ and $h = f_\Omega + g_{\mathcal{M}} \in H^{-1}(\Omega)$ in the sense that

$$\Big\langle h, \phi \Big\rangle_{-1,1} = \int_\Omega f\phi + \int_{\mathcal{M}} g\phi, \quad \phi \in H_0^1(\Omega).$$

Observe that when $h = 0$ problem (7.37) can be formally written as

$$U_t + AU = 0, \quad U(0) = U_0 = (u_0, v_0)^\perp$$

where $U = (u, u_t)^\perp \in E$

$$A = \left(\begin{array}{cc} 0 & -I \\ L + \gamma & \gamma \end{array} \right)$$

with

$$D(A) = \{U = (u, v)^\perp \in E,\ v \in H_0^1(\Omega),\ L(u) + \gamma(u) + \gamma(v) \in L^2(\Omega)\}$$

where L is defined as in (6.6).

Proposition 13. *The operator $-A$ generates a C_0 semigroup in $E = H^1(\Omega) \times L^2(\Omega)$, denoted $S(t)$, which is a semigroup of contractions for the norm E_0 in E given in (7.4).*

Therefore, if $U_0 = (u_0, v_0)^\perp \in E$, then the initial and boundary value problem (7.37), with $h = 0$, admits a unique **mild solution***, $S(t)U_0 = (u, u_t)^\perp \in C([0,\infty); E)$.*

Moreover, $U_0 = (u_0, v_0)^\perp \in D(A)$, the initial and boundary value problem (7.37) with $h = 0$, admits a unique **strict solution***, $S(t)U_0 = (u, u_t)^\perp \in C([0,\infty); D(A))$.*

Proof. First, we prove $D(A)$ is dense in E since $D(A)$ contains

$$V = \Big\{(u, v) \in (H^2(\Omega \setminus \mathcal{M}) \cap H_0^1(\Omega)) \times H_0^1(\Omega),\ \gamma(v) = -\Big[\frac{\partial u}{\partial \vec{n}}\Big] - \gamma(u)\Big\}.$$

For this, observe that if $(u, v) \in V$ and we denote $D(u, v) = L(u) + \gamma(u) + \gamma(v) \in H^{-1}(\Omega)$, then Lemma 7.4 implies that for $\phi \in H_0^1(\Omega)$ we have

$$< D(u, v), \phi >= \int_\Omega (-\Delta u + \lambda u)\phi$$

hence $D(u, v) \in L^2(\Omega)$ and then $V \subset D(A)$.

Now we prove V is dense in E. For this we use the scalar product associated to the norm E_0 in (7.4). Hence, if (ϕ,ψ) is orthogonal to V we have, for all $(u,v)\in V$,

$$0=\int_\Omega \nabla u\nabla\phi+\lambda\int_\Omega u\phi+\int_{\mathcal{M}}\gamma(u)\gamma(\phi)+\int_\Omega v\psi.$$

Then Lemma 7.4 implies

$$0=\int_\Omega(-\Delta u+\lambda u)\phi-\int_{\mathcal{M}}\gamma(v)\gamma(\phi)+\int_\Omega v\psi$$

for all $(u,v)\in V$. If, in particular we take $f\in L^2(\Omega)$ and $u\in H^2(\Omega)\cap H_0^1(\Omega)$ the solution of $-\Delta u+\lambda u=f$, then $(u,0)\in V$ and we get $\int_\Omega f\phi=0$ and hence $\phi=0$. This in turn, implies $\int_\Omega v\psi=0$ for all $(u,v)\in V$, which easily implies $\psi=0$.

Now we show that A is dissipative for the norm E_0 in (7.4). In fact, for $U=(u,v)^\perp\in D(A)$, using the associated scalar product, we get

$$A(U)\cdot U=\Big(-v,(L+\gamma)u+\gamma(v)\Big)\cdot(u,v)=\int_{\mathcal{M}}\gamma(v)^2\geq 0,$$

To finish the proof, it remains to show that $R(I+A)=E$. In effect, let $(j,k)\in E$ and consider the equation $(I+A)((u,v)^\perp)=(j,k)^\perp$ which can be written as

$$u-v=j \text{ and } v+Lu+\gamma(u)+\gamma(v)=k. \tag{7.38}$$

Hence $u=v+j$ and

$$v+L(v)+\gamma(v)=k-(L+\gamma)j\in H^{-1}(\Omega). \tag{7.39}$$

Taking into account that the properties of operator L and γ we get the operator $I+L+\gamma$ is onto from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ and the equation (7.39) has a unique solutions $v\in H_0^1(\Omega)$. Thus, with $u=v+j\in H_0^1(\Omega)$ from (7.38) we get $L(u)+\gamma(u)+\gamma(v)=k-v\in L^2(\Omega)$ so $(u,v)^\perp\in D(A)$. □

For the nonhomogeneous problem, i.e. $g\neq 0$, we proceed by transposition as in Section 6.2. For this first note that $-A^*$ generates in E' the C_0 semigroup $S^*(t)$, that is, the dual semigroup of $S(t)$. Now if we consider $U^*=(u,w)\in E'=L^2(\Omega)\times H^{-1}(\Omega)$ a solution in E' of

$$U_t^*+A^*U^*=H(t), \tag{7.40}$$

where $H(t)=(0,h(t))^\perp$ with $h(t):=f_\Omega(t)+g_{\mathcal{M}}(t)$ and A^* is given by $A^*=\begin{pmatrix}\gamma & -I\\ L+\gamma & 0\end{pmatrix}$ with

$$D(A^*)=\{(u,w)^\perp\in H^1(\Omega)\times H^{-1}(\Omega),\ \gamma(u)-w\in L^2(\Omega)\},$$

then $w=u_t+\gamma(u)\quad$ in $L^2(\Omega)$ and

$$(u_t+\gamma(u))_t+L(u)+\gamma(u)=h=f_\Omega+g_{\mathcal{M}}\quad \text{in } H^{-1}(\Omega).$$

Hence the mild solutions of the limit problem (7.37) are given by the strict solutions in E' of the dual equation.

Then we have the following result.

Theorem 7.5. *Let, $f\in L^1((0,T),L^2(\Omega))$, $g\in L^2((0,T),L^2(\mathcal{M}))$ and $U_0=(u_0,v_0)\in E=H^1(\Omega)\times L^2(\Omega)$. Let $U^*(t)=(u,w)^\perp$ be the mild solution of the dual equation (7.40) in $E'=L^2(\Omega)\times H^{-1}(\Omega)$*

$$U^*(t)=S^*(t)U_0^*+\int_0^t S^*(t-s)H(s)\,ds\quad 0<t<T$$

where $U_0^* = (u_0, w_0)^\perp \in E'$,*i.e.* $u_0 \in L^2(\Omega)$ *and* $w_0 = v_0 + \gamma(u_0) \in H^{-1}(\Omega)$, $H(t) = (0, h(t))^\perp$ *and* $h := f_\Omega + g_\mathcal{M} \in L^1((0,T), H^{-1}(\Omega))$.

Then $U^* = (u, w)^\perp \in C([0,T], E')$, $w = u_t + \gamma(u)$, *and* $U(t) = (u, u_t)^\perp$ *satisfies*
i) (Regularity). $U = (u, u_t)^\perp \in C([0,T], E)$,

$$\gamma(u) \in C([0,T], H^{\frac{1}{2}}(\mathcal{M})) \cap H^1((0,T), L^2(\mathcal{M})) \cap L^\infty((0,T), L^2(\mathcal{M}))$$

and $u_{tt} \in L^1((0,T), H^{-1}(\Omega))$.
ii) (Energy equality). U *satisfies the energy equality*

$$E_0(u(\tau), u_t(\tau)) + 2\int_0^\tau \int_\mathcal{M} \gamma(u)_t^2 = E_0(u_0, v_0) + 2\int_0^\tau \int_\mathcal{M} g\gamma(u)_t + 2\int_0^\tau \int_\Omega f u_t$$

for $0 < \tau < T$, *where* E_0 *is given in (7.4).*
iii) (The equation). The function $u(t)$ *satisfies the equation*

$$(u_t + \gamma(u))_t + L(u) + \gamma(u) = h = f_\Omega + g_\mathcal{M}$$

a.e. $[0,T]$, *as an equality in* $H^{-1}(\Omega)$.

8. **Some further research.** The problems presented here suggest some further research along the following lines. First, rates of convergence of solutions have been obtained for the resolvent estimates and the linear semigroups in Sections 2 and 3. It seems plausible to use some general techniques to derive rates of convergence for nonlinear problems and their attractors, see e.g. [13]. Also lower semicontinuity of attractors of parabolic problems, under generic hyperbolicity conditions of equilibria seem within reach; see [31] for an approach that has been successfully applied in different instances.

For problems in Sections 5, 6 and 7 it is interesting to study the spectral continuity (or stability) as $\varepsilon \to 0$. Observe that in the case of Section 5, the limit eigenvalues are given by the Steklov eigenvalue problem, see [62, 42]. In case of Section 6 a related important problem is that of the uniform stabilization of waves by the localized/boundary damping, see [32] and references therein. For Section 7 stronger convergence of solutions needs to be explored. Also, the convergence for the associated nonlinear problems seems relevant. Finally, problems in which also the second order time derivative concentrates may deserve some attention.

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