Dynamical Systems, Differential
Equations and Applications
AIMS Proceedings, 2015

doi:10.3934/proc.2015.0670

pp. **670–677**

LINEAR MODEL OF TRAFFIC FLOW IN AN ISOLATED NETWORK

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Abstract. We obtain a mathematical linear model which describes automatic operation of the traffic of material objects in a network. Existence and global solutions is obtained for such model. A related model which used outdated information is shown to collapse in finite time.

1. **Introduction.** In this paper we present a linear model of delay integral equations that describes the motion of physical objects on a spatial network.

To write done the equations describing this situation, the spatial network is assimilated to a graph, which is made of a set of "nodes" connected through oriented, non intersecting traffic paths, or oriented "edges", along which "objects" travel between nodes. Some objects could be "stored" in some node for some time before traveling to another node. The motion of objects along the network is denoted "traffic". We assume the network is isolated in the sense that any traffic originated at one node is arriving some other node of the network after some suitable travel time. Also, any incoming traffic into a node has origin is another node of the network.

We analyze a linear model describing an automatic operation of the traffic in the network in the sense that there are certain automatic rules that determine the traffic between nodes, from information of the number on objects stored at each node, see (1).

From these rules the corresponding initial value problem for the traffic in the network is obtained, see (7). Then we show that the initial value problem has a unique solution, defined for all times; see Theorem 3.

Later, we give a related model based on similar automatic rules, but which use some outdated information on the number of objects stored at each node, see (17). We show then that such rules may cause the model to collapse in finite time in the sense that its unique solution, although defined for all times, may become negative in finite time, see Section 4.

Similar models in discrete time for air traffic can be found in [5], [6] and references therein. Also, some related nonlinear problems can be found in [3], [4].

Note that our formulation of the model, see (6) or (7), is in terms of some delay integral systems, which relate to delay differential equations as in [1, 2].

2010 *Mathematics Subject Classification.* Primary: 34K06, 34A33; Secondary: 34A12, 34A30.

Key words and phrases. Networks, traffic flow, delay integral systems.

Partially suppported by grant MTM2012-31298 from Ministerio de Economia y Competitividad, Spain, GR58/08 Grupo 920894 BSCH-UCM, Grupo de Investigación CADEDIF and by Project FIS2009-12964-C05-03, SPAIN.

2. **Formulation of the equations of traffic in a network.** We consider a network and some objects which travel between nodes or could be stored in some node of the network. In order to obtain the system equations which governs the traffic of material objects in the network, we use the followings definitions and notations.

2.1. **Definitions and notations.** We assume the nodes in the graph are labeled by the indexes $i = 1, 2, \dots, N$.

For $i \neq j$, we will write $i \rightsquigarrow j$ if and only if the node i is connected to the node j and $i \not\rightsquigarrow j$ otherwise.

We denote by $N(i)$ the set of (indexes of) nodes to which the node i is connected to. That is, the set of j such that $i \rightsquigarrow j$.

Analogously, we denote $M(i)$ the set of (indexes of) nodes which connect to the node i. That is the set of j such that $j \rightsquigarrow i$.

Also we denote by $N_0 \leq N(N-1)$ the number of edges in the graph associated to network.

In order to determine the number of material objects in each node or edge of the network at each time, we will consider the following functions of time.

Definition 1. For $i, j = 1, 2, \dots, N$
i) If $i \rightsquigarrow j$, at a given time t, we denote by $f_{ij}(t) \geq 0$ the number of material objects that travel through the edge (i, j), from node i to j at time t.

If $i \not\rightsquigarrow j$, the the node i is not connected to the node j and since there is no traffic between the nodes, we agree in defining $f_{ij} = 0$.
ii) At a given time t, $x_i(t) = f_{ii}(t) \geq 0$, is the number of material objects that are stored in the node i at time t.
iii) If $i \rightsquigarrow j$, we denote by $\tau_{ij} > 0$ the travel time of a material object from node i to node j. We will assume τ_{ij} is known and constant for all the traffic between nodes i and j.

If $i \not\rightsquigarrow j$, we will agree in defining $\tau_{ij} = 0$.

We will also use the following

Definition 2. For $i, j = 1, 2, \dots, N$
i) If $i \rightsquigarrow j$ we denote by $T_{ij}(t) \geq 0$ the "departure rate" (or take–offs rate), that is, the number of material objects that depart from node i to node j, per unit time, at time t.

If $i \not\rightsquigarrow j$ we agree in defining that $T_{ij}(t) = 0$. We also agree that $T_{ii}(t) = 0$.

Also, $T_i(t) \geq 0$ represents the total rate of departures from node i, at time t.
ii) If $i \rightsquigarrow j$ we denote by $L_{ij}(t) \geq 0$ the "arrival rate" (or landings rate), that is, the number of material objects that arrive to node j from node i, per unit time, at time t.

If $i \not\rightsquigarrow j$ we agree that $L_{ij}(t) = 0$ and also we agree that $L_{ii}(t) = 0$.

Also, $L_i(t) \geq 0$ represents the total rate of arrivals to node i, at time t.

2.2. **System of linear equations for the traffic flow.** To obtain the equations that govern the traffic in a network, we assume that the take-offs from each node depend only in the number of objects in that node. This is, the take–off rate from node i would be a fraction of the number of objects in the node, which leads to

$$T_{ij}(t) = k_{ij} x_i(t), \quad \forall i \rightsquigarrow j, \tag{1}$$

where $x_i(t) = f_{ii}(t)$ is the number of objects in the node i and

$$0 \leq k_{ij} \quad \text{and} \quad 0 \leq \sum_{j \in N(i)} k_{ij}, \tag{2}$$

and $k_{ij} = 0$ if $i \not\rightsquigarrow j$ or $k_{ii} = 0$ (hence $T_{ij} = 0$, $T_{ii} = 0$ respectively).

Next, taking into account that the network is isolated, if $i \rightsquigarrow j$, then objects arriving to node j from node i, have previously departed that node, τ_{ij} units of time before. Thus,

$$L_{ij}(t) = T_{ij}(t-\tau_{ij}) = k_{ij}x_i(t-\tau_{ij}). \tag{3}$$

As a consequence, the number of objects in that edge satisfies for some time $t \geq s$,

$$f_{ij}(t) = f_{ij}(s) + \int_s^t T_{ij}(r)dr - \int_s^t L_{ij}(r)dr = f_{ij}(s) + k_{ij}\int_s^t x_i(r)dr - k_{ij}\int_s^t x_i(r-\tau_{ij})dr,$$

and using $\int_s^t x_i(r-\tau_{ij})dr = \int_{s-\tau_{ij}}^{t-\tau_{ij}} x_i(r)dr$, we get

$$f_{ij}(t) = f_{ij}(s) + k_{ij}\int_{t-\tau_{ij}}^t x_i(r)dr - k_{ij}\int_{s-\tau_{ij}}^s x_i(r)dr.$$

In particular, for any $t \geq s$ the number of objects in the edge between nodes i and j satisfies

$$f_{ij}(t) - k_{ij}\int_{t-\tau_{ij}}^t x_i(r)dr = f_{ij}(s) - k_{ij}\int_{s-\tau_{ij}}^s x_i(r)dr.$$

Moreover, $f_{ij}(t)$ is given by the departures from node i to node j, which have not yet arrived to destiny. Hence, using that the travel time is τ_{ij}, for every $i \rightsquigarrow j$ we get

$$f_{ij}(t) = k_{ij}\int_{t-\tau_{ij}}^t x_i(r)dr, \text{ for all } t \qquad i \rightsquigarrow j, \tag{4}$$

is completely determined by $x_i(t)$ in $[t-\tau_{ij}, t]$.

On the other hand, for any $i = 1, 2, .., N$ and for any $t \geq s$ the number of objects stored in node i satisfies

$$x_i(t) = x_i(s) + \int_s^t L_i(r)dr - \int_s^t T_i(r)dr. \tag{5}$$

Using that $M(i)$ is the set of nodes connected to node i, we have

$$L_i(t) = \sum_{j\in M(i)} L_{ji}(t) = \sum_{j\in M(i)} T_{ji}(t-\tau_{ji}) \geq 0$$

while

$$T_i(t) = \sum_{j\in N(i)} T_{ij}(t) \geq 0.$$

Hence, in (5) we have for all $t \geq s$,

$$x_i(t) = x_i(s) + \sum_{j\in M(i)} \int_s^t L_{ji}(r) - \sum_{j\in N(i)} \int_s^t T_{ij}(r)dr$$

and $\int_s^t L_{ji}(r) = \int_s^t T_{ji}(r-\tau_{ji}) = \int_{s-\tau_{ji}}^{t-\tau_{ji}} T_{ji}(r) = k_{ji}\int_{s-\tau_{ji}}^{t-\tau_{ji}} x_j(r)$. Therefore, for $t \geq s$ we get x_i satisfies

$$x_i(t) = x_i(s) + \sum_{j\in M(i)} k_{ji}\int_{s-\tau_{ji}}^{t-\tau_{ji}} x_j(r) - \sum_{j\in N(i)} k_{ij}\int_s^t x_i(r)dr; \qquad \text{for all } t \geq s. \tag{6}$$

Therefore the traffic in the network becomes determined by the node–traffic function x

$$t \mapsto x(t) \in (I\!R^N)^+ \qquad \text{where } x(t) = (x_i(t))_i, i \in \{1, 2, ..., N\}$$

which satisfies equations (6) and that determine the traffic in the edges of the network by (4) and $f_{ij}(t) = 0$ if $i \not\rightsquigarrow j$.

3. **The initial value problem for traffic flow.** Let a initial time $t_0 \in I\!R^+$, and an initial nonnegative vector $\theta^0 \in (I\!R^+)^N$ which describes the initial amount of objetcs in each node of the network. Then from (6) the node-traffic for $t \geq t_0$ with $x_i(t_0) = \theta_i^0$, is given by

$$x_i(t) = \theta_i^0 + \sum_{j \in M(i)} k_{ji} \int_{t_0-\tau_{ji}}^{t-\tau_{ji}} x_j(r) - \sum_{j \in N(i)} k_{ij} \int_{t_0}^{t} x_i(r)dr; \quad \text{for all } t \geq t_0. \tag{7}$$

First, we note that the part of x_i in $[t_0 - \tau_{ij}, t_0]$ is not determined by (7) and in fact it has to be given as the initial value for the function $x_i(t)$.

Now, consider

$$\tau_* = \max\{\tau_{ij},\ i, j = 1, \ldots, N\}, \tag{8}$$

the maximal travel time in the network. Therefore given an initial function $y \in L^\infty([t_0 - \tau_*, t_0], (I\!R^+)^N)$ and a vector $\theta^0 \in (I\!R^+)^N$, we are lead to find a function $x(t)$ in $[t_0 - \tau_*, T]$ for some $t_0 \leq T \leq \infty$, such that $x = y$ in $[t_0 - \tau_*, t_0]$, $x(t_0) = \theta^0$ and x satisfies (7), for $t \geq t_0$.

Observe that if $x(t)$ is continuous for $t \geq t_0 + \tau_*$, then it is actually differentiable and satisfies the delay differential system

$$x_i'(t) = \sum_{j \in M(i)} k_{ji}\, x_j(t - \tau_{ji}) - (\sum_{j \in N(i)} k_{ij}) x_i(t), \quad t > t_0 + \tau_*. \tag{9}$$

We thus have

Theorem 3. *With the assumptions above, assume given $\theta^0 \in (I\!R^+)^N$ and a function $y \in L^\infty([t_0 - \tau_*, t_0], (I\!R^N)^+)$.*

Then there exists a unique function $x \in L^\infty([t_0 - \tau_*, t_0], (I\!R^N)^+) \cap \mathcal{C}([t_0, \infty), (I\!R^N)^+)$ such that $x = y$ in $[t_0 - \tau_*, t_0]$, $x(t_0) = \theta^0$ and x satisfies (7), for $t \geq t_0$.

Assume moreover that $y \in \mathcal{C}([t_0 - \tau_*, t_0], (I\!R^N)^+)$. Then x is continuous at $t = t_0$ if and only if $y(t_0) = \theta^0$ and it has a finite jump at $t = t_0$ otherwise.

Proof. We observe the initial value problem (7) can be written as a functional relationship

$$x(t) = \mathcal{F}(x, t_0, \theta^0)(t), \quad t \geq t_0. \tag{10}$$

Moreover from (6) we also have

$$x(t) = \mathcal{F}(x, s, \theta)(t), \quad t \geq s \geq t_0, \quad \theta = x(s). \tag{11}$$

We note that $\mathcal{F}(x, s, \theta)$ (the right hand side of (11) and (6)) is a linear operator in (x, θ), that satisfies, for all $s \in I\!R$,

$$\mathcal{F}(\cdot, s, \cdot) : L^\infty_{loc}([s - \tau_*, \infty), I\!R^N) \times I\!R^N \longrightarrow \mathcal{C}\left([s, \infty), I\!R^N\right)$$

and even more, for every $\tau > 0$

$$\mathcal{F}(\cdot, s, \cdot) : L^\infty([s - \tau_*, s + \tau], I\!R^N) \times I\!R^N \longrightarrow \mathcal{C}\left([s, s + \tau], I\!R^N\right) \tag{12}$$

and is continuous. In particular, for $t \geq t_0$,

$$\mathcal{F}(\cdot, t_0, \cdot) : L^\infty([t_0 - \tau_*, t]), I\!R^N) \times I\!R^N \longrightarrow \mathcal{C}\left([t_0, t], I\!R^N\right). \tag{13}$$

Step 1. We prove from (10), there exists a unique solution x given by $x = \mathcal{F}(x, t_0, \theta^0)(t)$ for $t_0 \leq t \leq t_0 + \tau$ with suitable small $\tau > 0$, such that $x = y$ in $[t_0 - \tau_*, t_0]$.

First, we identify $X = L^\infty\left([t_0 - \tau_*, t_0 + \tau], I\!R^N\right)$ with

$$Y \times Z = L^\infty\left([t_0 - \tau_*, t_0]\,, I\!R^N\right) \times L^\infty\left([t_0, t_0 + \tau]\,, I\!R^N\right)$$

and define the operator H by

$$\begin{array}{rccl} H: & Z & \rightarrow & Z \\ & z & \mapsto & H(z) = \mathcal{F}((y, z), t_0, \theta^0). \end{array}$$

Thus the solution is $x=(y,z)\in L^{\infty}\left([t_0-\tau_*,t_0+\tau],\mathbb{R}^N\right)$ where $z=H(z)$, $z\in Z$, i.e. the solution is given by the point fix of the operator H.

From the properties of $\mathcal{F}$ we have that H is Lipschitz. Below we show that the Lipschitz constant of H is bounded by $\tau N_0 K_0$ where $K_0=\max\{k_{ij}\}_{ij}$.

For this, note that if $z^1,z^2\in Z=L^{\infty}([t_0,t_0+\tau],\mathbb{R}^N)$ we have that $x^1=(y,z^1)$, $x^2=(y,z^2)\in L^{\infty}\left([t_0-\tau_*,t_0+\tau],\mathbb{R}^N\right)$ coincide in $[t_0-\tau_*,t_0]$.

Hence ,we obtain for every $i\in\{1,2,..,N\}$

$$\begin{aligned}|\mathcal{F}_i((y,z^1),\theta^0)(t)-\mathcal{F}_i((y,z^2),\theta^0)(t)| \quad\leq\quad & \sum_{j\in M(i)} k_{ji}\int_{t_0-\tau_{ji}}^{t-\tau_{ji}}|x_j^1(r)-x_j^2(r)|dr \\ & +\sum_{j\in N(i)} k_{ij}\int_{t_0}^{t}|x_i^1(r)-x_i^2(r)|dr\end{aligned}$$

thus using that $x^1=x^2$ in $[t_0-\tau_*,t_0]$, we get

$$\|\mathcal{F}_i((y,z^1),\theta^0)-\mathcal{F}_i((y,z^2),\theta^0)\|_{L^{\infty}(t_0,t_0+\tau)}\leq N_0K_0\tau\|(y,z^1)-(y,z^2)\|_X \tag{14}$$

where $N_0\leq N(N-1)$ is the number or edges in the network. Therefore

$$\|H(z_1)-H(z_2)\|_Z=\|\mathcal{F}((y,z^1),\theta^0)-\mathcal{F}((y,z^2),\theta^0)\|_Z\leq N_0K_0\tau\|(y,z^1)-(y,z^2)\|_Y$$

since $(y,z^1)=(y,z^2)$ in $[t_0-\tau_*,t_0]$ we have $\|(y,z^1)-(y,z^2)\|_X\leq\|z^1-z^2\|_Z$ and we get $\|H(z_1)-H(z_2)\|_Z\leq N_0K_0\tau\|z^1-z^2\|_Z$.

Therefore, for suitable small τ such that $\tau N_0K_0<1$, there exists a unique solution of $z=H(z)$, with $z\in Z=L^{\infty}([t_0,t_0+\tau],\mathbb{R}^N)$ such that $x=(y,z)\in L^{\infty}([t_0-\tau_*,t_0+\tau],\mathbb{R}^N)$ satisfies (10) and thus is the unique solution of (7) in $[t_0,t_0+\tau]$. Moreover, from (12) we have $z\in\mathcal{C}([t_0,t_0+\tau],\mathbb{R}^N)$ and $z(t_0)=\theta^0$.

Hence, if $y\in\mathcal{C}([t_0-\tau_*,t_0],(\mathbb{R}^N)^+)$ we have that x is continuous in $[t_0-\tau_*,t_0+\tau]$ iff $z(t_0)=\theta^0$ or with a finite jump at t_0 otherwise.

Step 2. Observe that the existence time in Step 1 above, τ, is independent of t_0,θ^0 and y. Hence, we denote by x^1 the solution in $[t_0-\tau_*,t_0+\tau]$ given by step 1. We take now new initial time and data $\tilde{t}_0=t_0+\tau$, $\tilde{y}=x^1_{|[\tilde{t}_0-\tau_*,\tilde{t}_0]}$, $\tilde{\theta}^0=x^1(\tilde{t}_0)$. From (12), we have

$$\mathcal{F}:L^{\infty}([\tilde{t}_0-\tau_*,\tilde{t}_0+\tau]),\mathbb{R}^N)\times\mathbb{R}^N\to\mathcal{C}\left([\tilde{t}_0,t_0+\tau],\mathbb{R}^N\right)$$

and from the Step 1 we have a solution x^2 of (7) in $[\tilde{t}_0,\tilde{t}_0+\tau]$ with these initial data that satisfies

$$x^2(t)=\mathcal{F}((\tilde{y},x^2),\tilde{t}_0,\tilde{\theta}^0)(t),\quad \tilde{t}_0\leq t\leq\tilde{t}_0+\tau$$

with $x^2=\tilde{y}=x^1$ in $[\tilde{t}_0-\tau_*,\tilde{t}_0]$.

Since by construction, we have

$$x^2(\tilde{t}_0)=\tilde{\theta}^0=x^1(\tilde{t}_0)$$

we conclude that the function in $[t_0,t_0+2\tau]$ given by

$$x(t)=\begin{cases} x^1(t), & t_0\leq t\leq t_0+\tau\\ x^2(t), & t_0+\tau\leq t\leq t_0+2\tau\end{cases}$$

is continuous in $[t_0,t_0+2\tau]$.

Thus the traffic function

$$\widetilde{x}(t)=\begin{cases} y(t), & t_0-\tau_*\leq t<t_0\\ x^1(t), & t_0\leq t\leq t_0+\tau\\ x^2(t), & t_0+\tau\leq t\leq t_0+2\tau\end{cases}$$

satisfies

$$\widetilde{x}(t)=\mathcal{F}((y,x),t_0,\theta^0)(t),\quad t_0\leq t\leq t_0+2\tau$$

and it is therefore a solution of (7) in $[t_0,t_0+2\tau]$ and is continuous in $[t_0,t_0+2\tau]$.

Repeating this process we get that the solution of (7) is globally defined.

Step 3. We finally show that the solution is nonnegative, this is $x_i(t)$ is nonnegative for every $t \geq t_0$ and for every $i \in \{1, 2, .., N\}$.

For this notice that (7) can be written in the form

$$x_i(t) = x_i(t_0) + \int_{t_0}^{t} g_i(r)dr - \alpha_i \int_{t_0}^{t} x_i(r)dr, \quad t \geq t_0. \tag{15}$$

From this, first assuming g_i is smooth and then by density we get that

$$x_i(t) = \theta_i^0 e^{-\alpha_i(t-t_0)} + \int_{t_0}^{t} e^{-\alpha_i(t-s)} g_i(s)ds \quad t_0 \leq t \leq t_0 + \tau, \tag{16}$$

for any $\tau > 0$ with $\alpha_i = \sum_{j \in N(i)} k_{ij}$ and $g_i(t) = \sum_{j \in M(i)} k_{ji} x_j(t - \tau_{ji})$.

We take now the minimal travel time on any edge of the network, that is, $0 < \tau_{min} = \min_{i,j \in N(i)}\{\tau_{ij}\} \leq \tau_*$ and observe that if $j \rightsquigarrow i$, for $t_0 \leq t \leq t_0 + \tau_{min}$ we have $t_0 - \tau_* \leq t - \tau_{ji} \leq t_0$ and $x_j(t - \tau_{ji}) = y_j(t - \tau_{ji}) \geq 0$, then $g_i(t) = \sum_{j \in M(i)} k_{ji} y_j(t - \tau_{ji}) \geq 0$ since the initial data y is nonnegative.

Therefore in (16), since initial condition $\theta_i^0 \geq 0$, we get

$$x_i(t) \geq 0, \quad t_0 \leq t \leq t_0 + \tau_{min}$$

for all i.

Now, for $t_0 + \tau_{min} \leq t \leq t_0 + 2\tau_{min}$, similarly to (15), (7) can be written as

$$x_i(t) = x_i(\tilde{t}_0) + \int_{\tilde{t}_0}^{t} g_i(r)\, dr - \alpha_i \int_{\tilde{t}_0}^{t} x_i(r)\, dr, \quad \tilde{t}_0 \leq t \leq \tilde{t}_0 + \tau_{min},$$

with $\tilde{t}_0 = t_0 + \tau_{min}$ and we get as in (16)

$$x_i(t) = x_i(\tilde{t}_0) e^{-\alpha_i(t-\tilde{t}_0)} + \int_{\tilde{t}_0}^{t} e^{-\alpha_i(t-s)} g_i(s)ds \quad \tilde{t}_0 \leq t \leq \tilde{t}_0 + \tau_{min},$$

and now for $\tilde{t}_0 \leq t \leq \tilde{t}_0 + \tau_{min}$ we have $\tilde{t}_0 - \tau_* \leq t - \tau_{ji} \leq \tilde{t}_0$ and then $g_i(t) = \sum_{j \in M(i)} k_{ji} \widetilde{x}(t)_j (t - \tau_{ji}) \geq 0$ by the previous step.

Repeating this argument, we get $x_i(t) \geq 0$ for all $t \geq t_0$ and all i. $\square$

4. **Collapse in finite time of a model with out–dated information.** In this section we consider that the take-off rate at time t depends on the number of material objects at certain previous time. That is, the take-offs rate in (1) is replaced by

$$T_{ij}(t) = k_{ij} x_i(t - \tau_{ii}^1), \quad \forall i \rightsquigarrow j, \tag{17}$$

where $\tau_{ii}^1 \geq 0$ are some given numbers, and (2) holds. Note that (17) implies that the traffic put into the network is decided with old, out–dated information, that is disregarding the current value of traffic in the node.

We will show below that in such a case, the model may break in finite time, since solutions may become negative.

Under assumption (17) the node–traffic function satisfies

$$x_i(t) = \theta_i^0 + \sum_{j \in M(i)} k_{ji} \int_{t_0 - \tau_{ji}}^{t - \tau_{ji}} x_j(r - \tau_{jj}^1)dr - \sum_{j \in N(i)} k_{ij} \int_{t_0}^{t} x_i(r - \tau_{ii}^1)dr, \quad t \geq t_0,$$

or equivalenlty,

$$x_i(t) = \theta_i^0 + \sum_{j \in M(i)} k_{ji} \int_{t_0 - \tau_{ji} - \tau_{jj}^1}^{t - \tau_{ji} - \tau_{jj}^1} x_j(r)dr - \sum_{j \in N(i)} k_{ij} \int_{t_0 - \tau_{ii}^1}^{t - \tau_{ii}^1} x_i(r)dr, \quad t \geq t_0. \tag{18}$$

Now, we can use the similar techniques from the Theorem 3 to get a global solution. However, in contrast to Theorem 3 one may not guarantee that solutions of (18) remain nonnegative for all times. To see this, note that in this case we have, instead of (15),

$$x_i(t) = x_i(t_0) + \int_{t_0}^{t} g_i(r)\,dr - \alpha_i \int_{t_0-\tau^1}^{t-\tau^1} x_i(r)\,dr, \quad t \geq t_0, \tag{19}$$

for some $\tau^1 \geq 0$. When $\tau^1 > 0$, note that if we define $I_j = [t_0 + (j-1)\tau^1, t_0 + j\tau^1]$ and x_i^j is x_i restricted to I_j then we get

$$x_i^j(t) = x_i^{j-1}(t_0 + (j-1)\tau^1) + \int_{t_0+(j-1)\tau^1}^{t} g_i(r)dr - \alpha_i \int_{t_0+(j-2)\tau^1}^{t-\tau^1} x_i^{j-1}(r)dr, \quad t \in I_j,$$

$j = 1, 2, \ldots$, which we write as

$$x_i^j(t) = G(x_i^{j-1})(t), \quad t \in I_j.$$

Hence,

$$(x_i^j)'(t) = g_i(t) - \alpha_i x_i^{j-1}(t - \tau^1), \quad t \in I_j,$$

and one can not prevent that for example

$$g_i(t) \leq -k + \alpha_i x_i^{j-1}(t - \tau^1) \quad t \in I_j,$$

for some $k > 0$ and then x_i eventually becomes negative.

5. **Main conclusions.** In this article we have presented a linear integral–delay model to describe the traffic flow of material objects in a directed graph. We are interested in a situation that describes an automatic operation of the traffic in the network. That is, we assume that there are certain automatic rules that determine the traffic between nodes, from information on the number of objects stored at each node.

First we assume that, at each time, the traffic entering each edge of the graph is determined as a fraction of the number of material objects stored at the departing node, see (1). This leads to a system of integral–delay equations for the number of material objects in each node of the graph, see (7). Once this initial value problem is solved, we can compute the number of object in each edge, at any time; see (4). We then show that the system can be solved uniquely and for all times, see Theorem 3.

On the other hand, we show that if decission on the traffic is made with out–dated information, that is, the traffic entering each edge of the graph is determined as a fraction of the number of material objects stored at the departing node in an earlier time, see (17), then the model may break in finite time, since some component of the solution may become negative.

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Received September 2014; revised April 2015.

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